

# Convex spaces, affine spaces, and commutants for algebraic theories

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## Abstract

Certain axiomatic notions of *affine space* over a ring and *convex space* over a pre-ordered ring are examples of the notion of  $\mathcal{T}$ -algebra for an algebraic theory  $\mathcal{T}$  in the sense of Lawvere. Herein we study the notion of *commutant* for Lawvere theories that was defined by Wraith and generalizes the notion of *centralizer clone*. We focus on the Lawvere theory of *left  $R$ -affine spaces* for a ring or rig  $R$ , proving that this theory can be described as a commutant of the theory of pointed right  $R$ -modules. Further, we show that for a wide class of rigs  $R$  that includes all rings, these theories are commutants of one another in the full finitary theory of  $R$  in the category of sets. We define *left  $R$ -convex spaces* for a preordered ring  $R$  as left affine spaces over the positive part  $R_+$  of  $R$ . We show that for any *finitely archimedean* preordered algebra  $R$  over the dyadic rationals, the theories of left  $R$ -convex spaces and pointed right  $R_+$ -modules are commutants of one another within the full finitary theory of  $R_+$  in the category of sets. Applied to the ring of real numbers  $\mathbb{R}$ , this result shows that the connection between convex spaces and pointed  $\mathbb{R}_+$ -modules that is implicit in the integral representation of probability measures is a perfect ‘duality’ of algebraic theories.

## 1 Introduction

In 1963, Lawvere [3] introduced an elegant approach to Birkhoff’s universal algebra through category theory. Therein, an *algebraic theory* or *Lawvere theory* is by definition a category  $\mathcal{T}$  with a denumerable set of objects  $T^0, T^1, T^2, \dots$  in which  $T^n$  is an  $n$ -th power of the object  $T = T^1$ , and a  $\mathcal{T}$ -*algebra* is a functor  $A : \mathcal{T} \rightarrow \mathbf{Set}$  that is valued in the category of sets and preserves finite powers. We call  $|A| = A(T)$  the *carrier* of  $A$ , and we say that  $A$  is *normal* if  $A$  sends the powers  $T^n$  in  $\mathcal{T}$  to the canonical

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\*The author gratefully acknowledges financial support in the form of an AARMS Postdoctoral Fellowship, a Mount Allison University Research Stipend, and, earlier, an NSERC Postdoctoral Fellowship.

†Keywords: convex space; convex module; affine space; affine module; commutant; centralizer clone; commutation; algebraic theory; Lawvere theory; universal algebra; ring; rig; semiring; preordered ring; ordered ring; module; monad; commutative theory; semilattice; matrix; Kronecker product

‡2010 Mathematics Subject Classification: 18C10, 18C15, 18C20, 18C05, 08A62, 08B99, 08C05, 52A01, 52A05, 51N10, 16B50, 16B70, 16D10, 16D90, 16W80, 13J25, 13C99, 15A27, 15A69, 15A99, 15B51, 06A11, 06A12, 06F25

$n$ -th powers  $|A|^n$  in  $\text{Set}$  (2.4).  $\mathcal{T}$ -algebras and the natural transformations between them constitute a category  $\mathcal{T}\text{-Alg}$  with an equivalent full subcategory consisting of all normal  $\mathcal{T}$ -algebras (2.5). The morphisms  $\omega : T^n \rightarrow T$  in a Lawvere theory  $\mathcal{T}$  may be called *abstract operations*, and the mappings  $A(\omega) : |A|^n \rightarrow |A|$  associated to these by a given  $\mathcal{T}$ -algebra  $A$  are then called *concrete operations*. For convenience, we can take the objects  $T^n$  of  $\mathcal{T}$  to be just the finite cardinals  $n$  to which they correspond bijectively.

By a (*normal*) *faithful representation* of a Lawvere theory  $\mathcal{T}$  we mean a normal  $\mathcal{T}$ -algebra  $R : \mathcal{T} \rightarrow \text{Set}$  that is faithful as a functor. Writing simply  $R$  for the carrier of  $R$ , such a faithful representation presents  $\mathcal{T}$  as a subtheory  $\mathcal{T} \hookrightarrow \text{Set}_R$  of a larger theory  $\text{Set}_R$  called the *full finitary theory of  $R$  in  $\text{Set}$* , consisting of *all* the mappings between the  $n$ -th powers  $R^n$  of the set  $R$ . Such subtheories are essentially the *concrete clones* that appear in universal algebra, as contrasted with the (a priori) more general *abstract clones* of Hall, which correspond to arbitrary Lawvere theories.

One of the chief objectives of this paper is to study a phenomenon sometimes exhibited by a faithfully represented Lawvere theory  $\mathcal{T} \hookrightarrow \text{Set}_R$ , wherein the set  $R$  carries the structure of an  $\mathcal{S}$ -algebra for some other Lawvere theory  $\mathcal{S}$  and the mappings  $R^n \rightarrow R^m$  that lie within the subtheory  $\mathcal{T} \hookrightarrow \text{Set}_R$  are precisely those that are  $\mathcal{S}$ -homomorphisms with respect to the induced  $\mathcal{S}$ -algebra structures on  $R^n$  and  $R^m$ , so that

$$\mathcal{T}(n, m) \cong \mathcal{S}\text{-Alg}(R^n, R^m) .$$

We will in fact encounter situations in which, moreover, the  $\mathcal{S}$ -algebra structure on  $R$  is also a faithful representation of  $\mathcal{S}$  with the same property, such that the subtheory  $\mathcal{S} \hookrightarrow \text{Set}_R$  consists of exactly those mappings  $R^n \rightarrow R^m$  that are  $\mathcal{T}$ -homomorphisms. In symbols

$$\mathcal{S}(n, m) \cong \mathcal{T}\text{-Alg}(R^n, R^m) .$$

It is precisely this curious ‘duality’ of certain pairs of theories  $\mathcal{T}$  and  $\mathcal{S}$  that we seek to understand.

As a first example, let us consider the theories  $\mathcal{T}$  and  $\mathcal{S}$  of left and right  $R$ -modules, respectively, for a given ring<sup>1</sup>  $R$  (or even just a *rig* or *semiring*, 2.8). Concretely,  $\mathcal{T}$  is the category  $\text{Mat}_R$  whose objects are the natural numbers  $n$  and whose morphisms  $n \rightarrow m$  are  $m \times n$ -matrices with entries in  $R$ , with composition given by matrix multiplication. The category of normal  $\mathcal{T}$ -algebras is isomorphic to the category  $R\text{-Mod}$  of left  $R$ -modules (2.8). Similarly,  $\mathcal{S}$ -algebras for  $\mathcal{S} = \text{Mat}_{R^{\text{op}}}$  are right  $R$ -modules. Given a left  $R$ -module  $A$ , the corresponding normal  $\mathcal{T}$ -algebra  $\mathcal{T} \rightarrow \text{Set}$  is given on objects by  $n \mapsto A^n$  and associates to each  $m \times n$ -matrix  $w \in R^{m \times n}$  the mapping  $A^n \rightarrow A^m$  that sends a column vector  $x \in A^n$  to the matrix product  $wx \in A^m$ . A normal  $\mathcal{T}$ -algebra is uniquely determined by its carrier and its values on morphisms of the form  $w : n \rightarrow 1$  in  $\mathcal{T}$ , i.e. on row vectors  $w \in R^{1 \times n}$ , for which the associated maps

$$A^n \rightarrow A, \quad x \mapsto wx = \sum_{i=1}^n w_i x_i \tag{1.0.i}$$

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<sup>1</sup>Throughout this paper, we use the term *ring* to mean *unital ring*. A similar remark applies to the notion of *rig* or *semiring* employed herein, whose definition we recall in 2.8.

implement the taking of left  $R$ -linear combinations. In particular,  $R$  itself is a left  $R$ -module and so determines a normal  $\mathcal{T}$ -algebra  $R : \mathcal{T} \rightarrow \mathbf{Set}$  that is in fact a faithful representation. Similarly,  $R$  is a right  $R$ -module, so we have faithful representations

$$\mathcal{T} \hookrightarrow \mathbf{Set}_R \quad \mathcal{S} \hookrightarrow \mathbf{Set}_R .$$

Thus viewing  $\mathcal{T}$  as a subtheory of  $\mathbf{Set}_R$ , we find that the mappings  $R^n \rightarrow R^m$  that lie in  $\mathcal{T}$  are precisely the *right*  $R$ -linear maps (i.e. the  $\mathcal{S}$ -homomorphisms) whereas the mappings  $R^n \rightarrow R^m$  in  $\mathcal{S}$  are precisely the *left*  $R$ -linear maps (i.e. the  $\mathcal{T}$ -homomorphisms) (6.5).

This peculiar duality of pairs of theories  $\mathcal{T}, \mathcal{S}$  can be understood through the notion of *commutant* for Lawvere theories that was briefly introduced by Wraith in his lecture notes on algebraic theories [13] but was not studied to any substantial extent therein. Given a set  $R$ , a pair of mappings  $\mu : R^n \rightarrow R^m$  and  $\nu : R^{n'} \rightarrow R^{m'}$  is said to *commute* if the associated mappings

$$\begin{aligned} \mu * \nu &= \left( R^{n \times n'} \cong (R^n)^{n'} \xrightarrow{\mu^{n'}} (R^m)^{n'} \cong (R^{n'})^m \xrightarrow{\nu^m} (R^{m'})^m \cong R^{m \times m'} \right) \\ \mu \tilde{*} \nu &= \left( R^{n \times n'} \cong (R^{n'})^n \xrightarrow{\nu^n} (R^{m'})^n \cong (R^n)^{m'} \xrightarrow{\mu^{m'}} (R^m)^{m'} \cong R^{m \times m'} \right) \end{aligned}$$

are equal. Given a subtheory  $\mathcal{T} \hookrightarrow \mathbf{Set}_R$ , the *commutant* of  $\mathcal{T}$  in  $\mathbf{Set}_R$  is, by definition, the subtheory  $\mathcal{T}^\perp \hookrightarrow \mathbf{Set}_R$  consisting of those mappings  $\mu : R^n \rightarrow R^m$  that commute with every mapping  $\nu : R^{n'} \rightarrow R^{m'}$  in  $\mathcal{T}$ . A subtheory  $\mathcal{T} \hookrightarrow \mathbf{Set}_R$  is equivalently a theory  $\mathcal{T}$  admitting a faithful representation with carrier  $R$ , and the key observation is now that a mapping  $\mu : R^n \rightarrow R^m$  lies in the commutant  $\mathcal{T}^\perp$  if and only if  $\mu$  is a  $\mathcal{T}$ -homomorphism (5.9); i.e.,

$$\mathcal{T}^\perp(n, m) = \mathcal{T}\text{-Alg}(R^n, R^m) .$$

Hence the ‘duality’ observed above in pairs of faithfully represented theories  $\mathcal{T}, \mathcal{S}$  is equivalently the statement that  $\mathcal{S}$  and  $\mathcal{T}$  are commutants of one another within the full finitary theory  $\mathbf{Set}_R$  of a set  $R$ ; in symbols,

$$\mathcal{T} \cong \mathcal{S}^\perp \quad \mathcal{T}^\perp \cong \mathcal{S} .$$

In particular, given a ring or rig  $R$ , the theories  $\mathbf{Mat}_R$  and  $\mathbf{Mat}_{R^{\text{op}}}$  of left and right  $R$ -modules, respectively, are commutants of one another within  $\mathbf{Set}_R$ :

$$\mathbf{Mat}_R \cong (\mathbf{Mat}_{R^{\text{op}}})^\perp \quad (\mathbf{Mat}_R)^\perp \cong \mathbf{Mat}_{R^{\text{op}}} .$$

Wraith’s notion of commutant applies not only to subtheories  $\mathcal{T}$  of the full finitary theory  $\mathbf{Set}_R$  of a set  $R$  but also to subtheories  $\mathcal{T} \hookrightarrow \mathcal{U}$  of an arbitrary Lawvere theory  $\mathcal{U}$ . Indeed, in analogy with the above one can again define the notion of commutation of morphisms in  $\mathcal{U}$  (4.3), and the *commutant*

$$\mathcal{T}^\perp \hookrightarrow \mathcal{U}$$

of  $\mathcal{T}$  in  $\mathcal{U}$  is then defined in the analogous way (5.6). More generally, we can define the commutant  $\mathcal{T}_A^\perp \hookrightarrow \mathcal{U}$  of a morphism of Lawvere theories  $A : \mathcal{T} \rightarrow \mathcal{U}$  as the commutant of its image. The commutant is then characterized by a universal property, namely that a morphism of theories  $B : \mathcal{S} \rightarrow \mathcal{U}$  factors uniquely through the commutant  $\mathcal{T}_A^\perp \hookrightarrow \mathcal{U}$  if and only if  $A$  *commutes* with  $B$  in a suitable sense (5.1). Defining a *theory over  $\mathcal{U}$*  as a theory  $\mathcal{T}$  equipped with a morphism  $A : \mathcal{T} \rightarrow \mathcal{U}$ , it follows that the operation  $(-)^{\perp}$  on theories over  $\mathcal{U}$  gives rise to an adjunction between the category of theories over  $\mathcal{U}$  and its opposite (6.1), and this adjunction restricts to a Galois connection on subtheories of  $\mathcal{U}$ . Since a normal  $\mathcal{T}$ -algebra  $R : \mathcal{T} \rightarrow \mathbf{Set}$  is equivalently described as a morphism  $R : \mathcal{T} \rightarrow \mathbf{Set}_R$  into the full finitary theory of its carrier  $R$ , we recover the commutant of a faithful representation as a special case. In particular, when  $\mathcal{T}$  is the subtheory of  $\mathbf{Set}_R$  generated by a specified family of finitary operations on a given set  $R$ , we recover the notion of *centralizer clone* that has been studied to some extent in the literature on universal algebra. For example, the paper [12] characterizes those abstract clones or Lawvere theories  $\mathcal{T}$  for which there exists a set  $R$  equipped with a family of operations whose centralizer clone is isomorphic to  $\mathcal{T}$ .

In addition to a general study of the notion of commutant for Lawvere theories<sup>2</sup>, the present paper comprises an in-depth study of certain specific examples of commutants. In particular, we prove several theorems concerning the theory of  *$R$ -affine spaces* for a ring or rig  $R$  and, in particular,  *$R$ -convex spaces* for a preordered ring  $R$ . Several authors have studied axiomatic notions of affine space over a ring or rig, and the generality afforded by the use of a mere rig permits the consideration of the notion of convex space as a special case; for example, see [11] and the references there. Whereas a (left)  $R$ -module  $A$  is a set equipped with operations (1.0.i) that permit the taking of linear combinations, a (left)  $R$ -affine space or (left)  $R$ -affine module is a set equipped with operations (1.0.i) that permit the taking of *affine combinations*, i.e. those linear combinations  $\sum_{i=1}^n w_i x_i$  whose coefficients  $w_i$  sum to 1. More precisely, a left  $R$ -affine space is by definition a normal  $\mathcal{T}$ -algebra for a certain subtheory

$$\mathcal{T} = \mathbf{Mat}_R^{\text{aff}} \hookrightarrow \mathbf{Mat}_R$$

of the category of  $R$ -matrices, namely the subtheory consisting of all matrices in which each row sums to 1. This way of defining the notion of  $R$ -affine space was given in [4]. Letting  $\mathbb{R}_+$  denote the rig of non-negative reals,  $\mathbb{R}_+$ -affine spaces are usually called *convex spaces*, and  $\mathbb{R}_+$ -affine combinations are called *convex combinations*. This way of defining convex spaces was given in [10].

We pursue answers to the following questions:

1. Does  $\mathbf{Mat}_R^{\text{aff}}$  arise as a commutant of some theory over the full finitary theory  $\mathbf{Set}_R$  of  $R$  in  $\mathbf{Set}$ ?
2. What is the commutant of  $\mathbf{Mat}_R^{\text{aff}}$  in  $\mathbf{Set}_R$ ?

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<sup>2</sup>A further recent preprint [6] by the author of the present paper was made available subsequent to the initial version of the present paper and treats the general theory of commutants for  $\mathcal{V}$ -enriched algebraic theories for a system of arities  $\mathcal{J} \hookrightarrow \mathcal{V}$ . The development is much simpler in the present  $\mathbf{Set}$ -based case, and the present paper is principally concerned with specific examples.

We answer 1 in the affirmative for every rig  $R$ . Indeed, defining a *pointed right  $R$ -module* as a right  $R$ -module  $M$  equipped with a chosen element  $* \in M$ , the category of pointed right  $R$ -modules is isomorphic to the category of  $\mathcal{T}$ -algebras for a Lawvere theory  $\mathcal{T} = \text{Mat}_{R^{\text{op}}}^*$ . The pointed right  $R$ -module  $(R, 1)$  determines a morphism  $\text{Mat}_{R^{\text{op}}}^* \rightarrow \text{Set}_R$  by which  $\text{Mat}_{R^{\text{op}}}^*$  can be considered as a theory over  $\text{Set}_R$ , though not in general a subtheory, and we show in 7.2 that

$$\text{Mat}_R^{\text{aff}} \cong (\text{Mat}_{R^{\text{op}}}^*)^\perp$$

as theories over  $\text{Set}_R$ . Hence

*the Lawvere theory of left  $R$ -affine spaces is the commutant of the theory of pointed right  $R$ -modules when both are considered as theories over the full finitary theory of  $R$  in  $\text{Set}$ .*

Consequently  $\text{Mat}_R^{\text{aff}}$  is its own double-commutant  $(\text{Mat}_R^{\text{aff}})^{\perp\perp}$  over  $\text{Set}_R$ , so we say that  $\text{Mat}_R^{\text{aff}}$  is a *saturated* subtheory of  $\text{Set}_R$ .

It is illustrative to note that in the case of the rig  $\mathbb{R}_+$  one finds here a connection to the Kakutani-Markov-Riesz representation theorem, since for each finite cardinal  $n$  the resulting bijection  $\text{Mat}_{\mathbb{R}_+}^{\text{aff}}(n, 1) \cong (\text{Mat}_{\mathbb{R}_+}^*)^\perp(n, 1)$  is the correspondence between probability measures on the finite set  $n$  (on the left-hand side) and 1-preserving  $\mathbb{R}_+$ -linear functionals  $\mathbb{R}_+^n \rightarrow \mathbb{R}_+$  (on the right).

With regard to question 2 it is natural to ask also whether  $\text{Mat}_{R^{\text{op}}}^*$  is the commutant of  $\text{Mat}_R^{\text{aff}}$  over  $\text{Set}_R$ . When  $R$  is a *ring* we show that this is indeed the case (9.2), so that

$$(\text{Mat}_R^{\text{aff}})^\perp \cong \text{Mat}_{R^{\text{op}}}^* \tag{1.0.ii}$$

over  $\text{Set}_R$ . Hence

*if  $R$  is a ring, then the theories of left  $R$ -affine spaces and pointed right  $R$ -modules are commutants of one another within the full finitary theory of  $R$  in  $\text{Set}$ .*

However for arbitrary *rigs* this is no longer true. For example, when  $R$  is the two-element rig  $2 = (2, \vee, 0, \wedge, 1)$ , we show that (i) 2-modules are equivalently (*bounded*) *join semilattices* (2.10), (ii) 2-affine spaces are *unbounded join semilattices* (i.e., idempotent commutative semigroups, 3.3), and (iii) the commutant in  $\text{Set}_2$  of the theory of unbounded join semilattices is the theory of join semilattices with a *top* element (8.2).

Nevertheless, we show that (1.0.ii) *does* hold for many rigs other than rings. In particular, we show that it holds for the rig  $\mathbb{R}_+$  of non-negative reals (10.21), so that

*the theory of convex spaces (over  $\mathbb{R}$ ) and the theory of pointed right  $\mathbb{R}_+$ -modules are commutants of one another within the full finitary theory of  $\mathbb{R}_+$  in  $\text{Set}$ .* (1.0.iii)

This result shows that the connection between convex spaces and pointed  $\mathbb{R}_+$ -modules that is implicit in the integral representation of probability measures is in fact a perfect ‘duality’ of algebraic theories. Indeed, one of the purposes of the present paper is to provide an algebraic basis for a study of measure and distribution monads canonically determined by such dualities in the enriched context [5, 8].

In order to generalize this result, we study affine spaces over rigs of the form

$$R_+ = \{r \in R \mid r \geq 0\}$$

where  $R$  is a *preordered ring* (3.4). Preordered and partially ordered rings have been studied at various levels of generality in the literature on ordered algebra, and they can be defined equivalently as rigs  $R$  equipped with an arbitrary subrig  $R_+ \hookrightarrow R$ . The rigs that occur as the *positive part*  $R_+$  of some preordered ring  $R$  are precisely the *additively cancellative rigs* (3.4).

Given a preordered ring  $R$ , we call left  $R_+$ -affine spaces *left  $R$ -convex spaces* or *left  $R$ -convex modules*. In 10.10 we establish a characterization of the class of all preordered rings  $R$  for which the evident analogue of (1.0.iii) holds, and we then proceed to develop sufficient conditions that entail that a preordered ring  $R$  belongs to this class, as we now outline.

Whereas the *archimedean property* for totally ordered fields  $R$  can be expressed in several equivalent ways, certain of these statements become inequivalent when one passes to arbitrary preordered rings  $R$ . In particular, we define the notion of *firmly archimedean* preordered ring (10.18), noting that a nonzero totally ordered ring is firmly archimedean if and only if it is archimedean. Given any integer  $d > 1$ , we prove that if  $R$  is a firmly archimedean preordered ring and  $d$  is invertible in the rig  $R_+$ , then the relevant analogue of (1.0.iii) holds (10.20). But  $d$  is invertible in  $R_+$  if and only if there exists a (necessarily unique) morphism of preordered rings from the ring of  *$d$ -adic fractions*  $\mathbb{Z}[\frac{1}{d}]$  into  $R$  (10.11, 10.13), so this result can be stated as follows:

*Let  $R$  be a firmly archimedean preordered algebra over  $\mathbb{Z}[\frac{1}{d}]$ , for some integer  $d > 1$ . Then the Lawvere theory of left  $R$ -convex spaces and the Lawvere theory of pointed right  $R_+$ -modules are commutants of one another in the full finitary theory of  $R_+$  in Set.*

In particular, this applies to (i) the ring of real numbers  $R = \mathbb{R}$ , (ii) the ring of *dyadic rationals*  $R = \mathbb{Z}[\frac{1}{2}]$ , (iii) the ring  $R$  of all bounded real-valued functions on a set, or any sub- $\mathbb{Z}[\frac{1}{d}]$ -algebra thereof, and in particular (iv) the ring  $R = C(X)$  of all continuous functions on a compact space  $X$ .

We begin in §2 with a survey of basic material concerning Lawvere theories, and we discuss several examples of Lawvere theories for use in the sequel. In §3 we define the Lawvere theory of left  $R$ -affine spaces for a rig  $R$  and the Lawvere theory of left  $R$ -convex spaces for a preordered ring  $R$ . In §4 we provide a self-contained treatment of the notion of commutation of morphisms in a Lawvere theory  $\mathcal{T}$  by studying in detail the *first and second Kronecker products* of morphisms in  $\mathcal{T}$  (4.3). Noting that these Kronecker products in  $\mathcal{T}$  depend on a choice of binary product projections in the category of finite cardinals (4.1), we show that one specific such choice enables a rigorous proof that the first Kronecker product of morphisms in the category  $\text{Mat}_R$  of matrices over a rig  $R$  is the classical Kronecker product of matrices (4.4), as we are not aware of any statement or proof of this in the literature. In §5 we study the notion of commutant of a morphism of Lawvere theories  $A : \mathcal{T} \rightarrow \mathcal{U}$ , proving that it can be defined equivalently as the full finitary theory of  $A$  in the category of  $\mathcal{T}$ -algebras in  $\mathcal{U}$  (5.9), and we treat the example of the theory of left  $R$ -modules for a rig  $R$  (5.14).

In §6 we show that the passage from a theory  $\mathcal{T}$  over  $\mathcal{U}$  to its commutant is a ‘self-adjoint’ contravariant functor (6.1), and we study the notions of *saturated* and *balanced* subtheory (6.2). In §7, 8, 9, 10 we derive our main results concerning the theories of  $R$ -affine and  $R$ -convex spaces and their commutants (7.2, 8.2, 9.2, 10.10, 10.17, 10.20).

**Acknowledgement.** The author thanks the anonymous referee for helpful suggestions and remarks. In the original version of this paper, Theorems 10.17 and 10.20 were formulated for preordered algebras  $R$  over the dyadic rationals  $\mathbb{Z}[\frac{1}{2}]$ . However, the referee supplied an argument to the effect that equation (10.17.ii) still holds when one replaces  $\mathbb{Z}[\frac{1}{2}]$  with the ring of  $d$ -adic fractions  $\mathbb{Z}[\frac{1}{d}]$  for any integer  $d > 1$ , thus showing that Theorems 10.17 and 10.20 apply to the broader classes of preordered algebras for which they are now formulated herein.

## 2 Lawvere theories, their algebras, and several examples

**2.1 (Lawvere theories).** A **Lawvere theory** is a small category  $\mathcal{T}$  equipped with an identity-on-objects functor  $\tau : \text{FinCard}^{\text{op}} \rightarrow \mathcal{T}$  that preserves finite powers, where  $\text{FinCard}$  is the full subcategory of  $\text{Set}$  consisting of the finite cardinals. We may identify finite cardinals with natural numbers, so that  $\text{ob } \mathcal{T} = \mathbb{N}$  is the set of all natural numbers. When we want to emphasize that a natural number  $n$  is to be treated as an object of  $\mathcal{T}$ , we will sometimes denote it by  $\tau(n)$ .

Since  $\text{FinCard}$  has finite copowers, its opposite  $\text{FinCard}^{\text{op}}$  has finite powers and hence every Lawvere theory  $\mathcal{T}$  has finite powers, furnished by  $\tau$ . In particular, each finite cardinal  $n$  is an  $n$ -th copower of 1 in  $\text{FinCard}$ , so  $n$  is an  $n$ -th *power* of 1 in  $\mathcal{T}$ . In symbols,  $n = \tau(n) = \tau(1)^n$  in  $\mathcal{T}$ . For ease of notation we will sometimes write  $T = \tau(1)$  and correspondingly write  $T^n$  for the object  $n$  of  $\mathcal{T}$ . Choosing designated  $n$ -th copower cocones  $(\iota_i : 1 \rightarrow n)_{i=1}^n$  in  $\text{FinCard}$  in the evident way, we thus obtain designated  $n$ -th power cones  $(\pi_i = \tau(\iota_i) : T^n \rightarrow T)$  in  $\mathcal{T}$ . Moreover,  $\tau$  can then be characterized as the functor  $T^{(-)} : \text{FinCard}^{\text{op}} \rightarrow \mathcal{T}$  that is given on objects by  $n \mapsto T^n$  and sends each mapping  $f : m \rightarrow n$  in  $\text{FinCard}$  to the induced morphism  $T^f : T^n \rightarrow T^m$ .

We say that an object  $C$  of a category  $\mathcal{C}$  has **designated finite powers** if it is equipped with a specified choice of  $n$ -th power  $C^n$  in  $\mathcal{C}$  for each  $n \in \mathbb{N}$ . We say that these designated finite powers of  $C$  are **standard** if  $C^1 = C$ , with the identity morphism  $1_C$  as the designated projection  $\pi_1 : C^1 \rightarrow C$ . For example, the designated  $n$ -th powers of  $\tau(1)$  in a Lawvere theory  $(\mathcal{T}, \tau)$  are standard, since the designated morphism  $\iota_1 : 1 \rightarrow 1$  in  $\text{FinCard}$  is necessarily the identity. It shall be convenient to fix a choice of standard designated finite powers  $\tau(m)^n$  of each of the objects  $\tau(m) = m$  of  $\mathcal{T}$ , and we may assume that this choice of powers extends the basic choice  $\tau(1)^n = n$  in the case that  $m = 1$ .

In fact, a Lawvere theory is equivalently given by a small category  $\mathcal{T}$  with objects  $\text{ob } \mathcal{T} = \mathbb{N}$  in which each object  $n$  carries the structure of an  $n$ -th power of 1, such that these designated  $n$ -th powers of 1 are standard<sup>3</sup>.

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<sup>3</sup>Many authors drop the latter condition, with the immaterial consequence that a Lawvere theory  $\mathcal{T}$  may then carry an irrelevant specified automorphism  $\pi_1 : 1 \rightarrow 1$  of 1.

**Notation 2.2.** We will use each of the following notations interchangeably to denote a given Lawvere theory:

$$\mathcal{T}, \quad (\mathcal{T}, \tau), \quad (\mathcal{T}, T^{(-)}), \quad (\mathcal{T}, T).$$

We regard the last notation as a construct for naming both  $\mathcal{T}$  and the object  $T = 1$  of  $\mathcal{T}$ .

**2.3 (The category of Lawvere theories).** Given Lawvere theories  $\mathcal{T}$  and  $\mathcal{U}$ , a **morphism**  $M : \mathcal{T} \rightarrow \mathcal{U}$  is a functor that commutes with the associated functors  $\text{FinCard}^{\text{op}} \rightarrow \mathcal{T}$  and  $\text{FinCard}^{\text{op}} \rightarrow \mathcal{U}$ . Thus Lawvere theories are the objects of a category  $\text{Th}$ . Observe that  $\text{FinCard}^{\text{op}}$  is a Lawvere theory, when equipped with its identity functor, and hence is an initial object of  $\text{Th}$ . A morphism of theories may be equivalently defined as a functor that strictly preserves the designated  $n$ -th power projections  $\pi_i : n \rightarrow 1$  (and so, in particular, is identity-on-objects). Given a Lawvere theory  $\mathcal{U}$ , a **subtheory** of  $\mathcal{U}$  is a Lawvere theory  $\mathcal{T}$  equipped with a morphism  $\mathcal{T} \hookrightarrow \mathcal{U}$  that is faithful as a functor. Subtheories are the objects of a full subcategory  $\text{SubTh}(\mathcal{U})$  of the slice category  $\text{Th}/\mathcal{U}$ , and  $\text{SubTh}(\mathcal{U})$  is clearly a preordered set. A **concrete subtheory** is a subtheory  $\mathcal{T} \hookrightarrow \mathcal{U}$  for which the associated mapping  $\text{mor } \mathcal{T} \rightarrow \text{mor } \mathcal{U}$  is simply the inclusion of a subset  $\text{mor } \mathcal{T} \subseteq \text{mor } \mathcal{U}$ . Clearly every subtheory of  $\mathcal{U}$  is isomorphic to a concrete subtheory. A concrete subtheory of  $(\mathcal{U}, U)$  is equivalently given by a subset  $\mathcal{T} \subseteq \text{mor } \mathcal{U}$  such that (i)  $\mathcal{T}$  contains the designated projections  $\pi_i : U^n \rightarrow U$ , (ii)  $\mathcal{T}$  is closed under composition, and (iii) given a family of morphisms  $\{\omega_i : U^n \rightarrow U \mid i = 1, \dots, m\} \subseteq \mathcal{T}$ , the induced morphism  $\omega : U^n \rightarrow U^m$  lies in  $\mathcal{T}$ .

**2.4 ( $\mathcal{T}$ -algebras).** Given a Lawvere theory  $(\mathcal{T}, T)$  and a category  $\mathcal{C}$ , a  **$\mathcal{T}$ -algebra** in  $\mathcal{C}$  is a functor  $A : \mathcal{T} \rightarrow \mathcal{C}$  that preserves finite powers. We call  $|A| := A(T)$  the *carrier* of  $A$ . Hence for each natural number  $n$ , the object  $A(T^n)$  is simply an  $n$ -th power  $|A|^n$  of the carrier  $|A|$  in  $\mathcal{C}$ . However, when  $\mathcal{C}$  has designated finite powers of each of its objects, the  $n$ -th power  $A(T^n) = |A|^n$  need not be the designated  $n$ -th power. Assuming that  $\mathcal{C}$  has *standard* designated finite powers (2.1), we therefore define a **normal  $\mathcal{T}$ -algebra** to be a functor  $A : \mathcal{T} \rightarrow \mathcal{C}$  that sends the designated  $n$ -th power projections  $\pi_i : T^n \rightarrow T$  in  $\mathcal{T}$  to the designated  $n$ -th power projections  $\pi_i : |A|^n \rightarrow |A|$  in  $\mathcal{C}$ . Since a functor on  $\mathcal{T}$  preserves finite powers as soon as it preserves finite powers of  $T$ , every normal  $\mathcal{T}$ -algebra is necessarily a  $\mathcal{T}$ -algebra. Observe that a morphism of Lawvere theories  $A : (\mathcal{T}, T) \rightarrow (\mathcal{U}, U)$  is equivalently defined as a normal  $\mathcal{T}$ -algebra in  $\mathcal{U}$  with carrier  $A(T) = U$ . With this in mind, note also that a normal  $\mathcal{T}$ -algebra  $A : \mathcal{T} \rightarrow \mathcal{C}$  is uniquely determined by its carrier and its components  $A_{n,1} : \mathcal{T}(n, 1) \rightarrow \mathcal{C}(|A|^n, |A|)$ ,  $n \in \mathbb{N}$ . Related to this, observe that a morphism  $A : \mathcal{T} \rightarrow \mathcal{U}$  is an isomorphism (resp. a subtheory embedding) if and only if  $A_{n,1}$  is an isomorphism (resp. a monomorphism) in  $\text{Set}$  for each  $n \in \mathbb{N}$ .

**2.5 (The category of  $\mathcal{T}$ -algebras).** Letting  $(\mathcal{T}, T)$  be a Lawvere theory, observe that  $\mathcal{T}$ -algebras in a given category  $\mathcal{C}$  are the objects of a full subcategory  $\mathcal{T}\text{-Alg}_{\mathcal{C}}$  of the functor category  $[\mathcal{T}, \mathcal{C}]$ . We call natural transformations between  $\mathcal{T}$ -algebras  **$\mathcal{T}$ -homomorphisms**. When  $\mathcal{C}$  has standard designated finite powers, we denote by



$\mathcal{T}\text{-Alg}_{\mathcal{C}}^!$  the full subcategory of  $\mathcal{T}\text{-Alg}$  with objects all normal  $\mathcal{T}$ -algebras in  $\mathcal{C}$ . In fact we obtain an equivalence of categories

$$\mathcal{T}\text{-Alg}_{\mathcal{C}}^! \simeq \mathcal{T}\text{-Alg}_{\mathcal{C}}$$

between normal  $\mathcal{T}$ -algebras and arbitrary  $\mathcal{T}$ -algebras; this is proved in general context in [7, 5.14].

There is a canonical functor  $|-| = \text{Ev}_T : \mathcal{T}\text{-Alg}_{\mathcal{C}} \rightarrow \mathcal{C}$  given by evaluation at  $T$ , and in fact this functor is faithful. Indeed, given a  $\mathcal{T}$ -homomorphism  $\phi : A \rightarrow B$ , we know that for each  $n \in \mathbb{N}$ , the morphisms  $A(\pi_i) : A(T^n) \rightarrow A(T) = |A|$  present  $A(T^n)$  as an  $n$ -th power  $|A|^n$  of  $|A| = A(T)$  in  $\mathcal{C}$ , and similarly for  $B$ , and the naturality of  $\phi$  entails that the component  $\phi_{T^n} : A(T^n) \rightarrow B(T^n)$  is simply the morphism  $f^n : |A|^n \rightarrow |B|^n$  induced by  $f := \phi_T : |A| \rightarrow |B|$ . Hence the mapping  $\text{Ev}_T : \mathcal{T}\text{-Alg}_{\mathcal{C}}(A, B) \rightarrow \mathcal{C}(|A|, |B|)$  is injective. In fact, *via this injective map we can and will identify  $\mathcal{T}\text{-Alg}_{\mathcal{C}}(A, B)$  with the subset of  $\mathcal{C}(|A|, |B|)$  consisting of all morphisms  $f : |A| \rightarrow |B|$  such that*

$$\begin{array}{ccc} |A|^n & \xrightarrow{f^n} & |B|^n \\ A(\mu) \downarrow & & \downarrow B(\mu) \\ |A|^m & \xrightarrow{f^m} & |B|^m \end{array} \quad (2.5.i)$$

*commutes for every morphism  $\mu : T^n \rightarrow T^m$  in  $\mathcal{T}$ .* Hence we say that a morphism  $f : |A| \rightarrow |B|$  is a  **$\mathcal{T}$ -homomorphism** if it satisfies this condition. To assert merely that the square (2.5.i) commutes for a particular morphism  $\mu : T^n \rightarrow T^m$ , we say that  $f$  **preserves  $\mu$  (relative to  $A$  and  $B$ )**. Since each such morphism  $\mu$  is induced by a family of morphisms  $(\mu_i : T^n \rightarrow T)^{m}_{i=1}$ , it follows that  $f$  preserves  $\mu$  iff  $f$  preserves each of the  $\mu_i$ . Therefore  $f$  is a  $\mathcal{T}$ -homomorphism iff  $f$  preserves every morphism of the form  $\omega : T^n \rightarrow T$  in  $\mathcal{T}$ .

**2.6 (Algebraic categories).** We shall often call  $\mathcal{T}$ -algebras in  $\text{Set}$  simply  **$\mathcal{T}$ -algebras**. We write simply  $\mathcal{T}\text{-Alg}$  for the category of  $\mathcal{T}$ -algebras in  $\text{Set}$  and  $\mathcal{T}\text{-Alg}^!$  for its equivalent full subcategory consisting of normal  $\mathcal{T}$ -algebras. We say that a functor  $G : \mathcal{A} \rightarrow \text{Set}$  is **strictly finitary-algebraic** (or that  $\mathcal{A}$  is strictly finitary-algebraic over  $\text{Set}$ , via  $G$ ) if there exists a Lawvere theory  $\mathcal{T}$  and an isomorphism  $(\mathcal{A}, G) \cong (\mathcal{T}\text{-Alg}^!, |-|)$  in the slice category  $\text{CAT}/\text{Set}$ . It is well known that  $G$  is strictly finitary-algebraic if and only if  $G$  is strictly monadic for a finitary monad  $\mathbb{T}$  on  $\text{Set}$ , meaning that (i)  $G$  has a left adjoint  $F$ , (ii) the endofunctor  $T = GF$  preserves filtered colimits, and (iii) the comparison functor  $\mathcal{A} \rightarrow \text{Set}^{\mathbb{T}}$  for the associated monad  $\mathbb{T} = (T, \eta, \mu)$  is an isomorphism<sup>4</sup>. The associated theory  $\mathcal{T}$  is obtained by forming the Kleisli category  $\text{Set}_{\mathbb{T}}$  and defining  $\mathcal{T}$  to be the full subcategory of  $\text{Set}_{\mathbb{T}}^{\text{op}}$  with objects the finite cardinals, so that  $\mathcal{T}(n, m) = \text{Set}(m, T(n)) \cong (T(n))^m$ . Note then that  $\mathcal{T}(n, m) \cong \mathcal{A}(Fm, Fn)$ , with composition as in  $\mathcal{A}$ , so that we have a fully faithful functor  $\mathcal{T}^{\text{op}} \hookrightarrow \mathcal{A}$ . The associated isomorphism  $\mathcal{A} \rightarrow \mathcal{T}\text{-Alg}^!$  sends each object  $A$  of  $\mathcal{A}$  to a normal  $\mathcal{T}$ -algebra  $\underline{A} : \mathcal{T} \rightarrow \text{Set}$  that has carrier  $GA$  and associates to each

<sup>4</sup>Indeed, this follows from [7, 4.2, 11.3, 11.8, 11.14], noting that the isomorphisms in 11.14 there are isomorphisms in  $\text{CAT}/\text{Set}$  when  $\mathcal{V} = \text{Set}$ .

abstract operation  $\omega \in \mathcal{T}(n, 1)$  the mapping  $\underline{A}\omega : (GA)^n \rightarrow GA$  defined as follows. Regarding  $n$  as a cardinal, we have an associated object  $Fn$  of  $\mathcal{A}$  and an isomorphism  $\mathcal{A}(Fn, -) \cong \text{Set}(n, G-) = (G-)^n$ . Hence each element  $a = (a_1, \dots, a_n) \in (GA)^n$  determines a corresponding morphism  $a^\sharp : Fn \rightarrow A$  in  $\mathcal{A}$  and an underlying mapping  $Ga^\sharp : GFn \rightarrow GA$ . Recalling that  $GFn = Tn = \mathcal{T}(n, 1)$ , the associated mapping  $\underline{A}\omega : (GA)^n \rightarrow GA$  is given by

$$(\underline{A}\omega)(a) = (Ga^\sharp)(\omega) .$$

**2.7 (Varieties of algebras).** Let us now recall some points concerning the relation of Lawvere theories to classical universal algebra. Mac Lane [9, V.6, p. 120], for example, gives a concise introduction to the basic framework of classical universal algebra, defining the category  $\langle \Omega, E \rangle\text{-Alg}$  of  $\langle \Omega, E \rangle$ -algebras for what he calls simply a *type*  $\langle \Omega, E \rangle$ , where  $\Omega$  and  $E$  are suitable collections of formal operations and equations, respectively. Any category of the form  $\langle \Omega, E \rangle\text{-Alg}$  is called a **variety of finitary algebras**. Theorem 1 of [9, VI.8] shows that the forgetful functor  $|-| : \langle \Omega, E \rangle\text{-Alg} \rightarrow \text{Set}$  is strictly monadic, and, as noted in [9, IX.1, p. 209],  $|-|$  creates filtered colimits, so the induced monad on  $\text{Set}$  is finitary. Therefore  $|-| : \langle \Omega, E \rangle\text{-Alg} \rightarrow \text{Set}$  is strictly finitary-algebraic. In fact, it is well-known (and now straightforward to prove) that a functor  $G : \mathcal{A} \rightarrow \text{Set}$  is strictly finitary-algebraic if and only if the object  $(\mathcal{A}, G)$  of  $\text{CAT}/\text{Set}$  is isomorphic to  $(\langle \Omega, E \rangle\text{-Alg}, |-|)$  for some type  $\langle \Omega, E \rangle$ , though we shall not make use of this fact.

**Example 2.8 (Left  $R$ -modules).** The category  $R\text{-Mod}$  of left  $R$ -modules for a ring  $R$  is a variety of finitary algebras and so is isomorphic to the category of normal  $\mathcal{T}$ -algebras  $\mathcal{T}\text{-Alg}^!$  for a Lawvere theory  $\mathcal{T}$ . By 2.6, the associated theory  $\mathcal{T}$  has  $\mathcal{T}(n, m) = (|R|^n)^m$  since  $R^n$  is the free  $R$ -module on  $n$  generators, where we write  $|R|$  for the underlying set of  $R$ . By identifying  $(|R|^n)^m$  with the set  $|R|^{m \times n}$  of  $m \times n$ -matrices, we can conveniently describe composition in  $\mathcal{T}$  as matrix multiplication. Hence  $\mathcal{T}$  is the category  $\text{Mat}_R$  of  $R$ -**matrices**, whose objects are natural numbers and whose morphisms  $w : n \rightarrow m$  are  $m \times n$ -matrices with entries in  $R$ .

The normal  $\mathcal{T}$ -algebra  $\text{Mat}_R \rightarrow \text{Set}$  corresponding to a left  $R$ -module  $M$  necessarily sends each object  $n$  to the  $n$ -th power  $|M|^n$  of the underlying set  $|M|$  of  $M$ , and we identify  $|M|^n$  with the set  $|M|^{n \times 1}$  of  $n$ -element column vectors with entries in  $M$ . Given an  $m \times n$ -matrix  $w : n \rightarrow m$ , the associated mapping  $|M|^{n \times 1} \rightarrow |M|^{m \times 1}$  sends a column vector  $x$  to the matrix product  $wx$ , whose entries are the  $R$ -linear combinations  $(wx)_j = \sum_{i=1}^n w_{ji}x_i$  in  $M$ .

This all applies equally to the case where  $R$  is merely a **rig** (or *semiring*), i.e. a set  $R$  with two monoid structures  $(R, +, 0)$  and  $(R, \cdot, 1)$  with  $+$  commutative, such that  $\cdot : R \times R \rightarrow R$  preserves  $+$  and  $0$  in each variable separately.

Recall that we chose to regard  $(|R|^n)^m$  as the set of  $m \times n$ -matrices  $|R|^{m \times n}$ , whereas we could have considered  $n \times m$ -matrices instead. This line of inquiry is pursued below in the course of our discussion on *commutants* (5.13).

**Example 2.9 (The Lawvere theory of commutative  $k$ -algebras).** Given a commutative ring  $k$ , the category  $k\text{-CAlg}$  of commutative  $k$ -algebras is a variety of finitary algebras and so is isomorphic to the category  $\mathcal{T}\text{-Alg}^!$  of normal  $\mathcal{T}$ -algebras for

a Lawvere theory  $\mathcal{T}$ . Since the polynomial ring  $k[x_1, \dots, x_n]$  is the free commutative  $k$ -algebra on  $n$ -generators, we deduce by 2.6 that  $\mathcal{T}(n, 1) = k[x_1, \dots, x_n]$ , and more generally  $\mathcal{T}(n, m) = (\mathcal{T}(n, 1))^m$  is the set of  $m$ -tuples of  $n$ -variable polynomials.

**Example 2.10 (The Lawvere theory of semilattices).** A **(bounded) join semilattice** can be defined either as a partially ordered set with finite joins, or, equivalently, as a commutative monoid in which every element is idempotent. Hence the category  $\mathbf{SLat}_\vee$  of join semilattices (and their homomorphisms) is a variety of finitary algebras. The powerset  $2^n$  of a finite cardinal  $n$  is the free join semilattice on  $n$  generators, namely the singletons  $\{i\}$  with  $i \in n$ . By 2.6,  $\mathbf{SLat}_\vee$  is therefore isomorphic to the category  $\mathcal{T}\text{-Alg}^!$  of normal  $\mathcal{T}$ -algebras for a Lawvere theory  $\mathcal{T}$  with  $\mathcal{T}(n, m) = (2^n)^m = 2^{m \times n}$ , and we find that composition in  $\mathcal{T}$  is given by matrix multiplication when we view  $2 = \{0, 1\}$  as a rig (2.8) with underlying additive monoid  $(2, \vee, 0)$  and multiplicative monoid  $(2, \wedge, 1)$ . Hence  $\mathcal{T} = \mathbf{Mat}_2$  is the category of 2-matrices, and  $\mathbf{SLat}_\vee = 2\text{-Mod}$ , so that join semilattices are the same as 2-modules by 2.8.

Exchanging joins for meets, the category  $\mathbf{SLat}_\wedge$  of **(bounded) meet semilattices** is isomorphic to  $\mathbf{SLat}_\vee$ , via an isomorphism that commutes with the forgetful functors to  $\mathbf{Set}$ .

**Definition 2.11 (The full finitary theory of an object).** If a given object  $C$  of a locally small category  $\mathcal{C}$  has standard designated finite powers  $C^n$ ,  $n \in \mathbb{N}$ , then we obtain a Lawvere theory  $\mathcal{C}_C$ , called the **full finitary theory of  $C$  in  $\mathcal{C}$** , with

$$\mathcal{C}_C(n, m) = \mathcal{C}(C^n, C^m), \quad n, m \in \mathbf{ob} \mathcal{C}_C = \mathbb{N}$$

such that the mapping  $\mathbb{N} \rightarrow \mathbf{ob} \mathcal{C}$ ,  $n \mapsto C^n$ , extends to an identity-on-homs functor  $\mathcal{C}_C \rightarrow \mathcal{C}$ , which is evidently a  $\mathcal{C}_C$ -algebra in  $\mathcal{C}$  with carrier  $C$ .

In particular, given a Lawvere theory  $(\mathcal{T}, T)$ , any  $\mathcal{T}$ -algebra  $A : \mathcal{T} \rightarrow \mathcal{C}$  endows its carrier  $|A| = A(T)$  with standard designated finite powers  $|A|^n = A(T^n)$  (2.4), with respect to which we can form the full finitary theory of  $|A|$  in  $\mathcal{C}$ , which we shall denote by  $\mathcal{C}_A$ . The given  $\mathcal{T}$ -algebra  $A$  then factors uniquely as

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{A'} & \mathcal{C}_A \\ & \searrow A & \downarrow \\ & & \mathcal{C} \end{array}$$

where  $A'$  is a morphism of Lawvere theories, given on homs just as  $A$ . By abuse of notation, we write simply  $A$  to denote the morphism  $A'$ .

In the case that  $\mathcal{C}$  has standard designated finite powers, morphisms of Lawvere theories  $\mathcal{T} \rightarrow \mathcal{C}_C$  into the full finitary theory of an object  $C$  of  $\mathcal{C}$  are evidently in bijective correspondence with normal  $\mathcal{T}$ -algebras in  $\mathcal{C}$  with carrier  $C$ .

**Example 2.12 (The Lawvere theory of Boolean algebras).** The category  $\mathbf{Bool}$  of Boolean algebras is a variety of finitary algebras and so is isomorphic to the category  $\mathcal{T}\text{-Alg}^!$  of normal  $\mathcal{T}$ -algebras for a Lawvere theory  $\mathcal{T}$ . In fact, it is well-known that the morphism of theories  $\mathcal{T} \rightarrow \mathbf{Set}_2$  determined by the Boolean algebra  $2 = \{0, 1\}$  is an isomorphism between  $\mathcal{T}$  and the full finitary theory  $\mathbf{Set}_2$  of 2 in  $\mathbf{Set}$ . Indeed, this follows from [3, III.1, Example 4].

### 3 Affine and convex spaces

**3.1 (The affine core of a Lawvere theory).** Every Lawvere theory  $(\mathcal{T}, T)$  has a subtheory  $\mathcal{T}^{\text{aff}} \hookrightarrow \mathcal{T}$  called the **affine core** of  $\mathcal{T}$  [4, §3], consisting of all those morphisms  $\omega : T^n \rightarrow T^m$  for which the composite

$$T \xrightarrow{(1_T, \dots, 1_T)} T^n \xrightarrow{\omega} T^m$$

equals  $(1_T, \dots, 1_T) : T \rightarrow T^m$ . We say that  $\mathcal{T}$  is **affine** if  $\mathcal{T}$  equals its affine core.

**3.2 (Affine spaces over a ring or rig).** Let  $R$  be a ring, or more generally, a rig. Recall that the category of  $R$ -matrices  $\text{Mat}_R$  is the Lawvere theory of left  $R$ -modules (2.8). By definition, a **(left)  $R$ -affine space** (or **(left)  $R$ -affine module**) is a normal  $\mathcal{T}$ -algebra for the affine core  $\mathcal{T} = \text{Mat}_R^{\text{aff}}$  of  $\text{Mat}_R$ . Hence  $R$ -affine spaces are the objects of a category  $R\text{-Aff} = \text{Mat}_R^{\text{aff}}\text{-Alg}$ . Since the projection morphisms  $\pi_i : n \rightarrow 1$  in  $\text{Mat}_R$  are the standard basis vectors for  $R^{1 \times n}$ , one deduces that the affine part  $\text{Mat}_R^{\text{aff}}$  consists of the  $R$ -matrices in which each row sums to 1. By 2.4, an  $R$ -affine space  $E$  is therefore given by a set  $E$  (the carrier) equipped with a suitable family of mappings  $\text{Mat}_R^{\text{aff}}(n, 1) \rightarrow \text{Set}(E^n, E)$  that associate to each  $n$ -element row vector  $w = [w_1, \dots, w_n]$  with  $\sum_{i=1}^n w_i = 1$  a mapping  $E^n \rightarrow E$  whose value at a given column vector  $x \in E^n$  we write as  $\sum_{i=1}^n w_i x_i$  and call a **(left)  $R$ -affine combination** of the  $x_i$ . For example, since the morphism of theories  $\text{Mat}_R^{\text{aff}} \hookrightarrow \text{Mat}_R$  induces a functor  $R\text{-Mod} \rightarrow R\text{-Aff}$ , every left  $R$ -module  $M$  carries the structure of a left  $R$ -affine space. The morphisms in the category of (left)  $R$ -affine spaces  $R\text{-Aff}$  are **(left)  $R$ -affine maps**, i.e. those mappings that preserve left  $R$ -affine combinations.

**Example 3.3 (Unbounded semilattices as affine spaces).** An **unbounded join semilattice** may be defined as a poset in which every pair of elements has a join or, equivalently, as a commutative semigroup in which every element is idempotent. The set  $2^n \setminus \{0\}$  of all nonempty subsets of a given finite cardinal  $n$  is closed under binary joins in the semilattice  $2^n$  (2.10) and so carries the structure of an unbounded join semilattice, and the singletons  $\{i\}$  with  $i \in n$  exhibit  $2^n \setminus \{0\}$  as the free unbounded join semilattice on  $n$  generators. The category  $\text{USLat}_\vee$  of unbounded semilattices and their homomorphisms is a variety of finitary algebras and so is isomorphic to the category of normal  $\mathcal{T}$ -algebras for a Lawvere theory  $\mathcal{T}$  with  $\mathcal{T}(n, m) = (2^n \setminus \{0\})^m$ . The latter set may be identified with the subset of  $2^{m \times n}$  consisting of all  $m \times n$ -matrices in which each row is nonzero, whereupon we deduce that  $\mathcal{T}$  is a subtheory of the theory  $\text{Mat}_2$  of modules over the rig  $(2, \vee, 0, \wedge, 1)$ , i.e. (bounded) semilattices (2.10). Indeed,  $\mathcal{T}$  is precisely the theory  $\text{Mat}_2^{\text{aff}}$  of affine spaces over the rig  $(2, \vee, 0, \wedge, 1)$ . Hence  $\text{USLat}_\vee \cong 2\text{-Aff}$ , i.e. unbounded semilattices are the same as affine spaces over the rig  $(2, \vee, 0, \wedge, 1)$ .

**3.4 (Preordered abelian groups and preordered rings).** By definition, a **pre-ordered commutative monoid** is a commutative monoid object in the cartesian monoidal category  $\text{Ord}$  of preordered sets and monotone maps. Preordered commutative monoids are the objects of a category  $\text{CMon}(\text{Ord})$ , the category of commutative monoids in  $\text{Ord}$ . We say that a preordered commutative monoid  $M$  is a **preordered**

**abelian group** if its underlying commutative monoid (in  $\mathbf{Set}$ ) is an abelian group. Note then that the negation map  $- : M \rightarrow M$  is not monotone but rather is order-reversing. Preordered abelian groups form a full subcategory  $\mathbf{Ab}_{\leq}$  of the category of preordered commutative monoids.

It is well-known that the notion of preordered abelian group can be equivalently defined as an abelian group  $M$  equipped with a submonoid  $M_+ \hookrightarrow M$ . Indeed, given a preordered abelian group  $M$ , one takes  $M_+ = \{m \in M \mid 0 \leq m\}$ , and conversely, given a submonoid  $M_+$  of an abelian group  $M$  one defines a preorder  $\leq$  on  $M$  by  $m \leq m' \Leftrightarrow m' - m \in M_+$ . It is conventional to call  $M_+$  the **positive part** of  $M$  despite the fact that  $0 \in M_+$ . Morphisms of preordered abelian groups can be described equivalently as homomorphisms of the underlying abelian groups  $h : M \rightarrow N$  with the property that  $h(M_+) \subseteq N_+$ .

The category  $\mathbf{Ab}_{\leq}$  of preordered abelian groups is symmetric monoidal when we define the monoidal product  $M \otimes N$  of preordered abelian groups  $M$  and  $N$  to be the usual tensor product of abelian groups equipped with the submonoid  $(M \otimes N)_+ \hookrightarrow M \otimes N$  generated by the pure symbols  $m \otimes n$  with  $m \in M_+$ ,  $n \in N_+$ . The unit object is  $\mathbb{Z}$ , with the natural order.

By definition, a **preordered ring** is a monoid in the monoidal category of preordered abelian groups  $\mathbf{Ab}_{\leq}$ . Equivalently, a preordered ring is a ring  $R$  equipped with an arbitrary subrig  $R_+ \hookrightarrow R$ . Note that any rig  $S$  that occurs as the positive part  $R_+$  of some preordered ring  $R$  is necessarily **additively cancellative**, meaning that the commutative semigroup  $(S, +)$  is cancellative (i.e.,  $s + t = s + u \Rightarrow t = u$ ). Moreover, the rigs that occur as positive parts of preordered rings are precisely the additively cancellative rigs, since if  $S$  is additively cancellative then we can embed  $S$  into its *ring completion*, mimicking the usual construction of  $\mathbb{Z}$  from  $\mathbb{N}$ .

Preordered rings are the objects of a category  $\mathbf{Ring}_{\leq}$ , the category of monoids in the monoidal category  $\mathbf{Ab}_{\leq}$ , in which the morphisms are ring homomorphism that are also monotone.

**Definition 3.5 (Convex spaces over a preordered ring).** Given a preordered ring  $R$ , a **(left)  $R$ -convex space** (or **(left)  $R$ -convex module**) is a left  $R_+$ -affine space, i.e. a left affine space over the rig  $R_+$  obtained as the positive part of  $R$  (3.4). Hence an  $R$ -convex space is by definition a set equipped with operations that permit the taking of left  $R_+$ -affine combinations (3.2), which we call **(left)  $R$ -convex combinations**. We write  $R\text{-Cvx} := R_+\text{-Aff}$  for the category of  $R$ -convex spaces. Note that  $R$ -convex spaces are the normal  $\mathcal{T}$ -algebras for the Lawvere theory  $\mathcal{T} = \mathbf{Mat}_{R_+}^{\text{aff}}$ , whose morphisms are  $R_+$ -matrices in which each row sums to 1.

**Example 3.6 (Convex spaces over the reals).** For example, when  $R$  is the real numbers  $\mathbb{R}$  with the usual order, the notion of  $\mathbb{R}$ -convex space is the familiar notion of *convex space*. Observe that when  $n > 0$ ,  $\mathbf{Mat}_{\mathbb{R}_+}^{\text{aff}}(n, 1) \subseteq \mathbb{R}^{1 \times n}$  is the standard geometric  $(n - 1)$ -simplex, presented in terms of barycentric coordinates.

**Example 3.7 (Convex spaces over a ring of continuous functions).** Given a topological space  $X$ , let  $C(X)$  denote the ring of all real-valued continuous functions on  $X$ . The pointwise partial order on  $C(X)$  makes it a preordered ring whose positive part  $C(X)_+$  is the set  $C(X, \mathbb{R}_+)$  of all continuous  $\mathbb{R}_+$ -valued functions. Given any convex

subset  $S$  of  $\mathbb{R}^n$ , the set  $C(X, S)$  of all continuous  $S$ -valued functions on  $X$  carries the structure of a  $C(X)$ -convex space.

## 4 Commutation and Kronecker products of operations

**4.1.** Given a pair of finite cardinals  $(j, k)$ , regarded as sets, the cartesian product  $j \times k$  in  $\mathbf{Set}$  has cardinality  $jk$ , so  $jk$  serves as a product in  $\mathbf{FinCard}$  of the objects  $j$  and  $k$ . Despite this apparently simple way of forming binary products in  $\mathbf{FinCard}$ , one must also choose projections  $\pi_1^{(j,k)} : jk \rightarrow j$  and  $\pi_2^{(j,k)} : jk \rightarrow k$  if  $jk$  is to be equipped with the structure of a product of  $(j, k)$  in  $\mathbf{FinCard}$ , and there is more than one way to do this. In the present section, we must fix a determinate choice of such product projections—equivalently, we must fix designated bijections  $j \times k \rightarrow jk$  in  $\mathbf{Set}$  that let us *encode* elements of the cartesian product  $j \times k$  as elements of  $jk$ . For example, we shall see in 4.4 that the definition of the classical *Kronecker product* of matrices depends on one specific such encoding (4.4.ii). With such a choice,  $\mathbf{FinCard}$  is cartesian monoidal. When  $jk$  is to be regarded as a product of  $(j, k)$  via the chosen projections we shall denote it by  $j \times k$ . It is important to note that the symmetry isomorphism  $s_{j,k} : j \times k \rightarrow k \times j$  is *not* in general the identity map on  $jk = kj$ . We shall take the designated projections  $\pi_1 : j = j \times 1 \rightarrow j$  and  $\pi_2 : k = 1 \times k \rightarrow k$  to be the identity maps.

**4.2.** Let  $(\mathcal{T}, T)$  be a Lawvere theory. Since the product  $jk$  of natural numbers  $j$  and  $k$  carries the structure of a product  $j \times k$  in  $\mathbf{FinCard}$  (4.1), there is an associated canonical way of equipping the object  $T^{j \times k}$  of  $\mathcal{T}$  with the structure of both a  $j$ -th power of  $T^k$  and also a  $k$ -th power of  $T^j$ , which we may signify informally by writing

$$(T^k)^j = T^{j \times k} = (T^j)^k. \quad (4.2.i)$$

Writing  $p^{(j,k)} = (p_v^{(j,k)} : T^{j \times k} \rightarrow T^k)_{v=1}^j$  and  $q^{(j,k)} = (q_t^{(j,k)} : T^{j \times k} \rightarrow T^j)_{t=1}^k$  for the associated power cones, we call  $(T^{j \times k}, p^{(j,k)})$  the **left  $j$ -th power** of  $T^k$ , and we call  $(T^{j \times k}, q^{(j,k)})$  the **right  $k$ -th power**<sup>5</sup> of  $T^j$ . Writing  $T^j * T^k = T^{j \times k}$ , we therefore obtain evident functors

$$T^j * (-) : \mathcal{T} \rightarrow \mathcal{T}, \quad T^k \mapsto T^{j \times k} \quad (j \in \mathbb{N})$$

$$(-) * T^k : \mathcal{T} \rightarrow \mathcal{T}, \quad T^j \mapsto T^{j \times k} \quad (k \in \mathbb{N})$$

induced by left  $j$ -th and right  $k$ -th power structures, respectively, carried by the objects  $T^{j \times k}$  with  $(j, k) \in \mathbb{N} \times \mathbb{N}$ . This now begs the question as to whether these are the partial functors of a bifunctor

$$* : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$$

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<sup>5</sup>It is instructive to note that the right  $k$ -th power cone  $q^{(j,k)}$  is not in general equal to the left  $k$ -th power cone  $p^{(k,j)}$ , despite the fact that these cones both equip the same object  $T^{jk}$  with the structure of a  $k$ -th power of  $T^j$ . Indeed, the automorphism of  $T^{jk}$  induced by this pair of  $k$ -th power cones is the isomorphism  $T^{k \times j} \rightarrow T^{j \times k}$  determined by the symmetry isomorphism  $j \times k \rightarrow k \times j$  in  $\mathbf{FinCard}$  (4.1).

given on objects by  $(T^j, T^k) \mapsto T^{j \times k}$ . By [9, II.3, Prop. 1], this is the case if and only if for every pair of morphisms  $\mu : T^j \rightarrow T^{j'}$  and  $\nu : T^k \rightarrow T^{k'}$  in  $\mathcal{T}$  the composites

$$\begin{aligned} 1. \quad & T^j * T^k \xrightarrow{\mu * T^k} T^{j'} * T^k \xrightarrow{T^{j'} * \nu} T^{j'} * T^{k'} \\ 2. \quad & T^j * T^k \xrightarrow{T^j * \nu} T^j * T^{k'} \xrightarrow{\mu * T^{k'}} T^{j'} * T^{k'} \end{aligned} \tag{4.2.ii}$$

are equal. This leads to the following:

**Definition 4.3.**

1. Given morphisms  $\mu : T^j \rightarrow T^{j'}$  and  $\nu : T^k \rightarrow T^{k'}$  in  $\mathcal{T}$ , the **first and second Kronecker products**  $\mu * \nu$  and  $\mu \tilde{*} \nu$  of  $\mu$  with  $\nu$  are defined as the composites 1 and 2 in (4.2.ii), respectively, i.e.

$$\begin{aligned} \mu * \nu &= \left( T^{j \times k} \xrightarrow{\mu * T^k} T^{j' \times k} \xrightarrow{T^{j'} * \nu} T^{j' \times k'} \right), \\ \mu \tilde{*} \nu &= \left( T^{j \times k} \xrightarrow{T^j * \nu} T^{j \times k'} \xrightarrow{\mu * T^{k'}} T^{j' \times k'} \right). \end{aligned}$$

2. We say that  $\mu$  **commutes with**  $\nu$  if  $\mu * \nu = \mu \tilde{*} \nu$ .
3. We say that  $\mathcal{T}$  is **commutative** if  $\mu$  commutes with  $\nu$  for every pair of morphisms  $\mu$  and  $\nu$  in  $\mathcal{T}$ .

**Example 4.4 (The Kronecker product of matrices).** Given a ring  $R$ , or even just a rig  $R$ , consider the Lawvere theory of left  $R$ -modules, i.e. the category  $\text{Mat}_R$  of  $R$ -matrices (2.8). Recall that morphisms  $j \rightarrow j'$  in  $\text{Mat}_R$  are  $j' \times j$ -matrices. Since such matrices are usually indexed by pairs of *positive* integers, it shall be convenient here to depart from the usual von Neumann definition of the ordinals and instead identify each object  $j$  of  $\text{FinCard}$  with the set of all positive integers less than or equal to  $j$ , so that the above  $j' \times j$ -matrices are families indexed by the usual cartesian product<sup>6</sup>  $j' \times j$ .

Letting  $X \in \text{Mat}_R(j, j') = R^{j' \times j}$  and  $Y \in \text{Mat}_R(k, k') = R^{k' \times k}$ , the classical **Kronecker product** of  $Y$  by  $X$  is the  $j'k' \times jk$ -matrix  $Y \otimes X$  with entries

$$(Y \otimes X)_{\langle u, s \rangle \langle v, t \rangle} = Y_{st} X_{uv} \quad u \in j', s \in k', v \in j, t \in k. \tag{4.4.i}$$

where in general we write  $\langle v, t \rangle$  to denote the element

$$\langle v, t \rangle = v + j(t - 1) \tag{4.4.ii}$$

of  $jk$  associated to the pair  $(v, t) \in j \times k$ . The hidden reason behind the seemingly arbitrary convention (4.4.ii) is that it provides one standard way of assigning to each pair of finite cardinals  $(j, k)$  a bijection  $\langle -, - \rangle : j \times k \rightarrow jk$ , so that  $jk$  is thus equipped with the structure of a product of  $(j, k)$  in  $\text{FinCard}$ . Indeed, each element of  $jk$  can be written in the form  $\langle v, t \rangle$  for unique  $v \in j$  and  $t \in k$ , and the maps  $\pi_1 : jk \rightarrow j$  and

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<sup>6</sup>putting aside for the moment our similar notation for the product in  $\text{FinCard}$  (4.1).

$\pi_2 : jk \rightarrow k$  given by  $\pi_1(\langle v, t \rangle) = v$  and  $\pi_2(\langle v, t \rangle) = t$  present  $jk$  as a product  $j \times k$  in Set and hence in FinCard.

With this choice of binary products in FinCard, we claim that

$$X * Y = Y \otimes X,$$

i.e., the first Kronecker product  $X * Y$  in  $\text{Mat}_R$  is the usual Kronecker product of matrices  $Y \otimes X$ .

In order to prove this, recall that  $X * Y$  is defined as the composite

$$jk \xrightarrow{X * k} j'k \xrightarrow{j' * Y} j'k'$$

in  $\text{Mat}_R$ , where here we write the objects  $T^n$  of the theory  $\mathcal{T} = \text{Mat}_R$  simply as  $n$  (since concretely  $T^n = n$ ). In order to examine the entries of the matrices  $X * k$  and  $j' * Y$ , let us first note that for each object  $n$  of  $\text{Mat}_R$ , the designated  $n$ -th power cone  $(P_i : n \rightarrow 1)_{i \in n}$  in  $\text{Mat}_R$  consists of the standard basis row-vectors  $P_i \in R^{1 \times n}$ , having a 1 in the  $i$ -th position and zeros everywhere else. In particular,  $jk$  is a  $jk$ -th power of 1 in  $\text{Mat}_R$ , via the morphisms  $P_{\langle v, t \rangle} : jk \rightarrow 1$  with  $(v, t) \in j \times k$ . For fixed  $t \in k$ , these morphisms induce a unique morphism  $Q_t^{(j, k)} : jk \rightarrow j$  in  $\text{Mat}_R$  such that  $P_v Q_t^{(j, k)} = P_{\langle v, t \rangle}$  ( $v \in j$ ), and the resulting family<sup>7</sup>  $(Q_t^{(j, k)})_{t \in k}$  presents  $jk$  as a  $k$ -th power of  $j$  in  $\text{Mat}_R$ . Explicitly,  $Q_t^{(j, k)}$  is the  $j \times jk$ -matrix with entries

$$(Q_t^{(j, k)})_{vb} = \begin{cases} 1 & \text{if } b = \langle v, t \rangle, \\ 0 & \text{otherwise} \end{cases}$$

where  $v \in j$  and  $b \in jk$ . Similarly, morphisms  $(Q_t^{(j', k)} : j'k \rightarrow j')_{t \in k}$  present  $j'k$  as a  $k$ -th power of  $j'$  in  $\text{Mat}_R$ . By definition,  $X * k : jk \rightarrow j'k$  is the unique morphism such that  $jk \xrightarrow{X * k} j'k \xrightarrow{Q_t^{(j', k)}} j'$  equals  $jk \xrightarrow{Q_t^{(j, k)}} j \xrightarrow{X} j'$  for all  $t \in k$ . It is straightforward to show therefore that  $X * k$  is the  $j'k \times jk$ -matrix with entries

$$(X * k)_{\langle u, t_1 \rangle \langle v, t_2 \rangle} = \begin{cases} X_{uv} & \text{if } t_1 = t_2, \\ 0 & \text{otherwise,} \end{cases}$$

where  $u \in j'$ ,  $v \in j$ , and  $t_1, t_2 \in k$ . Analogously<sup>8</sup>,  $j' * Y : j'k \rightarrow j'k'$  is the  $j'k' \times j'k$ -matrix with entries

$$(j' * Y)_{\langle u_1, s \rangle \langle u_2, t \rangle} = \begin{cases} Y_{st} & \text{if } u_1 = u_2, \\ 0 & \text{otherwise,} \end{cases}$$

where  $u_1, u_2 \in j'$ ,  $s \in k'$  and  $t \in k$ . But  $X * Y$  is the matrix product  $(j' * Y)(X * k) \in R^{j'k' \times jk}$ , and one now readily computes that the entries of the latter product are exactly those of the Kronecker product  $Y \otimes X$  (cf. 4.4.i).

**Proposition 4.5 (Relation between the first and second Kronecker products).**

Given morphisms  $\mu : T^j \rightarrow T^{j'}$  and  $\nu : T^k \rightarrow T^{k'}$  in a Lawvere theory  $\mathcal{T}$ , the second

<sup>7</sup>This family is the *right  $k$ -th power cone*  $q^{(j, k)}$  in the terminology of 4.2.

<sup>8</sup>This time we use the *left  $j'$ -th power cones*  $p^{(j', k)}$  and  $p^{(j', k')}$  in the terminology of 4.2.



Kronecker product  $\mu \tilde{*} \nu$  can be expressed in terms of the first Kronecker product  $\nu * \mu$  via the commutativity of the diagram

$$\begin{array}{ccc} T^{j \times k} & \xrightarrow{\mu \tilde{*} \nu} & T^{j' \times k'} \\ \wr \downarrow & & \downarrow \wr \\ T^{k \times j} & \xrightarrow{\nu * \mu} & T^{k' \times j'} \end{array}$$

in which the left and right sides are the isomorphisms induced by the symmetry isomorphisms  $k \times j \rightarrow j \times k$  and  $k' \times j' \rightarrow j' \times k'$  in  $\text{FinCard}$  (4.1). As a consequence, the commutation relation is symmetric, i.e.

$\mu$  commutes with  $\nu$  if and only if  $\nu$  commutes with  $\mu$ .

*Proof.* This follows from the fact that for each fixed  $n \in \mathbb{N}$ , the symmetry isomorphisms  $m \times n \rightarrow n \times m$  in  $\text{FinCard}$  (4.1) with  $m \in \mathbb{N}$  induce isomorphisms  $T^{n \times m} \rightarrow T^{m \times n}$  in  $\mathcal{T}$  that constitute a natural isomorphism  $T^n * (-) \Rightarrow (-) * T^n$ .  $\square$

**Example 4.6 (The second Kronecker product of matrices).** Continuing Example 4.4, it now follows from 4.5 that the second Kronecker product  $X \tilde{*} Y : jk \rightarrow j'k'$  of morphisms  $X : j \rightarrow j'$  and  $Y : k \rightarrow k'$  in  $\text{Mat}_R$  is the  $j'k' \times jk$ -matrix with entries

$$(X \tilde{*} Y)_{\langle u, s \rangle \langle v, t \rangle} = (Y * X)_{\langle s, u \rangle \langle t, v \rangle} = (X \otimes Y)_{\langle s, u \rangle \langle t, v \rangle} = X_{uv} Y_{st}$$

where  $u \in j'$ ,  $s \in k'$ ,  $v \in j$ ,  $t \in k$ . Hence if  $R$  is commutative then  $X \tilde{*} Y = Y \otimes X = X * Y$ , showing that  $\text{Mat}_R$  is commutative. Conversely, if  $\text{Mat}_R$  is commutative then by taking  $j = j' = k = k' = 1$  we find that  $R$  is commutative. Hence we have proved the following:

*The Lawvere theory  $\text{Mat}_R$  of left  $R$ -modules for a rig  $R$  is commutative if and only if  $R$  is commutative.* (4.6.i)

In particular, the Lawvere theory  $\text{Mat}_2$  of semilattices is commutative.

Clearly any subtheory of a commutative theory is commutative. In particular, the following Lawvere theories are commutative, as each is a subtheory of a theory of the form  $\text{Mat}_R$  for a commutative rig  $R$ :

**Example 4.7.** The following Lawvere theories are commutative:

1. The theory  $\text{Mat}_R^{\text{aff}}$  of  $R$ -affine spaces for a commutative ring or rig  $R$ .
2. The theory of  $R$ -convex spaces  $\text{Mat}_{R_+}^{\text{aff}}$  for a commutative preordered ring  $R$ .
3. The theory of unbounded semilattices  $\text{Mat}_2^{\text{aff}}$ .

**4.8.** Let  $(\mathcal{T}, T)$  be a Lawvere theory. Given  $j \in \mathbb{N}$ , any choice of  $j$ -th powers in  $\mathcal{T}$  determines an endofunctor  $(-)^j : \mathcal{T} \rightarrow \mathcal{T}$ , and since this endofunctor  $(-)^j$  preserves finite powers it can be regarded as a  $\mathcal{T}$ -algebra in  $\mathcal{T}$ . In particular, if we employ the *left  $j$ -th powers* in  $\mathcal{T}$  (4.2), then the resulting endofunctor is the functor  $T^j * (-) : \mathcal{T} \rightarrow \mathcal{T}$  of 4.2, given on objects by  $T^k \mapsto T^{j \times k}$ . Therefore  $T^j * (-)$  is a  $\mathcal{T}$ -algebra in  $\mathcal{T}$  with carrier  $T^{j \times 1} = T^j$ . We employ this observation in the following:

**Proposition 4.9.** *Let  $\mu : T^j \rightarrow T^{j'}$  and  $\nu : T^k \rightarrow T^{k'}$  be morphisms in a Lawvere theory  $(\mathcal{T}, T)$ , and denote by  $(\mu_u : T^j \rightarrow T)_{u=1}^{j'}$  and  $(\nu_s : T^k \rightarrow T)_{s=1}^{k'}$  the families inducing  $\mu$  and  $\nu$ , respectively. Then we have  $\mathcal{T}$ -algebras  $A = T^j * (-)$  and  $B = T^{j'} * (-)$  in  $\mathcal{T}$  with carriers  $|A| = T^j$ ,  $|B| = T^{j'}$ , respectively, and the following are equivalent:*

1.  $\mu$  commutes with  $\nu$ .
2.  $\mu : |A| \rightarrow |B|$  preserves  $\nu$  relative to  $A$  and  $B$  (2.5).
3.  $\mu_u : T^j \rightarrow T$  commutes with  $\nu_s : T^k \rightarrow T$  for all indices  $u$  and  $s$ .
4.  $\mu$  commutes with each of the components  $\nu_s$  of  $\nu$ .

*Proof.* The equivalence  $1 \Leftrightarrow 2$  follows readily from the definitions. By 2.5, 2 is equivalent to the following statement:

5.  $\mu : |A| \rightarrow |B|$  preserves each of the components  $\nu_s$  of  $\nu$ .

Now invoking the equivalence  $1 \Leftrightarrow 2$  with respect to the morphisms  $\mu$  and  $\nu_s$ , we deduce that 5 is equivalent to 4. By symmetry, 4 holds iff each component  $\nu_s$  commutes with  $\mu$ . Having established the equivalence of 1 and 4 for an arbitrary pair of morphisms  $(\mu, \nu)$ , we can now invoke this equivalence with respect to each pair  $(\nu_s, \mu)$  to deduce that 4 holds if and only if each  $\nu_s$  commutes with each of the components  $\mu_u$  of  $\mu$ , and by symmetry this is equivalent to 3.  $\square$

**Remark 4.10.** Having reduced the notion of commutation of morphisms  $\mu$  and  $\nu$  in a Lawvere theory  $(\mathcal{T}, T)$  to the case of morphisms of the form  $\mu : T^j \rightarrow T$  and  $\nu : T^k \rightarrow T$ , observe that the definition of the first and second Kronecker products of such morphisms reduces to the following:

$$\mu * \nu = \left( T^{j \times k} \xrightarrow{\mu * T^k} T^k \xrightarrow{\nu} T \right), \quad \mu \tilde{*} \nu = \left( T^{j \times k} \xrightarrow{T^j * \nu} T^j \xrightarrow{\mu} T \right).$$

## 5 Commutants

**Definition 5.1.** Let  $\mathcal{U}$  be a Lawvere theory.

1. Letting  $A : \mathcal{T} \rightarrow \mathcal{U}$  and  $B : \mathcal{S} \rightarrow \mathcal{U}$  be morphisms of Lawvere theories, we say that  $A$  **commutes with**  $B$  (or that  $A$  and  $B$  commute) if  $A(\mu)$  commutes with  $B(\nu)$  in  $\mathcal{U}$  for all morphisms  $\mu$  in  $\mathcal{T}$  and  $\nu$  in  $\mathcal{S}$ .
2. A **Lawvere theory over**  $\mathcal{U}$  is a Lawvere theory  $\mathcal{T}$  equipped with a morphism  $\mathcal{T} \rightarrow \mathcal{U}$ . The **category of Lawvere theories over**  $\mathcal{U}$  is the slice category  $\text{Th}/\mathcal{U}$ .
3. Given Lawvere theories  $\mathcal{T}$  and  $\mathcal{S}$  over  $\mathcal{U}$ , we say that  $\mathcal{T}$  **commutes with**  $\mathcal{S}$  if the associated morphisms to  $\mathcal{U}$  commute.
4. Subtheories  $\mathcal{T}$  and  $\mathcal{S}$  of  $\mathcal{U}$  are said to **commute** if they commute as Lawvere theories over  $\mathcal{U}$ , i.e. if  $\mu$  commutes with  $\nu$  for all  $\mu \in \text{mor } \mathcal{T}$  and  $\nu \in \text{mor } \mathcal{S}$ .

**Proposition 5.2.** *Let  $A : (\mathcal{T}, T) \rightarrow (\mathcal{U}, U)$  and  $B : (\mathcal{S}, S) \rightarrow (\mathcal{U}, U)$  be morphisms of Lawvere theories. Then  $A$  commutes with  $B$  if and only if  $A(\mu)$  commutes with  $B(\nu)$  in  $\mathcal{U}$  for all morphisms of the form  $\mu : T^j \rightarrow T$  in  $\mathcal{T}$  and  $\nu : S^k \rightarrow S$  in  $\mathcal{S}$ .*

*Proof.* This follows from 4.9, since  $A$  and  $B$  strictly preserve the designated finite powers of  $T$  and  $S$ , respectively.  $\square$

**Definition 5.3.** A morphism of Lawvere theories  $A : \mathcal{T} \rightarrow \mathcal{U}$  is said to be **central** if it commutes with the identity morphism  $1_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}$ .

**Example 5.4.** Given a Lawvere theory  $(\mathcal{T}, T)$ , the unique morphism of Lawvere theories  $T^{(-)} : \mathbf{FinCard}^{\text{op}} \rightarrow \mathcal{T}$  is central. Indeed, a morphism of the form  $T^j \rightarrow T$  in  $\mathcal{T}$  lies in the image of  $T^{(-)}$  if and only if it is a projection  $\pi_i : T^j \rightarrow T$ , and given any operation  $\nu : T^k \rightarrow T$  in  $\mathcal{T}$ , the diagram

$$\begin{array}{ccc} T^{j \times k} & \xrightarrow{T^j * \nu} & T^j = T^{j \times 1} \\ \pi_i * T^k \downarrow & & \downarrow \pi_i \\ T^k & \xrightarrow{\nu} & T \end{array}$$

commutes since its left and right sides are equally the left  $j$ -th power projections  $p_i^{(j,k)}$  and  $p_i^{(j,1)}$  (4.2), respectively.

**Proposition 5.5.** Given a set of morphisms  $\Omega \subseteq \mathbf{mor} \mathcal{U}$  in a Lawvere theory  $(\mathcal{U}, U)$ , the set of morphisms

$$\Omega^\perp = \{\mu \in \mathbf{mor} \mathcal{U} \mid \mu \text{ commutes with every } \nu \in \Omega\}$$

is a concrete subtheory of  $\mathcal{U}$  (2.3).

*Proof.* Using the functoriality of  $(-) * U^k : \mathcal{U} \rightarrow \mathcal{U}$  for each  $k \in \mathbb{N}$ , one computes straightforwardly that  $\Omega^\perp$  is closed under composition in  $\mathcal{U}$ . By 5.4,  $\Omega^\perp$  contains all the projections  $\pi_i : U^j \rightarrow U$ . Lastly, given a family of morphisms  $\{\mu_i : U^j \rightarrow U \mid i = 1, \dots, j'\} \subseteq \Omega^\perp$ , the induced morphism  $\mu : U^j \rightarrow U^{j'}$  lies in  $\Omega^\perp$  by 4.9.  $\square$

**Definition 5.6.** Let  $\mathcal{U}$  be a Lawvere theory.

1. Given a set of morphisms  $\Omega \subseteq \mathbf{mor} \mathcal{U}$ , we call the subtheory  $\Omega^\perp \hookrightarrow \mathcal{U}$  of 5.5 the **commutant** of  $\Omega$  (in  $\mathcal{U}$ ).
2. Given a morphism of Lawvere theories  $A : \mathcal{T} \rightarrow \mathcal{U}$ , the **commutant**  $\mathcal{T}_A^\perp$  of  $A$  (or of  $\mathcal{T}$  with respect to  $A$ ) is defined as the commutant of the image  $A(\mathbf{mor} \mathcal{T}) \subseteq \mathbf{mor} \mathcal{U}$ .
3. Given a Lawvere theory  $\mathcal{T}$  over  $\mathcal{U}$ , the **commutant**  $\mathcal{T}^\perp$  of  $\mathcal{T}$  is defined as the commutant of the associated morphism  $\mathcal{T} \rightarrow \mathcal{U}$ .
4. The **commutant** of a subtheory  $\mathcal{T} \hookrightarrow \mathcal{U}$  is defined as the commutant  $\mathcal{T}^\perp$  of  $\mathcal{T}$ , considered as a theory over  $\mathcal{U}$ . Equivalently,  $\mathcal{T}^\perp$  is the commutant of  $\mathbf{mor} \mathcal{T} \subseteq \mathbf{mor} \mathcal{U}$ .

The following is immediate from the definitions:

**Proposition 5.7.** Let  $A : \mathcal{T} \rightarrow \mathcal{U}$  and  $B : \mathcal{S} \rightarrow \mathcal{U}$  be morphisms of Lawvere theories. Then  $A$  and  $B$  commute if and only if  $B$  factors through the commutant  $\mathcal{T}_A^\perp \hookrightarrow \mathcal{U}$  of  $A$ .

**Remark 5.8.** By 5.7, the commutant  $\mathcal{T}^\perp$  of a theory  $\mathcal{T}$  over  $\mathcal{U}$  is characterized, up to isomorphism, by a universal property. Hence we will sometimes also call any theory over  $\mathcal{U}$  isomorphic to  $\mathcal{T}^\perp$  **the commutant** of  $\mathcal{T}$ .

**Theorem 5.9.** *Given a morphism of Lawvere theories  $A : (\mathcal{T}, T) \rightarrow (\mathcal{U}, U)$ , let us regard  $A$  as a  $\mathcal{T}$ -algebra in  $\mathcal{U}$ . Then the commutant  $\mathcal{T}_A^\perp$  of  $A$  is isomorphic to the full finitary theory of  $A$  in the category  $\mathcal{T}\text{-Alg}_{\mathcal{U}}$  of  $\mathcal{T}$ -algebras in  $\mathcal{U}$ . In symbols,*

$$\mathcal{T}_A^\perp \cong (\mathcal{T}\text{-Alg}_{\mathcal{U}})_A$$

as theories over  $\mathcal{U}$ . Further, we can choose standard designated finite powers in  $\mathcal{T}\text{-Alg}_{\mathcal{U}}$  in such a way that this isomorphism is an identity.

*Proof.* Given an arbitrary object  $U^k$  of  $\mathcal{U}$ , note that the left  $j$ -th powers  $(U^k)^j = U^{j \times k}$  of  $U^k$  ( $j \in \mathbb{N}$ ) are standard (2.1), and for  $k = 1$  these are precisely the designated  $j$ -th powers  $U^j$  of  $U$  in  $\mathcal{U}$ . Let us now use these finite powers as our designated finite powers in  $\mathcal{U}$  (2.1), calling them the *left finite powers*. The resultant pointwise finite powers in  $\mathcal{T}\text{-Alg}_{\mathcal{U}}$  are standard, and we shall use them in forming the full finitary theory  $(\mathcal{T}\text{-Alg}_{\mathcal{U}})_A$  of  $A$  in  $\mathcal{T}\text{-Alg}_{\mathcal{U}}$ . Observe that for each  $j \in \mathbb{N}$ , the designated  $j$ -th power  $A^j$  of  $A$  in  $\mathcal{T}\text{-Alg}_{\mathcal{U}}$  is therefore the composite

$$\mathcal{T} \xrightarrow{A} \mathcal{U} \xrightarrow{U^{j*}(-)} \mathcal{U}$$

whose second factor is the endofunctor of  $\mathcal{U}$  induced by the left  $j$ -th powers in  $\mathcal{U}$  (4.2). In particular,  $A^j$  has carrier  $U^j$  and is given on objects by  $A^j(T^k) = U^{j \times k}$ .

The faithful functor  $|-| : \mathcal{T}\text{-Alg}_{\mathcal{U}} \rightarrow \mathcal{U}$  strictly preserves the designated finite powers and so induces a subtheory embedding  $(\mathcal{T}\text{-Alg}_{\mathcal{U}})_A \hookrightarrow \mathcal{U}_{|A|} = \mathcal{U}_U = \mathcal{U}$ , and we shall now show that this is precisely the subtheory inclusion  $\mathcal{T}^\perp \hookrightarrow \mathcal{U}$ . Fix a pair of natural numbers  $j, j'$ . Per 2.5, we have identified  $\mathcal{T}$ -homomorphisms  $A^j \rightarrow A^{j'}$  with certain morphisms  $\mu : |A^j| \rightarrow |A^{j'}|$  in  $\mathcal{U}$  between the carriers of the  $\mathcal{T}$ -algebras  $A^j$  and  $A^{j'}$ , namely those  $\mu$  that preserve each operation  $\nu : T^k \rightarrow T^{k'}$  in  $\mathcal{T}$ . But by using preceding description of the designated powers  $A^j$  and  $A^{j'}$ , we find that  $\mu$  preserves  $\nu$  iff the following diagram commutes

$$\begin{array}{ccc} U^{j \times k} & \xrightarrow{\mu * U^k} & U^{j' \times k} \\ U^j * A(\nu) \downarrow & & \downarrow U^{j'} * A(\nu) \\ U^{j \times k'} & \xrightarrow{\mu * U^{k'}} & U^{j' \times k'} \end{array}$$

i.e. iff  $\mu : U^j \rightarrow U^{j'}$  commutes with  $A(\nu) : U^k \rightarrow U^{k'}$ . □

**Definition 5.10.** Given a  $\mathcal{T}$ -algebra  $A : \mathcal{T} \rightarrow \mathcal{C}$  for a Lawvere theory  $\mathcal{T}$ , the **commutant**  $\mathcal{T}_A^\perp$  of  $A$  (or of  $\mathcal{T}$  with respect to  $A$ ) is defined as the commutant of the associated morphism of theories  $A' : \mathcal{T} \rightarrow \mathcal{C}_A$ , where  $\mathcal{C}_A$  is the full finitary theory of  $A$  in  $\mathcal{C}$  (2.11). By 5.9 we obtain the following equivalent definition:

**Corollary 5.11.** *The commutant  $\mathcal{T}_A^\perp$  of a  $\mathcal{T}$ -algebra  $A : \mathcal{T} \rightarrow \mathcal{C}$  is the full finitary theory of  $A$  in the category of  $\mathcal{T}$ -algebras in  $\mathcal{C}$ , i.e.  $\mathcal{T}_A^\perp \cong (\mathcal{T}\text{-Alg}_{\mathcal{C}})_A$  as theories over  $\mathcal{C}_A$ . For suitable choices of finite powers  $A^n$  in  $\mathcal{T}\text{-Alg}_{\mathcal{C}}$ , this isomorphism is an identity, so that  $\mathcal{T}_A^\perp(n, m) = \mathcal{T}\text{-Alg}_{\mathcal{C}}(A^n, A^m)$  for all  $n, m \in \mathbb{N}$ .*

*Proof.* This is verified straightforwardly by applying 5.9 and using the fact that the canonical functor  $\iota : \mathcal{C}_A \rightarrow \mathcal{C}$  is fully faithful and preserves finite powers (2.11). With reference to the proof of 5.9, the pointwise left finite powers of  $A' : \mathcal{T} \rightarrow \mathcal{C}_A$  in  $\mathcal{T}\text{-Alg}_{\mathcal{C}_A}$  induce standard designated finite powers of  $A$  in  $\mathcal{T}\text{-Alg}_{\mathcal{C}}$  by composition with  $\iota$ . Employing these in forming  $(\mathcal{T}\text{-Alg}_{\mathcal{C}})_A$ , we find that  $\iota$  induces an isomorphism of theories  $(\mathcal{T}\text{-Alg}_{\mathcal{C}_A})_{A'} \cong (\mathcal{T}\text{-Alg}_{\mathcal{C}})_A$  over  $\mathcal{C}_A$ , but the convention of (2.5.i) entails that the associated morphisms to  $\mathcal{C}_A$  are concrete subtheory embeddings, so the latter isomorphism is an equality of concrete subtheories.  $\square$

**Remark 5.12.** In the case where  $\mathcal{C} = \text{Set}$ , the commutant  $\mathcal{T}_A^\perp$  of a  $\mathcal{T}$ -algebra  $A : \mathcal{T} \rightarrow \text{Set}$  is (by 5.11) an instance of Lawvere's notion of the **algebraic structure** of a Set-valued functor [3, III.1].

**Example 5.13 (The Lawvere theory of  $R$ -modules).** Letting  $R$  be a ring or rig, recall that left  $R$ -modules are the same as normal  $\mathcal{T}$ -algebras for the Lawvere theory  $\mathcal{T} = \text{Mat}_R$  (2.8). In particular,  $R$  is a left  $R$ -module and so determines a morphism  $R : \mathcal{T} \rightarrow \text{Set}_R$  into the full finitary theory  $\text{Set}_R$  of  $R$  in  $\text{Set}$ . Thus regarding  $\mathcal{T}$  as a theory over  $\text{Set}_R$ , its commutant  $\mathcal{T}^\perp$  has  $\mathcal{T}^\perp(n, m) = R\text{-Mod}(R^n, R^m)$  by 5.11 once we identify  $R\text{-Mod}$  with the isomorphic category  $\mathcal{T}\text{-Alg}^\dagger$ . Moreover, we have an identity-on-homs functor  $\mathcal{T}^\perp \rightarrow R\text{-Mod}$  given on objects by  $n \mapsto R^n$ .

Let us first observe that  $\mathcal{T}^\perp \cong \mathcal{T}^{\text{op}}$  as categories. Indeed, as noted in 2.6 we have a fully faithful functor  $y : \mathcal{T}^{\text{op}} \hookrightarrow R\text{-Mod}$  sending  $n \in \text{ob } \mathcal{T} = \mathbb{N}$  to the free  $R$ -module  $R^n$ . The constituent isomorphisms  $R^{m \times n} = \mathcal{T}(n, m) \cong R\text{-Mod}(R^m, R^n)$  send each  $m \times n$ -matrix  $u$  to the left  $R$ -linear map  $R^m \rightarrow R^n$  given by *right* multiplication by  $u$ , i.e., we regard  $R^m$  and  $R^n$  as sets of *row vectors* so that the associated map  $R^{1 \times m} \rightarrow R^{1 \times n}$  is given by  $x \mapsto xu$ .

Next observe that the identity-on-objects functor  $(\text{Mat}_R)^{\text{op}} \rightarrow \text{Mat}_{R^{\text{op}}}$  given by transposition is an isomorphism of categories

$$(\text{Mat}_R)^{\text{op}} \cong \text{Mat}_{R^{\text{op}}}$$

so that

$$\mathcal{T}^\perp \cong \mathcal{T}^{\text{op}} \cong \text{Mat}_{R^{\text{op}}}$$

as categories. The composite isomorphism  $\text{Mat}_{R^{\text{op}}} \rightarrow \mathcal{T}^\perp$  commutes with the associated morphisms to  $\text{Set}_R = \text{Set}_{R^{\text{op}}}$ , as it associates to an  $m \times n$ -matrix  $u$  over  $R^{\text{op}}$  the (left  $R$ -linear) map  $R^n \rightarrow R^m$  given by  $x \mapsto ux$  when we regard each  $x \in R^n$  as a column vector with entries in  $R^{\text{op}}$ . We thus obtain the following, recalling that left  $R^{\text{op}}$ -modules are the same as right  $R$ -modules.

**Theorem 5.14.** *Let  $R$  be a ring or rig. Then the commutant  $(\text{Mat}_R)^\perp$  with respect to  $R$  of the theory of left  $R$ -modules  $\text{Mat}_R$  is the Lawvere theory of right  $R$ -modules  $\text{Mat}_{R^{\text{op}}}$ . Indeed, we have an isomorphism*

$$(\text{Mat}_R)^\perp \cong \text{Mat}_{R^{\text{op}}}$$

in the category of Lawvere theories over  $\text{Set}_R$ .

## 6 Saturated and balanced subtheories

**Proposition 6.1.** *Let  $\mathcal{U}$  be a Lawvere theory.*

1. *Lawvere theories  $\mathcal{T}$  and  $\mathcal{S}$  over  $\mathcal{U}$  commute if and only if there exists a (necessarily unique) morphism  $\mathcal{S} \rightarrow \mathcal{T}^\perp$  in the category of Lawvere theories over  $\mathcal{U}$ .*
2. *There is a unique functor  $(-)^\perp : (\text{Th}/\mathcal{U})^{\text{op}} \rightarrow \text{Th}/\mathcal{U}$  sending each theory  $\mathcal{T}$  over  $\mathcal{U}$  to its commutant  $\mathcal{T}^\perp$ .*
3. *The functor  $(-)^\perp$  in 2 is right adjoint to its formal dual  $(-)^\perp : \text{Th}/\mathcal{U} \rightarrow (\text{Th}/\mathcal{U})^{\text{op}}$ .*
4. *The adjunction in 3 restricts to a Galois connection on the preordered set  $\text{SubTh}(\mathcal{U})$  of subtheories of  $\mathcal{U}$  (2.3), i.e., an adjunction between  $\text{SubTh}(\mathcal{U})$  and its opposite.*

*Proof.* 1 follows from 5.7. Every theory  $\mathcal{T}$  over  $\mathcal{U}$  commutes with its commutant  $\mathcal{T}^\perp$ , so by 1 there is a unique morphism  $\mathcal{T} \rightarrow \mathcal{T}^{\perp\perp}$  in  $\text{Th}/\mathcal{U}$ . Given a morphism  $M : \mathcal{S} \rightarrow \mathcal{T}$  in  $\text{Th}/\mathcal{U}$ , we obtain a composite morphism  $\mathcal{S} \xrightarrow{M} \mathcal{T} \rightarrow \mathcal{T}^{\perp\perp}$ , so by 1  $\mathcal{T}^\perp$  and  $\mathcal{S}$  commute and hence there is a unique morphism  $\mathcal{T}^\perp \rightarrow \mathcal{S}^\perp$  in  $\text{Th}/\mathcal{U}$  and 2 follows. 3 and 4 now follow readily.  $\square$

**Definition 6.2.**

1. Given a Lawvere theory  $\mathcal{T}$  over  $\mathcal{U}$ , we say that
  - (a)  $\mathcal{T}$  is **saturated** if  $\mathcal{T}^{\perp\perp} \cong \mathcal{T}$  in  $\text{Th}/\mathcal{U}$ .
  - (b)  $\mathcal{T}$  is **balanced** if  $\mathcal{T}^\perp \cong \mathcal{T}$  in  $\text{Th}/\mathcal{U}$ .
2. Given a pair of Lawvere theories  $\mathcal{S}$  and  $\mathcal{T}$  over  $\mathcal{U}$ , we say that  $\mathcal{T}$  and  $\mathcal{S}$  are **mutual commutants** in  $\mathcal{U}$  if  $\mathcal{T} \cong \mathcal{S}^\perp$  and  $\mathcal{T}^\perp \cong \mathcal{S}$ .

We readily deduce the following:

**Proposition 6.3.**

1. *Any saturated theory  $\mathcal{T}$  over  $\mathcal{U}$  is necessarily a subtheory of  $\mathcal{U}$ .*
2. *A theory  $\mathcal{T}$  over  $\mathcal{U}$  is saturated if and only if it is (isomorphic to) a commutant  $\mathcal{S}^\perp$  of some theory  $\mathcal{S}$  over  $\mathcal{U}$ .*
3. *Theories  $\mathcal{T}$  and  $\mathcal{S}$  over  $\mathcal{U}$  are mutual commutants if and only if  $\mathcal{T}$  is saturated and  $\mathcal{S}$  is its commutant.*
4. *A subtheory  $\mathcal{T}$  of  $\mathcal{U}$  is commutative (as a Lawvere theory) if and only if  $\mathcal{T}$  is contained in its commutant.*

**Corollary 6.4.** *Every balanced theory over  $\mathcal{U}$  is a commutative, saturated subtheory of  $\mathcal{U}$ .*

**Theorem 6.5.** *Let  $R$  be a ring or rig.*

1. *The Lawvere theories  $\text{Mat}_R$  and  $\text{Mat}_{R^{\text{op}}}$  of left and right  $R$ -modules (respectively) are mutual commutants in the full finitary theory  $\text{Set}_R$  of  $R$  in  $\text{Set}$ .*
2. *The Lawvere theory of left  $R$ -modules  $\text{Mat}_R$  is a saturated subtheory of  $\text{Set}_R$ .*
3. *The subtheory  $\text{Mat}_R \hookrightarrow \text{Set}_R$  is balanced if and only if  $R$  is commutative.*

*Proof.* 1 is obtained by two applications of 5.14, and 2 then follows immediately. If  $R$  is commutative then  $\text{Mat}_R = \text{Mat}_{R^{\text{op}}}$  as theories over  $\text{Set}_R$ , so  $\text{Mat}_R$  is balanced, by 1. Conversely, if  $\text{Mat}_R$  is balanced over  $\text{Set}_R$ , then  $\text{Mat}_R$  is commutative by 6.4, so  $R$  is commutative by (4.6.i).  $\square$

**Example 6.6.** By 6.5, the Lawvere theory of semilattices  $\text{Mat}_2$  (2.10) is a balanced subtheory of the Lawvere theory of Boolean algebras  $\text{Set}_2$  (2.12).

**Example 6.7 (A non-saturated subtheory).** Let  $k$  be an infinite integral domain, and consider the Lawvere theory of commutative  $k$ -algebras  $\mathcal{T}$  (2.9). The domain  $k$  itself is a commutative  $k$ -algebra and so determines a morphism of Lawvere theories  $\kappa : \mathcal{T} \rightarrow \text{Set}_k$  into the full finitary theory  $\text{Set}_k$  of  $k$  in  $\text{Set}$  (2.11). For each natural number  $n$ , the associated component  $\kappa_{n,1} : \mathcal{T}(n, 1) \rightarrow \text{Set}_k(n, 1)$  is the mapping  $k[x_1, \dots, x_n] \rightarrow \text{Set}(k^n, k)$  that sends a polynomial  $f$  to the polynomial function  $k^n \rightarrow k$  determined by  $f$ . Since  $k$  is an infinite integral domain, this mapping  $\kappa_{n,1}$  is injective (e.g. by [1, III.4, Thm. 7]), so  $\kappa$  presents  $\mathcal{T}$  as a subtheory of  $\text{Set}_k$ . The commutant of this subtheory  $\mathcal{T}$  is the subtheory  $\mathcal{T}^\perp \hookrightarrow \text{Set}_k$  in which  $\mathcal{T}^\perp(n, 1) = k\text{-CAlg}(k^n, k)$  is the set of all  $k$ -algebra homomorphisms  $\varphi : k^n \rightarrow k$ . But any such homomorphism  $\varphi$  is  $k$ -linear and so is a linear combination  $\varphi = \sum_{i=1}^n c_i \pi_i$  of the projections  $\pi_i : k^n \rightarrow k$  ( $i = 1, \dots, n$ ), where  $c_i = \varphi(b_i)$  is the image of the  $i$ -th standard basis vector  $b_i$  for  $k^n$ . We also know that  $1 = \varphi(1) = \sum_i c_i$ . Hence since  $i \neq j$  implies  $b_i b_j = 0$  in  $k^n$  and  $\varphi$  preserves multiplication, it follows that  $\varphi = \pi_i$  for a unique  $i$ . Therefore  $\mathcal{T}^\perp(n, 1)$  is just the set of all  $n$  projections  $k^n \rightarrow k$ , and (since  $k$  has at least two elements) it follows that  $\mathcal{T}^\perp$  is isomorphic to the initial Lawvere theory  $\text{FinCard}^{\text{op}}$ . Therefore  $\mathcal{T}^\perp \hookrightarrow \text{Set}_k$  is central (by 5.4) and hence  $\mathcal{T}^{\perp\perp} = \text{Set}_k$ , but  $\text{Set}_k \not\cong \mathcal{T}$  by a cardinality argument:  $\#\text{Set}_k(1, 1) = (\#k)^{\#k} \geq 2^{\#k} > \#k = \#k[x] = \#\mathcal{T}(1, 1)$ .

## 7 The theories of affine and convex spaces as commutants

Let  $R$  be ring or, more generally, a rig.

**7.1 (Pointed  $R$ -modules).** By definition, a **pointed (left)  $R$ -module** is a (left)  $R$ -module  $M$  equipped with an arbitrary chosen element  $* \in M$ . Pointed  $R$ -modules are objects of a category  $R\text{-Mod}^*$  in which the morphisms are  $R$ -module homomorphisms that preserve the chosen points  $*$ . By 2.7,  $R\text{-Mod}^*$  is strictly finitary-algebraic over  $\text{Set}$ . Given a natural number  $n$ , the free pointed  $R$ -module on  $n$ -generators is the free  $R$ -module on  $1 + n$  generators  $R^{1+n}$ . Indeed, writing the successive standard basis vectors for  $R^{1+n}$  as  $\gamma_0, \gamma_1, \dots, \gamma_n$ , we find that  $R^{1+n}$  is a free pointed  $R$ -module on the  $n$  generators  $\gamma_1, \dots, \gamma_n$  when we take  $* = \gamma_0 = (1, 0, 0, \dots, 0)$ . By 2.6,  $R\text{-Mod}^*$  is therefore

isomorphic to the category of normal  $\mathcal{T}$ -algebras for a Lawvere theory  $\mathcal{T} = \text{Mat}_R^*$  with  $\text{Mat}_R^*(n, m) = (R^{1+n})^m = R^{m \times (1+n)}$ .

The notion of **pointed right  $R$ -module** is defined similarly, so that pointed right  $R$ -modules are the same as pointed left  $R^{\text{op}}$ -modules, equivalently, normal  $\text{Mat}_{R^{\text{op}}}^*$ -algebras. Given a pointed right  $R$ -module  $M$ , we shall now record a detailed description of the corresponding normal  $\text{Mat}_{R^{\text{op}}}^*$ -algebra  $\underline{M} : \text{Mat}_{R^{\text{op}}}^* \rightarrow \text{Set}$  for use in the sequel. By 2.6,  $\underline{M}$  has the same carrier as  $M$  and associates to each  $w \in \text{Mat}_{R^{\text{op}}}^*(n, 1) = R^{1+n}$  the mapping  $\Phi_w^M : M^n \rightarrow M$  defined as follows. Recalling that  $R^{1+n}$  is a free pointed right  $R$ -module on the  $n$  generators  $\gamma_1, \dots, \gamma_n \in R^{1+n}$ , each  $n$ -tuple  $x = (x_1, \dots, x_n) \in M^n$  induces a unique morphism of pointed right  $R$ -modules  $x^\sharp : R^{1+n} \rightarrow M$  with  $x^\sharp(\gamma_i) = x_i$  ( $i = 1, \dots, n$ ), and the associated mapping  $\Phi_w^M : M^n \rightarrow M$  is given by

$$\Phi_w^M(x) = x^\sharp(w) = * \cdot w_0 + \sum_{i=1}^n x_i w_i$$

where  $* \in M$  is the designated point and  $w = (w_0, w_1, \dots, w_n) \in R^{1+n}$ .

In particular, we can consider  $R$  itself as a pointed right  $R$ -module with chosen point  $1 \in R$ . Therefore  $R$  is the carrier of a normal  $\text{Mat}_{R^{\text{op}}}^*$ -algebra, and we can thus consider  $\text{Mat}_{R^{\text{op}}}^*$  as a Lawvere theory over the full finitary theory  $\text{Set}_R$  of  $R$  in  $\text{Set}$ . Explicitly, we have a morphism

$$\Phi^R : \text{Mat}_{R^{\text{op}}}^* \rightarrow \text{Set}_R \quad (7.1.i)$$

sending each  $w = (w_0, \dots, w_n) \in R^{1+n}$  to the mapping  $\Phi_w^R : R^n \rightarrow R$  given by

$$\Phi_w^R(x) = w_0 + \sum_{i=1}^n x_i w_i. \quad (7.1.ii)$$

**Theorem 7.2.** *The Lawvere theory of left  $R$ -affine spaces  $\text{Mat}_R^{\text{aff}}$  is the commutant with respect to  $R$  of the theory of pointed right  $R$ -modules  $\text{Mat}_{R^{\text{op}}}^*$ . Indeed,*

$$\text{Mat}_R^{\text{aff}} \cong (\text{Mat}_{R^{\text{op}}}^*)^\perp$$

as Lawvere theories over  $\text{Set}_R$  when  $\text{Mat}_R^{\text{aff}}$  is equipped with the morphism  $\text{Mat}_R^{\text{aff}} \rightarrow \text{Set}_R$  determined by the left  $R$ -affine space  $R$ .

*Proof.* By 6.5 we know that  $\text{Mat}_R \cong (\text{Mat}_{R^{\text{op}}}^*)^\perp$  over  $\text{Set}_R$ , so the theory  $\text{Mat}_R^{\text{aff}}$  is isomorphic to the affine core  $\mathcal{T}^{\text{aff}}$  of  $\mathcal{T} = (\text{Mat}_{R^{\text{op}}}^*)^\perp$ . This yields an isomorphism  $\text{Mat}_R^{\text{aff}} \cong \mathcal{T}^{\text{aff}}$  as theories over  $\text{Set}_R$  since the inclusion  $\text{Mat}_R^{\text{aff}} \hookrightarrow \text{Mat}_R$  is a morphism over  $\text{Set}_R$ . Recall that  $\mathcal{T}$  is the concrete subtheory of  $\text{Set}_R$  consisting of all right  $R$ -linear maps  $\varphi : R^n \rightarrow R^m$ . Therefore the affine core  $\mathcal{T}^{\text{aff}}$  of  $\mathcal{T}$  is the subtheory of  $\text{Set}_R$  consisting of all right  $R$ -linear maps  $\varphi : R^n \rightarrow R^m$  that commute with the ‘diagonal’ maps  $(1, \dots, 1) : R^1 \rightarrow R^j$  with  $j = n, m$ . But these are precisely the homomorphisms of pointed right  $R$ -modules  $\varphi : R^n \rightarrow R^m$ , where we regard the powers  $R^j$  of  $R$  as pointed right  $R$ -modules with chosen point  $1 = (1, \dots, 1) \in R^j$ . With this convention  $R^j$  is the  $j$ -th power of  $R = R^1$  in the category of pointed right  $R$ -modules, so the subtheory  $\mathcal{T}^{\text{aff}}$  of  $\text{Set}_R$  is precisely the commutant with respect to  $R$  of the theory of pointed right  $R$ -modules  $\text{Mat}_{R^{\text{op}}}^*$ . Hence  $\text{Mat}_R^{\text{aff}} \cong \mathcal{T}^{\text{aff}} = (\text{Mat}_{R^{\text{op}}}^*)^\perp$  over  $\text{Set}_R$ .  $\square$



**Corollary 7.3.**

1. The Lawvere theory of left  $R$ -affine spaces  $\text{Mat}_R^{\text{aff}}$  is a saturated subtheory of the full finitary theory  $\text{Set}_R$  of  $R$  in  $\text{Set}$ .
2. When  $R$  is commutative,  $\text{Mat}_R^{\text{aff}}$  is a commutative, saturated subtheory of  $\text{Set}_R$ .
3. If  $R \neq 0$ , then the subtheory  $\text{Mat}_R^{\text{aff}} \hookrightarrow \text{Set}_R$  is not balanced.

*Proof.* 1 and 2 follow immediately from 7.2, 6.3, and 4.7. For 3, suppose that  $R \neq 0$ , and fix some nonzero element  $r$  of  $R$ . Considering  $\text{Mat}_R^{\text{aff}}$  and  $\text{Mat}_{R^{\text{op}}}^*$  as Lawvere theories over  $\text{Set}_R$ , we know by 7.2 that  $\text{Mat}_R^{\text{aff}} \cong (\text{Mat}_{R^{\text{op}}}^*)^\perp$ , so in order to show that  $\text{Mat}_R^{\text{aff}}$  is not balanced it suffices to show that  $(\text{Mat}_R^{\text{aff}})^\perp \not\subseteq (\text{Mat}_{R^{\text{op}}}^*)^\perp$  as subtheories of  $\text{Set}_R$ . But the constant map  $R^n \rightarrow R$  with value  $r$  is left  $R$ -affine and is not right  $R$ -linear, so this constant map is an element of  $(\text{Mat}_R^{\text{aff}})^\perp(n, 1) = R\text{-Aff}(R^n, R)$  but is not an element of  $(\text{Mat}_{R^{\text{op}}}^*)^\perp(n, 1) = R^{\text{op}}\text{-Mod}^*(R^n, R)$ .  $\square$

**Example 7.4.** By 7.3, the theory  $\text{Mat}_2^{\text{aff}}$  of unbounded join semilattices (equivalently, 2-affine spaces, 3.3) is a non-balanced, commutative, saturated subtheory of the theory  $\text{Set}_2$  of Boolean algebras (2.12).

## 8 The commutant of the theory of unbounded semilattices

Given a ring or rig  $R$ , we showed in the previous section that the theory of left  $R$ -affine spaces  $\text{Mat}_R^{\text{aff}}$  is the commutant of the theory  $\text{Mat}_{R^{\text{op}}}^*$  of pointed right  $R$ -modules, considered as a theory over  $\text{Set}_R$ . In the remainder of the paper, we shall examine the commutant of  $\text{Mat}_R^{\text{aff}}$  over  $\text{Set}_R$ , and in particular we ask whether this commutant is  $\text{Mat}_{R^{\text{op}}}^*$ , i.e. whether the theories  $\text{Mat}_R^{\text{aff}}$  and  $\text{Mat}_{R^{\text{op}}}^*$  are mutual commutants in the full finitary theory of  $R$  in  $\text{Set}$ . We shall later show that this is indeed the case for all rings (9.2) and also for many rigs that are not rings (10.20). But first we will examine a notable example of a rig for which this is *not* the case.

Writing simply  $2$  to denote the rig  $(2, \vee, 0, \wedge, 1)$ , recall that 2-affine spaces are the same as unbounded join semilattices (3.3), equivalently, idempotent commutative semigroups, whereas 2-modules are (bounded) join semilattices (2.10). By 7.4, we know that the theory of unbounded join semilattices is a saturated subtheory of the theory  $\text{Set}_2$  of Boolean algebras, and in the present section we characterize its commutant.

**8.1 (Join semilattices with top element).** Let  $\text{SLat}_{\vee\top}$  denote the category whose objects are join semilattices with a top element and whose morphisms are homomorphisms of join semilattices that preserve the top element. Join semilattices with a top element can be described equivalently as idempotent commutative monoids  $S$  with an additional constant  $\top$  satisfying a single additional equation  $s \cdot \top = \top$  ( $s \in S$ ), so  $\text{SLat}_{\vee\top}$  is a variety of finitary algebras and hence (by 2.7) is isomorphic to the category of normal  $\mathcal{T}$ -algebras for a Lawvere theory  $\mathcal{T} = \mathcal{T}_{\perp\vee\top}$ .

In order to obtain a description of  $\mathcal{T}_{\perp\vee\top}$ , observe that for each finite cardinal  $n$ , the free join-semilattice-with-top-element  $F(n)$  on  $n$  generators can be obtained by artificially adjoining a new top element  $\top$  to the free join semilattice  $2^n = \mathcal{P}(n)$  on  $n$  generators (2.10), where as generators we take the standard basis vectors  $b_1, \dots, b_n \in 2^n$ ,

i.e., the singleton subsets of  $n$ . More precisely, we let  $F(n) := \mathcal{P}(n) + \{\top\}$  and observe that  $F(n)$  then carries a unique join semilattice structure such that  $\top$  is a top element of  $F(n)$  and such that the inclusion  $\mathcal{P}(n) \hookrightarrow F(n)$  is a homomorphism of join semilattices. Now let  $S$  be a join semilattice with a top element, and let  $x = (x_1, \dots, x_n) \in S^n$ . The universal property of  $\mathcal{P}(n)$  as a join semilattice now clearly entails that there is a unique morphism  $x^\sharp : F(n) \rightarrow S$  in  $\mathbf{SLat}_{\vee\top}$  with  $x^\sharp(b_i) = x_i$  for all  $i$ .

Hence by 2.6 the theory  $\mathcal{T}_{\perp\vee\top}$  has

$$\mathcal{T}_{\perp\vee\top}(n, 1) = F(n) = 2^n + \{\top\}.$$

The join semilattice  $2 = (2, \vee, 0)$  has top element 1 and so (by 2.11) determines a morphism of theories

$$\underline{2} : \mathcal{T}_{\perp\vee\top} \rightarrow \mathbf{Set}_2 \quad (8.1.i)$$

by means of which  $\mathcal{T}_{\perp\vee\top}$  can be considered as a theory over the full finitary theory  $\mathbf{Set}_2$  of 2 in  $\mathbf{Set}$ . Recalling that  $\mathbf{Set}_2$  is the theory of Boolean algebras (2.12), we shall prove the following:

**Theorem 8.2.** *The following Lawvere theories are mutual commutants in the theory of Boolean algebras:*

1. *The theory  $\mathbf{Mat}_2^{\text{aff}}$  of unbounded join semilattices (equivalently, 2-affine spaces);*
2. *the theory  $\mathcal{T}_{\perp\vee\top}$  of join semilattices with top element.*

*Proof.* By 7.2 we know that  $\mathbf{Mat}_2^{\text{aff}} \cong (\mathbf{Mat}_2^*)^\perp$  as theories over  $\mathbf{Set}_2$ , where  $\mathbf{Mat}_2^*$  is the theory of pointed 2-modules, equivalently, pointed join semilattices. Explicitly,  $(\mathbf{Mat}_2^*)^\perp$  is the concrete subtheory of  $\mathbf{Set}_2$  consisting of all homomorphisms of join semilattices  $\varphi : 2^n \rightarrow 2^m$  that preserve the designated points  $1 = (1, \dots, 1) \in 2^j$  with  $j = n, m$ . But the join semilattice 2 has top element  $1 \in 2$ , and the finite powers  $2^j$  of 2 in  $\mathbf{SLat}_{\vee\top}$  have top element  $1 = (1, \dots, 1) \in 2^j$ , so  $(\mathbf{Mat}_2^*)^\perp$  consists of all morphisms  $\varphi : 2^n \rightarrow 2^m$  in  $\mathbf{SLat}_{\vee\top}$ . Hence

$$\mathbf{Mat}_2^{\text{aff}} \cong (\mathbf{Mat}_2^*)^\perp = (\mathcal{T}_{\perp\vee\top})^\perp$$

as theories over  $\mathbf{Set}_2$ .

In particular,  $\mathcal{T}_{\perp\vee\top}$  and  $\mathbf{Mat}_2^{\text{aff}}$  commute over  $\mathbf{Set}_2$ , so the morphism (8.1.i) factors through the commutant  $(\mathbf{Mat}_2^{\text{aff}})^\perp \hookrightarrow \mathbf{Set}_2$ . We therefore have a morphism

$$\underline{2} : \mathcal{T}_{\perp\vee\top} \rightarrow (\mathbf{Mat}_2^{\text{aff}})^\perp$$

that sends each  $\omega \in \mathcal{T}_{\perp\vee\top}(n, 1) = F(n) = 2^n + \{\top\}$  to an element  $\underline{2}_\omega \in (\mathbf{Mat}_2^{\text{aff}})^\perp(n, 1) = \mathbf{USLat}_\vee(2^n, 2)$ , i.e. a homomorphism of unbounded join semilattices  $\underline{2}_\omega : 2^n \rightarrow 2$ . By 2.6, this map  $\underline{2}_\omega$  sends each  $x = (x_1, \dots, x_n) \in 2^n$  to  $\underline{2}_\omega(x) = x^\sharp(\omega)$ , recalling from 8.1 that  $x^\sharp : F(n) \rightarrow 2$  is the unique morphism in  $\mathbf{SLat}_{\vee\top}$  such that  $x^\sharp(b_i) = x_i$  for all  $i$ . Explicitly,

$$\underline{2}_\omega(x) = x^\sharp(\omega) = \begin{cases} \bigvee_{i=1}^n (\omega_i \wedge x_i) & \text{if } \omega = (\omega_1, \dots, \omega_n) \in 2^n \\ 1 & \text{if } \omega = \top \end{cases}.$$

Hence it suffices to show that each homomorphism of unbounded join semilattices  $\varphi : 2^n \rightarrow 2$  is equal to  $\underline{2}_\omega$  for a unique  $\omega \in 2^n + \{\top\}$ . If  $\varphi(0) = 0$  then  $\varphi$  is a homomorphism of join semilattices (or 2-modules) so since the theory of 2-modules  $\text{Mat}_2$  is a balanced subtheory of  $\text{Set}_2$  (6.6) it follows that  $\varphi$  is one of the 2-module operations (1.0.i) carried by 2. More precisely, there is a unique row vector  $\omega \in 2^n = \text{Mat}_2(n, 1)$  such that  $\varphi : 2^n \rightarrow 2$  is given by  $x \mapsto \bigvee_{i=1}^n (\omega_i \wedge x_i)$ , and  $\omega$  is then the unique element of  $2^n + \{\top\}$  with  $\underline{2}_\omega = \varphi$ . On the other hand, if  $\varphi(0) \neq 0$  then  $\varphi(0) = 1$ , but  $\varphi$  preserves binary joins and hence is monotone, so  $\varphi : 2^n \rightarrow 2$  must be the constant map with value 1 and hence we find that  $\omega = \top$  is the unique element  $\omega \in 2^n + \{\top\}$  with  $\underline{2}_\omega = \varphi$ .  $\square$

## 9 The commutant of the theory of affine spaces over a ring

The remainder of the paper is devoted to showing that the theories of left  $R$ -affine spaces and pointed right  $R$ -modules are mutual commutants in  $\text{Set}_R$  for many rigs  $R$ . In the present section, we show that this holds for all rings.

**9.1.** Given an arbitrary rig  $R$ , we will consider  $\text{Mat}_{R^{\text{op}}}^*$  and  $\text{Mat}_R^{\text{aff}}$  as theories over  $\text{Set}_R$ , via the morphisms to  $\text{Set}_R$  determined by  $R$  (7.1, 7.2). By 7.2 we know that these theories over  $\text{Set}_R$  commute, so the canonical morphism  $\Phi^R : \text{Mat}_{R^{\text{op}}}^* \rightarrow \text{Set}_R$  factors through the inclusion  $(\text{Mat}_R^{\text{aff}})^\perp \hookrightarrow \text{Set}_R$  via a unique morphism

$$\Phi^R : \text{Mat}_{R^{\text{op}}}^* \rightarrow (\text{Mat}_R^{\text{aff}})^\perp \quad (9.1.i)$$

for which we use the same notation  $\Phi^R$ . By 7.1, the components  $\text{Mat}_{R^{\text{op}}}^*(n, 1) \rightarrow (\text{Mat}_R^{\text{aff}})^\perp(n, 1)$  of this morphism are the maps

$$\Phi_{(-)}^R : R^{1+n} \rightarrow R\text{-Aff}(R^n, R) \quad (9.1.ii)$$

that send each element  $w = (w_0, \dots, w_n)$  of  $R^{1+n}$  to the  $R$ -affine map  $\Phi_w^R : R^n \rightarrow R$  given by

$$\Phi_w^R(x) = w_0 + \sum_{i=1}^n x_i w_i .$$

**Theorem 9.2.** *Given a ring  $R$ , the commutant  $(\text{Mat}_R^{\text{aff}})^\perp$  with respect to  $R$  of the theory  $\text{Mat}_R^{\text{aff}}$  of left  $R$ -affine spaces is the theory  $\text{Mat}_{R^{\text{op}}}^*$  of pointed right  $R$ -modules. Indeed, the morphism (9.1.i) is an isomorphism*

$$\text{Mat}_{R^{\text{op}}}^* \cong (\text{Mat}_R^{\text{aff}})^\perp$$

in the category of Lawvere theories over the full finitary theory  $\text{Set}_R$  of  $R$  in  $\text{Set}$ .

*Proof.* Let  $n \in \mathbb{N}$ . For each left  $R$ -affine map  $\psi : R^n \rightarrow R$  let us denote by

$$w^\psi = (w_0^\psi, w_1^\psi, \dots, w_n^\psi)$$

the element of  $R^{1+n}$  defined by

$$w_0^\psi = \psi(0), \quad w_i^\psi = \psi(b_i) - \psi(0) \quad (i = 1, \dots, n)$$

where  $b_i = (b_{i1}, \dots, b_{in}) \in R^n$  is the  $i$ -th standard basis vector (with  $b_{ii} = 1$  and  $b_{ij} = 0$  for  $j \neq i$ ). This defines a map

$$w^{(-)} : R\text{-Aff}(R^n, R) \rightarrow R^{1+n}$$

which we claim is inverse to the mapping  $\Phi_{(-)}^R$  of (9.1.ii). Indeed, it is easy to see that  $w^{(-)}$  is a retraction of  $\Phi_{(-)}^R$ , so it suffices to show that for any left  $R$ -affine map  $\psi : R^n \rightarrow R$ , if we let  $w = w^\psi$  then  $\Phi_w^R : R^n \rightarrow R$  is exactly  $\psi$ . To this end, observe that any element  $x = (x_1, \dots, x_n)$  of  $R^n$  can be expressed as a left  $R$ -affine combination

$$x = \left(1 - \sum_{i=1}^n x_i\right) 0 + \sum_{i=1}^n x_i b_i$$

of the elements  $0, b_1, \dots, b_n$  of  $R^n$ , so  $\psi$  necessarily sends  $x$  to

$$\begin{aligned} \psi(x) &= \left(1 - \sum_{i=1}^n x_i\right) \psi(0) + \sum_{i=1}^n x_i \psi(b_i) \\ &= \psi(0) + \sum_{i=1}^n x_i (\psi(b_i) - \psi(0)) = \Phi_w^R(x). \end{aligned}$$

□

By 7.2, we obtain the following:

**Corollary 9.3.** *For a ring  $R$ , the Lawvere theory of left  $R$ -affine spaces and the Lawvere theory of pointed right  $R$ -modules are mutual commutants in the full finitary theory of  $R$  in  $\text{Set}$ .*

## 10 The commutant of the theory of convex spaces for a preordered ring

Having shown that theories of left  $R$ -affine spaces and pointed right  $R$ -modules for a ring  $R$  are mutual commutants in  $\text{Set}_R$  (9.3), we now set ourselves to the task of widening the applicability of this result to include certain rigs that are not rings. In particular, we will focus on *additively cancellative* rigs  $S$  (3.4), which are precisely the positive parts  $S = R_+$  of preordered rings  $R$ . Our study of the commutant of the theory of  $R_+$ -affine spaces (which we also call  $R$ -convex spaces) begins with the following observations:

**Lemma 10.1.** *Let  $R$  be a preordered ring, and let  $\varphi : R_+^n \rightarrow R_+$  be a (left)  $R_+$ -affine map with  $n \in \mathbb{N}$ . Then  $\varphi$  is a restriction of at most one  $R$ -affine map  $\psi : R^n \rightarrow R$ . Further, the following are equivalent:*

1.  $\varphi$  is a restriction of some  $R$ -affine map  $\psi : R^n \rightarrow R$ .
2.  $\varphi$  is a restriction of the  $R$ -affine map  $\Phi_{w^\varphi}^R : R^n \rightarrow R$  (9.1) determined by

$$w^\varphi = (\varphi(0), \varphi(b_1) - \varphi(0), \dots, \varphi(b_n) - \varphi(0)) \in R^{1+n},$$

where  $b_i \in R_+^n$  is the  $i$ -th standard basis vector.

3. For every element  $x = (x_1, \dots, x_n) \in R_+^n$ ,

$$\varphi(x) = \varphi(0) + \sum_{i=1}^n x_i(\varphi(b_i) - \varphi(0)) \quad (10.1.i)$$

in  $R$ .

*Proof.* It is clear from the proof of 9.2 that an  $R$ -affine map  $\psi : R^n \rightarrow R$  is uniquely determined by its values on the elements  $0, b_1, \dots, b_n$  of  $R^n$  and so is uniquely determined by its restriction to  $R_+^n$ . The equivalence of 2 and 3 is immediate from the definitions, and clearly 2 implies 1. If 1 holds, then  $\varphi$ ,  $\psi$ , and  $\Phi_{w\varphi}^R$  all agree on the elements  $0, b_1, \dots, b_n$  of  $R^n$ , so  $\psi$  and  $\Phi_{w\varphi}^R$  are equal and hence 2 holds.  $\square$

**Definition 10.2.** We say that a preordered ring  $R$  has the **affine extension property** if every  $R_+$ -affine map  $\varphi : R_+^n \rightarrow R_+$  ( $n \in \mathbb{N}$ ) satisfies the equivalent conditions of 10.1.

We shall find that the affine extension property is a necessary condition for the Lawvere theories of left  $R$ -convex spaces and pointed right  $R_+$ -modules to be mutual commutants in  $\text{Set}_{R_+}$ . In order to obtain necessary and sufficient conditions, we will also need to impose a certain weakening of the *archimedean property*. The familiar archimedean property for totally ordered fields has been generalized to the context of partially ordered rings and abelian groups in a number of slightly different ways in the literature. We shall now recall one common definition, which appears for example in [2], and then proceed to define the weaker property that we shall require.

**Definition 10.3.** Let  $R$  be a preordered ring.

1. We say that  $R$  is **archimedean** provided that for each element  $r \in R$ , if  $\{nr \mid n \in \mathbb{N}\}$  has an upper bound in  $R$  then  $r \leq 0$ .
2. Given an element  $r \in R$ , we call the subset  $\{sr \mid s \in R_+\} \subseteq R$  the **(left) ray** of  $r$ .
3. We say that  $R$  is **(left) auto-archimedean** provided that for every element  $r$  of  $R$ , if the ray of  $r$  has an upper bound in  $R$ , then  $r \leq 0$ .

**Remark 10.4.** Observe that an archimedean preordered ring  $R$  is necessarily auto-archimedean.

**Remark 10.5.** By considering the additive inverse  $-r$  of each  $r \in R$ , we find that a preordered ring  $R$  is auto-archimedean precisely when for every  $r \in R$ , if the ray of  $r$  has a lower bound in  $R$ , then  $r \geq 0$ .

**Example 10.6 (The real numbers).** The totally ordered ring of real numbers  $\mathbb{R}$  is archimedean and hence auto-archimedean. It is well-known that  $\mathbb{R}$  also has the affine extension property, and in 10.17 we will prove a more general result that entails this fact.

**Example 10.7 (The integers).** The ring of integers  $\mathbb{Z}$  under the natural order is archimedean, but  $\mathbb{Z}$  does not have the affine extension property. Indeed,  $\mathbb{Z}_+ = \mathbb{N}$  and every mapping  $\mathbb{N}^n \rightarrow \mathbb{N}$  is  $\mathbb{N}$ -affine ( $n \in \mathbb{N}$ ), so for example the mapping  $\varphi = 2^{(-)} : \mathbb{N} \rightarrow \mathbb{N}$  is  $\mathbb{N}$ -affine but does not extend to a  $\mathbb{Z}$ -affine map  $\mathbb{Z} \rightarrow \mathbb{Z}$ .

**Lemma 10.8.** *For a preordered ring  $R$ , the following are equivalent:*

1.  $R$  is auto-archimedean.
2. For all  $r, c \in R$ , if the  $R$ -affine map  $R \rightarrow R$  given by  $x \mapsto c + xr$  maps  $R_+$  into  $R_+$ , then  $r, c \in R_+$ .
3. For every  $n \in \mathbb{N}$  and every  $w \in R^{1+n}$ , if the associated  $R$ -affine map  $\Phi_w^R : R^n \rightarrow R$  maps  $R_+^n$  into  $R_+$ , then  $w \in R_+^{1+n}$ .

*Proof.* By 10.5, 1 holds iff

$$\forall r, b \in R : (\forall s \in R_+ : b \leq sr) \Rightarrow r \in R_+,$$

and (by taking  $c = -b$ ) we find that this holds iff

$$\forall r, c \in R : (\forall s \in R_+ : c + sr \in R_+) \Rightarrow r \in R_+. \quad (10.8.i)$$

Hence 1 holds iff for all  $r, c \in R$ , if the map  $R \rightarrow R$  given by  $x \mapsto c + xr$  maps  $R_+$  into  $R_+$ , then  $r \in R_+$ . But if  $c + xr \in R_+$  for all  $x \in R_+$  then we necessarily have  $c \in R_+$ , so 1 is equivalent to 2. Also, 3 clearly implies 2. Assume 2 holds, and suppose that  $\Phi_w^R : R^n \rightarrow R$  maps  $R_+^n$  into  $R_+$ , where  $w = (w_0, w_1, \dots, w_n)$  is an element of  $R^{1+n}$ . For each  $i = 1, \dots, n$  we have a mapping  $(-) \cdot b_i : R \rightarrow R^n$  given by  $r \mapsto rb_i$  where  $b_i \in R^n$  is the  $i$ -th standard basis vector, and the composite

$$\varphi_i = \left( R \xrightarrow{(-) \cdot b_i} R^n \xrightarrow{\Phi_w^R} R \right)$$

is given by  $\varphi_i(x) = \Phi_w^R(xb_i) = w_0 + xw_i$ . But since  $\Phi_w^R$  maps  $R_+^n$  into  $R_+$  it follows that  $\varphi_i$  maps  $R_+$  into  $R_+$ , so by 2 we deduce that  $w_0, w_i \in R_+$ . Therefore if  $n \geq 1$  then  $w \in R_+^{1+n}$ , but if  $n = 0$  then it is readily seen that  $w = w_0 \in R_+^1$ . Hence 3 holds.  $\square$

**Lemma 10.9.** *For each  $n \in \mathbb{N}$ , the composite*

$$R_+^{1+n} \xrightarrow{\Phi_{(-)}^{R_+}} R_+\text{-Aff}(R_+^n, R_+) \xrightarrow{w^{(-)}} R^{1+n}$$

*is the inclusion  $R_+^{1+n} \hookrightarrow R^{1+n}$ , where  $\Phi_{(-)}^{R_+}$  is the mapping (9.1.ii) associated to the rig  $R_+$  and  $w^{(-)}$  is the mapping sending an  $R_+$ -affine map  $\varphi$  to the vector  $w^\varphi \in R^{1+n}$  of 10.1. Equivalently, if  $\varphi = \Phi_w^{R_+}$  then  $w = w^\varphi$ . In particular,  $\Phi_{(-)}^{R_+}$  is injective.*

*Proof.* Any  $R_+$ -affine map of the form  $\varphi = \Phi_w^{R_+}$  is a restriction of the  $R$ -affine map  $\Phi_w^R : R^n \rightarrow R$  and so by 10.1 we deduce that  $\Phi_w^R = \Phi_{w^\varphi}^R$ , whence  $w = w^\varphi$  by 9.2.  $\square$

**Theorem 10.10.** *The following are equivalent for a preordered ring  $R$ :*

1.  $R$  is auto-archimedean and has the affine extension property.
2. The commutant with respect to  $R_+$  of the theory  $\text{Mat}_{R_+}^{\text{aff}}$  of left  $R$ -convex spaces (i.e., left  $R_+$ -affine spaces) is isomorphic to the theory of pointed right  $R_+$ -modules  $\text{Mat}_{R_+^{\text{op}}}^*$ , where both theories are considered as Lawvere theories over  $\text{Set}_{R_+}$ .

*Proof.* By 9.1, we know that there is a unique morphism  $\text{Mat}_{R_+^{\text{op}}}^* \rightarrow (\text{Mat}_{R_+}^{\text{aff}})^\perp$  in  $\text{Th/Set}_{R_+}$ , namely  $\Phi^{R_+}$ , so 2 holds iff the mapping

$$\Phi_{(-)}^{R_+} : R_+^{1+n} \rightarrow R_+\text{-Aff}(R_+^n, R_+)$$

is a bijection for each  $n \in \mathbb{N}$ . But by 10.9 this mapping is necessarily injective and has as its image the set of all  $R_+$ -affine maps  $\varphi : R_+^n \rightarrow R_+$  such that the element  $w^\varphi$  of  $R_+^{1+n}$  lies in  $R_+^{1+n}$  and  $\varphi = \Phi_{w^\varphi}^{R_+}$ . Hence 2 holds iff

2'. For each  $R_+$ -affine map  $\varphi : R_+^n \rightarrow R_+$ ,  $w^\varphi \in R_+^{1+n}$  and  $\varphi = \Phi_{w^\varphi}^{R_+}$ .

This is clearly equivalent to the following:

2''. For each  $R_+$ -affine map  $\varphi : R_+^n \rightarrow R_+$ ,  $w^\varphi \in R_+^{1+n}$  and  $\varphi$  is a restriction of  $\Phi_{w^\varphi}^R : R^n \rightarrow R$ .

If 1 holds then this holds, since if  $\varphi : R_+^n \rightarrow R_+$  is  $R_+$ -affine then the affine extension property entails that  $\varphi$  is a restriction of  $\Phi_{w^\varphi}^R$  (10.1), and since  $R$  is auto-archimedean 10.8 entails that  $w^\varphi \in R_+^{1+n}$ . Conversely, suppose that 2'' holds. Then  $R$  clearly has the affine extension property, and we show by way of condition 3 in 10.8 that  $R$  is auto-archimedean. Suppose that  $\Phi_w^R : R^n \rightarrow R$  maps  $R_+^n$  into  $R_+$ . Then its restriction  $\varphi : R_+^n \rightarrow R_+$  is  $R_+$ -affine, and by 2'' we know that  $\varphi$  is also a restriction of  $\Phi_{w^\varphi}^R$ . Hence by 10.1 we deduce that  $\Phi_w^R = \Phi_{w^\varphi}^R$ , so by 9.2 we find that  $w = w^\varphi$ , and by 2'' we know that  $w^\varphi \in R_+^{1+n}$ .  $\square$

**10.11.** Letting  $d$  be a positive integer, we shall denote by  $\mathbb{Z}[\frac{1}{d}]$  the subring of  $\mathbb{Q}$  consisting of all rational numbers that can be expressed in the form  $\frac{p}{d^n}$  with  $p \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . Equivalently,  $\mathbb{Z}[\frac{1}{d}]$  is the localization of  $\mathbb{Z}$  at the element  $d \in \mathbb{Z}$ , i.e., the localization of  $\mathbb{Z}$  with respect to the multiplicative subset  $\{d^n \mid n \in \mathbb{N}\} \subseteq \mathbb{Z}$ . We shall call  $\mathbb{Z}[\frac{1}{d}]$  the ring of  **$d$ -adic fractions**. The ring  $\mathbb{Z}[\frac{1}{2}]$  is usually called the ring of **dyadic rationals**<sup>9</sup>. Under the natural order that  $\mathbb{Z}[\frac{1}{d}]$  inherits from  $\mathbb{Q}$ ,  $\mathbb{Z}[\frac{1}{d}]$  is a preordered ring.

**10.12.** For any preordered ring  $R$ , there is a unique morphism of preordered rings  $e : \mathbb{Z} \rightarrow R$ , and for each element  $n \in \mathbb{Z}$  we denote the associated element  $e(n)$  of  $R$  by  $n$ , in accordance with the usual abuse of notation.

**Proposition 10.13.** *For a preordered ring  $R$  and a positive integer  $d$ , the following are equivalent:*

1. *The element  $d$  of  $R$  is invertible and its inverse lies in  $R_+$ .*
2. *There is a unique morphism of preordered rings  $e^\sharp : \mathbb{Z}[\frac{1}{d}] \rightarrow R$ .*

*Proof.* The implication  $2 \Rightarrow 1$  is immediate since  $d$  is invertible in  $\mathbb{Z}[\frac{1}{d}]$  and its inverse lies in  $\mathbb{Z}[\frac{1}{d}]_+$ . Conversely if 1 holds then by using the universal property of the localization  $\mathbb{Z}[\frac{1}{d}]$  of  $\mathbb{Z}$  and the fact that  $\mathbb{Z}$  is an initial object of the category of rings, we find that there is a unique ring homomorphism  $e^\sharp : \mathbb{Z}[\frac{1}{d}] \rightarrow R$ . Further,  $e^\sharp$  is monotone since if  $\frac{p}{d^n} \in \mathbb{Z}[\frac{1}{d}]_+$  with  $p \in \mathbb{Z}$  then  $p \in \mathbb{Z}_+$  and hence  $e^\sharp(\frac{p}{d^n}) = p \cdot (d^{-1})^n \in R_+$  since  $p, d^{-1} \in R_+$ .  $\square$

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<sup>9</sup>However, the term *p-adic rational* for a prime  $p$  is often employed in other senses, in connection with the *p-adic numbers*. For this reason, we have chosen to use a different name for the localization  $\mathbb{Z}[\frac{1}{d}]$ .

**Definition 10.14.** Let  $d$  be a positive integer. A preordered ring  $R$  is said to be a **preordered algebra over the  $d$ -adic fractions**, or a **preordered  $\mathbb{Z}[\frac{1}{d}]$ -algebra**, if  $R$  satisfies the equivalent conditions of 10.13. Note that since  $e : \mathbb{Z} \rightarrow R$  maps  $\mathbb{Z}$  into the centre of  $R$ , it follows that  $e^\sharp : \mathbb{Z}[\frac{1}{d}] \rightarrow R$  maps  $\mathbb{Z}[\frac{1}{d}]$  into the centre of  $R$ . For each  $d$ -adic fraction  $q \in \mathbb{Z}[\frac{1}{d}]$  we write the element  $e^\sharp(q)$  of  $R$  simply as  $q$ .

**Example 10.15.** If we define a preordered ring  $R$  by taking the underlying ring of  $R$  to be  $\mathbb{Q}$  but taking  $R_+$  to be the subrig  $\mathbb{N} \subseteq R$ , then  $R$  is not a preordered  $\mathbb{Z}[\frac{1}{2}]$ -algebra in the above sense.

**Definition 10.16.** An element  $u$  of a preordered commutative monoid  $(M, +, 0)$  is an **order unit** for  $M$  if for every  $x \in M$  there exists some  $n \in \mathbb{N}$  such that  $x \leq nu$ . An element  $u$  of a preordered ring  $R$  is said to be an **order unit for the positive part** of  $R$  if  $u$  is an order unit for the additive monoid of  $R_+$ .

**Theorem 10.17.** *Let  $R$  be a preordered algebra over the  $d$ -adic fractions, for some integer  $d > 1$ , and suppose that 1 is an order unit for the positive part of  $R$ . Then  $R$  has the affine extension property.*

*Proof.* We need to show that if  $\varphi : R_+^n \rightarrow R_+$  is a left  $R_+$ -affine map then

$$\varphi(x) = \varphi(0) + \sum_{i=1}^n x_i(\varphi(b_i) - \varphi(0)) \quad (10.17.i)$$

in  $R$  for all  $x = (x_1, \dots, x_n) \in R_+^n$ , where  $b_i \in R_+^n$  is the  $i$ -th standard basis vector. Let us first treat the case where  $n = 1$ , so that  $\varphi : R_+ \rightarrow R_+$ . Letting  $\delta = \varphi(1) - \varphi(0) \in R$ , we must show that  $\varphi(x) = \varphi(0) + x\delta$  for all  $x \in R_+$ . To this end, we shall first prove that for each  $m \in \mathbb{N}$ ,

$$\varphi(m+2) - \varphi(m+1) = \varphi(m+1) - \varphi(m) \quad (10.17.ii)$$

(with the notational convention of 10.12). Indeed, the equation

$$m+1 = \frac{1}{d}m + \frac{d-2}{d}(m+1) + \frac{1}{d}(m+2)$$

holds in  $R$  and expresses  $m+1$  as a left  $R_+$ -affine combination of the elements  $m, m+1, m+2$  of  $R_+$ , so

$$\varphi(m+1) = \frac{1}{d}\varphi(m) + \frac{d-2}{d}\varphi(m+1) + \frac{1}{d}\varphi(m+2).$$

Multiplying both sides of this equation by  $d$ , we readily compute that (10.17.ii) holds. By induction on  $m \in \mathbb{N}$  the common difference (10.17.ii) is  $\delta = \varphi(1) - \varphi(0)$ , so by another induction on  $m \in \mathbb{N}$  we find that  $\varphi(m) = \varphi(0) + m\delta$  in  $R$ . Now for an arbitrary element  $x \in R_+$ , since 1 is an order unit for  $R_+$  there is some  $m \in \mathbb{N}$  such that  $x \leq m$  in  $R$ , and we can take  $m$  to be a power of  $d$  so that  $m$  has an inverse  $\frac{1}{m} \in R_+$  since  $R$  is a preordered  $\mathbb{Z}[\frac{1}{d}]$ -algebra. But then  $\frac{1}{m}x \leq 1$  in  $R$  and hence both  $1 - \frac{1}{m}x$  and  $\frac{1}{m}x$  lie in  $R_+$ , so we can express  $x$  as a left  $R_+$ -affine combination

$$x = (1 - \frac{1}{m}x) \cdot 0 + \frac{1}{m}x \cdot m$$



of the elements  $0, m \in R_+$  (using the fact that  $\frac{1}{m}, m$  lie in the centre of  $R$ ). Hence since  $\varphi$  is left  $R_+$ -affine we compute that

$$\begin{aligned}\varphi(x) &= \left(1 - \frac{1}{m}x\right)\varphi(0) + \frac{1}{m}x\varphi(m) &= \left(1 - \frac{1}{m}x\right)\varphi(0) + \frac{1}{m}x(\varphi(0) + m\delta) \\ &= \varphi(0) - \frac{1}{m}x\varphi(0) + \frac{1}{m}x\varphi(0) + \frac{1}{m}xm\delta &= \varphi(0) + x\delta\end{aligned}$$

since  $m, \frac{1}{m}$  lie in the centre of  $R$ .

Having thus established (10.17.i) in the case  $n = 1$ , we now treat the general case. Given a left  $R_+$ -affine map  $\varphi : R_+^n \rightarrow R_+$  and an element  $x \in R_+^n$ , note that the composite map

$$\varphi_x = \left(R_+ \xrightarrow{(-) \cdot x} R_+^n \xrightarrow{\varphi} R_+\right)$$

is left  $R_+$ -affine and is given by  $\varphi_x(r) = \varphi(rx)$ , so by what we have established above we find that

$$\begin{aligned}\varphi(rx) &= \varphi_x(r) &= \varphi_x(0) + r(\varphi_x(1) - \varphi_x(0)) \\ &= \varphi(0x) + r(\varphi(1x) - \varphi(0x)) &= \varphi(0) + r(\varphi(x) - \varphi(0)) .\end{aligned} \quad (10.17.iii)$$

for all  $r \in R_+$ . Next let  $\gamma = \sum_{i=1}^n x_i$ . Since  $\gamma$  lies in  $R_+$  and 1 is an order unit for  $R_+$  there is some  $m \in \mathbb{N}$  such that  $\gamma \leq m$  in  $R$ , and we can take  $m$  to be a power of  $d$  so that  $m$  has an inverse  $\frac{1}{m} \in R_+$ . Now  $\frac{1}{m}\gamma \leq 1$ , so  $1 - \frac{1}{m}\gamma \geq 0$ . The elements  $1 - \frac{1}{m}\gamma, \frac{1}{m}x_1, \dots, \frac{1}{m}x_n$  of  $R_+$  sum to 1, and so we can now express  $\frac{1}{m}x$  as a left  $R_+$ -affine combination

$$\frac{1}{m}x = \left(1 - \frac{1}{m}\gamma\right) \cdot 0 + \sum_{i=1}^n \frac{1}{m}x_i b_i$$

of the elements  $0, b_1, \dots, b_n$  of  $R_+^n$ . Hence since  $\varphi$  is left  $R_+$ -affine we find that

$$\varphi\left(\frac{1}{m}x\right) = \left(1 - \frac{1}{m}\gamma\right)\varphi(0) + \sum_{i=1}^n \frac{1}{m}x_i\varphi(b_i) .$$

But by (10.17.iii) we know that  $\varphi(\frac{1}{m}x) = \varphi(0) + \frac{1}{m}(\varphi(x) - \varphi(0))$ , so

$$\varphi(0) + \frac{1}{m}(\varphi(x) - \varphi(0)) = \left(1 - \frac{1}{m}\gamma\right)\varphi(0) + \sum_{i=1}^n \frac{1}{m}x_i\varphi(b_i) .$$

Multiplying both sides by  $m$ ,

$$m\varphi(0) + (\varphi(x) - \varphi(0)) = (m - \gamma)\varphi(0) + \sum_{i=1}^n x_i\varphi(b_i) ,$$

so

$$\begin{aligned}\varphi(x) &= (1 - \gamma)\varphi(0) + \sum_{i=1}^n x_i\varphi(b_i) \\ &= \left(1 - \sum_{i=1}^n x_i\right)\varphi(0) + \sum_{i=1}^n x_i\varphi(b_i) = \varphi(0) + \sum_{i=1}^n x_i(\varphi(b_i) - \varphi(0))\end{aligned}$$

as needed. □

We now focus on the following class of preordered rings, given by a slight weakening of the notion of *strongly archimedean* partially ordered ring from [2].

**Definition 10.18.** We say that a preordered ring  $R$  is **firmly archimedean** if  $R$  is archimedean and 1 is an order unit for the positive part of  $R$ .

**Remark 10.19.** Observe that a nonzero totally ordered ring  $R$  is firmly archimedean if and only if  $R$  is archimedean. Indeed, in a nonzero archimedean totally ordered ring, 1 is necessarily an order unit for  $(R, +, 0)$ .

**Theorem 10.20.** *Let  $R$  be a firmly archimedean preordered algebra over the  $d$ -adic fractions, for some integer  $d > 1$ . Then the Lawvere theory of left  $R$ -convex spaces (i.e., left  $R_+$ -affine spaces) and the Lawvere theory of pointed right  $R_+$ -modules are mutual commutants in the full finitary theory of  $R_+$  in  $\mathbf{Set}$ .*

*Proof.* This now follows from 10.17, 10.10, and 7.2. □

**Example 10.21.** Let us fix an integer  $d > 1$ .

1. Any subring  $R$  of  $\mathbb{R}$  containing the  $d$ -adic fractions is a firmly archimedean preordered  $\mathbb{Z}[\frac{1}{d}]$ -algebra. In particular, 10.20 applies to both  $R = \mathbb{R}$  and  $R = \mathbb{Z}[\frac{1}{d}]$ .
2. Given a set  $X$ , the ring  $R$  of all bounded real-valued functions on  $X$  is a firmly archimedean preordered  $\mathbb{Z}[\frac{1}{d}]$ -algebra under the pointwise order.
3. Let  $R$  be any subring of the ring of all bounded real-valued functions on a given set  $X$ , and suppose that  $R$  contains all constant functions with values in  $\mathbb{Z}[\frac{1}{d}]$ . Then  $R$  is a firmly archimedean preordered  $\mathbb{Z}[\frac{1}{d}]$ -algebra under the pointwise order.
4. Given a compact topological space, the ring  $R = C(X)$  of all continuous real-valued functions on  $X$  is a firmly archimedean preordered  $\mathbb{Z}[\frac{1}{d}]$ -algebra.

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