# ON ALGEBRAIC AND MORE GENERAL CATEGORIES WHOSE SPLIT EPIMORPHISMS HAVE UNDERLYING PRODUCT PROJECTIONS

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#### Abstract

We characterize those varieties of universal algebras where every split epimorphism considered as a map of sets is a product projection. In addition we obtain new characterizations of protomodular, unital and subtractive varieties as well as varieties of right  $\Omega$ -loops and biternary systems.

# Introduction

It is well known that in the category of groups if

 $0 \longrightarrow K \xrightarrow{\kappa} A \xrightarrow{\alpha} B \longrightarrow 0$ 

is a short exact sequence, then A and  $K \times B$  are bijective as sets, moreover when  $\alpha$  is split, i.e. for each split extension

$$K \xrightarrow{\kappa} A \xrightarrow{\alpha}_{\beta} B$$
,  $\alpha \beta = 1_B$ ,  $\kappa = \ker(\alpha)$ ,

this bijection becomes a natural bijection  $K \times B \to A$  such that the diagram

$$\begin{array}{c} K \xrightarrow{\langle 1,0 \rangle} K \times B \xrightarrow{\pi_2} B \\ \| & \downarrow^{\varphi} \\ K \xrightarrow{\kappa} A \xrightarrow{\alpha} B \end{array}$$

is a morphism of split extensions in the category **Set**, of sets, that is,  $\alpha \varphi = \pi_2$ ,  $\varphi \langle 0, 1 \rangle = \beta$ , and  $\varphi \langle 1, 0 \rangle = \kappa$ . As shown by E. B. Inyangala, these bijections exists in a more general setting of a variety of right  $\Omega$ -loops (see

[4, 5]), that is, a pointed variety of universal algebras  $\mathcal{V}$  with constant 0 and binary terms x + y and x - y satisfying the identities:

$$x + 0 = x \tag{1}$$

$$x - x = 0 \tag{2}$$

$$(x+y) - y = x \tag{3}$$

$$(x-y) + y = x \tag{4}$$

Moreover, he showed that if a pointed variety  $\mathcal{V}$  with constant 0 has binary terms x + y and x - y and there exist bijections (as above) constructed (in the same way as for groups) using those terms, i.e.  $\varphi(k,b) = \kappa(k) + \beta(b)$ and  $\varphi^{-1}(a) = (\lambda(a), \alpha(a))$ , where  $\lambda$  is the unique map such that  $\kappa\lambda(a) =$  $a - \beta\alpha(a)$ , then  $\mathcal{V}$  is a variety of right  $\Omega$ -loops and in particular the identities (1) - (4) hold for x + y and x - y. In this paper we prove that if for a pointed variety  $\mathcal{V}$  there exist natural bijections as above, then  $\mathcal{V}$  is a variety of right  $\Omega$ -loops (see Theorem 2.1).

For any category  $\mathbb{C}$  let  $\mathbf{Pt}(\mathbb{C})$  to be the category of split epimorphisms in  $\mathbb{C}$ : an object is a quadruple  $(A, B, \alpha, \beta)$  where A and B are objects in  $\mathbb{C}$  and  $\alpha : A \to B$  and  $\beta : B \to A$  are morphisms in  $\mathbb{C}$  with  $\alpha\beta = 1_B$ ; a morphism  $(A, B, \alpha, \beta) \to (A', B', \alpha', \beta')$  is a pair of morphisms  $(f : A \to A', g : B \to B')$  such that in the diagram

$$\begin{array}{c} A \xrightarrow{\alpha} B \\ f \downarrow & \downarrow g \\ A' \xrightarrow{\alpha'} B' \end{array}$$

 $\alpha' f = g\alpha$  and  $f\beta = \beta' g$ . Throughout this paper for any objects A and B we will denote by  $\pi_1$  and  $\pi_2$  the first and second product projections respectively. We will use the same notation for the first and second pullback projections and will write

$$(A_{\langle f,g\rangle} \times B, \pi_1, \pi_2)$$

for the pullback of  $f: A \to C$  and  $g: B \to C$  as in the diagram

$$\begin{array}{c} A \underset{\langle f,g \rangle}{\times} B \xrightarrow{\pi_2} B \\ \pi_1 \\ A \xrightarrow{\pi_1} C. \end{array}$$

For any morphisms  $u: W \to A$  and  $v: W \to B$  with fu = gv we will write

$$\langle u, v \rangle : W \to A_{\langle f, g \rangle} B$$

for the unique morphism with  $\pi_1 \langle u, v \rangle = u$  and  $\pi_2 \langle u, v \rangle = v$ .

We prove that for a pointed variety  $\mathcal{V}$ , if for each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathcal{V})$  there exists a natural bijection  $\varphi: K \times B \to A$ , where  $\kappa: K \to A$  is the kernel of  $\alpha$ , such that the diagram

$$\begin{array}{c} K \times B \xrightarrow{\pi_2} B \\ \varphi \downarrow & & \\ A \xrightarrow{\alpha} & B \end{array}$$

is a morphism in  $\mathbf{Pt}(\mathbf{Set})$ , then  $\mathcal{V}$  is a variety of right  $\Omega$ -loops (see Corollary 2.2). There is a *natural* generalization of this condition for any variety  $\mathcal{V}$ , namely asking for each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathcal{V})$  and for each morphism  $f: E \to B$  that there exists a bijection

$$\varphi: (A_{\langle \alpha, f \rangle} \times E) \times B \to E \times A$$

natural in both  $(A, B, \alpha, \beta)$  and  $f: E \to B$ , such that the diagram

$$\begin{array}{c|c} (A \times E) \times B \xrightarrow{\pi_2 \times 1} E \times B \\ & \swarrow \\ \varphi \\ E \times A \xrightarrow{1 \times \alpha} E \times B \end{array}$$

is a morphism in  $\mathbf{Pt}(\mathbf{Set})$ . It is clear that for a pointed variety this condition implies the previous condition, since taking E to be the zero object and fto be the unique morphism from E to B makes

$$\pi_1: A_{\langle \alpha, f \rangle} \xrightarrow{\times} E \to A$$

the kernel of  $\alpha$ . In Section 4 we prove that this condition is equivalent to the same condition under the restriction that each f as above is an identity morphism (see Theorem 4.6). We also prove that a variety satisfies this condition if and only if it is a *biternary system* [7] that is there exist ternary terms p(x, y, z) and q(x, y, z) satisfying the identities

$$p(x, x, y) = y \tag{5}$$

$$p(q(x, y, z), z, y) = x = q(p(x, y, z), z, y).$$
(6)

However, there are other generalizations that may be considered. In a variety  $\mathcal{V}$  with constants, for each X, let  $\theta_X : 1 \to X^n$  be a map (natural in X) such that the composite with each product projection  $\pi_i : X^n \to X$  gives a constant. We could then consider the following condition: for each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathcal{V})$  there exists a natural split epimorphism (in the category of sets)

$$\varphi: (A^n \underset{\langle \alpha^n, \theta_B \rangle}{\times} 1) \times B \to A$$

with splitting

$$\psi: A \to (A^n \underset{\langle \alpha^n, \theta_B \rangle}{\times} 1) \times B$$

such that in the diagram

$$\begin{array}{c} (A^n \times 1) \times B \xrightarrow{\pi_2} B \\ \langle \alpha^n, \theta_B \rangle \\ \psi \downarrow \varphi \\ A \xrightarrow{\alpha} B \end{array}$$

the upward and downward directed sub-diagrams are morphisms in  $\mathbf{Pt}(\mathbf{Set})$ . We prove in Section 3 that this condition is equivalent to  $\mathcal{V}$  being a protomodular variety [2] of *type* n, that is, a variety  $\mathcal{V}$  with constants  $e_1, \ldots, e_n$ , binary terms  $s_1(x, y), \ldots, s_n(x, y)$  and an n + 1-ary term  $p(x_1, \ldots, x_n, z)$ satisfying the identities:

$$s_i(x,x) = e_i \ i \in \{1,\dots,n\}$$
 (7)

$$p(s_1(x,z),\ldots,s_n(x,z),z) = x.$$
 (8)

Note that requiring  $\varphi$  to be a bijection gives the addition conditions

$$s_i(p(x_1, ..., x_n, y), y)) = x_i \text{ for all } i \in 1, ..., n.$$
 (9)

In order to study these conditions simultaneously we make a further generalization described in Section 1.

# 1 The general setting

In this section we replace a forgetful functor from a variety into the category of sets (or pointed sets) with an abstract functor (satisfying certain conditions) and consider a generalization allowing us to study simultaneously both generalizations discussed in the introduction.

For a set  $\mathbf{n}$ , a category  $\mathbb{D}$  with finite products and products indexed over  $\mathbf{n}$ , and for functors  $F, G, H : \mathbb{C} \to \mathbb{D}$  we denote by  $F^{\mathbf{n}}$  the  $\mathbf{n}$  indexed product of F with itself and by  $G \times H$  the product of G and H in the functor category  $\mathbb{D}^{\mathbb{C}}$ .

Throughout this section we will assume that:

- 1.  $\mathbb{A}$  is a category with finite products;
- 2. m and n are sets;
- 3. X is a category with finite limits and products indexed by the sets **m** and **n**;
- 4.  $U : \mathbb{A} \to \mathbb{X}$  is a functor preserving finite products;
- 5.  $\theta: U^{\mathbf{m}} \to U^{\mathbf{n}}$  is a natural transformation.

Let  $\Delta : \mathbb{A} \to \mathbf{Pt}(\mathbb{A})$  be the functor sending X in  $\mathbb{A}$  to  $(X \times X, X, \pi_2, \langle 1, 1 \rangle)$ and let  $D_{\mathbb{A}}$  be the functor  $\mathbf{Pt}(\mathbb{A}) \to \mathbb{A}$  taking  $(A, B, \alpha, \beta)$  to B. Let  $V : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$  and  $W : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$  be the functors sending  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  to

$$((U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}{\times} U(B)^{\mathbf{m}}) \times U(B), U(B)^{\mathbf{m}} \times U(B), \pi_2 \times 1, \langle U(\beta)^{\mathbf{n}} \theta_B, 1 \rangle \times 1)$$

and

$$(U(B)^{\mathbf{m}} \times U(A), U(B)^{\mathbf{m}} \times U(B), 1 \times U(\alpha), 1 \times U(\beta))$$

respectively.

From the beginning of the next section we will consider the case where  $\mathbb{A}$  is a variety,  $\mathbb{X}$  is the category of sets, U is the usual forgetful functor from the variety to the category of sets,  $\mathbf{m} = \{1, \ldots, m\}$ ,  $\mathbf{n} = \{1, \ldots, n\}$ , and  $\theta$  is constructed from n m-ary terms of  $\mathbb{A}$ . In particular when  $\mathbb{A}$  is pointed with constant  $0, \mathbf{n} = \{1\}, \mathbf{m} = \emptyset$ , and  $\theta : U^{\mathbf{m}} \to U^{\mathbf{n}}$  is the natural transformation with component at  $X \ \theta_X(1) = 0$  (where 1 is the unique element in  $U^{\mathbf{m}}(X)$ ), it can be seen that

$$\pi_1: U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}{\times} U(B)^{\mathbf{m}} \to U(A)$$

is up to isomorphism the image under U of the kernel of  $\alpha$  and the bijections mentioned at the start of the introduction become components of a natural transformation  $V \to W$ .

**Lemma 1.1.** Each of the following types of data uniquely determine each other:

- (a) a natural transformation  $\tau: V \to W$ ;
- (b) a natural transformation  $\overline{\tau}: V\Delta \to W\Delta;$
- (c) natural transformations  $\rho: (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$  and  $\zeta: U^{\mathbf{m}} \times U \to U^{\mathbf{m}}$ ;

*Proof.* For each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  and X in  $\mathbb{A}$ , let  $(\varphi_{1_{(A,B,\alpha,\beta)}}, \varphi_{0_{(A,B,\alpha,\beta)}}) = \tau_{(A,B,\alpha,\beta)}$  and  $(\overline{\varphi}_{1_X}, \overline{\varphi}_{0_X}) = \overline{\tau}_X$ . The diagram

in which

$$P_X = U(X \times X)^{\mathbf{n}} \underset{\langle U(\pi_2)^n, \theta_X \rangle}{\times} U(X)^{\mathbf{m}}$$

and

$$p_X = \langle \zeta_X(\pi_2 \times 1), \langle \rho_X, \rho_X(\langle \theta_X \pi_2, \pi_2 \rangle \times 1) \rangle \rangle,$$

is a commutative diagram of morphisms in  $\mathbf{Pt}(\mathbb{X})$ , and shows the relationship between  $\overline{\tau}$  and  $\rho$  and  $\zeta$ . The commutative diagrams

$$\begin{array}{c} (U(A)^{\mathbf{n}} \times U(B)^{\mathbf{m}}) \times U(B) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(B)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(B)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(B)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(B)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A \times A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A \times A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A \times A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A \times A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A \times A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(A)^{\mathbf{m}} \times U(A) \xrightarrow{\varphi_{1(A,B,\alpha,\beta)}} U(B)^{\mathbf{m}} \times U(A)$$

$$\begin{array}{c} U(B)^{\mathbf{m}} \times U(B) \xrightarrow{\varphi_{0}_{(A,B,\alpha,\beta)}} U(B^{m} \times U(B)) \\ \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) & U(\beta)^{\mathbf{m}} \times U(\beta) \\ \downarrow U(A)^{m} \times U(A) \xrightarrow{\varphi_{0}_{\Delta(A)}} U(A)^{\mathbf{m}} \times U(A) \\ \downarrow U(\alpha)^{\mathbf{m}} \times U(\alpha) & U(\alpha)^{\mathbf{m}} \times U(\alpha) \\ \downarrow U(B)^{\mathbf{m}} \times U(B) \xrightarrow{\varphi_{0}_{\Delta(B)}} U(B)^{\mathbf{m}} \times U(B) \\ \downarrow U(B)^{\mathbf{m}} \times U(B) \xrightarrow{\varphi_{0}_{\Delta(B)}} U(B)^{\mathbf{m}} \times U(B) \end{array}$$

show the relationships between  $\tau$  and  $\overline{\tau}$ , and  $\tau$  and  $\rho$  and  $\zeta$ .

**Lemma 1.2.** Each of the following types of data uniquely determine each other:

- (a) a natural transformation  $\gamma: W \to V$ ;
- (b) a natural transformation  $\overline{\gamma}: W\Delta \to V\Delta;$
- (c) natural transformations  $\sigma: U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}, \eta: U^{\mathbf{m}} \times U \to U^{\mathbf{m}}$  and  $\epsilon: U^{\mathbf{m}} \times U \to U$  with components at each X in A making the diagram

commute.

*Proof.* For each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  and X in  $\mathbb{A}$ , let  $(\psi_{1_{(A, B, \alpha, \beta)}}, \psi_{0_{(A, B, \alpha, \beta)}}) = \gamma_{(A, B, \alpha, \beta)}$  and  $(\overline{\psi}_{1_X}, \overline{\psi}_{0_X}) = \overline{\gamma}_X$ . The diagram

in which

$$P_X = U(X \times X)^{\mathbf{n}} \underset{\langle U(\pi_2)^n, \theta_X \rangle}{\times} U(X)^{\mathbf{m}}$$

and

$$q_X = \langle \langle \sigma_X, \eta_X(1 \times \pi_2) \rangle, \epsilon_X(1 \times \pi_2) \rangle,$$

is a commutative diagram of morphisms in  $\mathbf{Pt}(\mathbb{X})$ , and shows the relationship between  $\overline{\gamma}$  and  $\sigma$ ,  $\eta$  and  $\epsilon$ . The equations

$$\gamma_{\Delta(X)} = \overline{\gamma}_X$$

and

$$\psi_{1_{(A,B,\alpha,\beta)}} = \langle \langle \sigma_A(U(\beta)^{\mathbf{m}} \times U(\langle 1, \beta \alpha \rangle)), \eta_B(1 \times U(\alpha)) \rangle \epsilon_B(1 \times U(\alpha)) \rangle,$$

and the commutative diagram

$$\begin{array}{c} U(B)^{\mathbf{m}} \times U(B) & \xleftarrow{\psi_{0}_{(A,B,\alpha)}} U(B)^{\mathbf{m}} \times U(B) \\ & \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) & \downarrow U(\beta)^{\mathbf{m}} \times U(\beta) \\ & \downarrow U(A)^{\mathbf{m}} \times U(A) & \xleftarrow{\psi_{0}_{\Delta(A)}} U(A)^{\mathbf{m}} \times U(A) \\ & \downarrow U(\alpha)^{\mathbf{m}} \times U(\alpha) & \downarrow U(\alpha)^{\mathbf{m}} \times U(A) \\ & \downarrow U(B)^{\mathbf{m}} \times U(B) & \xleftarrow{\psi_{0}_{\Delta(B)}} U(B)^{\mathbf{m}} \times U(B) \end{array}$$

show the relationships between  $\gamma$  and  $\overline{\gamma}$ , and  $\gamma$  and  $\sigma$ ,  $\eta$  and  $\epsilon$ .

From the two lemmas above we easily prove the following corollaries.

**Corollary 1.3.** Each of the following types of data uniquely determine each other:

- (a) a natural transformation  $\tau: V \to W$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ ;
- (b) a natural transformation  $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$  with component at each X in  $\mathbb{C}$  making the diagram

commute.

**Corollary 1.4.** Each of the following types of data uniquely determine each other:

- (a) a natural transformation  $\gamma: W \to V$  with  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ ;
- (b) a natural transformation  $\sigma : U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$  with component at each X in  $\mathbb{C}$  making the diagram

$$U(X)^{\mathbf{m}} \times (U(X) \times U(X)) \xrightarrow{\theta_X} U(X)^{\mathbf{n}}$$

$$\downarrow^{1 \times \langle 1, 1 \rangle} \qquad \qquad \uparrow^{\theta_X} \qquad (12)$$

$$U(X)^{\mathbf{m}} \times U(X) \xrightarrow{\pi_1} U(X)^{\mathbf{m}}$$

commute.

**Corollary 1.5.** Each of the following types of data uniquely determine each other:

- (a) natural transformations  $\tau : V \to W$  and  $\gamma : W \to V$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^m \times D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^m \times D_{\mathbb{A}}}$  and such that  $\tau \gamma = 1_W$ ;
- (b) natural transformations  $\rho: (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$  and  $\sigma: U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$  with components at each X in  $\mathbb{C}$  making the diagrams (11), (12) and

commute.

**Corollary 1.6.** Each of the following types of data uniquely determine each other:

- (a) natural transformations  $\tau : V \to W$  and  $\gamma : W \to V$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathsf{m}} \times D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathsf{m}} \times D_{\mathbb{A}}}$  and such that  $\gamma \tau = 1_V$ ;
- (b) natural transformations  $\rho: (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$  and  $\sigma: U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$  with components at each X in  $\mathbb{C}$  making the diagrams (11), (12) and

commute.

**Corollary 1.7.** Each of the following types of data uniquely determine each other:

- (a) natural transformations  $\tau : V \to W$  and  $\sigma : W \to V$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathsf{m}} \times D_{\mathbb{A}}}$  and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathsf{m}} \times D_{\mathbb{A}}}$  and inverse to each other;
- (b) natural transformations  $\rho: (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$  and  $\sigma: U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$  with components at each X in  $\mathbb{C}$  making the diagrams (11), (12), (13) and (14) commute.

We now consider the case where  $\mathbf{m} = \emptyset$  and  $\mathbf{n} = \{1\}$ , the results proved here will be used in Section 2.

When  $\mathbf{m} = \emptyset$  and  $\mathbf{n} = \{1\}$ , the functors V and W are up to isomorphism the functors  $\tilde{V}, \tilde{W} : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$  sending  $(A, B, \alpha, \beta)$  to

$$((U(A)_{\langle U(\alpha),\theta_B \rangle}^{\times} 1) \times U(B), U(B), \pi_2, \langle \langle \theta_A, 1 \rangle !_{U(B)}, 1 \rangle)$$

and

$$(U(A), U(B), U(\alpha), U(\beta))$$

respectively.

**Corollary 1.8.** Each of the following types of data uniquely determine each other:

(a) a natural transformation  $\tau : \tilde{V} \to \tilde{W}$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}}$  and with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  such that the diagram

commutes;

(b) a natural transformation  $\rho: U \times U \to U$  with component at each X in A making the diagram

$$\begin{array}{c|c}
U(X) \\
\downarrow & \downarrow & \downarrow \\
U(X) \times U(X) \xrightarrow{I_{U(X)}} U(X) \\
\downarrow & \downarrow & \downarrow \\
U(X) \\
\downarrow & \downarrow \\
U(X) \\
\end{array}$$
(16)

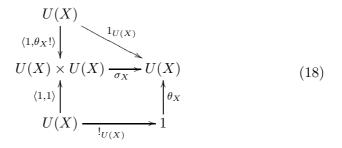
commute.

**Corollary 1.9.** Each of the following types of data uniquely determine each other:

(a) a natural transformation  $\gamma : \tilde{W} \to \tilde{V}$  with  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}}$  and with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  such that the diagram

commutes;

(b) a natural transformation  $\sigma: U \times U \to U$  with component at each X in A making the diagram



commute.

**Corollary 1.10.** Each of the following types of data uniquely determine each other:

- (a) natural transformations  $\tau : \tilde{V} \to \tilde{W}$  and  $\gamma : \tilde{W} \to \tilde{V}$  with  $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}}$ and  $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}}$  inverse to each other and with components at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  making the diagrams (15) and (17) commute;
- (b) natural transformations  $\rho : U \times U \to U$  and  $\sigma : U \times U \to U$  with component at each X in A making the diagrams (16), (18),

and

$$U(X) \times U(X)$$

$$\langle \rho_X, \pi_2 \rangle \bigvee_{X} \pi_1$$

$$U(X) \times U(X)_{\sigma_X} U(X)$$
(20)

commute.

In the sections that follows we use the fact that the set of natural transformation  $U^{\mathbf{n}} \to U$  (where  $\mathbf{n} = \{1, \ldots, n\}$  and U is the forgetful functor from a variety to sets) is in bijection with the set of *n*-ary terms of the variety. Since this is no longer true for arbitrary internal varieties (every term determines a natural transformation but not conversely) the results in the sections that follow hold only partially in arbitrary internal varieties, i.e. the existence of certain terms determine natural transformations between appropriate V and W but not conversely.

## 2 Pointed varieties

In this section we apply the results from Section 1 to the special case where  $\mathbb{A} = \mathcal{V}$  is a pointed variety,  $\mathbb{X} = \mathbf{Set}_*$  is the category of pointed sets, U is the usual forgetful functor,  $\mathbf{m} = \emptyset$ ,  $\mathbf{n} = \{1\}$ , and  $\theta$  is constructed using the constant of  $\mathcal{V}$ .

For any category  $\mathbb{C}$  we define **SplExt**( $\mathbb{C}$ ) to be the category of split extensions: an object is a sextuple  $(K, A, B, \kappa, \alpha, \beta)$  where K, A and B are objects in  $\mathbb{C}$  and  $\kappa : K \to B$ ,  $\alpha : A \to B$  and  $\beta : B \to A$  are morphisms in  $\mathbb{C}$ with  $(K, \kappa)$  the kernel of  $\alpha$  and  $\alpha\beta = 1_B$ ; a morphism  $(K, A, B, \kappa, \alpha, \beta) \to$  $(K', A', B', \kappa', \alpha', \beta')$  is a triple (u, v, w) of morphisms  $u : K \to K', v : A \to$ A' and  $w : B \to B'$  such that in the diagram

$$\begin{array}{ccc} K & \stackrel{\kappa}{\longrightarrow} & A & \stackrel{\alpha}{\longleftarrow} & B \\ u & & v & & \downarrow \\ u & & v & & \downarrow \\ K' & \stackrel{\kappa'}{\longrightarrow} & A' & \stackrel{\alpha'}{\longleftarrow} & B' \end{array}$$

 $v\kappa = \kappa' u, \, \alpha' v = w\alpha \text{ and } v\beta = \beta' w.$ 

**Theorem 2.1.** Let  $\mathcal{V}$  be a pointed variety and let P, Q: **SplExt**( $\mathcal{V}$ )  $\rightarrow$ **SplExt**(**Set**<sub>\*</sub>) be the functors taking ( $K, A, B, \kappa, \alpha, \beta$ ) to ( $U(K), U(K) \times U(B), \langle 1, 0 \rangle, \pi_2, \langle 0, 1 \rangle$ ) and ( $U(K), U(A), U(B), U(\kappa), U(\alpha), U(\beta)$ ) respectively.

(a)  $\mathcal{V}$  is a unital variety [1] if and only if there exists a natural transformation  $P \to Q$  with component at  $(K, A, B, \kappa, \alpha, \beta)$  of the form

(b)  $\mathcal{V}$  is a subtractive variety [6] if and only if there exists a natural transformation  $Q \to P$  with component at  $(K, A, B, \kappa, \alpha, \beta)$  of the form

(c)  $\mathcal{V}$  is a variety of right  $\Omega$ -loops if and only if there exists a natural isomorphism  $P \to Q$  with component at  $(K, A, B, \kappa, \alpha, \beta)$  of the form

$$\begin{array}{c} U(K) \longrightarrow U(K) \times U(B) \xrightarrow{\pi_2} U(B) \\ \| & | & | \\ \| & | \\ V \\ U(K) \longrightarrow U(A) \xrightarrow{U(\alpha)} U(B). \end{array}$$

*Proof.* It is easy to see that to give a natural transformation  $P \to Q$  as in (a) above is the same as to give a natural transformation  $\tilde{V} \to \tilde{W}$  as in (a) of Corollary 1.8 which, by Corollary 1.8, is uniquely determined by a natural transformation  $\rho : U \times U \to U$  with components making the diagram (16) commute. And, such a natural transformation determines and is determined by a binary term + such that for each x, y in X, an algebra,  $x + y = \rho_X(x, y)$ . The commutativity of (16) then implies that x + 0 = x = 0 + x. The statements (b) and (c) follow from Corollaries 1.9, and 1.10 in a similar way.

**Corollary 2.2.** Let  $\tilde{P}, \tilde{Q} : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$  be the functors sending  $(A, B, \alpha, \beta)$ in  $\mathbf{Pt}(\mathbb{A})$  to  $(U(K \times B), U(B), U(\pi_2), U(\langle 0, 1 \rangle))$  (where  $K = \text{Ker}(\alpha)$ ) and  $(U(A), U(B), U(\alpha), U(\beta))$  respectively.  $\mathcal{V}$  is a variety of right  $\Omega$ -loops if and only if there exists a natural bijection  $\tilde{P} \to \tilde{Q}$  with component  $(A, B, \alpha, \beta)$ of the form

*Proof.* It follows from Corollary 1.7 that a natural bijection  $\tilde{P} \to \tilde{Q}$  as above is completely determined by and determines binary terms  $\rho(x, y)$  and  $\sigma(x, y)$ satisfying the identities  $\sigma(x, x) = 0$ ,  $\rho(\sigma(x, y), y) = x$  and  $\sigma(\rho(x, y), y) = x$ . Setting  $x + y = \rho(\sigma(x, 0), y)$  and  $x - y = \rho(\sigma(x, y), 0)$  determines terms that satisfy the right loop identities.

**Remark 2.3.** In fact it can be shown that  $\mathcal{V}$  is a variety of right  $\Omega$ -loops if and only if there exists a natural isomorphism  $\tilde{P} \to \tilde{Q}$ .

### **3** Protomodular varieties

In this section we give a new classification of protomodular varieties by applying the results from Section 1 to the case where  $\mathbb{A} = \mathcal{V}$  is an arbitrary variety with constants,  $\mathbb{X} = \mathbf{Set}$  is the category of sets, and U is the usual forgetful functor.

**Theorem 3.1.**  $\mathcal{V}$  is a protomodular variety if and only if for some  $\mathbf{m} = \{1, \ldots, m\}$ ,  $\mathbf{n} = \{1, \ldots, n\}$  and  $\theta$  there exist natural transformations  $\tau : V \to W$  and  $\gamma : W \to V$  with  $\tau \gamma = 1_W$  and with components at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{C})$  of the form

*Proof.* It follows from Corollary 1.5 that natural transformations  $\tau : V \to W$  and  $\gamma : W \to V$  as above determine terms

 $\rho(x_1,\ldots,x_n,y_1,\ldots,y_m,z)$  and  $\sigma_i(y_1,\ldots,y_m,x,z)$   $i \in \mathbf{n}$ 

satisfying the identities

$$\sigma_i(y_1, \dots, y_m, x, x) = \theta_i(y_1, \dots, y_m) \ i \in \mathbf{n}$$
$$\rho(\sigma_1(y_1, \dots, y_m, x, z), \dots, \sigma_n(y_1, \dots, y_m, x, z), y_1, \dots, y_m, z) = x.$$

For any constant e we may form new terms  $e_i = \theta_i(e, \ldots, e)$   $i \in \mathbf{n}$ ,  $s_i(x, z) = \sigma_i(e, \ldots, e, x, z)$   $i \in \mathbf{n}$ , and  $p(x_1, \ldots, x_n, z) = \rho(x_1, \ldots, x_n, e, \ldots, e, z)$ . It easy to check that these terms make  $\mathcal{V}$  a protomodular variety. The converse follows from Corollary 1.5 with  $\mathbf{m} = \emptyset$ .

**Remark 3.2.** The results in this section can easily be extended to  $\mathcal{V}$  an infinitary variety, with  $\mathbf{m}$  and  $\mathbf{n}$  possibly infinite sets, giving, by Theorem 2.1 of [3], a new classification of infinitary protomodular varieties.

**Remark 3.3.** It could also be interesting to study when  $\gamma \tau = 1_V$  (without  $\tau \gamma = 1_W$ ) which can be seen to be equivalent to the existence of  $\rho$  and  $\sigma$  as above, satisfying the identities:

$$\sigma_i(y_1, \dots, y_m, x, x) = \theta_i(y_1, \dots, y_m) \ i \in \mathbf{n}$$
  
$$\rho(\theta_1(y_1, \dots, y_m), \dots, \theta_n(y_1, \dots, y_m), y_1, \dots, y_m, x) = x$$
  
$$\sigma_i(y_1, \dots, y_m, \rho(x_1, \dots, x_n, y_1, \dots, y_m, z), z) = x_i \ i \in \mathbf{n}$$

instead.

# 4 General varieties

In this section we consider the case where  $\mathbb{A} = \mathcal{V}$  is a variety,  $\mathbb{X} = \mathbf{Set}$  is the category sets, and U is the usual forgetful functor.

For a variety  $\mathcal{V}$  consider the condition:

**Condition 4.1.** There exist ternary terms p and q satisfying the identities: p(x, x, y) = y and p(q(x, y, z), z, y) = x = q(p(x, y, z), z, y).

It is easy to see that q(x, x, y) = y follows from the conditions above, as remarked in [7], where such a variety was called a *biternary system*.

**Remark 4.2.** It is easy to see that if a variety  $\mathcal{V}$  satisfies Condition 4.1 then every regular epimorphism  $f: E \to B$  is up to bijection a product projection  $\pi_2: X \times B \to B$  for some X (since for each b and b' choosing e and e' in  $f^{-1}(\{b\})$  and  $f^{-1}(\{b'\})$  respectively gives a bijection  $p(-, e, e'): f^{-1}(\{b\}) \to f^{-1}(\{b\}))$ .

**Proposition 4.3.** For a variety  $\mathcal{V}$  the following conditions are equivalent:

- 1. V satisfies Condition 4.1;
- 2. There exist ternary terms  $\tilde{p}$  and  $\tilde{p}$  satisfying the identities:  $\tilde{p}(x, x, y) = y = \tilde{q}(x, x, y), \ \tilde{p}(x, y, y) = x = \tilde{q}(x, y, y) \ and \ \tilde{p}(\tilde{q}(x, y, z), z, y) = x = \tilde{q}(\tilde{p}(x, y, z), z, y);$
- 3. There exists a quaternary term u satisfying the identities: u(a, b, b, a) = b and u(u(a, b, c, d), b, d, c) = a;
- 4. There exists a quaternary term  $\tilde{u}$  satisfying the identities:  $\tilde{u}(a, b, b, a) = b = \tilde{u}(a, a, b, a)$  and  $\tilde{u}(a, b, c, c) = a = \tilde{u}(\tilde{u}(a, b, c, d), b, d, c);$

If in addition  $\mathcal{V}$  has at least one constant, those conditions are further equivalent to:

5. For each constant e there exist binary terms x + y and x - y satisfying the right loop identities (for that constant e).

Proof. The implications  $2 \Rightarrow 1$  and  $4 \Rightarrow 3$  are trivial.  $1 \Rightarrow 2$ : Given p and q define  $\tilde{p}(x, y, z) = p(q(x, y, y), y, z)$  and  $\tilde{q}(x, y, z) = p(q(x, y, z), z, z)$ .  $2 \Rightarrow 4$ : Given  $\tilde{p}$  and  $\tilde{q}$  define  $\tilde{u}(a, b, c, d) = \tilde{p}(\tilde{q}(a, b, c), d, b)$ .  $3 \Rightarrow 1$ : Given u define p(x, y, z) = u(x, z, z, y) and q(x, y, z) = u(x, y, z, y). If in addition  $\mathcal{V}$  has at least one constant.  $2 \Rightarrow 5$ : Given  $\tilde{p}$  and  $\tilde{q}$  for each constant e define  $x + y = \tilde{p}(x, e, y)$  and  $x - y = \tilde{q}(x, y, e)$ .  $5 \Rightarrow 1$ : Given x + y and x - y for some constant e define p(x, y, z) = q(x, y, z) = (x - y) + z.

**Remark 4.4.** It follows that a variety satisfying Condition 4.1 is a Mal'tsev variety.

**Theorem 4.5.** (a) If  $\mathcal{V}$  satisfies Condition 4.1, then for  $\mathbf{n} = \{1\}$ ,  $\mathbf{m} = \{1\}$  and  $\theta = 1_U$  there exists a natural isomorphism  $\tau : V \to W$  with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{C})$  of the form

(b) If for some  $\mathbf{n} = \{1, ..., n\}$ ,  $\mathbf{m} = \{1, ..., m\}$  and  $\theta$  there exists a natural isomorphism  $\tau : V \to W$  with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{C})$  of the form

$$\begin{array}{cccc} (U(A)^{\mathbf{n}} & \underset{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}{\times} U(B)^{\mathbf{m}} \times U(B) \xrightarrow[\langle U(\beta)^{\mathbf{m}} \theta_B, 1 \rangle \times 1]{} U(B)^{\mathbf{m}} \times U(B) \\ & & & & \\ & & & \\ & & & \\ & & & \\ &$$

then  $\mathcal{V}$  satisfies Condition 4.1.

- *Proof.* (a) Let  $\mathbf{n} = \mathbf{m} = \{1\}$  and  $\theta = 1_U$ . Given ternary terms p and q as in Condition 4.1, it is easy to check that  $\rho = p$  and  $\sigma(x, y, z) = q(y, z, x)$  define natural transformations making the diagrams (11), (12), (13) and (14) commute. Therefore by Corollary 1.7 determine a natural isomorphism  $V \to W$ , as required.
- (b) If for some  $\mathbf{n} = \{1, \ldots, n\}$ ,  $\mathbf{m} = \{1, \ldots, m\}$  and  $\theta$  there exists an isomorphism  $V \to W$  then by Corollary 1.7 there exist terms  $\rho(x_1, \ldots, x_n, y_1, \ldots, y_m, z)$  and  $\sigma_i(y_1, \ldots, y_m, x, z)$   $i \in \mathbf{n}$  satisfying the identities:

$$\sigma_i(y_1, \dots, y_m, x, x) = \theta_i(y_1, \dots, y_m)$$
  

$$\rho(\sigma_1(y_1, \dots, y_m, x, z), \dots, \sigma_n(y_1, \dots, y_m, x, z), y_1, \dots, y_m, z) = x$$
  

$$\sigma_i(y_1, \dots, y_m, \rho(x_1, \dots, x_n, y_1, \dots, y_m, z), z) = x_i.$$

Let p and q be the terms defined by

$$p(x, y, z) = \rho(\sigma_1(y, \dots, y, x, y), \dots, \sigma_n(y, \dots, y, x, y), y, \dots, y, z)$$
  
$$q(x, y, z) = \rho(\sigma_1(z, \dots, z, x, y), \dots, \sigma_n(z, \dots, z, x, y), z, \dots, z, z).$$

It is easy to check that p and q satisfy the desired identities as in Condition 4.1.

Recall that for any category  $\mathbb{C}$  the functor  $D_{\mathbb{C}}$  is the functor  $\mathbf{Pt}(\mathbb{C}) \to \mathbb{C}$  taking  $(A, B, \alpha, \beta)$  to B.

**Theorem 4.6.** Let V and W be the functors defined in Section 1 with  $\mathbb{A} = \mathcal{V}, \mathbb{X} = \mathbf{Set}, U$  the usual forgetful functor,  $\mathbf{n} = \mathbf{m} = \{1\}$  and  $\theta = 1_U$ . Let  $P, Q : (\mathbb{A} \downarrow D_{\mathbb{A}}) \to \mathbf{Pt}(\mathbb{X})$  be the functors sending  $(E, (A, B, \alpha, \beta), f)$  to

$$(U(A_{\langle \alpha, f \rangle} \times E) \times U(B), U(E) \times U(B), U(\pi_2) \times 1, U(\langle \beta f, 1 \rangle) \times 1)$$

and

$$(U(E) \times U(A), U(E) \times U(B), 1 \times U(\alpha), 1 \times U(\beta))$$

respectively. The following are equivalent:

1. There exists an isomorphism  $\tau : V \to W$  with component at each  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  of the form

2. There exists an isomorphism  $\chi : P \to Q$  with component at each  $(E, (A, B, \alpha, \beta), f)$  in  $(\mathbb{A} \downarrow D_{\mathbb{A}})$  of the form

$$\begin{array}{c} (U(A_{\langle \alpha, f \rangle}^{\times} E) \times U(B) \xrightarrow{U(\pi_2) \times 1} U(E) \times U(B) \\ \downarrow \\ \downarrow \\ U(\langle \beta f, 1 \rangle) \times 1 \\ \downarrow \\ \downarrow \\ U(E) \times U(A) \xrightarrow{1 \times U(\alpha)} U(E) \times U(B); \end{array}$$

#### 3. V satisfies Condition 4.1.

*Proof.* The equivalence of 1 and 3 follows from Theorem 4.5. It is easy to show that  $2 \Rightarrow 1$  since P and Q composed with the functor sending  $(A, B, \alpha, \beta)$  in  $\mathbf{Pt}(\mathbb{A})$  to  $(B, (A, B, \alpha, \beta), \mathbf{1}_B)$  in  $(\mathbb{A} \downarrow D_{\mathbb{A}})$  are up to natural isomorphism the functors V and W respectively. We will show that  $3 \Rightarrow 2$ .

Let p and q be ternary terms as in Condition 4.1. It is easy to check that  $\chi$  with component at each  $(E, (A, B, \alpha, \beta), f)$  defined by  $\chi_{(E, (A, B, \alpha, \beta), f)} = (\varphi_{(E, (A, B, \alpha, \beta), f)}, 1_{U(B)})$  where  $\varphi_{(E, (A, B, \alpha, \beta), f)}((a, e), b) = (e, p(a, \beta f(e), \beta(b)))$  is an isomorphism with inverse  $\chi_{(E, (A, B, \alpha, \beta), f)}^{-1} = (\psi_{(E, (A, B, \alpha, \beta), f)}, 1_U(B))$  where  $\psi_{(E, (A, B, \alpha, \beta), f)}(e, a) = ((q(a, \beta \alpha(a), \beta f(e)), e), \alpha(a)).$ 

# References

- D. Bourn, Mal'cev categories and fibration of pointed objects, Applied Categorical Structures, 4 (2-3), 1996, 307-327.
- [2] D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras, Theory and Applications of Categories, 11 (6), 2003, 143-147.
- [3] M. Gran and J. Rosicky, Semi-abelian monadic categories, Theory and Applications of Categories, 13 (6), 2004, 106-113.
- [4] E. B. Inyangala, *Categorical semi-direct products in varieties of groups* with multiple operators, PhD Thesis, UCT, 2010.
- [5] E. B. Inyangala, Semidirect products and crossed modules in varieties of right Ω-loops, Theory and Applications of Categories, 25 (16), 2011, 426-435.
- [6] Z. Janelidze, Subtractive categories, Applied Categorical Structures, 13 (4), 2005, 343-350.
- [7] A. Mal'tsev, Towards the general theory of algebraic systems, Mat. Sb., N.S. 35 (77), 1954, 3-20.