# MAXIMAL IDEALS IN MODULE CATEGORIES AND APPLICATIONS

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Abstract. We study the existence of maximal ideals in preadditive categories defining an order $\preceq$ between objects, in such a way that if there do not exist maximal objects with respect to $\preceq$, then there is no maximal ideal in the category. In our study, it is sometimes sufficient to restrict our attention to suitable subcategories. We give an example of a category $\mathbf{C}_F$ of modules over a right noetherian ring $R$ in which there is a unique maximal ideal. The category $\mathbf{C}_F$ is related to an indecomposable injective module $F$, and the objects of $\mathbf{C}_F$ are the $R$-modules of finite $F$-rank.

## INTRODUCTION

This paper is related to the study of ideals in preadditive categories. Recall that an *ideal* in a preadditive category $\mathbf{C}$ is an additive subfunctor $\mathcal{I}$ of the additive bifunctor $\mathrm{Hom}_{\mathbf{C}}\colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups.

Let us mention two motivations for our study. The first is related to extensions of the classical Krull-Schmidt theorem to additive categories. In [8], the second author proved that the class of all uniserial right modules over a ring $R$ does not satisfy the Krull-Schmidt theorem, thus answering a question posed by Warfield in 1975, but that nevertheless a weak version of the Krull-Schmidt theorem for uniserial modules holds [8, Theorem 1.9]. This weak version of the Krull-Schmidt theorem was extended as follows, in [6, Theorem 6.4], to any additive category $\mathbf{A}$ with a pair of ideals $\mathcal{I}$ and $\mathcal{J}$ satisfying suitable conditions: if $U_1, \dots, U_n$ and $V_1, \dots, V_n$ are objects in $\mathbf{A}$ with local endomorphism rings in the quotient categories $\mathbf{A}/\mathcal{I}$ and $\mathbf{A}/\mathcal{J}$, then $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots V_m$ if and only if $n = m$ and there exist two permutations $\sigma$ and $\tau$ of $\{1, \dots, n\}$ such that $U_i$ and $V_{\sigma(i)}$ are isomorphic in $\mathbf{A}/\mathcal{I}$, and $U_i$ and $V_{\tau(i)}$ are isomorphic in $\mathbf{A}/\mathcal{J}$, for every $i = 1, \dots, n$.

Our second motivation is related to the problem of approximating objects by morphisms belonging to some ideal. This idea first appeared in [12], where the author introduced phantom maps in module categories, considered the ideal consisting of all such maps and proved that each module $M$ has a phantom cover (that is, a phantom map $\varphi\colon P \to M$ such that every phantom map $\psi\colon Q \to M$ factors through $\varphi$, and minimal with respect to this property). This particular situation was extended in [11], where it was characterized when an ideal $\mathcal{I}$ in an exact category provides approximations in this sense. Notice that this theory contains, as a particular case, the classical one about precovers and covers by objects, see [4].

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As in the case of ideals of rings, one can consider minimal and maximal ideals in a preadditive category $\mathbf{C}$. In [5, Theorem 3.1], it is proved that the minimal ideals in a module category are in one-to-one correspondence with the simple modules. Hence we have a complete description of the minimal ideals of the category. A similar description of maximal ideals is not known (the best description of maximal ideals is Prihoda's result [9, Lemma 2.1]). One of the main results of our paper is now that there do not exist maximal ideals in module categories Mod-$R$ (actually, in Grothendieck categories). The idea of the proof is to define a order $\preceq$ in the class of objects and relate the existence of maximal ideals with the existence of non-zero maximal objects with respect to this order. More precisely, we prove (Theorem 3.1) that if for each object $A$ in the category there exists an object $B$ such that $A \prec B$, then there do not exist maximal ideals. Since a Grothendieck category has this property (Proposition 2.4), we conclude that there are no maximal ideals in Grothendieck categories.

If $\mathbf{C}$ is a preadditive category, we can consider the full subcategory $\mathbf{M}(\mathbf{C})$ of $\mathbf{C}$ consisting of all objects $C$ of $\mathbf{C}$ for which there do not exist objects $B$ in $\mathbf{C}$ with $C \prec B$. Then the maximal ideals of $\mathbf{M}(\mathbf{C})$ determine those of $\mathbf{C}$ (Proposition 3.9). Using these ideas, the last part of the paper is devoted to describing the maximal ideals in a full subcategory $\mathbf{C}_F$ constructed starting from an indecomposable injective module $F$ over a right noetherian ring.

All rings in this paper are associative with unit and not necessarily commutative. If $R$ is such a ring, module will mean right $R$-module and we will denote by Mod-$R$ the category whose objects are all right $R$-modules.

## 1. PRELIMINARIES

By a *preadditive category*, we mean a category together with an abelian group structure on each of its hom-sets such that composition is bilinear. An *additive category* is a preadditive category with finite products. Let $\mathbf{C}$ be a preadditive category and $A$ an object of $\mathbf{C}$. We will denote by $\operatorname{add}(A)$ the class of the objects $X$ of $\mathbf{C}$ for which there exist an integer $n > 0$ and morphisms $f_1, \dots, f_n \in \operatorname{Hom}_{\mathbf{C}}(A, X)$ and $g_1, \dots, g_n \in \operatorname{Hom}_{\mathbf{C}}(X, A)$ such that $1_X = \sum_{i=1}^n f_i g_i$. If $\mathbf{C}$ is additive and idempotents split in $\mathbf{C}$, then $X \in \operatorname{add}(A)$ if and only if $X$ is isomorphic to a direct summand of $A^n$ for some integer $n \geq 0$. If, moreover, $\mathbf{C}$ has arbitrary direct sums, we will denote by $\operatorname{Add}(A)$ the class of all objects that are isomorphic to direct summmands of arbitrary direct sums of copies of $A$.

An *ideal* in $\mathbf{C}$ is an additive subfunctor $\mathcal{I}$ of the additive bifunctor $\operatorname{Hom}_{\mathbf{C}} \colon \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. Thus $\mathcal{I}$ associates to every pair $A$ and $B$ of objects in $\mathbf{C}$ a subgroup $\mathcal{I}(A, B)$ of $\operatorname{Hom}_{\mathbf{C}}(A, B)$ so that if $f \colon X \to A$ and $g \colon B \to Y$ are morphisms in $\mathbf{C}$ and $i \in \mathcal{I}(A, B)$, then $gif \in \mathcal{I}(X, Y)$. An ideal in $\mathbf{C}$ is *maximal* if it is proper, that is, it is not equal to $\operatorname{Hom}_{\mathbf{C}}$, and is not properly contained in any other proper ideal. For instance, it is easy to see that the zero ideal is a maximal ideal in the full subcategory of Mod-$K$ whose objects are all finite-dimensional vector spaces over a field $K$.

Given an object $A$ in $\mathbf{C}$ and any two-sided ideal $I$ of $\operatorname{End}_{\mathbf{C}}(A)$, we will denote by $\mathcal{A}_I$ the ideal of the category $\mathbf{C}$ defined, for each pair of objects $X, Y \in \mathbf{C}$, by

$$\mathcal{A}_I(X, Y) = \{ f \in \operatorname{Hom}_{\mathbf{C}}(X, Y) : \beta f \alpha \in I \text{ for } \\ \text{all } \alpha \in \operatorname{Hom}_{\mathbf{C}}(A, X) \text{ and } \beta \in \operatorname{Hom}_{\mathbf{C}}(Y, A) \}.$$

This ideal is called the *ideal associated to* $I$ ([7, Section 2] and [10, Section 3]). The ideal $\mathcal{A}_I$ contains any ideal $\mathcal{I}$ in $\mathbf{C}$ satisfying $\mathcal{I}(A,A) \subseteq I$. As proved in [9, Lemma 2.4], there is a strong relation between ideals associated to maximal ideals of the endomorphism ring of an object, and maximal ideals in the preadditive category. For instance, the same argument as [9, Proposition 2.5] gives:

**Example 1.1.** Let $\mathbf{C}$ be an additive category in which idempotent splits and $C$ any object of $\mathbf{C}$. Then the maximal ideals in the category $\operatorname{add}(C)$ are the ideals associated to maximal ideals of $\operatorname{End}_{\mathbf{C}}(C)$.

The following easy lemma will be useful to compute ideals in the endomorphism ring of a finite direct sum of objects.

**Lemma 1.2.** *Let* $\mathbf{C}$ *be an additive category,* $A$ *an object of* $\mathbf{C}$ *and* $I$ *an ideal in* $\operatorname{End}_{\mathbf{C}}(A)$. *Given any finite family* $B_1, \dots, B_n$ *of objects of* $\mathbf{C}$, *denote by* $\iota_l$ *and* $\pi_l$ *the inclusion and the projection corresponding to the* $l$*-th component of* $B = \bigoplus_{i=1}^n B_i$ *for each* $l = 1, \dots, n$. *Then*

$$\mathcal{A}_I(B,B) = \{\, f \in \operatorname{End}_{\mathbf{C}}(B) : \pi_m f \iota_l \in \mathcal{A}_I(B_l, B_m) \text{ for every } l,m = 1,2,\dots,n \,\}.$$

Note that, as a consequence of this result, if $M_1$ and $M_2$ are objects in an additive category $\mathbf{C}$ and $I$ is an ideal in the endomorphism ring of an object $A$ of $\mathbf{C}$, then $\mathcal{A}_I(M_1 \oplus M_2, M_1 \oplus M_2) = \operatorname{End}_R(M_1 \oplus M_2)$ if and only if $\mathcal{A}_I(M_i, M_i) = \operatorname{End}_R(M_i)$ for $i = 1, 2$.

## 2. The strict order $\prec$ and its corresponding partial order $\preceq$.

The existence of maximal ideals in preadditive categories is related to an order $\preceq$ between objects. In this section, we define the partial order $\preceq$ and give a number of examples.

**Definition 2.1.** *Let* $\mathbf{C}$ *be a preadditive category and* $A$, $B$ *objects of* $\mathbf{C}$. *Set* $A \prec B$ *if there exists an infinite subset* $E \subseteq \operatorname{Hom}_{\mathbf{C}}(B,A) \times \operatorname{Hom}_{\mathbf{C}}(A,B)$ *with the following properties:*

*(1)* $fg = 1_A$ *for every* $(f,g) \in E$.
*(2) For each* $\varphi \in \operatorname{Hom}_{\mathbf{C}}(A,B)$, $|\{(f,g) \in E : f\varphi \neq 0\}| < |E|$.

*We shall write* $A \preceq B$ *if either* $A \prec B$ *or* $A = B$.

Here we are using the well known one-to-one correspondence between strict orders and partial orders. For any partial order $\leq$, the corresponding strict order $<$ is defined by $A < B$ if $A \leq B$ and $A \neq B$.

Let $\mathbf{C}$ be a preadditive category, $\mathbf{A}$ a subcategory of $\mathbf{C}$ and $A$ and $B$ objects of $\mathbf{A}$. Notice that it can occur that $A \prec B$ in $\mathbf{C}$ but not in $\mathbf{A}$. However, if $\mathbf{A}$ is full, $A \prec B$ in $\mathbf{C}$ if and only if $A \prec B$ in $\mathbf{A}$.

**Example 2.2.** Let $\mathbf{C}$ be any preadditive category and $A, B \in \mathbf{C}$ objects. If both $\operatorname{Hom}_{\mathbf{C}}(A,B)$ and $\operatorname{Hom}_{\mathbf{C}}(B,A)$ are finite, then $A \not\prec B$. In particular, if $\mathbf{C}$ has a zero object 0, then $0 \not\prec B$ and $B \not\prec 0$ for every object $B$.

Let us see some properties of the order $\preceq$.

**Lemma 2.3.** *Let* $\mathbf{C}$ *be a preadditive category and* $A$, $B$ *and* $C$ *objects of* $\mathbf{C}$.

*(1) If $A \prec B$ and $B$ is a retract of $C$, then $A \prec C$.*
*(2) If $B \in \operatorname{add}(A)$, then $A \not\prec B$.*

*Proof.* *(1)* Denote by $\iota_B\colon B \to C$ and $\pi_B\colon C \to B$ the morphisms satisfying $\pi_B \iota_B = 1_B$. Since $A \prec B$, there exists a set $E \subseteq \operatorname{Hom}_{\mathbf{C}}(B, A) \times \operatorname{Hom}_{\mathbf{C}}(A, B)$ satisfying the conditions of Definition 2.1. Then $E' = \{(f\pi_B, \iota_B g) : (f, g) \in E\}$ is a subset of $\operatorname{Hom}_{\mathbf{C}}(B \oplus C, A) \times \operatorname{Hom}_{\mathbf{C}}(A, B \oplus C)$ that has cardinality equal to $|E|$ and that trivially verifies the conditions of Definition 2.1. Thus $A \prec C$.

*(2)* Let $n > 0$ be an integer and

$$f_1, \dots, f_n \in \operatorname{Hom}_{\mathbf{C}}(A, B), \quad g_1, \dots, g_n \in \operatorname{Hom}_{\mathbf{C}}(B, A)$$

be such that $\sum_{i=1}^n f_i g_i = 1_B$. Suppose, in order to get a contradiction, that $A \prec B$. Let $E \subseteq \operatorname{Hom}_{\mathbf{C}}(B, A) \times \operatorname{Hom}_{\mathbf{C}}(A, B)$ be the set satisfying the conditions of Definition 2.1. By Definition 2.1*(2)*, the set

$$E_k := \{(f, g) \in E : ff_k \neq 0\}$$

has cardinality smaller than $|E|$ for each $k = 1, \dots, n$. But, for each morphism $\varphi : B \to A$, $\varphi \neq 0$ if and only if $\varphi f_k \neq 0$ for some $k = 1, \dots, n$. This implies that $E = \bigcup_{k=1}^n E_k$ as $f \neq 0$ for each $(f, g) \in E$. Since $E$ is infinite, we conclude that at least one of the sets $E_k$ has the same cardinality as $E$, which is a contradiction. □

Let $\mathbf{C}$ be a preadditive category. The main consequence of the preceeding result is that the relation $\prec$ is a strict order, since it is irreflexive by *(2)* and transitive by *(1)*. As we have already said, we denote by $\preceq$ the partial order associated to the strict order $\prec$.

Now we will consider a relation between large direct sums of copies of a non-zero object in a Grothendieck category and the strict order $\prec$ of Definition 2.1. Let $\mathbf{G}$ be a Grothendieck category, $A$ an object of $\mathbf{G}$ and $\kappa$ an infinite regular cardinal. Recall that $A$ is said to be $< \kappa$*-generated* [1, Definition 1.67] if $\operatorname{Hom}_{\mathbf{G}}(A, -)$ commutes with $\kappa$-directed colimits with all morphisms in the direct system being monomorphisms (a $\kappa$-directed colimit is the colimit of a $\kappa$-system in $\mathbf{G}$, $(A_i, f_{ij})_I$, the latter meaning that each subset of $I$ of cardinality smaller than $\kappa$ has an upper bound [1, Definition 1.13]).

**Proposition 2.4.** *Let $\mathbf{G}$ be a Grothendieck category and $\kappa$ an infinite regular cardinal.*

*(1) Let $A$ be a non-zero $< \kappa$-generated object of $\mathbf{G}$. Then $A \prec A^{(\kappa)}$.*
*(2) For each non-zero object $A$ of $\mathbf{G}$, there exists an object $B$ of $\mathbf{G}$ such that $A \prec B$.*

*Proof.* *(1)* Denote by $\iota_\alpha\colon A \to A^{(\kappa)}$ and $\pi_\alpha\colon A^{(\alpha)} \to A$ the injection and the projection corresponding to the $\alpha$-component of $A^{(\kappa)}$ for each $\alpha < \kappa$. Consider the subset $\{(\pi_\alpha, \iota_\alpha) : \alpha < \kappa\}$ of $\operatorname{Hom}_{\mathbf{G}}\left(A^{(\kappa)}, A\right) \times \operatorname{Hom}_{\mathbf{G}}\left(A, A^{(\kappa)}\right)$. Then $E$ satisfies *(1)* of Definition 2.1 since $\pi_\alpha \iota_\alpha = 1_A$ for each $\alpha < \kappa$.

In order to prove condition *(2)* of Definition 2.1, note that $A^{(\kappa)}$ is the colimit of the $\kappa$-direct system $(A^{(\alpha)}, \iota_{\alpha\beta})_\kappa$, where $\iota_{\alpha\beta}\colon A^{(\alpha)} \to A^{(\beta)}$ is the inclusion for each $\alpha < \beta$ in $\kappa$. The colimit maps are the inclusions $\iota_\alpha\colon A^{(\alpha)} \to A^{(\kappa)}$ for each $\alpha < \kappa$. Let $\varphi\colon A \to A^{(\kappa)}$ be any morphism. Since $A$ is $< \kappa$-generated and the morphism $\iota_{\alpha\beta}$ is monic for every $\alpha < \beta$ in $\kappa$, there exists $\alpha_0 < \kappa$ and $\overline{\varphi}\colon A \to A^{(\alpha_0)}$ such that

$\varphi = \iota_{\alpha_0}\overline{\varphi}$. In particular, we get that

$$|\{(\pi_\alpha, \iota_\alpha) \colon \pi_\alpha\varphi \neq 0\}| \leq |\alpha_0| < \kappa = |E|.$$

*(2)* Notice that, for each object $A$ in $\mathbf{G}$, there exists an infinite regular cardinal $\kappa$ such that $A$ is $< \kappa$-generated [14, Lemma A.1]. $\square$

Using these results, we can characterize when $V \prec W$ for vector spaces $V$ and $W$.

**Corollary 2.5.** *Let $V$ and $W$ be two vector spaces over a field $K$. Then $V \prec W$ if and only if $W$ is infinite dimensional and $0 \neq \dim(V) < \dim(W)$.*

*Proof.* Suppose $V \prec W$. First of all, note that $\dim(V) < \dim(W)$ since, otherwise, there would exist an epimorphism $\varphi \colon V \to W$. This would imply that, for any subset $E$ of $\operatorname{Hom}_K(W, V) \times \operatorname{Hom}_K(V, W)$, $\{(f, g) \in E : f\varphi \neq 0\} = E$. That is, $V \not\prec W$. Furthermore, $W$ has to be infinite dimensional, since finite dimensional vector spaces belong to $\operatorname{add}(V)$ and, by Lemma 2.3, for any vector space $W$ in $\operatorname{add}(V)$, we have that $V \not\prec W$.

Conversely, suppose that $W$ is infinite dimensional and that $0 \neq \dim(V) < \dim(W)$. Set $\dim W = \kappa$ and $\dim V = \lambda$ and take an infinite regular cardinal $\mu$ with $\lambda < \mu \leq \kappa$ (if $\kappa$ is regular, take $\mu = \kappa$; otherwise, set $\mu = \lambda^+$, the successor cardinal of $\lambda$). By Proposition 2.4 and Lemma 2.3, $V \prec V^{(\mu)} \prec V^{(\mu)} \oplus V^{(\kappa)}$. Since $\lambda < \mu \leq \kappa$, $\dim\left(V^{(\mu)} \oplus V^{(\kappa)}\right) = \kappa$ and $V^{(\mu)} \oplus V^{(\kappa)} \cong W$. Consequently, $V \prec W$. $\square$

Let $R$ be a ring and $A$ and $B$ right $R$-modules. If $A \prec B$ and $E$ is the set of Definition 2.1, then, for each $(f, g) \in E$, $\operatorname{Im} g$ is a direct summand of $B$ isomorphic to $A$. That is, $B$ contains many direct summands isomorphic to $A$. In view of the preceding result, a natural question arises: is $B$ isomorphic to a direct sum of copies of $A$? The following example shows that the answer to this question is negative in general.

**Example 2.6.** *Let $\kappa$ be an infinite regular cardinal and consider the abelian group $M = \mathbb{Z}^{(\kappa)} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. Then $\mathbb{Z} \prec M$ by Proposition 2.4 and Lemma 2.3, while $M$ is not isomorphic to a direct sum of copies of $\mathbb{Z}$ since it is not free.*

## 3. MAXIMAL IDEALS

In this section, using the order $\preceq$, we prove that there do not exist maximal ideals in Grothendieck categories. We will prove a more general result: if $\mathbf{C}$ is a preadditive category such that there is no non-zero maximal object with respect to $\preceq$, then $\mathbf{C}$ does not have maximal ideals. The proof is based on the following theorem:

**Theorem 3.1.** *Let $\mathbf{C}$ be a preadditive category, $A$ and $B$ objects of $\mathbf{C}$ such that $A \prec B$, and $I$ a proper ideal of $\operatorname{End}_{\mathbf{C}}(A)$. Then $\mathcal{A}_I(B, B)$ is a proper ideal of $\operatorname{End}_{\mathbf{C}}(B)$ which is not maximal.*

*Proof.* Since $A \prec B$, there exists $E \subseteq \operatorname{Hom}_{\mathbf{C}}(B, A) \times \operatorname{Hom}_{\mathbf{C}}(A, B)$ satisfying the conditions of Definition 2.1. Let $J$ be the ideal of $\operatorname{End}_{\mathbf{C}}(B)$ generated by the set of all the endomorphisms of $B$ that factors through $A$. We claim that $J + \mathcal{A}_I(B, B)$ is a proper ideal of $\operatorname{End}_{\mathbf{C}}(B)$ strictly containing $\mathcal{A}_I(B, B)$.

First of all, note that $\mathcal{A}_I(B,B)$ is not equal to $J+\mathcal{A}_I(B,B)$. In fact, for each $(f,g)\in E$, $gf$ is an element of $J$ not belonging to $\mathcal{A}_I(B,B)$, since $fgfg=1_A\notin I$ because $I$ is proper.

Now we will prove that $J+\mathcal{A}_I(B,B)$ is a proper ideal. Fix any element $\psi\in J+\mathcal{A}_I(B,B)$. Let $\varphi\in J$ and $\varphi'\in\mathcal{A}_I(B,B)$ be such that $\psi=\varphi+\varphi'$. Since $\varphi\in J$, $\varphi=\sum_{i=1}^n f_ig_i$ for morphisms $f_1,\dots,f_n\in\operatorname{Hom}_{\mathbf{C}}(A,B)$ and $g_1,\dots,g_n\in\operatorname{Hom}_{\mathbf{C}}(B,A)$. The set $\{(f,g)\in E: f\psi g\notin I\}$ is contained in the set $\{(f,g)\in E: f\varphi\neq 0\}$, which is contained in

$$\bigcup_{i=1}^n\{(f,g)\in E: ff_i\neq 0\}.$$

Since $A\prec B$ and $E$ is infinite, this set has cardinality smaller than $|E|$. The conclusion is that, for each $\psi\in J+\mathcal{A}_I(B,B)$, the set $\{(f,g)\in E: f\psi g\notin I\}$ has cardinality smaller than $|E|$. But this implies that $1_B$ does not belong to $J+\mathcal{A}_I(B,B)$, as $\{(f,g)\in E: f1_Bg\notin I\}=E$. Consequently, $J+\mathcal{A}_I(B,B)$ is a proper ideal. □

This theorem has a number of consequences.

**Corollary 3.2.** *Let* $\mathbf{C}$ *be a preadditive category such that there do not exist non-zero maximal objects with respect to* $\preceq$*. Then* $\mathbf{C}$ *does not have maximal ideals.*

*Proof.* Let $\mathcal{I}$ be any proper ideal in $\mathbf{C}$. Then there exists an object $A$ such that $\mathcal{I}(A,A)\neq\operatorname{End}_{\mathbf{C}}(A)$. Set $I=\mathcal{I}(A,A)$. Let $B$ be an object such that $A\prec B$. Then $\mathcal{I}(B,B)\subseteq\mathcal{A}_I(B,B)$ which, as a consequence of the previous result, is properly contained in a proper ideal of $\operatorname{End}_{\mathbf{C}}(B)$. This means that $\mathcal{I}(B,B)$ is not a maximal ideal of $\operatorname{End}_{\mathbf{C}}(B)$, and $\mathcal{I}$ is not a maximal ideal in $\mathbf{C}$ by [9, Lemma 2.4]. □

Combining this result with Proposition 2.4, we obtain that maximal ideals do not exist in any Grothendieck category (in particular, in any module category).

**Corollary 3.3.** *Let* $\mathbf{G}$ *be a Grothendieck category. Then there do not exist maximal ideals in* $\mathbf{G}$.

Another remarkable consequence of Theorem 3.1 is the following.

**Corollary 3.4.** *Let* $\mathbf{C}$ *be a preadditive category,* $\mathcal{M}$ *a maximal ideal of* $\mathbf{C}$ *and* $A$ *an object of* $\mathbf{C}$*. If* $\mathcal{M}(A,A)\neq\operatorname{End}_R(A)$*, then* $A$ *is maximal with respect to* $\preceq$.

*Proof.* Set $I=\mathcal{M}(A,A)$. Suppose that there exists an object $B$ such that $A\prec B$. Then $\mathcal{M}(B,B)=\mathcal{A}_I(B,B)$ by [9, Lemma 2.4]. But, by Theorem 3.1, $\mathcal{A}_I(B,B)$ is a proper ideal that is not maximal. This contradicts the maximality of $\mathcal{M}$. □

**Remark 3.5.** Notice that, equivalently, if $A$ is an object of a preadditive category $\mathbf{C}$ and there exists an object $B$ with $A\prec B$, then $\mathcal{M}(A,A)=\operatorname{End}_{\mathbf{C}}(A)$ for every maximal ideal $\mathcal{M}$ in $\mathbf{C}$.

As a consequence of Corollary 3.2, if a preadditive category $\mathbf{C}$ has maximal ideals, then there exist maximal objects with respect to the partial order $\preceq$. However, not all objects have to be maximal. That is, there can exist objects $A$ for which there are objects $B$ with $A\prec B$. For example, let $\kappa$ a cardinal and $\kappa^+$ be its successor cardinal. Let $K$ be a field and $\mathbf{C}$ the full subcategory of Mod-$K$ whose objects are all vector spaces of dimension smaller than $\kappa^+$. As is proved in [9, Example 4.1],

$\mathbf{C}$ has one maximal ideal. However, for each vector space $M$ of dimension smaller than $\kappa$, there exists spaces $V$ with $M \prec V$ by Proposition 2.5.

Despite this observation, we are going to see that, in order to determine if a preadditive category has maximal ideals, we can restrict our attention to a full subcategory in which each object is maximal with respect to $\preceq$.

**Definition 3.6.** *Let $\mathbf{C}$ be a preadditive category.*

*(1) We will denote by $\mathbf{M}(\mathbf{C})$ the full subcategory of $\mathbf{C}$ consisiting of all maximal objects with respec to $\preceq$, that is,*

$$\{\, C \in \mathbf{C} : \text{ there does not exist } A \in \mathbf{C} \text{ with } C \prec A \,\}$$

*(2) We will denote by $\mathbf{S}(\mathbf{C})$ the full subcategory of $\mathbf{C}$ whose class of objects is*

$$\{C \in \mathbf{C} : \text{there exists } A \in \mathbf{M}(\mathbf{C}) \text{ with } C \prec A\}$$

Let $\mathbf{C}$ be a preadditive category and $\mathbf{D}$ a full subcategory of $\mathbf{C}$. We now define how to restrict an ideal of $\mathbf{C}$ to $\mathbf{D}$ and, conversely, how to extend a maximal ideal of $\mathbf{D}$ to $\mathbf{C}$.

**Definition 3.7.** *Let $\mathbf{C}$ be a preadditive category and $\mathbf{D}$ a full subcategory of $\mathbf{C}$.*

*(1) Given $\mathcal{I}$ an ideal of $\mathbf{C}$, define its restriction $\mathcal{I}^{\mathrm{r}}$ to $\mathbf{D}$ by*

$$\mathcal{I}^{\mathrm{r}}(D, D') = \mathcal{I}(D, D')$$

*for every $D, D' \in \mathbf{D}$.*

*(2) Given any maximal ideal $\mathcal{M}$ of $\mathbf{D}$, there exists an object $D \in \mathbf{D}$ such that $\mathcal{M}(D, D) \neq \mathrm{End}_{\mathbf{D}}(D)$. Define the extension $\mathcal{M}^{\mathrm{e}}$ of $\mathcal{M}$ to $\mathbf{C}$ to be the ideal of $\mathbf{C}$ associated to $\mathcal{M}(D, D)$.*

**Lemma 3.8.** *Let $\mathbf{C}$ be a preadditive category, $\mathbf{D}$ a full subcategory of $\mathbf{C}$ and $\mathcal{M}$ a maximal ideal in $\mathbf{D}$.*

*(1) Let $D$ and $D'$ be objects of $\mathbf{D}$ such that $I := \mathcal{M}(D, D)$ and $I' := \mathcal{M}(D', D')$ are maximal ideals of $\mathrm{End}_{\mathbf{C}}(D)$ and $\mathrm{End}_{\mathbf{C}}(D')$ respectively. Then the ideals $\mathcal{A}_I$ and $\mathcal{A}_{I'}$ coincide in $\mathbf{C}$. In particular, the definition of $\mathcal{M}^{\mathrm{e}}$ does not depend on the choice of the object $D$ with $\mathcal{M}(D, D) \neq \mathrm{End}_{\mathbf{D}}(D)$.*

*(2) For any objects $D, D'$ of $\mathbf{D}$, $\mathcal{M}^{\mathrm{e}}(D, D') = \mathcal{M}(D, D')$.*

*Proof. (1)* By [9, Lemma 2.4], $\mathcal{M} = \mathcal{A}_I$ in $\mathbf{D}$. Then $\mathcal{A}_I(D', D') \subseteq I'$, which implies that $\mathcal{A}_I$ is contained in $\mathcal{A}_{I'}$ (in $\mathbf{C}$). Using the same argument, $\mathcal{M} = \mathcal{A}_{I'}$ in $\mathbf{D}$ and, consequently, $\mathcal{A}_{I'}(D, D) \subseteq I$. Thus $\mathcal{A}_{I'}$ is contained in $\mathcal{A}_I$ (in $\mathbf{C}$).

*(2)* By [9, Lemma 2.4]. □

Now we can establish, for any preadditive category $\mathbf{C}$, the relation between the maximal ideals of $\mathbf{C}$ and those of $\mathbf{M}(\mathbf{C})$.

**Theorem 3.9.** *Let $\mathbf{C}$ be a preadditive category. Then the assignments $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{r}}$ and $\mathcal{M} \mapsto \mathcal{M}^{\mathrm{e}}$ define bijective correspondences between the following classes of ideals:*

*(1) Maximal ideals of $\mathbf{C}$.*

*(2) Maximal ideals $\mathcal{M}$ of $\mathbf{M}(\mathbf{C})$ satisfying $\mathcal{M}^{\mathrm{e}}(C, C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ not belonging to $\mathbf{M}(\mathbf{C}) \cup \mathbf{S}(\mathbf{C})$, that is, for each object $C$ with no maximal $N$ with $C \preceq N$.*

*Proof.* Let $\mathcal{M}$ be a maximal ideal of $\mathbf{C}$. We will now prove that $\mathcal{M}^{\mathrm{r}}$ is a maximal ideal of $\mathbf{M}(\mathbf{C})$ satisfying $\mathcal{M}^{\mathrm{r\,e}}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ not belonging to $\mathbf{M}(\mathbf{C}) \cup \mathbf{S}(\mathbf{C})$. Since $\mathcal{M}$ is proper, there exists an object $C_0$ of $\mathbf{C}$ such that $\mathcal{M}(C_0, C_0) \neq \mathrm{End}_{\mathbf{C}}(C_0)$. By Theorem 3.1, $C_0$ must belong to $\mathbf{M}(\mathbf{C})$. This means that $\mathcal{M}^{\mathrm{r}}$ is a proper ideal in $\mathbf{M}(\mathbf{C})$, which is trivially maximal, as $\mathcal{M}$ is maximal in $\mathbf{C}$. Moreover, note that $\mathcal{M}^{\mathrm{r\,e}}$ is the ideal of $\mathbf{C}$ associated to $\mathcal{M}(C_0, C_0)$, which is equal to $\mathcal{M}$ by [9, Lemma 2.4]. Then, again by Theorem 3.1, $\mathcal{M}^{\mathrm{r\,e}}(C,C) = \mathcal{M}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for every object $C$ not belonging to $\mathbf{M}(\mathbf{C})$.

Conversely, let $\mathcal{M}$ be a maximal ideal of $\mathbf{M}(\mathbf{C})$ satisfying $\mathcal{M}^{\mathrm{e}}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ not belonging to $\mathbf{M}(\mathbf{C}) \cup \mathbf{S}(\mathbf{C})$. We claim that $\mathcal{M}^{\mathrm{e}}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ belonging to $\mathbf{S}(\mathbf{C})$. To prove the claim, let $C$ be an object of $\mathbf{S}(\mathbf{C})$ and suppose that $D \in \mathbf{M}(\mathbf{C})$ satisfies $C \prec D$. If $\mathcal{M}^{\mathrm{e}}(C,C) \neq \mathrm{End}_{\mathbf{C}}(C)$, then $\mathcal{M}^{\mathrm{e}}(D,D)$ is a proper ideal of $\mathrm{End}_{\mathbf{C}}(D)$, which is not maximal by Theorem 3.1. By Lemma 3.8, $\mathcal{M}^{\mathrm{e}}(D,D) = \mathcal{M}(D,D)$ and, consequently, $\mathcal{M}(D,D)$ is a proper ideal of $\mathrm{End}_{\mathbf{M}(\mathbf{C})}(D)$ that is not maximal. Since $\mathcal{M}$ is maximal in $\mathbf{M}(\mathbf{C})$, this contadicts [9, Lemma 2.4]. The contradiction proves the claim.

Now let $\mathcal{N}$ be an ideal of $\mathbf{C}$ properly containing $\mathcal{M}^{\mathrm{e}}$. We will prove that $\mathcal{N} = \mathrm{Hom}_{\mathbf{C}}$. Since $\mathcal{N}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ not belonging to $\mathbf{M}(\mathbf{C})$, it follows that $\mathcal{N}^{\mathrm{r}}$ properly contains $\mathcal{M}^{\mathrm{e\,r}}$. But $\mathcal{M}^{\mathrm{e\,r}} = \mathcal{M}$ by Lemma 3.8 and, since $\mathcal{M}$ is maximal in $\mathbf{M}(\mathbf{C})$, we get that $\mathcal{N}^{\mathrm{r}} = \mathrm{Hom}_{\mathbf{M}(\mathbf{C})}$. This fact with the previous claim gives that $\mathcal{N} = \mathrm{Hom}_{\mathbf{C}}$.

Finally, it is easy to see that the two assignments are mutually inverse. □

**Remark 3.10.** Let $\mathbf{C}$ be a preadditive category, $A$ and $C$ objects of $\mathbf{C}$ and $I$ an ideal of $\mathrm{End}_{\mathbf{C}}(C)$. Then, $\mathcal{A}_I(A,A) = \mathrm{End}_{\mathbf{C}}(A)$ if and only if each endomorphism of $C$ factoring through $A$ belongs to $I$. Consequently, if $\mathcal{M}$ is a maximal ideal in $\mathbf{M}(\mathbf{C})$ and $C$ is an object with no maximal $N$ satisfying $C \preceq N$, then the following conditions are equivalent:

(1) $\mathcal{M}^{\mathrm{e}}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$.
(2) There exists an object $A \in \mathbf{M}(\mathbf{C})$ with $\mathcal{M}(A,A) \neq \mathrm{End}_{\mathbf{C}}(A)$ such that each endomorphism of $A$ factoring through $C$ belongs to $\mathcal{M}(A,A)$.
(3) For each object $A \in \mathbf{M}(\mathbf{C})$ with $\mathcal{M}(A,A) \neq \mathrm{End}_{\mathbf{C}}(A)$, every endomorphism of $A$ factoring through $C$ belongs to $\mathcal{M}(A,A)$.

Proposition 3.9 says that, in order to compute the maximal ideals in a category $\mathbf{C}$, we can *(1)* determine the subcategories $\mathbf{M}(\mathbf{C})$ and $\mathbf{S}(\mathbf{C})$, and *(2)* find the maximal ideals $\mathcal{M}$ of $\mathbf{M}(\mathbf{C})$ such that $\mathcal{M}^{\mathrm{e}}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ with no maximal $N$ satisfying $C \preceq N$. We will use this procedure in the following example.

**Example 3.11.** Let $R$ be a simple non-artinian ring with $\mathrm{Soc}(R_R)$ non-projective as a right $R$-module. Then there exists a non projective simple right module $S$ contained in $R$. Consider the full subcategory $\mathbf{C} = \mathrm{add}(R_R) \cup \mathrm{Add}(S)$ of Mod-$R$. Then $\mathbf{M}(\mathbf{C}) = \mathrm{add}(R_R)$ and $\mathbf{S}(\mathbf{C}) = \emptyset$. Since $R$ is simple, Example 1.1 says that the unique maximal ideal of $\mathbf{M}(\mathbf{C})$ is the ideal $\mathcal{A}_0$ associated to the zero ideal of $R$. However, $\mathcal{A}_0(S,S) \neq \mathrm{End}_R(S)$ since, if we take $f : R_R \to S$ an epimorphism and we denote by $g : S \to R_R$ the inclusion, we have that $g1_Sf \neq 0$, which means that $1_S \notin \mathcal{A}_0(S,S)$. Then, by Theorem 3.9, $\mathbf{C}$ does not have maximal ideals.

**Remark 3.12.** Let $\mathbf{C}$ be a preadditive category. As the preceding example shows, there does not exist a bijective correspondence between maximal ideals in $\mathbf{C}$ and maximal ideals in $\mathbf{M}(\mathbf{C})$. This is due to the fact that there can exist maximal ideals

$\mathcal{M}$ of $\mathbf{M}(\mathbf{C})$ such that $\mathcal{M}^{\mathrm{e}}(C,C) \neq \mathrm{End}_{\mathbf{C}}(C)$ for some object $C$ with no maximal $N$ satisfying $C \preceq N$. More precisely, the map $(-)^{\mathrm{r}}$ from the class of maximal ideals of $\mathbf{C}$ to the class of maximal ideals of $\mathbf{M}(\mathbf{C})$ is injective, and its image consists of those maximal ideals $\mathcal{M}$ of $\mathbf{M}(\mathbf{C})$ for which $\mathcal{M}^{\mathrm{r}}(C,C) = \mathrm{End}_{\mathbf{C}}(C)$ for each object $C$ with no maximal object $M$ with $C \preceq M$. The map $(-)^{\mathrm{r}}$ is not, in general, surjective .

## 4. MAXIMAL IDEALS INDUCED BY AN INDECOMPOSABLE INJECTIVE MODULE

We conclude the paper computing the maximal ideals of a certain subcategory of a module category. The main idea in this computation is to apply the results of the previous sections to describe the ideal associated to a maximal ideal in the endomorphism ring of an indecomposable injective module.

Let $R$ be a ring that we fix through the rest of the section and let $F$ be an injective $R$-module. Recall that $F$ is indecomposable if and only if $F$ has a local endomorphism ring [2, Theorem 25.4].

**Lemma 4.1.** *Let $F$ be an indecomposable injective module, and let $I$ be the maximal ideal of $\mathrm{End}_R(F)$. The following conditions are equivalent for modules $A$, $B$ and $f \in \mathrm{Hom}_R(A,B)$:*

*(1) $f \notin \mathcal{A}_I(A,B)$.*
*(2) There exists $\alpha\colon F \to A$ and $\beta\colon B \to F$ such that $\beta f \alpha = 1_F$.*
*(3) There exists a submodule $C$ of $A$ such that $C \cong F$ and $C \cap \ker f = 0$.*

*Proof.* *(1) ⇒ (2).* If $f \notin \mathcal{A}_I$, there exists $\alpha\colon F \to A$ and $\beta\colon A \to F$ such that $\beta f \alpha \notin I$. Since the endomorphisms of $F$ not belonging to $I$ are isomorphisms, there exists an inverse $\gamma \in \mathrm{End}_R(F)$ of $\beta f \alpha$. Then $\gamma\beta f \alpha = 1_F$.

*(2) ⇒ (1)* is trivial.

*(2) ⇒ (3).* Set $C = \alpha(F)$, which is isomorphic to $F$ as $\alpha$ is monic. Then $A = C \oplus \ker(\beta f)$. In particular, $C \cap \ker f \subseteq C \cap \ker(\beta f) = 0$.

*(3) ⇒ (2).* Let $\alpha\colon F \to A$ be a monomorphism with image $C$. Since $C \cap \ker f = 0$, $f\alpha$ is a monomorphism. Since $F$ is an injective, this implies the existence of $\beta\colon B \to F$ with $\beta f \alpha = 1_F$, as desired. □

As a byproduct of this result we get:

**Corollary 4.2.** *Let $F$ be an indecomposable injective module, and let $I$ be the maximal ideal of $\mathrm{End}_R(F)$. Then, for any pair $A, B$ of modules, $\mathcal{A}_I(A,B) \neq \mathrm{Hom}_R(A,B)$ if and only if both $A$ and $B$ contain a submodule isomorphic to $F$.*

*Proof.* If $\mathcal{A}_I(A,B) \neq \mathrm{Hom}_R(A,B)$ and $f\colon A \to B$ does not belong to $\mathcal{A}_I(A,B)$, then, by the previous lemma, there exists $C \leq A$ with $C \cong F$ and $C \cap \ker f = 0$. This implies that $f(C)$ is isomorphic to $F$. Thus $C$ and $f(C)$ are submodules with the desired property.

Conversely, assume $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1 \cong B_1 \cong F$. Let $f : A_1 \to B_1$ be an isomorphism and let $F : A \to B$ be the morphism $f \oplus 0$. Then $F$ trivially satisfies *(3)* of the previous lemma and, consequently, $F$ does not belong to $\mathcal{A}_I(A,B)$. □

It follows that, in order to determine $\mathcal{A}_I$, we only have to look at the modules $A$ containing isomorphic copies of $F$. These modules have a nice description if $R$ is right noetherian as we will prove next. We shall use the following well known facts about an injective module $F$:

(1) If $K$ is a non-zero submodule of $F$, then there exists an injective submodule $G$ of $F$ containing $K$ such that the inclusion $K \leq G$ is an injective envelope. In particular, if $F$ is indecomposable, then the inclusion $K \leq F$ is an injective envelope of $K$.
(2) $F$ satisfies the exchange property, which means that for each module $N$ and each decomposition $F \oplus N = \bigoplus_{\alpha<\kappa} A_\alpha$, there exists a submodule $B_\alpha \leq A_\alpha$ for each $\alpha < \kappa$ such that $F \oplus N = F \oplus \left(\bigoplus_{\alpha<\kappa} B_\alpha\right)$.

**Theorem 4.3.** *Suppose $R$ right noetherian. Let $F$ be an indecomposable injective module. Then every module $M$ has a decomposition $M = M_1 \oplus M_2$ where:*

*(1) $M_1 \cong F^{(\Gamma)}$ for some set $\Gamma$.*
*(2) $M_2$ does not contain submodules isomorphic to $F$.*

*Moreover, $\Gamma$ is uniquely determined up to cardinality and $M_2$ is uniquely determined up to isomorphism.*

*Proof.* If $M$ does not have submodules isomorphic to $F$, there is nothing to prove. So suppose that $M$ has submodules isomorphic to $F$ and consider the non-empty family of submodules

$$\mathcal{S} = \{N \leq M : N \cong F^{(\Gamma)} \text{ for some set } \Gamma\}$$

Let us show that $\mathcal{S}$ is inductive. Take a chain $\mathcal{S}' = \{\, M_\lambda : \lambda \in \Lambda \,\}$ in $\mathcal{S}$. We will prove that $M' := \bigcup_{\lambda\in\Lambda} M_\lambda \in \mathcal{S}$. Since $R$ is right noetherian, directed colimits of injective modules are injective [3, Exercise 8 of Chapter I], so that $M'$ is injective. Hence, $M'$ has a direct-sum decomposition, $M' = \bigoplus_{i\in I} F_i$, where the submodules $F_i$ of $M'$ are injective and indecomposable [13, Theorem 3.48, p. 82]. Let $i \in I$ and $x$ be a non-zero element of $F_i$; note that $F_i$ is the injective envelope of $xR$. Since $x \in M_\lambda$ for some $\lambda \in \Lambda$, and $M_\lambda \in \mathcal{S}$, $x$ belongs to a direct summand of $M_\lambda$ isomorphic to $F^n$ for some $n$. But, as $F_i$ is the injective envelope of $xR$, $F^n$ must contain a direct summand isomorphic to $F_i$. This implies that $F_i \cong F$ because all indecomposable direct summands of $F^n$ are isomorphic to $F$. Consequently, $M' \in \mathcal{S}$.

The first part of the statement now follows taking a maximal element $M_1$ of $\mathcal{S}$ and a submodule $M_2$ of $M$ with $M_1 \oplus M_2 = M$.

In order to prove the last part of the statement, suppose that $M = M_1' \oplus M_2'$ is another decomposition of $M$ satisfying *(1)* and *(2)*. Write $M_1 = \bigoplus_{\beta<\kappa} G_\beta$ and $M_1' = \bigoplus_{\alpha<\lambda} F_\alpha$ for suitable families of submodules $\{G_\beta : \beta < \kappa\}$ and $\{F_\alpha : \alpha < \lambda\}$ of $M_1$ and $M_1'$ respectively, and cardinals $\kappa$ and $\lambda$, satisfying $G_\beta \cong F_\alpha \cong F$ for each $\beta < \kappa$ and $\alpha < \lambda$.

Since $M_1$ satisfies the exchange property, there exist submodules $H_\alpha \leq F_\alpha$ for each $\alpha < \lambda$ and $N_2' \leq M_2'$ such that $M = M_1 \oplus \left(\bigoplus_{\alpha<\lambda} H_\alpha\right) \oplus N_2'$. Since $F_\alpha$ is indecomposable for each $\alpha < \lambda$, it follows that $H_\alpha = 0$ or $H_\alpha = F_\alpha$. But, as $\left(\bigoplus_{\alpha<\lambda} H_\alpha\right) \oplus N_2' \cong M_2$ and $M_2$ does not contain any submodule isomorphic to $F$, we get that $H_\alpha = 0$ for each $\alpha < \lambda$. That is, $M = M_1 \oplus N_2'$.

Applying the modular law, we see that $M_2' = N_2' \oplus (M_2' \cap M_1)$. We claim that $M_2' \cap M_1 = 0$. Assume the contrary, i. e., that $M_1 \cap M_2' \neq 0$. Since $M_1 \cap M_2'$ is a direct summand of $M$, it is a direct summand of $M_1$ and, consequently, it is injective. As it is non-zero, there exists $\beta < \kappa$ such that $G_\beta \cap M_1 \cap M_2' \neq 0$. Let $x$ be a non-zero element in this intersection. Notice that $xR \leq G_\beta$ is an injective envelope. Since $M_1 \cap M_2'$ is injective, there exists an injective envelope $C$ of $xR$ contained in $M_1 \cap M_2'$. But $C$ is isomorphic to $F$ and $M_2'$ does not contain any submodule isomorphic to $F$, a contradiction. This proves the claim.

As a consequence, $M = M_1 \oplus M_2'$. Then $M_1 \cong M_1'$ and $M_2 \cong M_2'$. By Azumaya's Theorem [2, Theorem 12.6], $\kappa = \lambda$ and we are done. □

We can use this result to define the $F$-rank of a module $M$, for any indecomposable injective module $F$ and any module $M$ over a right noetherian ring.

**Definition 4.4.** *Suppose $R$ is right noetherian. Let $F$ be an indecomposable injective module. Given any module $M$ and any cardinal $\kappa$, we say that $M$* has $F$-rank equal to $\kappa$ *(written $\mathrm{r}_F(M) = \kappa$) if $M = M_1 \oplus M_2$, where $M_1 \cong F^{(\kappa)}$ and $M_2$ has no direct summand isomorphic to $F$. We will denote by $\mathbf{C}_F$ the full subcategory of Mod-$R$ whose objects are all modules of finite $F$-rank.*

Now we can compute the maximal ideals in the category $\mathbf{C}_F$ for an indecomposable injective module $F$. First of all, we compute the subcategories $\mathbf{M}(\mathbf{C}_F)$ and $\mathbf{S}(\mathbf{C}_F)$.

**Proposition 4.5.** *Suppose $R$ is right noetherian. Let $F$ be an indecomposable injective module. Then:*

*(1)* $\mathbf{M}(\mathbf{C}_F) = \{M \in \mathbf{C}_F : \mathrm{r}_F(M) > 0\}$.
*(2)* $\mathbf{S}(\mathbf{C}_F) = \{M \in \mathbf{C}_F : \mathrm{r}_F(M) = 0\}$.

*Proof. (1)* Let $M$ be a module in $\mathbf{C}_F$ with $\mathrm{r}_F(M) = 0$. We can find an infinite cardinal $\kappa$ such that $M$ is $< \kappa$-generated in Mod-$R$. By Proposition 2.4, $M \prec M^{(\kappa)}$ in Mod-$R$. Since $\mathbf{C}_F$ is a full subcategory of Mod-$R$ and $M^{(\kappa)} \in \mathbf{C}_F$, we get that $M \prec M^{(\kappa)}$ in $\mathbf{C}_F$. Consequently, $M$ does not belong to $\mathbf{M}(\mathbf{C}_F)$. This proves the inclusion $\mathbf{M}(\mathbf{C}_F) \subseteq \{M \in \mathbf{C}_F : \mathrm{r}_F(M) > 0\}$.

In order to prove the inverse inclusion, let $M$ be any module with $\mathrm{r}_F(M) > 0$ and suppose, by contradiction, that $M \notin \mathbf{M}(\mathbf{C}_F)$. Then there exists $N \in \mathbf{C}_F$ such that $M \prec N$. Let $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ be the decompositions given by Theorem 4.3, and let $E$ be the set of Definition 2.1. Write $M_1 = \bigoplus_{i=1}^n G_i$ and $N_1 = \bigoplus_{j=1}^m F_j$ for modules $G_i$ and $F_j$ isomorphic to $F$ for each $i = 1, \dots, n$ and $j = 1, \dots, m$.

We claim that $f(N_1) \neq 0$ for each $(f, g) \in E$. Given $i = 1, \dots, n$, $g(G_i)$ is isomorphic to $F$. Then $g(E_i) \cap N_1 \neq 0$ since, otherwise, $N_1 \oplus g(E_i)$ would be a direct summand of $N$, and $N_2$ would contain a submodule isomorphic to $F$. Now, taking $y \in g(E_i) \cap N_1$ non-zero and $x \in E_i$ with $g(x) = y$, we have that $f(y) = x \neq 0$. This proves the claim.

For each $i = 1, \dots, m$, let $q_i : G_1 \to F_i$ be an isomorphism. Then $q_i$ extends to a morphism $p_i \colon M \to N$. The preceeding claim says that for each $(f, g) \in E$, $fp_i \neq 0$ for some $i = 1, \dots, m$. Consequently, if

$$E_i = \{(f, g) \in E : fp_i \neq 0\}$$

for each $i = 1, \dots, m$, we conclude that $E \subseteq \bigcup_{i=1}^m E_i$. This is a contradiction, because the second set has cardinality smaller than $|E|$ by Definition 2.1*(2)*.

The conclusion is that there is no object $N \in \mathbf{C}_F$ with $M \prec N$, so that $M \in \mathbf{M}(\mathbf{C}_F)$.

*(2)* As a direct consequence of *(1)* we have $\mathbf{S}(\mathbf{C}_F) \subseteq \{M \in \mathbf{C}_F : \mathrm{r}_F(M) = 0\}$. In order to see the other inclusion, fix $M \in \mathbf{C}_F$ with $\mathrm{r}_F(M) = 0$. Then, as in the proof of *(1)*, there exists an infinite cardinal $\kappa$ such that $M \prec M^{(\kappa)}$ in $\mathbf{C}_F$. By Lemma 1.2, $M \prec F \oplus M^{(\kappa)}$. Then $M \in \mathbf{S}(\mathbf{C}_F)$ because $F \oplus M^{(\kappa)} \in \mathbf{M}(\mathbf{C}_F)$ by *(1)*. $\square$

Finally, we can determine all maximal ideals of the category $\mathbf{C}_F$ for an indecomposable injective module $F$ over a right noetherian ring.

**Proposition 4.6.** *Suppose $R$ is right noetherian. Let $F$ be an indecomposable injective module and $I$ be the maximal ideal of $\mathrm{End}_R(F)$. Then $\mathcal{A}_I$ is the unique maximal ideal of $\mathbf{C}_F$.*

*Proof.* By Theorem 3.9, we only have to compute the maximal ideals of $\mathbf{M}(\mathbf{C}_F)$. First, we prove that $\mathcal{A}_I$ is a maximal ideal of $\mathbf{M}(\mathbf{C}_F)$. Given $M \in \mathbf{M}(\mathbf{C}_F)$, since $\mathcal{A}_I(M, M) \neq \mathrm{End}_R(M)$ by Corollary 4.2, we have to see, applying [9, Lemma 2.4], that

*(a)* $\mathcal{A}_I(M, M)$ is maximal in $\mathrm{End}_R(M)$ and,
*(b)* if $J_0 = \mathcal{A}_I(M, M)$, then $\mathcal{A}_I = \mathcal{A}_{J_0}$.

Let $M = M_1 \oplus M_2$ be the decomposition of $M$ given in Theorem 4.3. Note that, by Lemma 1.2 and Corollary 4.2,

$$\mathcal{A}_I(M, M) = \{f \in \mathrm{End}_R(M) : \pi_1 f \iota_1 \in \mathcal{A}_I(M_1, M_1)\},$$

where $\pi_i : M \to M_i$ and $\iota_i : M_i \to M$ are the corresponding projections and inclusions for $i = 1, 2$. As $M_1 \in \mathrm{add}(F)$ and $\mathcal{A}_I$ is a maximal ideal in this category by Example 1.1, $\mathcal{A}_I(M_1, M_1)$ is a maximal ideal in $\mathrm{End}_R(M_1, M_1)$. In order to see that $\mathcal{A}_I(M, M)$ is maximal, let $J$ be an ideal of $\mathrm{End}_R(M)$ strictly containing $\mathcal{A}_I(M, M)$. Let $f \in J$ not belonging to $\mathcal{A}_I(M, M)$. Then $\pi_1 f \iota_1$ does not belong to $\mathcal{A}_I(M_1, M_1)$ and, by the maximality of this ideal in $\mathrm{End}_R(M_1, M_1)$, there exist $g \in \mathcal{A}_I(M_1, M_1)$ and $\alpha, \beta \in \mathrm{End}_R(M_1)$ such that $1_{M_1} = g + \alpha \pi_1 f \iota_1 \beta$. Then we have the identity

$$1_{M_1} \oplus 0 = g \oplus 0 + (\alpha \oplus 0) f (\beta \oplus 0)$$

in $\mathrm{End}_R(M)$, with both $g$ and $(\alpha \oplus 0) f (\beta \oplus 0)$ in $J$. Consequently, $1_{M_1} \oplus 0 \in J$. Now use $0 \oplus 1_{M_2} \in J$ to get that $1_M = 1_{M_1} \oplus 0 + 0 \oplus 1_{M_2} \in J$ and that $J = \mathrm{End}_R(M)$.

Let us prove *(b)*. Since $\mathcal{A}_{J_0}$ is the greatest of all the ideals $\mathcal{I}'$ of $\mathbf{C}_F$ such that $\mathcal{I}'(M, M) \leq J_0$, we conclude that $\mathcal{A}_I \subseteq \mathcal{A}_{J_0}$. In order to prove the other inclusion, we only have to see, by the same argument, that $\mathcal{A}_{J_0}(F, F) \leq I$. Let $f \in \mathcal{A}_{J_0}(F, F)$. Fix a monomorphism $\alpha_1 \colon F \to M_1$, which, as $\mathrm{Im}\, \alpha_1$ is a direct summand, has an splitting $\beta_1 : M_1 \to F$. Then note that $\iota_1 \alpha_1 f \beta_1 \pi_1 \in J_0$, because $f \in \mathcal{A}_{J_0}(F, F)$. Then $\pi_1 \iota_1 \alpha_1 f \beta_1 \iota_1 \alpha_1 \in \mathcal{A}_I(M_1, M_1)$ and, consequently, $\beta_1 \pi_1 \iota_1 \alpha_1 f \beta_1 \iota_1 \alpha_1 \in \mathcal{A}_I(F, F) = I$. Since

$$f = \beta_1 \pi_1 \iota_1 \alpha_1 f \beta_1 \iota_1 \alpha_1,$$

we conclude that $f \in I$.

To finish the proof, we will see that $\mathcal{A}_I$ is the unique maximal ideal of $\mathbf{M}(\mathbf{C}_F)$. Let $\mathcal{M}$ be any maximal ideal of $\mathbf{M}(\mathbf{C}_F)$ and $M \in \mathbf{M}(\mathbf{C}_F)$ be such that $\mathcal{M}(M, M) \neq$

$\mathrm{End}_R(M)$. If $J = \mathcal{M}(M, M)$, then $\mathcal{M} = \mathcal{A}_J$ by [9, Lemma 2.4]. Let $M = M_1 \oplus M_2$ be the decomposition of $M$ given by Theorem 4.3. By Lemma 1.2, either $\mathcal{M}(M_1, M_1)$ or $\mathcal{M}(M_2, M_2)$ have to be proper. But $M_2 \in \mathbf{S}(\mathbf{C}_F)$ by Proposition 4.5, so that $\mathcal{M}(M_2, M_2) = \mathrm{End}_R(M_2)$ by Remark 3.5. Thus $\mathcal{M}(M_1, M_1) \neq \mathrm{End}_R(M_1)$ which implies, again by Lemma 1.2, that $\mathcal{M}(F, F) \neq \mathrm{End}_R(F)$. Since $I$ is the unique maximal ideal of $\mathrm{End}_R(F)$ and $\mathcal{M}(E, E)$ is maximal, we conclude that $\mathcal{M}(F, F) = I$. Now $\mathcal{M} = \mathcal{A}_I$ by [9, Lemma 2.4], which concludes the proof. □

**Example 4.7.** The category $\mathbf{C}_F$ has maximal ideals and objects $M, N$ with $M \prec N$ since, if $M$ is an object in $\mathbf{C}_F$ with $F$-rank 0, then each direct sum of copies of $M$ belongs to $\mathbf{C}_F$. By Proposition 2.4, there exist objects $N$ in $\mathbf{C}_F$ with $M \prec N$.

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