# ON ALGEBRAIC AND MORE GENERAL CATEGORIES WHOSE SPLIT EPIMORPHISMS HAVE UNDERLYING PRODUCT PROJECTIONS 

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#### Abstract

We characterize those varieties of universal algebras where every split epimorphism considered as a map of sets is a product projection. In addition we obtain new characterizations of protomodular, unital and subtractive varieties as well as varieties of right $\Omega$-loops and biternary systems.


## Introduction

It is well known that in the category of groups if

$$
0 \longrightarrow K \xrightarrow{\kappa} A \xrightarrow{\alpha} B \longrightarrow 0
$$

is a short exact sequence, then $A$ and $K \times B$ are bijective as sets, moreover when $\alpha$ is split, i.e. for each split extension

$$
K \xrightarrow{\kappa} A \underset{\beta}{\stackrel{\alpha}{\rightleftarrows}} B, \alpha \beta=1_{B}, \kappa=\operatorname{ker}(\alpha),
$$

this bijection becomes a natural bijection $K \times B \rightarrow A$ such that the diagram

is a morphism of split extensions in the category Set, of sets, that is, $\alpha \varphi=$ $\pi_{2}, \varphi\langle 0,1\rangle=\beta$, and $\varphi\langle 1,0\rangle=\kappa$. As shown by E. B. Inyangala, these bijections exists in a more general setting of a variety of right $\Omega$-loops (see
[4, [5]), that is, a pointed variety of universal algebras $\mathcal{V}$ with constant 0 and binary terms $x+y$ and $x-y$ satisfying the identities:

$$
\begin{array}{r}
x+0=x \\
x-x=0 \\
(x+y)-y=x \\
(x-y)+y=x \tag{4}
\end{array}
$$

Moreover, he showed that if a pointed variety $\mathcal{V}$ with constant 0 has binary terms $x+y$ and $x-y$ and there exist bijections (as above) constructed (in the same way as for groups) using those terms, i.e. $\varphi(k, b)=\kappa(k)+\beta(b)$ and $\varphi^{-1}(a)=(\lambda(a), \alpha(a))$, where $\lambda$ is the unique map such that $\kappa \lambda(a)=$ $a-\beta \alpha(a)$, then $\mathcal{V}$ is a variety of right $\Omega$-loops and in particular the identities (1) - (4) hold for $x+y$ and $x-y$. In this paper we prove that if for a pointed variety $\mathcal{V}$ there exist natural bijections as above, then $\mathcal{V}$ is a variety of right $\Omega$-loops (see Theorem 2.1).

For any category $\mathbb{C}$ let $\operatorname{Pt}(\mathbb{C})$ to be the category of split epimorphisms in $\mathbb{C}$ : an object is a quadruple $(A, B, \alpha, \beta)$ where $A$ and $B$ are objects in $\mathbb{C}$ and $\alpha: A \rightarrow B$ and $\beta: B \rightarrow A$ are morphisms in $\mathbb{C}$ with $\alpha \beta=1_{B}$; a morphism $(A, B, \alpha, \beta) \rightarrow\left(A^{\prime}, B^{\prime}, \alpha^{\prime}, \beta^{\prime}\right)$ is a pair of morphisms $\left(f: A \rightarrow A^{\prime}, g: B \rightarrow\right.$ $B^{\prime}$ ) such that in the diagram

$\alpha^{\prime} f=g \alpha$ and $f \beta=\beta^{\prime} g$. Throughout this paper for any objects $A$ and $B$ we will denote by $\pi_{1}$ and $\pi_{2}$ the first and second product projections respectively. We will use the same notation for the first and second pullback projections and will write

$$
\left(A_{\langle f, g\rangle}^{\times} B, \pi_{1}, \pi_{2}\right)
$$

for the pullback of $f: A \rightarrow C$ and $g: B \rightarrow C$ as in the diagram


For any morphisms $u: W \rightarrow A$ and $v: W \rightarrow B$ with $f u=g v$ we will write

$$
\langle u, v\rangle: W \rightarrow A_{\langle f, g\rangle}^{\times} B
$$

for the unique morphism with $\pi_{1}\langle u, v\rangle=u$ and $\pi_{2}\langle u, v\rangle=v$.
We prove that for a pointed variety $\mathcal{V}$, if for each $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathcal{V})$ there exists a natural bijection $\varphi: K \times B \rightarrow A$, where $\kappa: K \rightarrow A$ is the kernel of $\alpha$, such that the diagram

is a morphism in $\operatorname{Pt}(\mathbf{S e t})$, then $\mathcal{V}$ is a variety of right $\Omega$-loops (see Corollary 2.2 ). There is a natural generalization of this condition for any variety $\mathcal{V}$, namely asking for each $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathcal{V})$ and for each morphism $f: E \rightarrow B$ that there exists a bijection

$$
\varphi:\left(A_{\langle\alpha, f\rangle}^{\times} E\right) \times B \rightarrow E \times A
$$

natural in both $(A, B, \alpha, \beta)$ and $f: E \rightarrow B$, such that the diagram

is a morphism in $\mathbf{P t}(\mathbf{S e t})$. It is clear that for a pointed variety this condition implies the previous condition, since taking $E$ to be the zero object and $f$ to be the unique morphism from $E$ to $B$ makes

$$
\pi_{1}: A_{\langle\alpha, f\rangle}^{\times} E \rightarrow A
$$

the kernel of $\alpha$. In Section 4 we prove that this condition is equivalent to the same condition under the restriction that each $f$ as above is an identity morphism (see Theorem 4.6). We also prove that a variety satisfies this condition if and only if it is a biternary system [7] that is there exist ternary terms $p(x, y, z)$ and $q(x, y, z)$ satisfying the identities

$$
\begin{array}{r}
p(x, x, y)=y \\
p(q(x, y, z), z, y)=x=q(p(x, y, z), z, y) . \tag{6}
\end{array}
$$

However, there are other generalizations that may be considered. In a variety $\mathcal{V}$ with constants, for each $X$, let $\theta_{X}: 1 \rightarrow X^{n}$ be a map (natural in $X$ ) such that the composite with each product projection $\pi_{i}: X^{n} \rightarrow X$
gives a constant. We could then consider the following condition: for each $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathcal{V})$ there exists a natural split epimorphism (in the category of sets)

$$
\varphi:\left(A_{\left\langle\alpha^{n}, \theta_{B}\right\rangle}^{n} 1\right) \times B \rightarrow A
$$

with splitting

$$
\psi: A \rightarrow\left(A_{\left\langle\alpha^{n}, \theta_{B}\right\rangle}^{\times} 1\right) \times B
$$

such that in the diagram
the upward and downward directed sub-diagrams are morphisms in $\mathbf{P t}($ Set $)$. We prove in Section 3 that this condition is equivalent to $\mathcal{V}$ being a protomodular variety [2] of type $n$, that is, a variety $\mathcal{V}$ with constants $e_{1}, \ldots, e_{n}$, binary terms $s_{1}(x, y), \ldots, s_{n}(x, y)$ and an $n+1$-ary term $p\left(x_{1}, \ldots, x_{n}, z\right)$ satisfying the identities:

$$
\begin{array}{r}
s_{i}(x, x)=e_{i} i \in\{1, \ldots, n\} \\
p\left(s_{1}(x, z), \ldots, s_{n}(x, z), z\right)=x \tag{8}
\end{array}
$$

Note that requiring $\varphi$ to be a bijection gives the addition conditions

$$
\begin{equation*}
\left.s_{i}\left(p\left(x_{1}, \ldots, x_{n}, y\right), y\right)\right)=x_{i} \text { for all } i \in 1, \ldots, n \tag{9}
\end{equation*}
$$

In order to study these conditions simultaneously we make a further generalization described in Section 1.

## 1 The general setting

In this section we replace a forgetful functor from a variety into the category of sets (or pointed sets) with an abstract functor (satisfying certain conditions) and consider a generalization allowing us to study simultaneously both generalizations discussed in the introduction.

For a set $\mathbf{n}$, a category $\mathbb{D}$ with finite products and products indexed over $\mathbf{n}$, and for functors $F, G, H: \mathbb{C} \rightarrow \mathbb{D}$ we denote by $F^{\mathbf{n}}$ the $\mathbf{n}$ indexed product of $F$ with itself and by $G \times H$ the product of $G$ and $H$ in the functor category $\mathbb{D}^{\mathbb{C}}$.

Throughout this section we will assume that:

1. $\mathbb{A}$ is a category with finite products;
2. $\mathbf{m}$ and $\mathbf{n}$ are sets;
3. $\mathbb{X}$ is a category with finite limits and products indexed by the sets $\mathbf{m}$ and $\mathbf{n}$;
4. $U: \mathbb{A} \rightarrow \mathbb{X}$ is a functor preserving finite products;
5. $\theta: U^{\mathrm{m}} \rightarrow U^{\mathrm{n}}$ is a natural transformation.

Let $\Delta: \mathbb{A} \rightarrow \operatorname{Pt}(\mathbb{A})$ be the functor sending $X$ in $\mathbb{A}$ to $\left(X \times X, X, \pi_{2},\langle 1,1\rangle\right)$ and let $D_{\mathbb{A}}$ be the functor $\operatorname{Pt}(\mathbb{A}) \rightarrow \mathbb{A}$ taking $(A, B, \alpha, \beta)$ to $B$. Let $V: \operatorname{Pt}(\mathbb{A}) \rightarrow \mathbf{P t}(\mathbb{X})$ and $W: \operatorname{Pt}(\mathbb{A}) \rightarrow \mathbf{P t}(\mathbb{X})$ be the functors sending $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathbb{A})$ to
$\left(\left(U(A)^{\mathbf{n}} \underset{\left\langle U(\alpha)^{\mathbf{n}}, \theta_{B}\right\rangle}{\times} U(B)^{\mathbf{m}}\right) \times U(B), U(B)^{\mathbf{m}} \times U(B), \pi_{2} \times 1,\left\langle U(\beta)^{\mathbf{n}} \theta_{B}, 1\right\rangle \times 1\right)$
and

$$
\left(U(B)^{\mathbf{m}} \times U(A), U(B)^{\mathbf{m}} \times U(B), 1 \times U(\alpha), 1 \times U(\beta)\right)
$$

respectively.
From the beginning of the next section we will consider the case where $\mathbb{A}$ is a variety, $\mathbb{X}$ is the category of sets, $U$ is the usual forgetful functor from the variety to the category of sets, $\mathbf{m}=\{1, \ldots, m\}, \mathbf{n}=\{1, \ldots, n\}$, and $\theta$ is constructed from $n m$-ary terms of $\mathbb{A}$. In particular when $\mathbb{A}$ is pointed with constant $0, \mathbf{n}=\{1\}, \mathbf{m}=\emptyset$, and $\theta: U^{\mathbf{m}} \rightarrow U^{\mathbf{n}}$ is the natural transformation with component at $X \theta_{X}(1)=0$ (where 1 is the unique element in $\left.U^{\mathbf{m}}(X)\right)$, it can be seen that

$$
\pi_{1}: U(A)^{\mathbf{n}} \underset{\left\langle U(\alpha)^{\mathbf{n}}, \theta_{B}\right\rangle}{\times} U(B)^{\mathbf{m}} \rightarrow U(A)
$$

is up to isomorphism the image under $U$ of the kernel of $\alpha$ and the bijections mentioned at the start of the introduction become components of a natural transformation $V \rightarrow W$.

Lemma 1.1. Each of the following types of data uniquely determine each other:
(a) a natural transformation $\tau: V \rightarrow W$;
(b) a natural transformation $\bar{\tau}: V \Delta \rightarrow W \Delta$;
(c) natural transformations $\rho:\left(U^{\mathbf{n}} \times U^{\mathbf{m}}\right) \times U \rightarrow U$ and $\zeta: U^{\mathbf{m}} \times U \rightarrow U^{\mathrm{m}}$;

Proof. For each $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathbb{A})$ and $X$ in $\mathbb{A}$, let $\left(\varphi_{(A, B, \alpha, \beta)}, \varphi_{0_{(A, B, \alpha, \beta)}}\right)=$ $\tau_{(A, B, \alpha, B)}$ and $\left(\bar{\varphi}_{1_{X}}, \bar{\varphi}_{0_{X}}\right)=\bar{\tau}_{X}$. The diagram

in which

$$
P_{X}=U(X \times X)^{\mathbf{n}} \underset{\left\langle U\left(\pi_{2}\right)^{n}, \theta_{X}\right\rangle}{\times} U(X)^{\mathbf{m}}
$$

and

$$
p_{X}=\left\langle\zeta_{X}\left(\pi_{2} \times 1\right),\left\langle\rho_{X}, \rho_{X}\left(\left\langle\theta_{X} \pi_{2}, \pi_{2}\right\rangle \times 1\right)\right\rangle\right\rangle,
$$

is a commutative diagram of morphisms in $\operatorname{Pt}(\mathbb{X})$, and shows the relationship between $\bar{\tau}$ and $\rho$ and $\zeta$. The commutative diagrams


show the relationships between $\tau$ and $\bar{\tau}$, and $\tau$ and $\rho$ and $\zeta$.

Lemma 1.2. Each of the following types of data uniquely determine each other:
(a) a natural transformation $\gamma: W \rightarrow V$;
(b) a natural transformation $\bar{\gamma}: W \Delta \rightarrow V \Delta$;
(c) natural transformations $\sigma: U^{\mathbf{m}} \times(U \times U) \rightarrow U^{\mathbf{n}}, \eta: U^{\mathbf{m}} \times U \rightarrow U^{\mathbf{m}}$ and $\epsilon: U^{\mathbf{m}} \times U \rightarrow U$ with components at each $X$ in $\mathbb{A}$ making the diagram

commute.
Proof. For each $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathbb{A})$ and $X$ in $\mathbb{A}$, let $\left(\psi_{1_{(A, B, \alpha, \beta)}}, \psi_{0_{(A, B, \alpha, \beta)}}\right)=$ $\gamma_{(A, B, \alpha, \beta)}$ and $\left(\bar{\psi}_{1_{X}}, \bar{\psi}_{0_{X}}\right)=\bar{\gamma}_{X}$. The diagram

in which

$$
P_{X}=U(X \times X)^{\mathbf{n}} \underset{\left\langle U\left(\pi_{2}\right)^{n}, \theta_{X}\right\rangle}{\times} U(X)^{\mathbf{m}}
$$

and

$$
q_{X}=\left\langle\left\langle\sigma_{X}, \eta_{X}\left(1 \times \pi_{2}\right)\right\rangle, \epsilon_{X}\left(1 \times \pi_{2}\right)\right\rangle
$$

is a commutative diagram of morphisms in $\mathbf{P t}(\mathbb{X})$, and shows the relationship between $\bar{\gamma}$ and $\sigma, \eta$ and $\epsilon$. The equations

$$
\gamma_{\Delta(X)}=\bar{\gamma}_{X}
$$

and

$$
\psi_{1_{(A, B, \alpha, \beta)}}=\left\langle\left\langle\sigma_{A}\left(U(\beta)^{\mathbf{m}} \times U(\langle 1, \beta \alpha\rangle)\right), \eta_{B}(1 \times U(\alpha))\right\rangle \epsilon_{B}(1 \times U(\alpha))\right\rangle
$$

and the commutative diagram

show the relationships between $\gamma$ and $\bar{\gamma}$, and $\gamma$ and $\sigma, \eta$ and $\epsilon$.
From the two lemmas above we easily prove the following corollaries.
Corollary 1.3. Each of the following types of data uniquely determine each other:
(a) a natural transformation $\tau: V \rightarrow W$ with $1_{D_{\mathrm{X}}} \circ \tau=1_{D_{\mathrm{A}} \times D_{\mathbb{A}}}$;
(b) a natural transformation $\rho:\left(U^{\mathbf{n}} \times U^{\mathbf{m}}\right) \times U \rightarrow U$ with component at each $X$ in $\mathbb{C}$ making the diagram

$$
\begin{gather*}
\left(U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}\right) \times U(X) \xrightarrow{\rho_{X}} U(X)  \tag{11}\\
\quad\left\langle\theta_{X}, 1\right\rangle \times 1 \\
U(X)^{\mathbf{m}} \times U(X)
\end{gather*}
$$

commute.
Corollary 1.4. Each of the following types of data uniquely determine each other:
(a) a natural transformation $\gamma: W \rightarrow V$ with $1_{D_{\mathbb{X}}} \circ \gamma=1_{D_{\mathbb{A}}}^{\mathrm{m} \times D_{\mathbb{A}}}$;
(b) a natural transformation $\sigma: U^{\mathbf{m}} \times(U \times U) \rightarrow U^{\mathbf{n}}$ with component at each $X$ in $\mathbb{C}$ making the diagram

commute.

Corollary 1.5. Each of the following types of data uniquely determine each other:
(a) natural transformations $\tau: V \rightarrow W$ and $\gamma: W \rightarrow V$ with $1_{D_{\mathbb{X}}} \circ \tau=$ $1_{D_{\mathbb{A}} \mathbf{m}} \times D_{\mathbb{A}}$ and $1_{D_{\mathbb{X}}} \circ \gamma=1_{D_{\mathbb{A}}} \times D_{\mathbb{A}}$ and such that $\tau \gamma=1_{W}$;
(b) natural transformations $\rho:\left(U^{\mathbf{n}} \times U^{\mathbf{m}}\right) \times U \rightarrow U$ and $\sigma: U^{\mathbf{m}} \times(U \times U) \rightarrow$ $U^{\mathbf{n}}$ with components at each $X$ in $\mathbb{C}$ making the diagrams (11), (12) and

$$
\begin{align*}
& U(X)^{\mathbf{m}} \times(U(X) \times U(X))  \tag{13}\\
& \quad\left\langle\left\langle\sigma, \pi_{1}\right\rangle, \pi_{2} \pi_{2}\right\rangle \\
& \left(U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}\right) \times U(X) \xrightarrow[\rho_{X}]{\pi_{1} \pi_{2}} U(X)
\end{align*}
$$

commute.
Corollary 1.6. Each of the following types of data uniquely determine each other:
(a) natural transformations $\tau: V \rightarrow W$ and $\gamma: W \rightarrow V$ with $1_{D_{\mathbb{X}}} \circ \tau=$ $1_{D_{\mathbb{A}}^{\mathrm{m}} \times D_{\mathbb{A}}}$ and $1_{D_{\mathbb{X}}} \circ \gamma=1_{D_{\mathbb{A}}^{\mathrm{m}} \times D_{\mathbb{A}}}$ and such that $\gamma \tau=1_{V}$;
(b) natural transformations $\rho:\left(U^{\mathbf{n}} \times U^{\mathbf{m}}\right) \times U \rightarrow U$ and $\sigma: U^{\mathbf{m}} \times(U \times U) \rightarrow$ $U^{\mathbf{n}}$ with components at each $X$ in $\mathbb{C}$ making the diagrams (11), (12) and

$$
\begin{align*}
& \left(U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}\right) \times U(X)  \tag{14}\\
& \left\langle\pi_{2} \pi_{1},\left\langle\rho_{X}, \pi_{2}\right\rangle\right\rangle \mid \\
& U(X)^{\mathbf{m}} \times\left(U(X) \times U\left(X_{\sigma}\right)\right) \longrightarrow \\
& \pi_{1} \pi_{1} \\
& \longrightarrow
\end{align*}(X)^{\mathbf{n}}
$$

commute.
Corollary 1.7. Each of the following types of data uniquely determine each other:
(a) natural transformations $\tau: V \rightarrow W$ and $\sigma: W \rightarrow V$ with $1_{D_{\mathbb{X}}} \circ \tau=$ $1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ and $1_{D_{\mathbb{X}}} \circ \gamma=1_{D_{\mathbb{A}}} \times D_{\mathbb{A}}$ and inverse to each other;
(b) natural transformations $\rho:\left(U^{\mathbf{n}} \times U^{\mathbf{m}}\right) \times U \rightarrow U$ and $\sigma: U^{\mathbf{m}} \times(U \times U) \rightarrow$ $U^{\mathbf{n}}$ with components at each $X$ in $\mathbb{C}$ making the diagrams (11), (12), (13) and (14) commute.

We now consider the case where $\mathbf{m}=\emptyset$ and $\mathbf{n}=\{1\}$, the results proved here will be used in Section 2,

When $\mathbf{m}=\emptyset$ and $\mathbf{n}=\{1\}$, the functors $V$ and $W$ are up to isomorphism the functors $\tilde{V}, \tilde{W}: \mathbf{P t}(\mathbb{A}) \rightarrow \mathbf{P t}(\mathbb{X})$ sending $(A, B, \alpha, \beta)$ to

$$
\left(\left(U(A)\left\langle U(\alpha), \theta_{B}\right\rangle\right) \times U(B), U(B), \pi_{2},\left\langle\left\langle\theta_{A}, 1\right\rangle!_{U(B)}, 1\right\rangle\right)
$$

and

$$
(U(A), U(B), U(\alpha), U(\beta))
$$

respectively.
Corollary 1.8. Each of the following types of data uniquely determine each other:
(a) a natural transformation $\tau: \tilde{V} \rightarrow \tilde{W}$ with $1_{D_{\mathrm{X}}} \circ \tau=1_{D_{\mathrm{A}}}$ and with component at each $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{A})$ such that the diagram

commutes;
(b) a natural transformation $\rho: U \times U \rightarrow U$ with component at each $X$ in $\mathbb{A}$ making the diagram

commute.
Corollary 1.9. Each of the following types of data uniquely determine each other:
(a) a natural transformation $\gamma: \tilde{W} \rightarrow \tilde{V}$ with $1_{D_{\mathbb{X}}} \circ \gamma=1_{D_{\mathbb{A}}}$ and with component at each $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{A})$ such that the diagram

commutes;
(b) a natural transformation $\sigma: U \times U \rightarrow U$ with component at each $X$ in $\mathbb{A}$ making the diagram

commute.
Corollary 1.10. Each of the following types of data uniquely determine each other:
(a) natural transformations $\tau: \tilde{V} \rightarrow \tilde{W}$ and $\gamma: \tilde{W} \rightarrow \tilde{V}$ with $1_{D_{\mathbb{X}}} \circ \tau=1_{D_{\mathbb{A}}}$ and $1_{D_{\mathbb{X}}} \circ \gamma=1_{D_{\mathbb{A}}}$ inverse to each other and with components at each $(A, B, \alpha, \beta)$ in $\operatorname{Pt}(\mathbb{A})$ making the diagrams (15) and (17) commute;
(b) natural transformations $\rho: U \times U \rightarrow U$ and $\sigma: U \times U \rightarrow U$ with component at each $X$ in $\mathbb{A}$ making the diagrams (16), (18),

and

$$
\begin{align*}
& U(X) \times U(X)  \tag{20}\\
& \left\langle\rho_{X}, \pi_{2}\right\rangle \\
& U(X) \times U(X)_{\sigma_{X}} \xrightarrow{\pi_{1}} U(X)
\end{align*}
$$

commute.
In the sections that follows we use the fact that the set of natural transformation $U^{\mathbf{n}} \rightarrow U$ (where $\mathbf{n}=\{1, \ldots, n\}$ and $U$ is the forgetful functor from a variety to sets) is in bijection with the set of $n$-ary terms of the variety. Since this is no longer true for arbitrary internal varieties (every term determines a natural transformation but not conversely) the results in the sections that follow hold only partially in arbitrary internal varieties, i.e. the existence of certain terms determine natural transformations between appropriate $V$ and $W$ but not conversely.

## 2 Pointed varieties

In this section we apply the results from Section 1 to the special case where $\mathbb{A}=\mathcal{V}$ is a pointed variety, $\mathbb{X}=\mathbf{S e t}_{*}$ is the category of pointed sets, $U$ is the usual forgetful functor, $\mathbf{m}=\emptyset, \mathbf{n}=\{1\}$, and $\theta$ is constructed using the constant of $\mathcal{V}$.

For any category $\mathbb{C}$ we define $\operatorname{SplExt}(\mathbb{C})$ to be the category of split extensions: an object is a sextuple $(K, A, B, \kappa, \alpha, \beta)$ where $K, A$ and $B$ are objects in $\mathbb{C}$ and $\kappa: K \rightarrow B, \alpha: A \rightarrow B$ and $\beta: B \rightarrow A$ are morphisms in $\mathbb{C}$ with $(K, \kappa)$ the kernel of $\alpha$ and $\alpha \beta=1_{B}$; a morphism $(K, A, B, \kappa, \alpha, \beta) \rightarrow$ $\left(K^{\prime}, A^{\prime}, B^{\prime}, \kappa^{\prime}, \alpha^{\prime}, \beta^{\prime}\right)$ is a triple $(u, v, w)$ of morphisms $u: K \rightarrow K^{\prime}, v: A \rightarrow$ $A^{\prime}$ and $w: B \rightarrow B^{\prime}$ such that in the diagram

$v \kappa=\kappa^{\prime} u, \alpha^{\prime} v=w \alpha$ and $v \beta=\beta^{\prime} w$.
Theorem 2.1. Let $\mathcal{V}$ be a pointed variety and let $P, Q: \operatorname{SplExt}(\mathcal{V}) \rightarrow$ $\operatorname{SplExt}\left(\mathbf{S e t}_{*}\right)$ be the functors taking $(K, A, B, \kappa, \alpha, \beta)$ to $(U(K), U(K) \times$ $\left.U(B),\langle 1,0\rangle, \pi_{2},\langle 0,1\rangle\right)$ and $(U(K), U(A), U(B), U(\kappa), U(\alpha), U(\beta))$ respectively.
(a) $\mathcal{V}$ is a unital variety [1] if and only if there exists a natural transformation $P \rightarrow Q$ with component at $(K, A, B, \kappa, \alpha, \beta)$ of the form

(b) $\mathcal{V}$ is a subtractive variety [6] if and only if there exists a natural transformation $Q \rightarrow P$ with component at $(K, A, B, \kappa, \alpha, \beta)$ of the form

(c) $\mathcal{V}$ is a variety of right $\Omega$-loops if and only if there exists a natural isomorphism $P \rightarrow Q$ with component at $(K, A, B, \kappa, \alpha, \beta)$ of the form


Proof. It is easy to see that to give a natural transformation $P \rightarrow Q$ as in (a) above is the same as to give a natural transformation $\tilde{V} \rightarrow \tilde{W}$ as in (a) of Corollary 1.8 which, by Corollary 1.8 , is uniquely determined by a natural transformation $\rho: U \times U \rightarrow U$ with components making the diagram (16) commute. And, such a natural transformation determines and is determined by a binary term + such that for each $x, y$ in $X$, an algebra, $x+y=\rho_{X}(x, y)$. The commutativity of (16) then implies that $x+0=x=0+x$. The statements (b) and (c) follow from Corollaries 1.9, and 1.10 in a similar way.
Corollary 2.2. Let $\tilde{P}, \tilde{Q}: \operatorname{Pt}(\mathbb{A}) \rightarrow \mathbf{P t}(\mathbb{X})$ be the functors sending $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{A})$ to $\left(U(K \times B), U(B), U\left(\pi_{2}\right), U(\langle 0,1\rangle)\right.$ ) (where $\left.K=\operatorname{Ker}(\alpha)\right)$ and $(U(A), U(B), U(\alpha), U(\beta))$ respectively. $\mathcal{V}$ is a variety of right $\Omega$-loops if and only if there exists a natural bijection $\tilde{P} \rightarrow \tilde{Q}$ with component $(A, B, \alpha, \beta)$ of the form

$$
\begin{gathered}
U(K \times B) \underset{U(\langle 0,1\rangle)}{\stackrel{U\left(\pi_{2}\right)}{\leftrightarrows}} U(B) \\
\vdots \\
\vdots \\
U(A) \underset{U(\beta)}{U(\alpha)} U(B) .
\end{gathered}
$$

Proof. It follows from Corollary 1.7]that a natural bijection $\tilde{P} \rightarrow \tilde{Q}$ as above is completely determined by and determines binary terms $\rho(x, y)$ and $\sigma(x, y)$ satisfying the identities $\sigma(x, x)=0, \rho(\sigma(x, y), y)=x$ and $\sigma(\rho(x, y), y)=x$. Setting $x+y=\rho(\sigma(x, 0), y)$ and $x-y=\rho(\sigma(x, y), 0)$ determines terms that satisfy the right loop identities.

Remark 2.3. In fact it can be shown that $\mathcal{V}$ is a variety of right $\Omega$-loops if and only if there exists a natural isomorphism $\tilde{P} \rightarrow \tilde{Q}$.

## 3 Protomodular varieties

In this section we give a new classification of protomodular varieties by applying the results from Section $\square$ to the case where $\mathbb{A}=\mathcal{V}$ is an arbitrary variety with constants, $\mathbb{X}=$ Set is the category of sets, and $U$ is the usual forgetful functor.

Theorem 3.1. $\mathcal{V}$ is a protomodular variety if and only if for some $\mathbf{m}=$ $\{1, \ldots, m\}, \mathbf{n}=\{1, \ldots, n\}$ and $\theta$ there exist natural transformations $\tau$ : $V \rightarrow W$ and $\gamma: W \rightarrow V$ with $\tau \gamma=1_{W}$ and with components at each $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{C})$ of the form

Proof. It follows from Corollary 1.5 that natural transformations $\tau: V \rightarrow W$ and $\gamma: W \rightarrow V$ as above determine terms

$$
\rho\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}, z\right) \text { and } \sigma_{i}\left(y_{1}, \ldots, y_{m}, x, z\right) i \in \mathbf{n}
$$

satisfying the identities

$$
\begin{gathered}
\sigma_{i}\left(y_{1}, \ldots, y_{m}, x, x\right)=\theta_{i}\left(y_{1}, \ldots, y_{m}\right) i \in \mathbf{n} \\
\rho\left(\sigma_{1}\left(y_{1}, \ldots, y_{m}, x, z\right), \ldots, \sigma_{n}\left(y_{1}, \ldots, y_{m}, x, z\right), y_{1}, \ldots, y_{m}, z\right)=x
\end{gathered}
$$

For any constant $e$ we may form new terms $e_{i}=\theta_{i}(e, \ldots, e) i \in \mathbf{n}, s_{i}(x, z)=$ $\sigma_{i}(e, \ldots, e, x, z) i \in \mathbf{n}$, and $p\left(x_{1}, \ldots, x_{n}, z\right)=\rho\left(x_{1}, \ldots, x_{n}, e, \ldots, e, z\right)$. It easy to check that these terms make $\mathcal{V}$ a protomodular variety. The converse follows from Corollary 1.5 with $\mathbf{m}=\emptyset$.

Remark 3.2. The results in this section can easily be extended to $\mathcal{V}$ an infinitary variety, with $\mathbf{m}$ and $\mathbf{n}$ possibly infinite sets, giving, by Theorem 2.1 of [3], a new classification of infinitary protomodular varieties.

Remark 3.3. It could also be interesting to study when $\gamma \tau=1_{V}$ (without $\tau \gamma=1_{W}$ ) which can be seen to be equivalent to the existence of $\rho$ and $\sigma$ as above, satisfying the identities:

$$
\begin{array}{r}
\sigma_{i}\left(y_{1}, \ldots, y_{m}, x, x\right)=\theta_{i}\left(y_{1}, \ldots, y_{m}\right) i \in \mathbf{n} \\
\rho\left(\theta_{1}\left(y_{1}, \ldots, y_{m}\right), \ldots, \theta_{n}\left(y_{1}, \ldots, y_{m}\right), y_{1}, \ldots, y_{m}, x\right)=x \\
\sigma_{i}\left(y_{1}, \ldots, y_{m}, \rho\left(x_{1}, . ., x_{n}, y_{1}, . ., y_{m}, z\right), z\right)=x_{i} i \in \mathbf{n}
\end{array}
$$

instead.

## 4 General varieties

In this section we consider the case where $\mathbb{A}=\mathcal{V}$ is a variety, $\mathbb{X}=$ Set is the category sets, and $U$ is the usual forgetful functor.

For a variety $\mathcal{V}$ consider the condition:

Condition 4.1. There exist ternary terms $p$ and $q$ satisfying the identities: $p(x, x, y)=y$ and $p(q(x, y, z), z, y)=x=q(p(x, y, z), z, y)$.

It is easy to see that $q(x, x, y)=y$ follows from the conditions above, as remarked in [7, where such a variety was called a biternary system.

Remark 4.2. It is easy to see that if a variety $\mathcal{V}$ satisfies Condition 4.1 then every regular epimorphism $f: E \rightarrow B$ is up to bijection a product projection $\pi_{2}: X \times B \rightarrow B$ for some $X$ (since for each $b$ and $b^{\prime}$ choosing $e$ and $e^{\prime}$ in $f^{-1}(\{b\})$ and $f^{-1}\left(\left\{b^{\prime}\right\}\right)$ respectively gives a bijection $p\left(-, e, e^{\prime}\right): f^{-1}(\{b\}) \rightarrow$ $\left.f^{-1}(\{b\})\right)$.

Proposition 4.3. For a variety $\mathcal{V}$ the following conditions are equivalent:

1. $\mathcal{V}$ satisfies Condition 4.1;
2. There exist ternary terms $\tilde{p}$ and $\tilde{p}$ satisfying the identities: $\tilde{p}(x, x, y)=$ $y=\tilde{q}(x, x, y), \tilde{p}(x, y, y)=x=\tilde{q}(x, y, y)$ and $\tilde{p}(\tilde{q}(x, y, z), z, y)=x=$ $\tilde{q}(\tilde{p}(x, y, z), z, y)$;
3. There exists a quaternary term $u$ satisfying the identities: $u(a, b, b, a)=$ $b$ and $u(u(a, b, c, d), b, d, c)=a$;
4. There exists a quaternary term $\tilde{u}$ satisfying the identities: $\tilde{u}(a, b, b, a)=$ $b=\tilde{u}(a, a, b, a)$ and $\tilde{u}(a, b, c, c)=a=\tilde{u}(\tilde{u}(a, b, c, d), b, d, c) ;$

If in addition $\mathcal{V}$ has at least one constant, those conditions are further equivalent to:
5. For each constant e there exist binary terms $x+y$ and $x-y$ satisfying the right loop identities (for that constant e).

Proof. The implications $2 \Rightarrow 1$ and $4 \Rightarrow 3$ are trivial.
$1 \Rightarrow 2$ : Given $p$ and $q$ define $\tilde{p}(x, y, z)=p(q(x, y, y), y, z)$ and $\tilde{q}(x, y, z)=$ $p(q(x, y, z), z, z)$.
$2 \Rightarrow 4$ : Given $\tilde{p}$ and $\tilde{q}$ define $\tilde{u}(a, b, c, d)=\tilde{p}(\tilde{q}(a, b, c), d, b)$.
$3 \Rightarrow 1$ : Given $u$ define $p(x, y, z)=u(x, z, z, y)$ and $q(x, y, z)=u(x, y, z, y)$. If in addition $\mathcal{V}$ has at least one constant.
$2 \Rightarrow 5:$ Given $\tilde{p}$ and $\tilde{q}$ for each constant $e$ define $x+y=\tilde{p}(x, e, y)$ and $x-y=\tilde{q}(x, y, e)$.
$5 \Rightarrow 1$ : Given $x+y$ and $x-y$ for some constant $e$ define $p(x, y, z)=$ $q(x, y, z)=(x-y)+z$.

Remark 4.4. It follows that a variety satisfying Condition 4.1 is a Mal'tsev variety.

Theorem 4.5. (a) If $\mathcal{V}$ satisfies Condition 4.1, then for $\mathbf{n}=\{1\}, \mathbf{m}=$ $\{1\}$ and $\theta=1_{U}$ there exists a natural isomorphism $\tau: V \rightarrow W$ with component at each $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{C})$ of the form
(b) If for some $\mathbf{n}=\{1, \ldots, n\}, \mathbf{m}=\{1, \ldots, m\}$ and $\theta$ there exists a natural isomorphism $\tau: V \rightarrow W$ with component at each $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{C})$ of the form
then $\mathcal{V}$ satisfies Condition 4.1.
Proof. (a) Let $\mathbf{n}=\mathbf{m}=\{1\}$ and $\theta=1_{U}$. Given ternary terms $p$ and $q$ as in Condition 4.1, it is easy to check that $\rho=p$ and $\sigma(x, y, z)=q(y, z, x)$ define natural transformations making the diagrams (11), (12), (13) and (14) commute. Therefore by Corollary 1.7 determine a natural isomorphism $V \rightarrow W$, as required.
(b) If for some $\mathbf{n}=\{1, \ldots, n\}, \mathbf{m}=\{1, \ldots, m\}$ and $\theta$ there exists an isomorphism $V \rightarrow W$ then by Corollary 1.7 there exist terms $\rho\left(x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}, z\right)$ and $\sigma_{i}\left(y_{1}, \ldots, y_{m}, x, z\right) i \in \mathbf{n}$ satisfying the identities:

$$
\begin{gathered}
\sigma_{i}\left(y_{1}, \ldots, y_{m}, x, x\right)=\theta_{i}\left(y_{1}, \ldots, y_{m}\right) \\
\rho\left(\sigma_{1}\left(y_{1}, \ldots, y_{m}, x, z\right), \ldots, \sigma_{n}\left(y_{1}, \ldots, y_{m}, x, z\right), y_{1}, \ldots, y_{m}, z\right)=x \\
\sigma_{i}\left(y_{1}, \ldots, y_{m}, \rho\left(x_{1}, . ., x_{n}, y_{1}, . ., y_{m}, z\right), z\right)=x_{i} .
\end{gathered}
$$

Let $p$ and $q$ be the terms defined by

$$
\begin{aligned}
p(x, y, z) & =\rho\left(\sigma_{1}(y, \ldots, y, x, y), \ldots, \sigma_{n}(y, \ldots, y, x, y), y, \ldots, y, z\right) \\
q(x, y, z) & =\rho\left(\sigma_{1}(z, \ldots, z, x, y), \ldots, \sigma_{n}(z, \ldots, z, x, y), z, \ldots, z, z\right) .
\end{aligned}
$$

It is easy to check that $p$ and $q$ satisfy the desired identities as in Condition 4.1

Recall that for any category $\mathbb{C}$ the functor $D_{\mathbb{C}}$ is the functor $\operatorname{Pt}(\mathbb{C}) \rightarrow \mathbb{C}$ taking $(A, B, \alpha, \beta)$ to $B$.

Theorem 4.6. Let $V$ and $W$ be the functors defined in Section $\square$ with $\mathbb{A}=\mathcal{V}, \mathbb{X}=\mathbf{S e t}, U$ the usual forgetful functor, $\mathbf{n}=\mathbf{m}=\{1\}$ and $\theta=1_{U}$. Let $P, Q:\left(\mathbb{A} \downarrow D_{\mathbb{A}}\right) \rightarrow \mathbf{P t}(\mathbb{X})$ be the functors sending $(E,(A, B, \alpha, \beta), f)$ to

$$
\left(U\left(A_{\langle\alpha, f\rangle}^{\times} E\right) \times U(B), U(E) \times U(B), U\left(\pi_{2}\right) \times 1, U(\langle\beta f, 1\rangle) \times 1\right)
$$

and

$$
(U(E) \times U(A), U(E) \times U(B), 1 \times U(\alpha), 1 \times U(\beta))
$$

respectively. The following are equivalent:

1. There exists an isomorphism $\tau: V \rightarrow W$ with component at each $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{A})$ of the form
2. There exists an isomorphism $\chi: P \rightarrow Q$ with component at each $(E,(A, B, \alpha, \beta), f)$ in $\left(\mathbb{A} \downarrow D_{\mathbb{A}}\right)$ of the form

$$
\begin{gathered}
\left(U\left(A_{\langle\alpha, f\rangle}^{\times} E\right) \times U\left(\underset{U(\langle\beta f, 1\rangle) \times 1}{\stackrel{U\left(\pi_{2}\right) \times 1}{\rightleftarrows}} U(E) \times U(B)\right.\right. \\
\vdots \\
\vdots \\
U(E) \times U(A) \underset{1 \times U(\beta)}{\stackrel{1}{\gtrless}} \stackrel{1 \times U(\alpha)}{\underset{1 \times(\alpha)}{\rightleftarrows}} U(E) \times U(B)
\end{gathered}
$$

3. $\mathcal{V}$ satisfies Condition 4.1.

Proof. The equivalence of 1 and 3 follows from Theorem 4.5, It is easy to show that $2 \Rightarrow 1$ since $P$ and $Q$ composed with the functor sending $(A, B, \alpha, \beta)$ in $\mathbf{P t}(\mathbb{A})$ to $\left(B,(A, B, \alpha, \beta), 1_{B}\right)$ in $\left(\mathbb{A} \downarrow D_{\mathbb{A}}\right)$ are up to natural isomorphism the functors $V$ and $W$ respectively. We will show that $3 \Rightarrow 2$.

Let $p$ and $q$ be ternary terms as in Condition 4.1. It is easy to check that $\chi$ with component at each $(E,(A, B, \alpha, \beta), f)$ defined by $\chi_{(E,(A, B, \alpha, \beta), f)}=$ $\left(\varphi_{(E,(A, B, \alpha, \beta), f)}, 1_{U(B)}\right)$ where $\varphi_{(E,(A, B, \alpha, \beta), f)}((a, e), b)=(e, p(a, \beta f(e), \beta(b)))$ is an isomorphism with inverse $\chi_{(E,(A, B, \alpha, \beta), f)}^{-1}=\left(\psi_{(E,(A, B, \alpha, \beta), f)}, 1_{U}(B)\right)$ where $\psi_{(E,(A, B, \alpha, \beta), f)}(e, a)=((q(a, \beta \alpha(a), \beta f(e)), e), \alpha(a))$.

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