Dedekind complete posets from sheaves on von Neumann algebras

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Abstract

We show that for any two von Neumann algebras M and N, the space of non-unital normal homomorphisms $N \to M$ with finite support, modulo conjugation by unitaries in M, is Dedekind complete with respect to the partial order coming from the addition of homomorphisms with orthogonal ranges; this ties in with work by Brown and Capraro, where the corresponding objects are given Banach space structures under various niceness conditions on M and N. More generally, we associate to M a Grothendieck site-type category, and show that presheaves satisfying a sheaf-like condition on this category give rise to Dedekind complete lattices upon modding out unitary conjugation. Examples include the above-mentioned spaces of morphisms, as well as analogous spaces of completely positive or contractive maps. We also study conditions under which these posets can be endowed with cone structures extending their partial additions.

Keywords: von Neumann algebra, poset, lattice, Dedekind complete, sheaf, sketch, cone

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Introduction

Organizing objects one is interested in into "moduli spaces" and then endowing such spaces with as much structure as possible is one of the more familiar themes in mathematics. As an instance of this, with the objects in question being operator algebra homomorphisms, Brown studies in [2] the space Hom(A, B) of normal unital homomorphisms between two W^* -algebras A, B up to conjugation by unitaries in B (and similarly a C^* version). As it turns out, when B is an ultrapower R^{ω} of the hyperfinite Π_1 factor and A is a separable, embeddable Π_1 factor (so as to define away Connes' embedding problem, stating that any separable Π_1 factor embeds in such a manner), this space exhibits a rich structure indeed: It can be given a complete metric and a convex structure compatible with this metric in the sense of [2, 2.1].

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This analysis is continued in [3]. The authors show that the similarly-defined moduli space of (this time non-unital) homomorphisms $N \to M \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ with finite supporting projection can be organized naturally as a cancellable semigroup. Moreover, when N is a separable Π_1 factor and $M = X^{\omega}$ for some McDuff factor X (i.e. $X \cong X \otimes R$, for the spatial von Neumann algebra tensor product), this semigroup can be made into a cone, and its enveloping group then becomes a vector space. Furthermore, the same type of metric as in [2] makes this vector space into a Banach space.

The motivation for the present paper is the following question: Given that the above constructions provide (a) a Banach space and (b) a cone sitting naturally inside of it, is it the case that the resulting partial order makes the Banach space into a Banach lattice? Once one is led to ask this though, the problem quickly becomes dissociated into two more or less independent halves: There is the Banach space aspect, perhaps requiring technical tools like McDuff-ness, and there is the question of whether or not the order on $\operatorname{Hom}(N, M \otimes B(\ell^2(\mathbb{N})))$ coming from its semigroup structure (which makes sense under very general circumstances) is complete. The main goal here is to answer this latter question affirmatively, making the connection with the former problem only towards the end.

As we explain in Section 2 (at the very end, after Proposition 2.9), the results on completeness may be regarded as providing some small measure of evidence that the Banach spaces introduced in [3] have preduals. This, in turn, would answer affirmatively a question by Sorin Popa, to the effect that embeddable factors have embeddings in R^{ω} with factorial commutant. Secondly, the same completeness results have consequences on the construction of the vector space structure itself, simplifying it somewhat; this is taken up in Section 3.

The contents of the paper are as follows:

In an attempt to isolate the main abstract features of the problem, in Section 1 we set up a framework where invariants of a von Neumann algebra M such as $\operatorname{Hom}(N,M)$ mentioned above or similar moduli spaces of completely positive or completely bounded maps $N \to M$ can be treated simultaneously. This will require associating to M a category \mathcal{C}_M with distinguished cocones (a structure referred to as a 'sketch'; see below), and recasting these invariants as "sheaves" on this sketch. This is spelled out and clarified in Section 1, but sheaves are essentially presheaves on \mathcal{C}_M which "respect the cocone structure". This will be more of a preliminary section, aimed at fixing the general setup; the only result we actually state here is Proposition 1.16, and for the proof we refer to that of [3, 9], which can easily be paraphrased into the present setting.

Section 2 contains the first main result of the paper, Theorem 2.5. It is stated in the general language of sheaves on sketches, but the connection to the previous discussion is as follows: For any two W^* -algebras M, N whatsoever, the set of non-unital morphisms $N \to M$ with finite supporting projection modulo conjugation by unitaries in M is a Dedekind complete poset with respect to the ordering induced by its partial abelian monoid structure (Proposition 1.16). This means that any subset, regardless of cardinality, has a largest lower (least upper) bound as soon as it is bounded from below (respectively above).

In Section 3 we move to the "continuous" side of the problem, addressing the issue of when the monoids from the preceding section can be endowed with something like a vector space structure. The precise notion we need is that of partial cone structure, i.e. action of the partial semiring ($[0,1],+,\cdot$), and the relevant result is Theorem 3.7. It follows, only from the results on the completeness of the ordering, that such cone structures are unique when they exist. In conclusion, not much care is required when constructing one.

Finally, Section 4 contains only announcements of results to be proven elsewhere. Since the somewhat abstract categorical machinery of Section 1 was partly justified by the natural desire to study completely positive or contractive maps along with W^* morphisms, it seems fair to point out that in fact, these do not give rise to fundamentally new invariants. Specifically (Proposition 4.1),

for von Neumann algebras M, N, the set of completely positive maps $N \to M$ is (functorially in M and N) in bijection with W^* morphisms $\widetilde{N} \to M$ for some von Neumann algebra \widetilde{N} ; similarly for completely contractive maps.

The point of Section 1, however, is precisely that there is an abstract framework that will instantly accommodate all of these invariants (and similar ones that might come up); one need not go through results in the spirit of Proposition 4.1, which are maybe a bit surprising and not immediately obvious.

Acknowledgements

I would like to thank Valerio Capraro for bringing many of these problems to my attention (my interest was sparked following a talk he gave at UC Berkeley in the spring of 2013) and for his patient explanations on the contents [3]. Thanks to Andre Kornell are also in order, for numerous discussions on the results covered in Section 4.

1 The canonical sketch and sheaves on it

Throughout, capital letters such as M, N, etc. denote W^* -algebras. We will be taking for granted the basics of the theory of von Neumann algebras (a term used here as a synonym for ' W^* -algebra'), as in [14], say. There are portions of the paper (e.g. the end of Section 2), offered mostly as motivation, where some of the main characters relating to Connes' embedding problem enter the discussion (the hyperfinite II_1 factor, ultrafilters, etc.). For these I recommend the preliminary section of [4], where all of the relevant notions are gathered together and presented very accessibly and to the point.

All maps between von Neumann algebras are assumed to be weak* continuous (or normal), an assumption that will always be implicit, even when not repeated explicitly. W^* morphisms or homomorphisms are *-preserving, multiplicative maps. They may or may not be unital, depending on context; it will always be clear which case we are considering.

Whenever the symbol ' \otimes ' appears between two W^* -algebras, it denotes spatial tensor products (e.g. [14, IV.5]). The matrix algebra $M_n = M_n(\mathbb{C})$ is given the usual *-structure, via the transpose conjugate.

We mention completely positive and completely contractive maps between W^* -algebras only very briefly, in Section 2 and Section 4. Recall that a map $\theta: N \to M$ between W^* -algebras is a complete contraction [15, p. 167] if the maps $\theta \otimes \operatorname{id}: N \otimes M_n(\mathbb{C}) \to M \otimes M_n(\mathbb{C})$ all have norm ≤ 1 . Similarly, θ is completely positive [14, IV.3] if $\theta \otimes \operatorname{id}$ are all positive, in the sense that they turn positive elements $x^*x \in N \otimes M_n$ into positive elements $y^*y \in M \otimes M_n$. Just like W^* morphisms, completely positive or contractive maps may or may not be unital (i.e. send 1 to 1), and we will specify or otherwise make it clear when they do and when they do not.

We also assume some of the standard background on category theory, as covered by [9]; this means functors, natural transformations, (co)limits, adjoint functors etc. In addition, [1] is a natural source for some of the concepts we are about to review. We write C(c, c') for the set of morphisms $c \to c'$ in a category C.

Let \mathcal{C} be a category, and $\mathcal{D} \subseteq \mathcal{C}$ a subcategory. A *cocone* on \mathcal{D} in \mathcal{C} (simply called a cone in [9, III.3]) consists of an object $\bullet \in \mathcal{C}$ and a natural transformation from the inclusion $\mathcal{D} \subseteq \mathcal{C}$ to the functor $\mathcal{D} \to \mathcal{C}$ constantly equal to \bullet . Explicitly, this means each object $d \in \mathcal{D}$ is endowed with an

arrow $d \to \bullet$ in \mathcal{C} , and for each arrow $d \to d'$ in \mathcal{D} the resulting triangle

$$d \xrightarrow{} d'$$

is commutative. Cones are defined dually, and consist of arrows $\bullet \to d$, etc. Finally, a cone is *limiting* if it makes \bullet into the limit of the functor $\mathcal{D} \subseteq \mathcal{C}$ ([9, III.4]).

1.1 Definition A *sketch* consists of a small category \mathcal{C} together with a collection of distinguished cocones. We typically abuse terminology and refer to \mathcal{C} alone as being the sketch, with the cocones being implicit.

A sheaf on a sketch \mathcal{C} is a contravariant functor $\mathcal{C} \to \operatorname{Set}$ which turns the distinguished cocones into limiting cones.

Because the data on C consists of cocones only (as opposed to both cones and cocones), our sketches are referred to as *colimit* sketches in [1, 2.55]. The current usage is consistent with that in [6], where colimit sketches are similarly the main characters.

1.2 Example Some of the more ubiquitous examples of sketches are the so-called sites, also known as Grothendieck topologies [10, III]. These are small categories whose objects and arrows are supposed to be thought of as the open sets of some kind of "space", and inclusions between them, respectively. By definition, they come endowed with distinguished cocones, which heuristically indicate which collections of open sets "cover" other open sets. Finally, what's typically referred to as a sheaf on a site [10, III.4] is precisely a sheaf in the sense of Definition 1.1.

This is the example that motivated the term 'sheaf' in [6, 2.2.1] on a sketch, even in cases (such as that of Definition 1.5 below) when the sketch is not necessarily a site.

- **1.3 Definition** Let M be a von Neumann algebra. The *finite subalgebra category* \mathcal{C}_M of M has as objects the not-necessarily-unital W^* -subalgebras of M with finite supporting projection. A morphism $A \to B$ between such objects is a partial isometry $u \in M$ such that:
 - the initial space u^*u of u is the supporting projection 1_A of A;
 - the final space uu^* is dominated by the supporting projection 1_B of B (i.e. $uu^* \leq 1_B$ in the usual partial ordering on projections);
 - u conjugates A into B, in the sense that $uAu^* \leq B$.

Composition of morphisms is defined in the obvious manner, as multiplication of the underlying partial isometries.

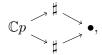
1.4 Remark Definition 1.3 works just fine for arbitrary subalgebras, with perhaps infinite supporting projection, but finiteness is a technical tool need below (e.g. it is crucial to the cancellability results [3, 9] and Proposition 1.16; see also Example 1.14).

We sometimes drop the adjective 'finite' and just refer to \mathcal{C}_M as the subalgebra category.

Finally, we come to the main definition of this section:

1.5 Definition The *canonical sketch* on the subalgebra category \mathcal{C}_M of M consists of the following cocones:

(a) pushout-type ones of the form



where $p \in M$ is a projection, the \sharp 's are ≤ 2 -dimensional algebras generated by subprojections $q \leq p$ and p-q (possibly different q's for the two \sharp 's), \bullet is the subalgebra generated by the two \sharp 's, and the arrows are inclusions, i.e. as partial isometries, they are all equal to p.;

(b) coproduct-type ones, consisting of the inclusions $A_i \to \bigoplus A_i$ into the direct sum of a family of mutually orthogonal subalgebras $A_i \leq M$, $i \in I$. I can be any set whatsoever, including empty; the direct sum is then understood to be the zero subalgebra of M.

Recall that sketches are supposed to be cones in \mathcal{C}_M . Hence, it is implicit in Definition 1.5 that all supporting projections in sight (such as that of $\bigoplus A_i$) are finite in M.

1.6 Remark Definition 1.5 follows a familiar pattern when dealing with (co)limits of functors: All colimits can be built out of just pushouts and arbitrary coproducts (and dually, limits can be constructed using pullbacks and products; this is a variant of [9, V.2]), so cocones of the types (a) and (b) are what one might expect to work with.

Note that when referring to (a) and (b) as 'pushout-type' and 'coproduct-type', the terms are only intended to suggest analogies; these cocones are not, in general, genuine colimits in \mathcal{C}_M . The sheaf condition (Definition 1.1) will demand that they be turned into limits in SET however.

- **1.7 Definition** For a W^* -algebra M, a sheaf on M is a sheaf in the sense of Definition 1.1 on the sketch \mathcal{C}_M .
- **1.8 Remark** In Definition 1.5, we allowed the indexing set I of a coproduct-type cocone to be empty. The corresponding condition on a sheaf is that it turn the zero algebra into the terminal object in Set, i.e. the singleton.

There are a few examples of sheaves on M that the reader should have in mind, and which provided the motivation for these definitions to begin with:

1.9 Example Let N be another W^* -algebra, and consider the sheaf W_N^* sending a finitely-supported subalgebra $A \leq M$ to the set of unital W^* morphisms $N \to 1_A M 1_A$ whose image commutes with A (we will simply say that the maps themselves commute with A).

This is easily seen to define a persheaf on \mathcal{C}_M (i.e. a contravariant functor into SET) essentially because whatever commutes with an algebra also commutes with a smaller algebra. We leave it to the reader to check that the presheaf turns the cocones of Definition 1.1 into limits in SET.

- **1.10 Example** We can play the exact same game with completely positive maps rather than W^* morphisms. Indeed, for N as before, we can define the sheaf CP_N (CP for completely positive) sending $A \in \mathcal{C}_M$ to the set of unital completely positive maps $N \to 1_A M 1_A$ commuting with $A \blacklozenge$
- 1.11 Example Repeat previous example with completely contractive maps to get a sheaf CC_N .
- **1.12 Remark** If C_M is heuristically a site, we ought to be able to make sense of more sophisticated notions usually associated to sites, such as, say, stacks. These would be "categorified" sheaves, in

the sense of being contravariant functors from \mathcal{C}_M to the bicategory of categories, functors, and natural transformations, with an additional compatibility with the cocones (e.g. [8]).

Associating to $A \in \mathcal{C}_M$ the category of $A-1_AM1_A$ Hilbert space bimodules (i.e. correspondences from 1_AM1_A to A in the sense of Connes' [7, Chapter 5, Appendix B]) produces precisely such a stack-like object. We will not elaborate, as these ideas play no further role in the paper.

Finally, we define objects analogous to the morphisms-up-to-conjugation sets of [2, 3].

1.13 Definition A morphism $u: A \to B$ in \mathcal{C}_M is a *pseudoisomorphism* if the final space uu^* of the underlying partial isometry u is precisely 1_B (as opposed to just being dominated by 1_B ; cf. Definition 1.3).

Let \mathcal{F} be a sheaf on M. The moduli space of \mathcal{F} -objects $\pi(\mathcal{F})$ is a set defined as follows: The elements are equivalence classes of elements in the disjoint union

$$\coprod_{A\in\mathcal{C}_M}\mathcal{F}(A),$$

where $x \in \mathcal{F}(A)$ and $y \in \mathcal{F}(B)$ are equivalent if there is some pseudoisomorphism $u : A \to B$ in \mathcal{C}_M such that $x = \mathcal{F}(u)(y)$.

We write either [x] or just plain x for the class of $x \in \mathcal{F}(A)$ in $\pi(\mathcal{F})$.

1.14 Example Applying π to the sheaf W_N^* of Example 1.9 produces exactly the set of non-unital morphisms $N \to M$ up to conjugation by unitaries in M. This is because even though in Definition 1.13 we are conjugating by partial isometries, an equivalence by partial isometries between *finite* projections can always be extended to a unitary equivalence [14, V.1.38].

So far $\pi(\mathcal{F})$ is only a set, but it is naturally endowed with additional structure.

Suppose first that $A, B \in \mathcal{C}_M$ are such that the projections 1_A and 1_B are orthogonal (we simply say that A and B themselves are orthogonal). Then, by the sheaf conditions, $\mathcal{F}(A \oplus B)$ is the product of the sets $\mathcal{F}(A)$ and $\mathcal{F}(B)$. Hence, for every $x \in \mathcal{F}(A)$, $y \in \mathcal{F}(B)$, it makes sense to define $x + y \in \mathcal{F}(A \oplus B)$ as the unique element projecting on x and y through the canonical maps.

More generall, if $u1_Bu^*$ is orthogonal to 1_A for some partial isometry u, $u^*u = 1_B$, then one can still make sense of x + y at the level of $\pi(\mathcal{F})$ by defining it as above in $\mathcal{F}(A \oplus uBu^*)$. The definition, it turns out, does not depend on u as long as B is moved orthogonally to A (cf. [3, Definition 4]). We record all of this in the next proposition. The only perhaps non-obvious claim is the cancellability of the partial addition; we do not repeat the argument from [3, Proposition 9], which contains all of the needed ingredients and can easily be adapted here.

First, one more piece of terminology. This is a very guessable definition, but in any event:

1.15 Definition Let G be a set endowed with a partial binary operation '+' and an element $0 \in G$. We say that (G, +, 0) is a partial abelian monoid if

- Whenever x + y is defined so is y + x, and the two are equal;
- Whenever one of (x + y) + z and x + (y + z) is defined so is the other, and they are equal;
- 0 + x = x for all $x \in G$ (in particular, 0 + x is always defined).

A partial monoid G as above (dropping the adjective 'abelian' and abusing notation by omitting '+' and '0') is cancellable or has cancellation if x + y = x + z implies y = z.

Recall from Remark 1.8 that for any sheaf \mathcal{F} , the image of the zero algebra $\{0\}$ is a singleton.

1.16 Proposition For any sheaf \mathcal{F} on M, the partial operation '+' defined above on $\pi(\mathcal{F})$ makes the latter into a partial abelian monoid, with 0 being the (class in $\pi(\mathcal{F})$ of the) single element of $\mathcal{F}(\{0\})$. Moreover, $\pi(\mathcal{F})$ is cancellable.

Cancellation, in particular, will play a crucial role in the sequel.

- **1.17 Remark** The supporting projection of an element $x \in \pi(\mathcal{F})$ is well defined up to unitary equivalence, and associating it to x consistutes a morphism (in the obvious sense) from the partial monoid $\pi(\mathcal{F})$ to that of equivalence classes of finite projections in M.
- **1.18 Remark** Note that in fact there is even more structure on $\pi(\mathcal{F})$ than Proposition 1.16 lets on: It makes sense to add *arbitrary* families $x_i \in \pi(\mathcal{F})$, so long as their supporting projections can be moved inside M so that they form an orthogonal family.

We will also talk freely about possibly infinite sums of elements $x_i \in \mathcal{F}(A_i)$ (so at the level of \mathcal{F} rather than $\pi(\mathcal{F})$) when the algebras A_i are mutually orthogonal.

2 Sheaves give rise to Dedekind complete posets

We fix a von Neumann algebra M and a sheaf \mathcal{F} on it for the remainder of the paper, and freely resort to the notation introduced in the previous section, including '+' for the partial addition on $\pi(\mathcal{F})$. For projections $p \in M$, we write $\mathcal{F}(p)$ as short for $\mathcal{F}(\mathbb{C}p)$.

As mentioned at the beginning, the paper is mostly about the partial order $(\pi(\mathcal{F}), \leq)$ resulting from the monoid structure:

2.1 Definition For $x, y \in \pi(\mathcal{F})$, we write $x \leq y$ if there is some $z \in \pi(\mathcal{F})$ such that x + z = y.

A priori, ' \leq ' is only a binary relation. As expected however, we have:

2.2 Lemma $(\pi(\mathcal{F}), \leq)$ is a poset.

Proof Reflexivity and transitivity are clear (the former because of the existence of an additive zero for '+', and the former by associativity). We only need to argue that ' \leq ' is antisymmetric.

Suppose $x \le y \le x$. By definition, we have y = x + z and x = y + w = x + z + w. By cancellation we get z + w = 0, and hence (as can easily be seen just from the definition of $\pi(\mathcal{F})$) z = w = 0.

We now start focusing on the completeness properties of the poset $(\pi(\mathcal{F}), \leq)$ that are the main topic of the paper. First:

2.3 Definition A poset (P, \leq) (or its underlying order \leq) is *Dedekind complete* if every non-empty subset bounded from below has a largest lower bound and dually, every non-empty subset bounded from above has a least upper bound.

We write $\bigwedge S$ and $\bigvee S$ for the largest lower bound and the least upper bound of a set S respectively.

2.4 Remark This is a somewhat non-standard definition: Dedekind completeness is usually considered only in the context of lattices, when pairs of elements are assumed to always have greatest lower and least upper bounds. This is not implicit in Definition 2.3, and for good reason.

As we will see later (Theorem 2.5), $(\pi(\mathcal{F}), \leq)$ always has largest lower bounds for arbitrary subsets, but not least upper bounds, even for pairs of elements. This is why the relevant notion here is that of Definition 2.3.

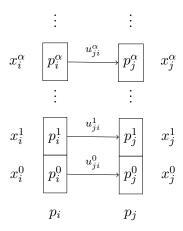
2.5 Theorem Keeping the notations fixed above, $(\pi(\mathcal{F}), \leq)$ is Dedekind complete.

The proof consists of a series of lemmas, which we then assemble together to get the conclusion. First, we tackle the issue of greatest lower bounds. The next result gets us only partly there.

- **2.6 Lemma** Let $x_i \in \mathcal{F}(p_i)$, $i \in I$ be an arbitary non-empty family for projections $p_i \in M$. The set of lower bounds for the classes $[x_i] \in \pi(\mathcal{F})$ has maximal elements.
- **2.7 Remark** The assertion is not that a greatest lower bound exists, which is the ultimate goal, but rather that maximal elements exist. These are elements not dominated strictly by any other element, but they themselves need not dominate everyone else, a priori.

Proof We construct such a maximal element by a transfinite recursive process. The steps are indexed by ordinal numbers α , and the α 'th step produces ever-increasing families p_i^{β} , $\beta < \alpha$ of projections and elements $x_i^{\beta} \in \mathcal{F}(p_i^{\beta})$ such that

- For each $i \in I$, the projections $p_i^{\beta} \leq p, \, \beta < \alpha$ are mutually orthogonal;
- The x_i^{β} 's are summands of the x_i 's in the sense of Remark 1.18;
- For every ordinal number $\beta < \alpha$ and every pair $i, j \in I$ of indices, there are isomorphisms $u_{ji}^{\beta} : \mathbb{C}p_i^{\beta} \cong \mathbb{C}p_j^{\beta}$ in \mathcal{C}_M (u_{ji}^{β} being partial isometries);
- The isomorphisms u_{ji}^{β} intertwine the objects x_i^{β} , in the sense that x_i^{β} is equal to $\mathcal{F}(u_{ji}^{\beta})(x_j^{\beta})$.



In step $\alpha = 0$ there is nothing to do: We already have an empty family of projections. Next, we need to describe what to do in step α , having completed all previous steps.

Case 1: Assume first that α is a successor ordinal, with predecessor α_- . We have the families p_i^{β} , x_i^{β} for $\beta < \alpha_-$ from the previous inductive step. By mutual orthogonality, we can make sense, for each i, of the infinite sum y_i of all x_i^{β} 's (see Remark 1.18). It is an element of $\mathcal{F}(q_i)$ for $q_i = \bigoplus_{\beta} A_i^{\beta}$.

Now denote $r_i = p_i - q_i$. By the sheaf condition demanding that \mathcal{F} turn direct sums of orthogonal subalgebras into direct products, we can find $z_i \in \mathcal{F}(r_i)$ such that $x_i = y_i + z_i$. If the set $[z_i]$, $i \in I$ has no non-zero lower bounds, terminate the induction. If there is some non-zero lower bound, then we would be able to find non-zero subprojections $r'_i \leq r_i$, elements $w_i \in \mathcal{F}(r'_i)$, and isometries

 $u_{ji}^{\alpha_-}$ in \mathcal{C}_M between the r_i' 's intertwining the w_i 's such that the common class $[w_i]$ is a lower bound for all z_i . The prescription for the recursion is to take $x_i^{\alpha_-} = w_i$ and $p_i^{\alpha_-} = r_i'$.

Case 2: Now suppose α is a limit ordinal. Since the previous steps $\beta < \alpha$ already provide a family of subalgebras indexed as desired (by all ordinals less than α), simply do nothing and move on to the next ordinal number.

Since successor ordinals increase the family of mutually orthogonal non-zero subalgebras strictly, the process must stop at some point. When it does denote

$$\mathcal{F}\left(\bigoplus_{\beta} p_i^{\beta}\right) \ni y_i = \sum_{\alpha} x_i^{\alpha}.$$

The resulting elements $[y_i] \in \pi(\mathcal{F})$ coincide because the y_i 's are intertwined by the \mathcal{C}_M -isomorphisms $\bigoplus_{\beta} u_{ji}^{\beta}$. Moreover, this common $[y_i]$ is by construction majorized by all $[x_i]$'s and is maximal with this property (or else we could continue the recursion).

We are now ready to complete the greatest lower bound half of Theorem 2.5.

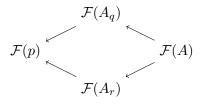
2.8 Lemma For any family $x_i \in \mathcal{F}(p_i)$, the set $\{[x_i]\} \subseteq \pi(\mathcal{F})$ has a greatest lower bound.

Proof Let [y] be a maximal lower bound for the $[x_i]$'s, provided by Lemma 2.6. We can think of [y] as the common class $[y_i]$ for some $y_i \in \mathcal{F}(q_i)$, $q_i \leq p_i$. Letting $[z] = [z_i] \leq [x_i]$ be the common class of $z_i \in \mathcal{F}(r_i)$ for some subprojections $r_i \leq p_i$, we have to show that $[z] \leq [y]$ in $\pi(\mathcal{F})$.

Cancellability allows us to subtract a maximal lower bound of [y] and [z] from all $[x_i]$, and hence to assume without loss of generality that $[y] \wedge [z] = 0$. Under this hypothesis, the new goal is to show that [z] = 0.

Now fix $i \in I$. We will be working with x_i , y_i , p_i , etc., but drop the subscript to streamline the notation.

The fact that [x] has both [y] and [z] as summands means that $x \in \mathcal{F}(p)$ is in the image of both $\mathcal{F}(\mathbb{C}q \oplus \mathbb{C}(p-q)) \to \mathcal{F}(p)$ and $\mathcal{F}(\mathbb{C}r \oplus \mathbb{C}(p-r)) \to \mathcal{F}(p)$ (the maps associated by \mathcal{F} to the inclusions of $\mathbb{C}p$ into the two \leq 2-dimensional algebras A_q and A_r obtained by cutting p by q and r respectively). The pushout-type sheaf condition (a) of Definition 1.5 says that the diagram



is a pullback in SET, where A is the algebra generated by p, q and r. By a slight abuse of notation, we write $x - y \in \mathcal{F}(p - q)$ and $x - z \in \mathcal{F}(p - r)$ for the summands of x complementary to y and z respectively. The pairs

$$(y, x - y) \in \mathcal{F}(A_q) \cong \mathcal{F}(q) \times \mathcal{F}(p - q)$$

and

$$(z, x - z) \in \mathcal{F}(A_r) \cong \mathcal{F}(r) \times \mathcal{F}(p - r)$$

both get mapped to $x \in \mathcal{F}(p)$ through the left hand side of the above diagram, and so, by the pullback property, there is a unique $w \in \mathcal{F}(A)$ mapped to (y, x - y) and (z, x - z) by the right hand side of the same diagram.

Observe that q and r intersect trivially. Indeed, the intersection is a central projection in A, and the summand of w corresponding to this central projection is a common summand of y and z. The claim follows from our assumption $[y] \wedge [z] = 0$.

Now, since q, r are projections with trivial intersection in a von Neumann algebra A, there is a partial isometry $u \in A$ moving r orthogonally to q: Simply take u to be the partial isometry component in the polar decomposition of (p-q)r (this is a familiar trick; see e.g. [14, V.1.6]). Since u implements an automorphism of $A \in \mathcal{C}_M$, it doesn't change the class [w] = [x]. But then it follows that $[y] + [z] \leq [x]$, and so, by the maximality of [y] = [z] = 0.

We are now ready to prove the main result of this section.

Proof of Theorem 2.5 Lemma 2.8 takes care of half the theorem. The result still left to prove is that for an arbitrary family $x_i \in \mathcal{F}(p_i)$, $i \in I$ such that $[x_i]$ have a common upper bound [x], $x \in \mathcal{F}(p)$, the set $\{[x_i]\}$ has a least upper bound.

By Lemma 2.8, the $[x] - [x_i]$'s (which make unambiguous sense by the cancellability claim in Proposition 1.16) have a greatest lower bound, say [x] - [y] for some $[y] \le [x]$. The claim now is that [y] is a least upper bound for the $[x_i]$'s. In other words, we want to show

$$[z] \ge [x_i], \ \forall i \Rightarrow [z] \ge [y].$$

Substituting $[z] \wedge [x]$ for [z] (the former exists by Lemma 2.8 again), we may as well assume $[z] \leq [x]$. But then $[z] \geq [x_i]$ is equivalent to $[x] - [z] \leq [x] - [x_i]$, which in turn, by the greatest lower bound property of [x] - [y], is equivalent to $[x] - [z] \leq [x] - [y]$. In turn, this gets us the desired result $[z] \geq [y]$.

Finally, the following observation is needed in the next section.

2.9 Proposition For any sheaf \mathcal{F} on a von Neumann algebra M and any $y, z \in \pi(\mathcal{F})$ with a common upper bound, we have

$$y \lor z - y = z - y \land z. \tag{1}$$

As usual, subtraction makes sense by cancellation. The greatest lower bound always exists by Lemma 2.8, while the least upper bound exists by Theorem 2.5 and the assumption that y and z have at least one common upper bound.

Proof First, we reduce to the case $y \wedge z = 0$.

Note that $y \lor z - y \land z$ is greater than both $y - y \land z$ and $z - y \land z$, and hence

$$y \lor z - y \land z \ge (y - y \land z) \lor (z - y \land z).$$

in particular, the sum

$$(y - y \wedge z) \vee (z - y \wedge z) + y \wedge z \tag{2}$$

is defined, and it is at most $y \vee z$. On the other hand, (2) is clearly greater than (or equal to) both y and z, and so

$$(y - y \wedge z) \vee (z - y \wedge z) = y \vee z - y \wedge z. \tag{3}$$

Now, denoting $y - y \wedge z$ by y' and $z - y \wedge z$ by z', (1) becomes (via (3)) $y' \vee z' = y' + z'$, under the assumption $y' \wedge z' = 0$. We henceforth revert to y and z, make the additional assumption $y \wedge z = 0$ (as announced at the beginning of the proof), and seek to show $y \vee z = y + z$.

The proof of the latter claim is essentially contained in that of Lemma 2.8. Realize y, z and some common upper bound ($y \lor z$, say) as classes of elements in $\mathcal{F}(q)$, $\mathcal{F}(r)$ and $\mathcal{F}(p)$ respectively,

where p is a projection in M and $q, r \leq p$ are sub-projections. Then, as in the proof of the lemma, r can be moved orthogonally to q by some partial isometry in the algebra A generated by q, r and p. The conclusion, as in the last line of the proof of the lemma, is $y + z \leq y \vee z$.

We end this section by pointing to one possible interpretation of Theorem 2.5.

Specialize for the remainder of the section to the main case studied in [2]: M is an ultrapower R^{ω} of the hyperfinite II_1 factor R (see [4, §2]) and N is a separable II_1 factor. The invariant studied by Brown is a close relative of $\pi(W_N^*)$; it is defined in precisely the same way, out of only *unital* morphisms $N \to M$. For this reason, we denote it by π_u .

Although a priori it might not be clear what kind of structure one should expect to have on π_u , [2, 4.6] makes it into a convex space, in the sense that one can define linear combinations with non-negative real coefficients adding up to 1 in a consistent manner (see [2, 2.1]). Now that convex combinations make sense, so does the notion of extreme point, and [2, 5.2] argues that a unital morphism $N \to M$ has factorial commutant $N' \cap M$ if and only if the class of the morphism in π_u is an extreme point. In conclusion, there is a connection between (a) a question by Sorin Popa of whether factors N admitting an embedding into M also admit one with factorial commutant and (b) the geometry of the convex set π_u .

Consider now the sheaf W_N^* based on $M = R^\omega \otimes \mathcal{B}(\ell^2)$, the corresponding monoid π (no longer just partial; addition is always defined becaue tensoring with $\mathcal{B}(\ell^2)$ produces enough room to add any two projections), and the enveloping group $K(\pi)$ of the partial obtained by formally adjoining additive inverses to elements of π . As a continuation of Brown's work, the convex structure on π_u is extended in [3] to a Banach space structure on $K(\pi)$ (more on this circle of ideas in the next section). π_u sits inside this Banach space as a closed, convex, bounded subset. One possible approach to proving the existence of extreme points, then, would be to show that $K(\pi)$ is a dual Banach space; π_u would then be compact in the weak* topology, and it would have plenty of extreme point by the Krein-Milman theorem.

Theorem 2.5 can then be viewed as pointing roughly in the correct direction on the predual front. It implies that $K(\pi)$, with all of the structure just mentioned, is a Dedekind complete Banach lattice. This has some (admittedly flimsy) connection to being a dual, as the following result illustrates (originally due to Nakano; [14, 1.7] or [12, 2.1.4]):

2.10 Proposition The Banach lattice $C_{\mathbb{R}}(K)$ of real-valued continuous functions on a compact Hausdorff space K is Dedekind complete in the sense of Definition 2.3 if and only if K is Stonean, i.e. closures of open sets are open.

The reason why this is relevant is that being Stonean is only slightly weaker than $\mathcal{C}(K)$ being a dual Banach space. Indeed, this latter condition is equivalent to the algebra $\mathcal{C}_{\mathbb{C}}(K)$ of complex-valued continuous functions on K being W^* (this is more or less [14, III.3.5]), which in turn is equivalent to K being hyperstonean. This means that the space is stonean and in some sense has "enough measures" ([14, 1.14, 1.18]).

As acknowledged above, this is not strong evidence for $K(\pi)$ being a dual Banach space, but these ideas do seem to somehow fit together quite naturally, and the order-theoretic aspect of invariants such as π complement the metric side of things (Banach space structures, etc.) very nicely.

3 Divisibility and cone structures

In some sense, the main theme in [3] is the promotion of the "discrete" addition + on a partial monoid of the form $\pi(W_N^*)$ (notation as in Example 1.9) to a "continuous addition" (roughly

speaking an action of the semiring $(\mathbb{R}_+, +, \cdot)$ of non-negative reals with the usual addition and multiplication) under certain conditions on the W^* -algebras M and N. In a slightly less accurate way, this is also true of [2], where invariants analogous to $\pi(W_N^*)$ constructed using only unital morphisms are endowed with convex structures (i.e. linear combinations with non-negative real coefficients adding up to 1 are made sense of). We take up this same theme in the present section.

First, we fix some terminology:

- **3.1 Definition** Let (G, +, 0) be a partial (abelian) monoid as in Definition 1.15. A partial cone structure on G is an operation $[0, 1] \times G \to G$, $(r, g) \mapsto rg$ such that for $g, h \in G$ and $s, t \in [0, 1]$:
 - (a) $0g = 0 \in G$;
 - (b) 1g = g;
 - (c) s(g+h) = sg + sh when either of the two sides is defined;
 - (d) (s+t)g = sg + tg when $s+t \le 1$;

(e)
$$(st)g = s(tg)$$
.

3.2 Remark This is the kind of structure considered in [3, Appendix]. A cone is more appropriately thought of as an action by the semiring $(\mathbb{R}_+, +, \cdot)$ of non-negative integers rather than just [0, 1], hence the adjective 'partial'. Nevertheless, since all cone structures considered here are of the kind introduced in Definition 3.1, we sometimes drop that adjective.

Since cones allow for "division" by positive reals ≥ 1 (and in particular by positive integers), the following term seems appropriate:

- **3.3 Definition** An element g of a partial abelian monoid G is *divisible* if there exists an element $h \in G$ such that h + h = g. The monoid G is divisible if all of its elements are.
- **3.4 Remark** A more precise term in the spirit of [13] (a paper that will be relevant below, when we specialize to $G = \pi(\mathcal{F})$) would be '2-divisible'; *n*-divisibility would substitute $h + \ldots + h$ (*n* times) for h + h in Definition 3.3. There will never be any occasion to use the more elaborate notion, hence the simpler term 'divisible'.

Let us return to the setup from before, with M being a von Neumann algebra and \mathcal{F} a sheaf on it.

- **3.5 Definition** The sheaf \mathcal{F} is divisible if the monoid $\pi(\mathcal{F})$ with its partial monoid structure from Proposition 1.16 is divisible in the sense of Definition 3.3.
- **3.6 Remark** Unpacking the definition somewhat, \mathcal{F} is divisible if and only if for every projection $p \in M$, every element $x \in \mathcal{F}(p)$ is in the image of $\mathcal{F}(\iota) : \mathcal{F}(A) \to \mathcal{F}(p)$ for some unital inclusion $\iota : \mathbb{C}p \to A \cong M_2$.

As observed above, putting a cone structure on a partial monoid requires divisibility as a bare minimum. In the specific examples we are analyzing though, it turns out this is also enough:

3.7 Theorem Let M be a W^* -algebra, and \mathcal{F} a sheaf on it. The monoid $\pi = \pi(\mathcal{F})$ admits a cone structure if and only if \mathcal{F} is divisible. Moreover, when a cone structure exists, it is unique.

We prove a few auxiliary results first.

3.8 Lemma For an element $x \in \pi = \pi(\mathcal{F})$, there is at most one $y \in \pi$ such that y + y = x.

Proof This follows rather quickly from the results of the previous section. Let $y, z \in \pi$ both satisfy the condition:

$$y + y = x = z + z.$$

Subtracting $y \wedge z$ from each of y and z, and $y \wedge z + y \wedge z$ from x (which makes perfect sense by cancellability), we may as well assume $y \wedge z = 0$. But then Proposition 2.9 says that $y \vee z = y + z$, and hence x, which dominates both y and z, must also dominate their sum: $x \geq y + z$. This can be recast as either $y + y \geq y + z$ or $z + z \geq y + z$. The former implies (via cancellation) that $y \geq z$, while the latter shows the opposite inequality. Since ' \geq ' is a partial order, we are done: y = z.

The following simple observation will be handy to cite directly, so we record it. First, in order to make sense of the statement, we point out that (by Lemma 3.8) expressions such as $\frac{x}{2^n}$ are unambiguous for $x \in \pi(\mathcal{F})$ and positive integers n whenever the sheaf \mathcal{F} is divisible.

3.9 Lemma Let \mathcal{F} be a divisible sheaf on a W*-algebra, and $x \in \pi(\mathcal{F})$ an element. The greatest lower bound of the set of all $\frac{x}{2^n}$ for n ranging over the positive integers is $0 \in \pi$.

Proof The greatest lower bound (y, say) exists by Theorem 2.5. Since

$$y + y \le \frac{x}{2^n} + \frac{x}{2^n} = \frac{x}{2^{n-1}}$$

for all positive integers n, it follows that $y + y \leq y$. This implies y = 0, as desired.

We next take on the uniqueness part of Theorem 3.7.

3.10 Lemma In the setting of Theorem 3.7, there is at most one cone structure on $\pi = \pi(\mathcal{F})$.

Proof We know from Lemma 3.8 and condition (d) of Definition 3.1 that multiplication by $\frac{1}{2}$ is uniquely determined, and hence, by repreated applications of the same argument, multiplication by any dyadic rational number (i.e. with a power-of-two denominator) is unique. We will be done if we manage to show that in any cone structure, any multiple sx for $x \in \pi$ and $s \in (0,1)$ is the supremum of tx over all dyadic rationals $t \leq s$. We focus on this claim for the remainder of the proof.

Denote by y the supremum in the claim, so that $sx \geq y$. Assume s is not a dyadic rational, or there is nothing to prove. For any positive integer n, we can find a dyadic rational t such that $s \in \left(t, t + \frac{1}{2^n}\right)$. This makes difference sx - y at most

$$\left(t + \frac{1}{2^n}\right)x - tx = \frac{x}{2^n}.$$

Since this holds for all positive integers n, the conclusion follows by Lemma 3.9.

Proof of Theorem 3.7 One implication (cone \Rightarrow divisible) is clear, and uniqueness is taken care of by Lemma 3.10. We focus on the other implication of the first statement, and so start out with a divisible sheaf \mathcal{F} on M.

Divisibility allows us to divide and $x \in \pi$ by powers of 2 and then, by adding such terms of the form $\frac{x}{2^n}$, to define multiplication by dyadic rationals. Properties (a) - (e) are easily checked for dyadic rational coefficients using only Lemma 3.10. For non-dyadic coefficients s, set

$$sx = \sup_{t \le s} tx = \inf_{t \ge s} tx,$$

both ranging over dyadics. That the supremum and the infimum coincide is essentially shown in the course of the proof of Lemma 3.10 (the argument is the same: One places s inside a small dyadic interval of length $\frac{1}{2^n}$, etc.). Checking the properties is now rather routine, and we illustrate by proving only (d) of Definition 3.1.

As in the definition, let $s, t \in (0, 1)$ satisfy $s + t \le 1$, and $x \in \pi$ arbitrary. Assume for simplicity that s + t is strictly less than 1. Placing both s and t inside tiny dyadic intervals $(s - \varepsilon_1, s + \varepsilon_2)$ and $(t - \delta_1, t + \delta_2)$, note

$$(t - \delta_1 - \varepsilon_1 - \varepsilon_2)x = (s + t - \varepsilon_1 - \delta_1)x - (s + \varepsilon_2)x \le$$

$$\le (s + t)x - sx \le (s + t + \varepsilon_2 + \delta_2)x - (s - \varepsilon_1)x = (t + \delta_2 + \varepsilon_1 + \varepsilon_2)x.$$

Both sides are dyadic multiples of x, and making the ε 's and δ 's approach zero squeezes both sides to tx. A similar argument works for s+t=1, approximating only s by dyadics close to it.

Let's again specialize to sheaves of the form W_N^* for separable II₁ factors N. The divisibility hypothesis of Theorem 3.7 would be satisfied if, for instance, we knew that for any projection $p \in M$, the relative commutant of any separable von Neumann subalgebra of pMp contains M_2 unitally (in which case we say that M has property (DIV)). Indeed, applying property (DIV) to images of N through not-necessarily-unital morphisms $N \to M$, we recover the characterization of divisibility for W_N^* noted in Remark 3.6.

When M is an ultrapower X^{ω} of a separable II₁ factor X, (DIV) is in fact equivalent to X being McDuff (i.e. $X \otimes R \cong X$). The direction McDuff \Rightarrow (DIV) follows from [3, Lemma 17] (see also [2, 3.2.3], which is a precursor for unital morphisms when X = R, but contains the main ideas of the proof). For the opposite implication, it is known that McDuffness follows from the non-commutativity of the diagonal morphism $X \to X^{\omega}$; this is the reformulation given in [13, 2.7] to one of the main results of [11]. This same discussion applies to $M = X^{\omega} \otimes \mathcal{B}(\ell^2)$, which is the choice relevant to [3].

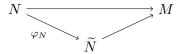
In any event, this particular case of Theorem 3.7 recovers the cone structure on $\pi(\mathcal{F})$ from a somewhat different perspective: We still need to check divisibility (by 2, in Sherman's more refined terminology [13]), but the rest of the structure and the cone conditions in Definition 3.1 follow mechanically and uniquely, and for very general reasons that are not specific to being McDuff or having a large fundamental group.

4 Completely positive / contractive maps as homomorphisms

As outlined briefly in the introduction, the aim of this short section is to explain why Examples 1.9 to 1.11 are actually the same kind of sheaf (Corollary 4.2). There are no proofs in this section; although not difficult, they would take us too far afield, and will appear elsewhere [5].

Let us focus on Example 1.10, to fix ideas. The claim is that any von Neumann algebra N comes equipped with a canonical completely positive unital (and as always, normal) map $\varphi_N: N \to \widetilde{N}$

such that any completely positive unital map $N \to M$ factors uniquely as



for a W^* morphism $\widetilde{N} \to M$. The uniqueness part of the assertion then makes the whole thing functorial: $N \mapsto \widetilde{N}$ is a functor from the category (W^* -algebras, completely positive unital maps) to (W^* -algebras, W^* morphisms), and in fact the universality property of φ_N says that this functor is left adjoint to the inclusion of categories going in the opposite direction (a W^* morphism being completely positive). The analogous discussion applies to completely contractive maps.

Denoting by W*, CPU, and CCU the categories of W^* -agebras and unital W^* morphisms, completely positive maps, and completely contractive maps respectively, the statement is:

4.1 Proposition The inclusion functors $W^* \to CPU$ and $W^* \to CCU$ both have left adjoints.

Denote by $N \mapsto \widehat{N}$ the left adjoint to the second inclusion $W^* \to CCU$ of the statement. Consider the bijection $CPU(\widetilde{N} \to M) \cong W^*(N \to M)$ induced by φ_N , and similarly for CCU. Applying these bijections to all corner subalgebras pMp instead of just M, we get:

4.2 Corollary For any von Neumann algebra M, the canonical completely positive map $N \to \widetilde{N}$ induces an isomorphism $\operatorname{CP}_N \cong \operatorname{W}_{\widetilde{N}}^*$ of sheaves on M. Similarly, we have a canonical isomorphism $\operatorname{CC}_N \cong \operatorname{W}_{\widehat{N}}^*$.

It seems unlikely that all sheaves as in Definition 1.7 are of the form W_N^* , but I do not know any counterexamples. Other natural examples, such as the sheaf associating to $A \in \mathcal{C}_M$ the set of unitaries or contractions in the relative commutant $A' \cap 1_A M 1_A$, are even more obviously of that form than Examples 1.10 and 1.11.

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