Open maps of involutive quantales^{*}

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Abstract

By a map $p: Q \to X$ of involutive quantales is meant a homomorphism $p^*: X \to Q$. Calling a map p weakly open if p^* has a left adjoint $p_!$ which satisfies the Frobenius reciprocity condition (i.e., $p_!$ is a homomorphism of X-modules), we say that p is open if it is stably weakly open. We also study a two-sided version, FR2, of the Frobenius reciprocity condition, and show that the weakly open surjections that satisfy FR2 are open. Maps of the latter kind arise in the study of Fell bundles on groupoids.

Keywords: Involutive quantales, open maps, Frobenius reciprocity condition.

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1 Introduction

Quantales, in particular involutive quantales, are generalizations of locales which often can be regarded as generalized spaces. For instance, there are several notions of point of a quantale [10, 18, 20], and there are correspondences between quantales and other types

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of generalized space such as Grothendieck toposes [5–7,24], C*-algebras [1,11,18,19,28], or groupoids [21,23]. To some extent the geometric meaning of quantales is borrowed from such structures, and also from the functoriality of the correspondences, which in turn depends on which notion of morphism of quantales we adopt.

No choice of morphism works for all purposes. For instance, inverse quantal frames [23] form a bicategory which is biequivalent to that of étale groupoids [25], and this suggests that "geometric maps" of these quantales can be defined in terms of quantale bimodules so as to correspond to those groupoid bi-actions which in turn yield geometric morphisms of étendues [4, 14, 15]. (A similar notion of morphism of groupoids is relevant in the context of strong Morita equivalence for C*-algebras [13, 16, 17].) Another more immediate (and in general different) notion is based on homomorphisms of involutive quantales, which correspond, for groupoids, to the algebraic morphisms of [2, 3]. See [25]. Homomorphisms are also natural when relating C*-algebras and quantales, since *-homomorphisms of C*-algebras translate functorially to involutive homomorphisms between their quantales [18]. And another C*-algebra-related context, in fact the one which motivates the present paper, is that of Fell bundles on groupoids [12], where again quantale homomorphisms arise naturally [26].

Here we shall be concerned with general involutive homomorphisms of involutive quantales or, rather, with their formal duals as in locale theory [8]: by a *(continuous)* map of involutive quantales $p: Q \to X$ will be meant an involutive homomorphism $p^*: X \to Q$, i.e., a mapping that satisfies the following conditions for all $x, y \in X$ and all families (x_i) in X:

$$p^{*}(xy) = p^{*}(x)p^{*}(y) p^{*}(x^{*}) = (p^{*}(x))^{*} p^{*}(\bigvee_{i} x_{i}) = \bigvee_{i} p^{*}(x_{i})$$

This is referred to as the *inverse image homomorphism* of p. Following the terminology for locales in [22], if p^* has a (necessarily involution preserving) left adjoint $p_!$ we say that p is *semiopen*, and refer to $p_!$ as the *direct image homomorphism* of p.

The main purpose of this short paper is to examine conditions under which a semiopen map of involutive quantales can be regarded as being open. This is not for the sake of generalization per se, but rather because examples of open-like maps arise from some Fell bundles on groupoids (see section 3), and it is worth understanding their properties in particular as regards stability under pullbacks.

2 Preliminaries

This section is mostly for fixing terminology and notation.

Quantic subspaces. We shall denote by Qu the usual category of involutive quantales whose arrows are the involutive homomorphisms, and by QSp the opposite category Qu^{op} , whose arrows are the maps. We refer to QSp as the category of quantic spaces, in order to disambiguate our terminology, for instance when referring to subobjects: whereas by an *involutive subquantale* is usually meant a subobject of an involutive quantale in the algebraic sense, in this paper a quantic subspace of an involutive quantale Q is defined to be an equivalence class of regular monomorphisms $m: S \to Q$ in QSp; that is, an equivalence class of maps m such that m^* is a surjective homomorphism, where two such maps $m: S \to Q$ and $m': S' \to Q$ are equivalent if there is an isomorphism $S \cong S'$ commuting with m and m'.¹

A quantic subspace of Q can also be identified with an *involutive quantic nucleus* on Q, i.e., a closure operator on Q that satisfies $j(a)j(b) \leq j(ab)$ and $j(a^*) = j(a)^*$ for all $a, b \in Q$, as we now explain. Let Q be an involutive quantale, and let $j : Q \to Q$ be an involutive quantic nucleus. The set of closed elements

$$Q_j = j(Q) = \{a \in Q \mid j(a) = a\}$$

is closed under meets and involution, and it is an involutive quantale with multiplication defined by $(a, b) \mapsto j(ab)$ and the join of each family (a_i) in Q_j being $j(\bigvee_i a_i)$. Moreover, $j : Q \to Q_j$ is a surjective homomorphism of involutive quantales, and, up to isomorphisms, every surjective homomorphism of involutive quantales arises like this. For a proof of this (for non involutive quantales) see [27]. Hence, the involutive quantic nuclei on Q are in bijective correspondence with quantic subspaces of Q: given a regular monomorphism $m : S \to Q$ in QSp we have $S \cong Q_j$, where $j = m_* \circ m^*$ and m_* is the right adjoint of m^* .

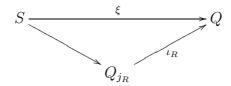
Quantic subspaces presented by relations. Let Q be an involutive quantale. If $R \subseteq Q \times Q$ is a binary relation on Q, the least (in the pointwise order) involutive quantic nucleus j on Q such that j(r) = j(s) for all $(r, s) \in R$ is denoted by j_R , and

 $\iota_R:Q_{j_R}\to Q$

is the regular monomorphism defined by $\iota_R^* = j_R$. Any regular monomorphism

 $\xi:S\to Q$

such that $\xi^*(r) = \xi^*(s)$ for all $(r, s) \in R$ factors (uniquely) through ι_R :



¹We note that the terminology 'quantic space versus quantale' is consistent with the terminology 'space versus locale' of Joyal and Tierney [9], albeit not with the more common terminology 'locale versus frame'. An alternative would be to use 'quantale' in the topological sense and, say, 'quantic frame' in the algebraic one. However, the ensuing meaning of 'subquantale' would go against the usual terminology in quantale theory, and, moreover, 'quantic frame' is misleadingly close to 'quantal frame', which has been consistently used by more than one author in order to refer to a quantale whose order satisfies the locale distributivity law.

This provides us with the construction of equalizers in QSp: the equalizer of two maps $f, g: Q \to X$ is the quantic subspace $\iota_R: Q_{j_R} \to Q$ for $R = \{(f^*(x), g^*(x)) \mid x \in X\}$.

The following facts are easily derived from [9] and are useful in calculations in order to translate the universal property of Q_{i_B} into sup-lattices:

2.1 Lemma. Let Q be an involutive quantale, let $R \subseteq Q \times Q$ be a binary relation on Q, and let $\widetilde{R} \subseteq Q \times Q$ be the least binary relation on Q such that the following conditions hold:

- $R \subseteq \widetilde{R}$,
- $(r^*, s^*) \in \widetilde{R}$ for all $(r, s) \in \widetilde{R}$,
- $(ar, as) \in \widetilde{R}$ for all $(r, s) \in \widetilde{R}$ and all $a \in Q$.

The following equivalent conditions hold:

- 1. The quantic nucleus j_R coincides with the least closure operator j on Q such that j(r) = j(s) for all $(r, s) \in \widetilde{R}$.
- 2. Q_{j_R} consists of those $\alpha \in Q$ such that for all $(r,s) \in \widetilde{R}$ the following condition holds:

$$r \leq \alpha \quad \Longleftrightarrow \quad s \leq \alpha \;.$$

3. Any sup-lattice homomorphism $h : Q \to L$, where L is a sup-lattice, factors through the quotient homomorphism $j_R : Q \to Q_{j_R}$ if and only if for all $(r, s) \in \widetilde{R}$ we have h(r) = h(s).

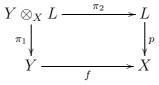
3 Open maps

Locales. Let us first recall open maps of locales (see [9]). A map of locales $p: L \to X$ is open if the image of any open sublocale of L is an open sublocale of X. Equivalently, p is open if and only if the inverse image homomorphism $p^*: X \to L$ has a left adjoint $p_!: L \to X$ which is a homomorphism of X-modules; that is, such that for all $x \in X$ and $a \in L$ the Frobenius reciprocity condition holds:

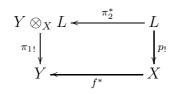
$$p_!(a \wedge p^*(x)) = p_!(a) \wedge x$$
.

Then it is also true that a sublocale of X is open if and only if its inclusion into X in $Loc = Frm^{op}$ is an open map. Moreover, the open sublocales of X can be identified with elements $u \in X$: the quotient that determines the corresponding open sublocale can be taken to be $(-) \wedge u : X \to \downarrow u$. Finally, one important property of open maps

is their stability under pullbacks: if the following is a pullback diagram in Loc and p is open then so is π_1 ,



and the following diagram is commutative (Beck–Chevalley condition):



Frobenius reciprocity conditions. A straightforward generalization of the facts above does not exist for an involutive quantale X, as there is no a priori known notion of open quantic subspace of X, and the elements $u \in X$ do not determine quotients of X in Qu in a canonical way. So we shall look at maps of involutive quantales satisfying conditions that mimick the Frobenius reciprocity condition of locales, at the same time keeping in mind that any reasonable notion of open map should be such that pullbacks of open maps are open. Namely, we shall examine the following conditions for a semiopen map $p: Q \to X$ of involutive quantales:

FR1:
$$p_!(ap^*(x)) = p_!(a)x$$
 for all $x \in X$ and $a \in Q$;

FR2: $p_!(ap^*(x)b) = p_!(a)xp_!(b)$ for all $x \in X$ and $a, b \in Q$.

The first condition, FR1, which we refer to as the one-sided Frobenius reciprocity condition, is an immediate generalization of the Frobenius reciprocity condition of locales, stating that p_1 is a homomorphism of left X-modules. We shall call a semiopen map that satisfies FR1 weakly open, following [26]. We note that, due to the involution, FR1 is equivalent to the analogous condition applied to right X-modules:

3.1 Lemma. Let $p: Q \to X$ be a weakly open map. Then for all $a \in Q$ and $x \in X$ we have

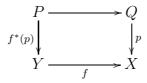
$$p_!(p^*(x)a) = xp_!(a) .$$

Proof. Let $a \in Q$ and $x \in X$. Then $p_!(p^*(x)a) = p_!(p^*(x)a)^{**} = p_!((p^*(x)a)^*)^* = p_!(a^*p^*(x)^*)^* = (p_!(a^*)x^*)^* = (p_!(a)^*x^*)^* = xp_!(a)$.

Another simple property of weakly open maps is the following:

3.2 Lemma. Let $p: Q \to X$ be a weakly open map with X a unital involutive quantale. Then p is a surjection if and only if $p_!(p^*(e)) = e$. *Proof.* p is a surjection if and only if for all $x \in X$ we have $p_!(p^*(x)) = x$, so one implication is trivial. Let then $p_!(p^*(e)) = e$ and assume that p is weakly open. Then for all $x \in X$ we have $p_!(p^*(x)) = p_!(p^*(ex)) = p_!(p^*(e)p^*(x)) = p_!(p^*(e))x = ex = x$. \Box

Contrary to the situation with locales, it is not to be expected that weakly open maps should be stable under pullbacks in QSp (cf. section 5). Hence, as a working definition of openness, even if a naive one, let us say that a semiopen map $p: Q \to X$ is *open* if for all maps $f: Y \to X$ the pullback $f^*(p)$ in QSp of p along f is weakly open:



It is clear that identity maps are open, and that the class of open maps is closed under composition, so the open maps define a subcategory of QSp. And, since any pullback of $f^*(p)$ is itself isomorphic to a pullback of p, this subcategory is closed under pullbacks along arbitrary maps:

3.3 Lemma. Let $p: Q \to X$ be an open map of involutive quantales, and let $f: Y \to X$ be an arbitrary map of involutive quantales. Then the pullback $f^*(p)$ is an open map.

Note that it is not implied that open maps of locales (in the usual sense) are necessarily open when regarded as maps in QSp.

The second condition, FR2, will be referred to as the *two-sided Frobenius reciprocity* condition. It is not a generalization of the Frobenius reciprocity condition of locales, for in general it is not satisfied by open maps of locales, as the following shows:

3.4 Lemma. Let $p: L \to X$ be a semiopen map of locales that satisfies FR2. Then for all $a, b \in L$ we have $p_!(a \land b) = p_!(a) \land p_!(b)$.

Proof. Let $a, b \in L$. Then $p_!(a \wedge b) = p_!(a \wedge 1_L \wedge b) = p_!(a \wedge p^*(1_X) \wedge b) = p_!(a) \wedge 1_X \wedge p_!(b) = p_!(a) \wedge p_!(b)$.

3.5 Example. Let $f: X \to Y$ be a continuous open map of topological spaces, with X Hausdorff. The map of locales $\Omega(f): \Omega(X) \to \Omega(Y)$ defined by $\Omega(f)^* = f^{-1}$ satisfies FR2 if and only if f is injective.

This example also shows that in general FR1 does not imply FR2, whereas in some situations (in particular those coming from Fell bundles — see below) FR2 implies FR1:

3.6 Lemma. Let $p : Q \to X$ be a semiopen map of involutive quantales satisfying FR2. If X is unital and $p_!(p^*(e)) = e$ then p is surjective and it satisfies FR1.

Proof. Let e be the multiplicative unit of X. For all $a \in Q$ and $x \in X$ we have, using FR2,

$$p_!(ap^*(x)) = p_!(ap^*(xe)) = p_!(ap^*(x)p^*(e))$$

= $p_!(a)xp_!(p^*(e)) = p_!(a)xe = p_!(a)x$,

and thus FR1 holds. Surjectivity of p follows from Lemma 3.2.

As we shall see in section 5, the conjunction of FR1 and FR2 is interesting, at least in the case of surjective maps, because the class of semiopen surjections satisfying FR1 and FR2 is closed under pullbacks. In particular, it follows that such surjections are examples of open maps according to the definition above.

Fell bundles. A Fell bundle $\pi : E \to G$ on a (suitable) topological étale groupoid G is a Banach bundle on G equipped with additional structure such that, among other things, E is an involutive semicategory and p is functorial [12]. Associated to a Fell bundle $\pi : E \to G$ there is a convolution algebra of sections $C_c(G, E)$ (this generalizes the usual convolution algebra $C_c(G)$ of continuous compactly supported functions $G \to \mathbb{C}$) and for a large class of C*-completions A of $C_c(G, E)$ we obtain maps of involutive quantales $p : \operatorname{Max} A \to \Omega(G)$, where $\operatorname{Max} A$ and $\Omega(G)$ are the involutive quantales associated to A and G, respectively (Max A consists of all the norm-closed linear subspaces of A with multiplication given by the topological closure of the linear span of the pointwise multiplication, and $\Omega(G)$ is the topology of G under pointwise multiplication). The properties of p are closely related to properties of π and A. In particular, the situations where p is semiopen are closely related to G being Hausdorff and A being the reduced C*-algebra $C_r^*(G, E)$. Further imposing on p conditions that approach FR2 has the effect of restricting the bundle to be a line bundle, or even force G to be a principal groupoid (an equivalence relation). See [26].

3.7 Example. As a simple illustration, take G to be the discrete pair groupoid (= total binary relation) on the set $\{1, \ldots, n\}$, and let $\pi := \pi_1 : G \times \mathbb{C} \to G$. The convolution algebra of π can be identified with the matrix algebra $A = M_n(\mathbb{C})$, and the quantale $\Omega(G)$ is the quantale of binary relations on $\{1, \ldots, n\}$. For each binary relation $U \in \Omega(G)$ let $p^*(U)$ be the set of matrices M such that $m_{ij} = 0$ whenever $(i, j) \notin U$. Then $p^*(U) \in \text{Max } A$. The mapping $p^* : \Omega(G) \to \text{Max } A$ is an injective homomorphism of involutive quantales, and it has a left adjoint p_1 which to each linear subspace $V \subseteq A$ assigns the relation

$$\{(i,j) \in G \mid m_{ij} \neq 0 \text{ for some } M \in V\}$$
.

The semiopen map p thus defined is a surjection in QSp. It satisfies FR2, and therefore also FR1 because $\Omega(G)$ is unital.

3.8 Example. Let again $\pi = \pi_1 : G \times \mathbb{C} \to G$, now for G a non-trivial finite discrete group. Then the convolution algebra A can be identified with the group algebra $\mathbb{C}G$,

and again we obtain a surjective semiopen map $p : \text{Max } A \to \Omega(G)$, such that $p^*(U) = \mathbb{C}U$ for all $U \subseteq G$ and

 $p_!(V) = \{g \in G \mid v_g \neq 0 \text{ for some } v \in V\}$

for all $V \in Max A$. Now p satisfies FR1 but not FR2.

4 Pullbacks of quantic spaces

Given two sup-lattices L and M we write $L \otimes M$ for their tensor product, as in [9], and $L \oplus M$ (= $L \times M$) for their coproduct, which we shall refer to as the *direct sum* of L and M. Similarly, $\bigoplus_i L_i$ is the coproduct of a family (L_i) of sup-lattices.

Products. Let Y and Q be involutive quantales. The product Y * Q in QSp can be constructed concretely as being the following sup-lattice:

$$Y * Q := \bigoplus_{n=1}^{\infty} T_n$$

$$\cong Y \oplus Q \oplus (Y \otimes Q) \oplus (Q \otimes Y) \oplus (Y \otimes Q \otimes Y) \oplus (Q \otimes Y \otimes Q) \oplus \cdots,$$

where

$$T_{4k+1} = Y \otimes (Q \otimes Y)^{\otimes k}$$

$$T_{4k+2} = Q \otimes (Y \otimes Q)^{\otimes k}$$

$$T_{4k+3} = (Y \otimes Q)^{\otimes (k+1)}$$

$$T_{4k+4} = (Q \otimes Y)^{\otimes (k+1)}$$

for all $k \in \{0, 1, 2, ...\}$ and we use the notation $Z^{\otimes k}$ for the tensor product $Z \otimes \cdots \otimes Z$ of k copies of Z, with $Z^{\otimes 0}$ being $\Omega = \mathcal{P}(\{*\})$. Hence,

$$(Y * Q) \otimes (Y * Q) \cong \bigoplus_{m,n=1}^{\infty} T_m \otimes T_n .$$

The quantale multiplication on Y * Q is given by a sup-lattice homomorphism

$$(Y * Q) \otimes (Y * Q) \rightarrow Y * Q$$
,

which, due to the universal property of the sup-lattice coproduct, is equivalent to specifying, for each m and n, a sup-lattice homomorphism

$$\mu_{m,n}: T_m \otimes T_n \to Y * Q .$$

Succintly, the homomorphisms $\mu_{m,n}$ are computed using the following rules, for all $y, y' \in Y$ and $a, a' \in Q$:

$$(\dots \otimes y)(a \otimes \dots) = \dots \otimes y \otimes a \otimes \dots$$
$$(\dots \otimes a)(y \otimes \dots) = \dots \otimes a \otimes y \otimes \dots$$
$$(\dots \otimes y)(y' \otimes \dots) = \dots \otimes yy' \otimes \dots$$
$$(\dots \otimes a)(a' \otimes \dots) = \dots \otimes aa' \otimes \dots$$

For instance,

$$\mu_{5,6}: (Y \otimes (Q \otimes Y)) \otimes (Q \otimes (Y \otimes Q)) \to Y * Q$$

is given by

$$(y \otimes (a \otimes y')) \otimes (b \otimes (z \otimes b')) \mapsto (y \otimes a) \otimes (y' \otimes b) \otimes (z \otimes b') ,$$

which means $\mu_{5,6}$ is an isomorphism

$$T_5 \otimes T_6 \stackrel{\cong}{\to} T_{11}$$

composed with the inclusion $T_{11} \to Y * Q$. Another example, now one that uses the multiplication of Y, is

$$\mu_{5,5}: (Y \otimes (Q \otimes Y)) \otimes (Y \otimes (Q \otimes Y)) \to Y * Q ,$$

which is given by

$$(y \otimes (a \otimes y')) \otimes (z \otimes (b \otimes z')) \mapsto y \otimes (a \otimes y'z) \otimes (b \otimes z')$$
,

so $\mu_{5,5}$ is a homomorphism

$$T_5 \otimes T_5 \to T_9$$

following by the inclusion $T_9 \to Y * Q$. We note that this homomorphism is well defined because the distributivity of the quantale product of Y ensures that $\mu_{5,5}$, regarded as a multilinear map, preserves joins in each variable separately. A complete specification of the multiplication involves sixteen different cases, corresponding to the four types of direct summands in the construction of Y * Q, and we omit it. We note that, similarly to the two examples just seen, for each m and n there is k such that the homomorphism $\mu_{m,n}$ is the composition of a homomorphism $T_m \otimes T_n \to T_k$ with the inclusion $T_k \to Y * Q$.

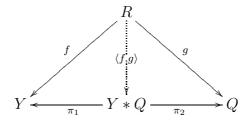
Due to the universal property of the tensor product the multiplication thus obtained on Y * Q preserves joins in each variable. It is also associative due to the associativity of both Y and Q, and due to the fact that if a product of pure tensors $\boldsymbol{\tau} \otimes (\boldsymbol{\tau}' \otimes \boldsymbol{\tau}'')$ is valued in T_n then so is $(\boldsymbol{\tau} \otimes \boldsymbol{\tau}') \otimes \boldsymbol{\tau}''$, and thus no bracketing mismatches occur. Therefore Y * Q is a quantale. It is also an involutive quantale, with the involution $Y * Q \to Y * Q$ defined again separately for each T_n by the generic rule

$$(\cdots a \otimes y \otimes b \otimes z \cdots)^* \quad = \quad \cdots z^* \otimes b^* \otimes y^* \otimes a^* \cdots .$$

The embeddings $Y = T_1 \rightarrow Y * Q$ and $Q = T_2 \rightarrow Y * Q$ are therefore homomorphisms of involutive quantales, and they provide the projections

$$Y \xleftarrow{\pi_1} Y * Q \xrightarrow{\pi_2} Q$$

of the product in QS. The pairing $\langle f, g \rangle$ of two maps f and g

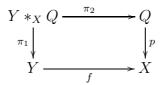


is again defined separately on each T_n ; that is, its inverse image $\langle f, g \rangle^*$ is obtained by linear extension from the assignments

$$\cdots Y \otimes Q \otimes Y \otimes Q \cdots \xrightarrow{\langle f,g \rangle^* = [f^*,g^*]} R$$
$$\cdots y \otimes a \otimes y' \otimes a' \cdots \longmapsto \cdots f^*(y)g^*(a)f^*(y')g^*(a') \cdots$$

It is straightforward to verify that $\langle f, g \rangle$ is a map of involutive quantales, and that it is the unique map making the above diagram commute, so Y * Q is a product in QS as intended.

Pullbacks. Let the following be a pullback diagram in *QSp*:



Of course, $Y *_X Q$ equals $(Y * Q)_j$ where j is the least quantic nucleus on Y * Q such that $j(p^*(x)) = j(f^*(x))$ for all $x \in X$ (for notational convenience we shall usually identify $f^*(x)$ with $\pi_1^*(f^*(x))$ and $p^*(x)$ with $\pi_2^*(p^*(x))$). By Lemma 2.1, taking \widetilde{R} to be the closure of R under involution and multiplication by elements of Y * Q, we obtain

$$Y *_X Q = \{ \alpha \in Y * Q \mid \forall_{(r,s) \in \widetilde{R}} \ r \le \alpha \Leftrightarrow s \le \alpha \} .$$

More explicitly, and taking into account that in this case R is already closed under involution, $Y*_XQ$ consists of those $\alpha \in Y*Q$ such that for all $x \in X$ and all $z, w \in Y*Q$ the following conditions hold:

$$p^*(x) \le \alpha \quad \Longleftrightarrow \quad f^*(x) \le \alpha$$

$$(4.1)$$

$$zp^*(x) \le \alpha \quad \Longleftrightarrow \quad zf^*(x) \le \alpha$$

$$(4.2)$$

$$p^*(x)w \le \alpha \quad \Longleftrightarrow \quad f^*(x)w \le \alpha$$

$$(4.3)$$

 $zp^*(x)w \le \alpha \quad \Longleftrightarrow \quad zf^*(x)w \le \alpha$ (4.4)

Moreover, it suffices to take z and w to be pure tensors

$$\cdots y \otimes a \otimes y' \otimes a' \cdots$$

and thus $Y *_X Q$ consists of those $\alpha \in Y * Q$ that satisfy the following nine types of conditions, for all $a, a' \in Q$ and all $y, y' \in Y$, and all pure tensors $\tau, \tau' \in Y * Q$ that are appropriate in the sense that they yield \otimes -strings $\tau \otimes a, y \otimes \tau'$, etc., that alternate the elements of Q and Y:

$$p^*(x) \le \alpha \iff f^*(x) \le \alpha$$
 (4.5)

$$f^*(x)a \otimes \boldsymbol{\tau} \leq \alpha \quad \Longleftrightarrow \quad f^*(x) \otimes a \otimes \boldsymbol{\tau} \leq \alpha$$

$$(4.6)$$

$$p^{*}(x)a \otimes \boldsymbol{\tau} \leq \alpha \iff f^{*}(x) \otimes a \otimes \boldsymbol{\tau} \leq \alpha$$

$$p^{*}(x) \otimes y \otimes \boldsymbol{\tau} \leq \alpha \iff f^{*}(x)y \otimes \boldsymbol{\tau} \leq \alpha$$

$$\boldsymbol{\tau} \otimes ap^{*}(x) \leq \alpha \iff \boldsymbol{\tau} \otimes a \otimes f^{*}(x) \leq \alpha$$

$$(4.6)$$

$$(4.7)$$

$$\boldsymbol{\tau} \otimes ap^{*}(x) \leq \alpha \iff \boldsymbol{\tau} \otimes a \otimes f^{*}(x) \leq \alpha$$

$$(4.8)$$

$$\tau \otimes ap^*(x) \le \alpha \quad \Longleftrightarrow \quad \tau \otimes a \otimes f^*(x) \le \alpha \tag{4.8}$$

$$\boldsymbol{\tau} \otimes \boldsymbol{y} \otimes \boldsymbol{p}^*(\boldsymbol{x}) \le \alpha \quad \Longleftrightarrow \quad \boldsymbol{\tau} \otimes \boldsymbol{y} \boldsymbol{f}^*(\boldsymbol{x}) \le \alpha \tag{4.9}$$

$$\boldsymbol{\tau} \otimes ap^*(x)a' \otimes \boldsymbol{\tau}' \leq \alpha \quad \Longleftrightarrow \quad \boldsymbol{\tau} \otimes a \otimes f^*(x) \otimes a' \otimes \boldsymbol{\tau}' \leq \alpha$$
 (4.10)

$$\begin{split} \boldsymbol{\tau} &\otimes y \otimes p^*(x) a \otimes \boldsymbol{\tau}' &\leq \alpha \quad \Longleftrightarrow \quad \boldsymbol{\tau} \otimes y f^*(x) \otimes a \otimes \boldsymbol{\tau}' &\leq \alpha \\ \boldsymbol{\tau} &\otimes a p^*(x) \otimes y \otimes \boldsymbol{\tau}' &\leq \alpha \quad \Longleftrightarrow \quad \boldsymbol{\tau} \otimes a \otimes f^*(x) y \otimes \boldsymbol{\tau}' &\leq \alpha \end{split}$$
(4.11)

$$\otimes ap^{*}(x) \otimes y \otimes \tau' \leq \alpha \quad \Longleftrightarrow \quad \tau \otimes a \otimes f^{*}(x)y \otimes \tau' \leq \alpha$$

$$(4.12)$$

$$\boldsymbol{\tau} \otimes y \otimes p^*(x) \otimes y' \otimes \boldsymbol{\tau}' \leq \alpha \quad \Longleftrightarrow \quad \boldsymbol{\tau} \otimes y f^*(x) y' \otimes \boldsymbol{\tau}' \leq \alpha$$

$$(4.13)$$

Stability under pullbacks 5

5.1 Theorem. Let $p: Q \to X$ be a semiopen surjective map of involutive quantales satisfying both FR1 and FR2, and let the following be a pullback diagram in QSp:

$$\begin{array}{ccc} Y \ast_X Q & \xrightarrow{\pi_2} & Q \\ & & & & \\ \pi_1 & & & & \\ & & & & \\ Y & \xrightarrow{f} & X \end{array} \tag{5.1}$$

Then π_1 is a semiopen surjection satisfying both FR1 and FR2, and the following diagram in the category of sup-lattices is commutative (Beck-Chevalley condition):

Proof. Consider the pullback diagram (5.1) with p a semiopen surjection satisfying FR1 and FR2. In order to show that π_1 is semiopen we begin by defining a sup-lattice homomorphism $Y *_X Q \to Y$ which will then be shown to be the required direct image homomorphism of π_1 . First let us recall the following notation from the definition of the product in QSp, where $k \in \{0, 1, 2, ...\}$:

$$Y * Q = \bigoplus_{n=1}^{\infty} T_n$$

$$T_{4k+1} = Y \otimes (Q \otimes Y)^{\otimes k}$$

$$T_{4k+2} = Q \otimes (Y \otimes Q)^{\otimes k}$$

$$T_{4k+3} = (Y \otimes Q)^{\otimes (k+1)}$$

$$T_{4k+4} = (Q \otimes Y)^{\otimes (k+1)}.$$

For each *n* define a sup-lattice homomorphism $h_n: T_n \to Y$:

$$h_1(y) = y \tag{5.3}$$

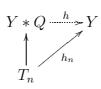
$$h_2(a) = f^*(p_!(a))$$
(5.4)
$$h_2(a) = h^*(p_!(a))$$
(5.5)

$$h_3(y \otimes a) = y f^*(p_!(a))$$
 (5.5)

$$h_4(a \otimes y) = f^*(p_!(a))y \tag{5.6}$$

$$h_n(\cdots y \otimes a \otimes y' \otimes a' \cdots) = \cdots y f^*(p_!(a)) y' f^*(p_!(a')) \cdots$$
(5.7)

$$h = [h_n] : Y * Q \to Y : \tag{5.8}$$



We prove that h factors through the surjection $q: Y * Q \to Y *_X Q$ by showing that it satisfies the following nine conditions [cf. (4.5)–(4.13)] for all $a, a' \in Q$ and all $y, y' \in Y$, and all appropriate alternated pure tensors $\tau, \tau' \in Y * Q$ as in (4.5)–(4.13):

$$h(p^*(x)) = h(f^*(x))$$
 (5.9)

$$h(p^*(x)a \otimes \boldsymbol{\tau}) = h(f^*(x) \otimes a \otimes \boldsymbol{\tau})$$
(5.10)

$$h(p^*(x) \otimes y \otimes \boldsymbol{\tau}) = h(f^*(x)y \otimes \boldsymbol{\tau})$$
(5.11)

$$h(\boldsymbol{\tau} \otimes ap^*(x)) = h(\boldsymbol{\tau} \otimes a \otimes f^*(x))$$
(5.12)

$$h(\boldsymbol{\tau} \otimes \boldsymbol{y} \otimes \boldsymbol{p}^*(\boldsymbol{x})) = h(\boldsymbol{\tau} \otimes \boldsymbol{y} f^*(\boldsymbol{x}))$$
(5.13)

$$h(\boldsymbol{\tau} \otimes ap^*(x)a' \otimes \boldsymbol{\tau}') = h(\boldsymbol{\tau} \otimes a \otimes f^*(x) \otimes a' \otimes \boldsymbol{\tau}')$$
(5.14)

$$h(\boldsymbol{\tau} \otimes y \otimes p^*(x)a \otimes \boldsymbol{\tau}') = h(\boldsymbol{\tau} \otimes yf^*(x) \otimes a \otimes \boldsymbol{\tau}')$$
(5.15)

$$h(\boldsymbol{\tau} \otimes ap^*(x) \otimes y \otimes \boldsymbol{\tau}') = h(\boldsymbol{\tau} \otimes a \otimes f^*(x)y \otimes \boldsymbol{\tau}')$$
(5.16)

$$h(\boldsymbol{\tau} \otimes y \otimes p^*(x) \otimes y' \otimes \boldsymbol{\tau}') = h(\boldsymbol{\tau} \otimes y f^*(x) y' \otimes \boldsymbol{\tau}').$$
(5.17)

Condition (5.9) is a consequence of surjectivity:

$$h(p^*(x)) = h_2(p^*(x)) = f^*(p_!(p^*(x))) = f^*(x) = h_1(f^*(x)) = h(f^*(x))$$

Similarly, (5.11), (5.13) and (5.17) follow from surjectivity: for instance, for (5.17) we obtain, applying (5.3)–(5.8),

$$\begin{aligned} h(\boldsymbol{\tau} \otimes y \otimes p^*(x) \otimes y' \otimes \boldsymbol{\tau}') &= h(\boldsymbol{\tau})yf^*(p_!(p^*(x)))y'h(\boldsymbol{\tau}') = h(\boldsymbol{\tau})yf^*(x)y'h(\boldsymbol{\tau}') \\ &= h(\boldsymbol{\tau} \otimes yf^*(x)y' \otimes \boldsymbol{\tau}') \;. \end{aligned}$$

Again applying (5.3)–(5.8), conditions (5.10), (5.12), (5.15) and (5.16) follow from FR1: for instance, for (5.10) we have

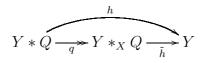
$$h(p^*(x)a \otimes \boldsymbol{\tau}) = f^*(p_!(p^*(x)a))h(\boldsymbol{\tau}) = f^*(xp_!(a))h(\boldsymbol{\tau})$$

= $f^*(x)f^*(p_!(a))h(\boldsymbol{\tau}) = h(f^*(x) \otimes a \otimes \boldsymbol{\tau}).$

Finally, still applying (5.3)–(5.8), (5.14) follows from FR2:

$$\begin{aligned} h(\boldsymbol{\tau} \otimes ap^*(x)a' \otimes \boldsymbol{\tau}') &= h(\boldsymbol{\tau})f^*(p_!(ap^*(x)a'))h(\boldsymbol{\tau}') = h(\boldsymbol{\tau})f^*(p_!(a)xp_!(a'))h(\boldsymbol{\tau}') \\ &= h(\boldsymbol{\tau})f^*(p_!(a))f^*(x)f^*(p_!(a'))h(\boldsymbol{\tau}') \\ &= h(\boldsymbol{\tau} \otimes a \otimes f^*(x) \otimes a' \otimes \boldsymbol{\tau}') . \end{aligned}$$

Since h respects all the conditions (5.9)–(5.17) it factors through $Y *_X Q$ via a suplattice homomorphism \tilde{h} :



Now we show that \tilde{h} is left adjoint to π_1^* . The counit of the adjunction is immediate, since $\tilde{h}(\pi_1^*(y)) = y$ for all $y \in Y$. This also shows that π_1 is a surjection. In order to prove the unit of the adjunction let us use abbreviations such as

$$\cdots a \,\hat{\otimes}\, y \,\hat{\otimes}\, a' \,\hat{\otimes}\, y' \cdots \quad := \quad q(\cdots a \otimes y \otimes a' \otimes y' \cdots) \,,$$

and let us consider a "word"

$$w := a_1 \,\hat{\otimes}\, y_1 \,\hat{\otimes} \cdots \,\hat{\otimes}\, a_n \,\hat{\otimes}\, y_n \quad \in \quad q(T_{4n}) \quad \subseteq \quad Y *_X Q \;.$$

We have

$$w = a_1 \otimes y_1 \otimes \cdots \otimes a_n \otimes y_n$$

$$\leq p^*(p_!(a_1)) \otimes y_1 \otimes \cdots \otimes p^*(p_!(a_n)) \otimes y_n$$

$$= \pi_1^* \underbrace{\left(f^*(p_!(a_1))y_1 \cdots f^*(p_!(a_n))y_n\right)}_{\in T_1 = Y}$$

$$= \pi_1^* \Big(h(a_1 \otimes y_1 \otimes \cdots \otimes a_n \otimes y_n)\Big)$$

$$= \pi_1^*(\tilde{h}(w)).$$

Similar reasoning applies to any $z_1 \otimes \cdots \otimes z_m$ with the $z'_i s$ taken alternately from Q and Y, so we conclude that

$$\alpha \le \pi_1^*(h(\alpha))$$

for all $\alpha \in Y *_X Q$, thus showing that \tilde{h} is left adjoint to π_1^* , so π_1 is semiopen. It is also clear that \tilde{h} satisfies FR1 because for all $a, a' \in Q$ and $y, y' \in Y$, with τ being an image by q of an appropriate pure tensor in Y * Q, we have

$$\begin{split} \tilde{h}(\underbrace{y\pi_1^*(y')}_{\in T_1=Y}) &= \tilde{h}(yy') \\ &= \tilde{h}(y)y' , \\ \tilde{h}((\boldsymbol{\tau} \otimes y)\pi_1^*(y')) &= \tilde{h}(\boldsymbol{\tau} \otimes yy') \\ &= \tilde{h}(\boldsymbol{\tau})yy' \\ &= \tilde{h}(\boldsymbol{\tau} \otimes y)y' , \end{split}$$

and

$$\begin{split} \tilde{h}\big(a\pi_1^*(y)\big) &= \tilde{h}(a\,\hat{\otimes}\,y) \\ &= f^*(p_!(a))y \\ &= \tilde{h}(a)y \;, \\ \tilde{h}\big((\boldsymbol{\tau}\,\hat{\otimes}\,a)\pi_1^*(y)\big) &= \tilde{h}(\boldsymbol{\tau}\,\hat{\otimes}\,a\,\hat{\otimes}\,y) \\ &= \tilde{h}(\boldsymbol{\tau})f^*(p_!(a))y \end{split}$$

FR2 is proved in a similar way, now computing
$$\tilde{h}([\boldsymbol{\tau} \otimes] z \pi_1^*(y) z'[\otimes \boldsymbol{\tau}'])$$
 for a total of sixteen combinations with $z, z' \in Q \cup Y$. To conclude, the commutativity of the diagram (5.2) is just the statement that $\tilde{h}(a) = f^*(p_!(a))$ for all $a \in Q$.

 $= \tilde{h}(\boldsymbol{\tau} \, \hat{\otimes} \, a) y$.

5.2 Corollary. Any weakly open surjection satisfying FR2 is an open map.

References

- F. Borceux, J. Rosický, and G. Van den Bossche, *Quantales and C*-algebras*, J. London Math. Soc. (2) 40 (1989), no. 3, 398–404. MR1053610 (91d:46075)
- [2] M. R. Buneci, *Groupoid categories*, Perspectives in operator algebras and mathematical physics, Theta Ser. Adv. Math., vol. 8, Theta, Bucharest, 2008, pp. 27–40. MR2433025 (2010b:22007)
- [3] M. R. Buneci and P. Stachura, Morphisms of locally compact groupoids endowed with Haar systems (2005), available at arXiv:math.OA/0511613v1.
- M. Bunge, An application of descent to a classification theorem for toposes, Math. Proc. Cambridge Philos. Soc. 107 (1990), no. 1, 59–79, DOI 10.1017/S0305004100068365. MR1021873 (90k:18002)

- [5] S. Henry, Des topos à la géométrie non commutative par l'étude des espaces de Hilbert internes, Thèse de Doctorat, École Doctorale de Science Mathématiques de Paris Centre, September 25, 2014.
- [6] H. Heymans, Sheaves on involutive quantales: Grothendieck quantales, Fuzzy Sets and Systems 256 (2014), 117–148, DOI 10.1016/j.fss.2013.07.008. MR3263495
- H. Heymans and I. Stubbe, Modules on involutive quantales: canonical Hilbert structure, applications to sheaf theory, Order 26 (2009), no. 2, 177–196, DOI 10.1007/s11083-009-9116-x. MR25255366
- [8] P. T. Johnstone, Stone Spaces, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition. MR861951 (87m:54001)
- [9] A. Joyal and M. Tierney, An extension of the Galois theory of Grothendieck, Mem. Amer. Math. Soc. 51 (1984), no. 309, vii+71. MR756176 (86d:18002)
- [10] D. Kruml, Spatial quantales, Appl. Categ. Structures 10 (2002), no. 1, 49–62. MR1883084 (2002m:06010)
- [11] D. Kruml and P. Resende, On quantales that classify C^{*}-algebras, Cah. Topol. Géom. Différ. Catég. 45 (2004), no. 4, 287–296 (English, with French summary). MR2108195 (2006b:46096)
- [12] A. Kumjian, Fell bundles over groupoids, Proc. Amer. Math. Soc. 126 (1998), no. 4, 1115–1125, DOI 10.1090/S0002-9939-98-04240-3. MR1443836 (98i:46055)
- N. P. Landsman, Operator algebras and Poisson manifolds associated to groupoids, Comm. Math. Phys. 222 (2001), no. 1, 97–116, DOI 10.1007/s002200100496. MR1853865 (2002f:46142)
- [14] I. Moerdijk, The classifying topos of a continuous groupoid. II, Cahiers Topologie Géom. Différentielle Catég. 31 (1990), no. 2, 137–168 (English, with French summary). MR1080241 (92c:18003)
- [15] _____, Toposes and groupoids, Categorical algebra and its applications (Louvain-La-Neuve, 1987), Lecture Notes in Math., vol. 1348, Springer, Berlin, 1988, pp. 280–298, DOI 10.1007/BFb0081366. MR975977 (89m:18003)
- [16] J. Mrčun, Functoriality of the bimodule associated to a Hilsum-Skandalis map, K-Theory 18 (1999), no. 3, 235–253, DOI 10.1023/A:1007773511327. MR1722796 (2001k:22004)
- [17] P. S. Muhly, J. N. Renault, and D. P. Williams, Equivalence and isomorphism for groupoid C^{*}-algebras, J. Operator Theory 17 (1987), no. 1, 3–22. MR873460 (88h:46123)
- [18] C. J. Mulvey and J. W. Pelletier, On the quantisation of points, J. Pure Appl. Algebra 159 (2001), no. 2-3, 231–295. MR1828940 (2002g:46126)
- [19] _____, On the quantisation of spaces, J. Pure Appl. Algebra **175** (2002), no. 1-3, 289–325. Special volume celebrating the 70th birthday of Professor Max Kelly. MR1935983 (2003m:06014)
- [20] C. J. Mulvey and P. Resende, A noncommutative theory of Penrose tilings, Internat. J. Theoret. Phys. 44 (2005), no. 6, 655–689. MR2150184 (2006a:58011)
- [21] M. C. Protin and P. Resende, Quantales of open groupoids, J. Noncommut. Geom. 6 (2012), no. 2, 199–247, DOI 10.4171/JNCG/90. MR2914865
- [22] P. Resende and S. Vickers, *Localic sup-lattices and tropological systems*, Theoret. Comput. Sci. **305** (2003), no. 1-3, 311–346, DOI 10.1016/S0304-3975(02)00702-8. Topology in computer science (Schloß Dagstuhl, 2000). MR2013577 (2004i:68130)
- [23] P. Resende, Étale groupoids and their quantales, Adv. Math. 208 (2007), no. 1, 147–209. MR2304314 (2008c:22002)

- [24] _____, Groupoid sheaves as quantale sheaves, J. Pure Appl. Algebra 216 (2012), no. 1, 41–70, DOI 10.1016/j.jpaa.2011.05.002. MR2826418
- [25] _____, Functoriality of groupoid quantales. I, J. Pure Appl. Algebra 219 (2015), no. 8, 3089– 3109, DOI 10.1016/j.jpaa.2014.10.004. MR3320209
- [26] _____, Quantales and Fell bundles, 2017, arXiv:1701.08653v2.
- [27] K. I. Rosenthal, Quantales and Their Applications, Pitman Research Notes in Mathematics Series, vol. 234, Longman Scientific & Technical, Harlow, 1990. MR1088258 (92e:06028)
- [28] J. Rosický, Multiplicative lattices and C*-algebras, Cahiers Topologie Géom. Différentielle Catég.
 30 (1989), no. 2, 95–110 (English, with French summary). MR1004734 (91e:46079)

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