

FROM OBSERVABLES AND STATES TO HILBERT SPACE AND BACK: A 2-CATEGORICAL ADJUNCTION

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ABSTRACT. Given a representation of a C^* -algebra, thought of as an abstract collection of physical observables, together with a unit vector, one obtains a state on the algebra via restriction. We show that the Gelfand-Naimark-Segal (GNS) construction furnishes a left adjoint of this restriction. To properly formulate this adjoint, it must be viewed as a weak natural transformation, a 1-morphism in a suitable 2-category, rather than as a functor between categories. Weak naturality encodes the functoriality and the universal property of adjunctions encodes the characterizing features of the GNS construction. Mathematical definitions and results are accompanied by physical interpretations.

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1. INTRODUCTION AND OUTLINE

There is a familiar construction whose input consists of a representation of a C^* -algebra on a Hilbert space together with a unit vector and whose output is a state on the C^* -algebra via restriction. Namely, given an algebra \mathcal{A} , a representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ to bounded operators on a Hilbert space \mathcal{H} , and a unit vector $\psi \in \mathcal{H}$, one obtains a state on \mathcal{A} given by the expectation values of observables in \mathcal{A} sending $a \in \mathcal{A}$ to $\langle \psi, \pi(a)\psi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathcal{H} . We show that this construction, denoted

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by **rest**, can be expressed categorically as a natural transformation

$$(1.1) \quad \begin{array}{ccc} & \text{States} & \\ \text{C}^*\text{-Alg}^{\text{op}} & \begin{array}{c} \xrightarrow{\quad} \\ \uparrow \text{rest} \\ \xleftarrow{\quad} \end{array} & \text{Cat} \\ & \text{Rep}^\bullet & \end{array}$$

Here, **Cat** is the category of categories, **C^{*}-Alg** is the category of *C^{*}*-algebras, **States** is the functor that associates a category of states to every *C^{*}*-algebra, and **Rep[•]** is the functor that associates the category of representations of *C^{*}*-algebras (the **•** is to denote the additional choice of a unit vector).

Our purpose here is to prove that the natural transformation **rest** has a left adjoint

$$(1.2) \quad \begin{array}{ccc} & \text{States} & \\ \text{C}^*\text{-Alg}^{\text{op}} & \begin{array}{c} \xrightarrow{\quad} \\ \uparrow \text{rest} \\ \downarrow \text{GNS}^\bullet \end{array} & \text{Cat} \\ & \text{Rep}^\bullet & \end{array}$$

denoted by **GNS[•]** because its ingredients are composed of constructions due to Gelfand, Naimark, and Segal [1], [2], which we review. We therefore call **GNS[•]** the *GNS construction*. This allows us to view the GNS construction as a morphism/process in some appropriate category rendering it accessible to the techniques and tools of category theory. Although not quite a functor, this provides a precise sense in which the GNS construction is functorial with respect to ***-homomorphisms of *C^{*}*-algebras. Furthermore, the key properties of the GNS construction are shown to be naturally described in terms of categorical concepts characterizing it as a left adjoint to the restriction map from representations to states. By the essential uniqueness of adjoints, this offers a *definition* of the GNS construction so that one can view the standard GNS construction as exhibiting the *existence* of such an adjoint.

However, there are subtleties in this description. First, the GNS construction is a certain 2-categorical natural transformation (utilizing the fact that **Cat** is a 2-category) instead of a natural transformation in the usual sense of ordinary category theory [3]. Second, the category of states is not the naive one that one might think of—one must view the states of a fixed *C^{*}*-algebra as a discrete category to obtain an appropriate functorial description. Third, for a robust statement with physical applications, the morphisms in the representation category associated to a *C^{*}*-algebra must include all intertwiners that are isometries and not only the unitary equivalences.

The outline of this paper is as follows. Section 2 defines all relevant notions from *C^{*}*-algebras as well as the states functor and the representation functor. Section 3 describes the GNS construction as is usually found in the literature but framed in a categorical setting. For simplicity, we ignore the cyclic vector and focus only on the fact that the GNS construction produces a representation. In particular, we prove that the GNS construction is an oplax-natural transformation (though not a natural transformation) in

Theorem 3.26. Section 4 explains why the category of states (introduced in Section 2) must have no non-trivial morphisms for our purposes. Section 5 properly accounts for the fact that the GNS construction produces a *cyclic* representation. The statement that the GNS construction is left adjoint to the restriction to states natural transformation is proved in Theorem 5.64. Definition 5.70 gives a categorical definition for the GNS construction motivated by our results. A summary of our results characterizing the GNS construction is given after this definition. In Section 6, we illustrate several of the constructions and results in terms of a simple example of a bipartite system familiar (to physicists) from the Einstein-Podolsky-Rosen (EPR) setup. Throughout, we provide physical interpretations of most definitions, constructions, and results, though some interpretations are heuristic rather than rigorous. Although we assume the reader is familiar with some basics of category theory [3], we include a short appendix on 2-categories and 2-categorical adjunctions. Otherwise, we aim to be mostly self-contained.

2. STATES AND REPRESENTATIONS OF C^* -ALGEBRAS

For more details on C^* -algebras, the reader is referred to [4] and [5].

Definition 2.1. A unital Banach algebra is a vector space \mathcal{A} together with

- i) a binary multiplication operation $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$,
- ii) a norm $\|\cdot\| : \mathcal{A} \longrightarrow \mathbb{R}_{\geq 0}$, and
- iii) an element $1_{\mathcal{A}} \in \mathcal{A}$.

The multiplication must be distributive over vector addition, the scalar multiplication must satisfy $k(ab) = (ka)b = a(kb)$ for all $k \in \mathbb{C}$ and $a, b \in \mathcal{A}$, the element $1_{\mathcal{A}}$ must satisfy $a1_{\mathcal{A}} = 1_{\mathcal{A}}a = a$ for all $a \in \mathcal{A}$, and finally, all Cauchy sequences must converge.

Definition 2.2. A unital C^* -algebra is a unital Banach algebra \mathcal{A} with an involution $*$: $\mathcal{A} \longrightarrow \mathcal{A}$ that is an anti-homomorphism for the multiplication and satisfies $\|aa^*\| = \|a\|^2$ for all $a \in \mathcal{A}$. An element $a \in \mathcal{A}$ is self-adjoint if $a^* = a$, an isometry if $a^*a = 1_{\mathcal{A}}$, and unitary if $a^*a = 1_{\mathcal{A}} = aa^*$.

Definition 2.3. Let \mathcal{A} and \mathcal{A}' be two unital C^* -algebras. A map/morphism of unital C^* -algebras from \mathcal{A}' to \mathcal{A} is a bounded linear map $f : \mathcal{A}' \longrightarrow \mathcal{A}$ such that $f(a'^*) = f(a')^*$, $f(a'_1a'_2) = f(a'_1)f(a'_2)$, and $f(1_{\mathcal{A}'}) = 1_{\mathcal{A}}$ for all $a', a'_1, a'_2 \in \mathcal{A}'$.

Definition 2.4. Let $\mathbf{C}^*\text{-Alg}$ be the category of unital C^* -algebras. Namely, the objects of $\mathbf{C}^*\text{-Alg}$ are unital C^* -algebras and the morphisms are maps of unital C^* -algebras.

Throughout this article, *all* C^* -algebras and their morphisms will be assumed unital and we will avoid overuse of this adjective unless it is necessary to stress it.

Definition 2.5. Given a C^* -algebra \mathcal{A} , a state on \mathcal{A} is a bounded linear function $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ such that $\omega(1_{\mathcal{A}}) = 1$ and $\omega(a^*a) \geq 0$ for all $a \in \mathcal{A}$. Denote the set of states on a C^* -algebra \mathcal{A} by $\mathcal{S}(\mathcal{A})$.¹

¹ $\mathcal{S}(\mathcal{A})$ is more than just a set—it is a convex set, though this is irrelevant for our present discussion.

Definition 2.6. Let $\mathbf{Rep}(\mathcal{A})$ be the category of representations of the C^* -algebra \mathcal{A} on Hilbert spaces. This means the objects are pairs (π, \mathcal{H}) with \mathcal{H} a Hilbert space and $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a map of C^* -algebras. Here $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on \mathcal{H} —the involution on $\mathcal{B}(\mathcal{H})$ is taking the Hilbert space adjoint. Morphisms $(\pi, \mathcal{H}) \rightarrow (\pi', \mathcal{H}')$ are intertwiners, i.e. bounded linear operators $L : \mathcal{H} \rightarrow \mathcal{H}'$ such that

$$(2.7) \quad L \circ \pi(a) = \pi'(a) \circ L \quad \text{for all } a \in \mathcal{A}.$$

Remark 2.8. It is very important that we assume the morphisms in $\mathbf{Rep}(\mathcal{A})$ are intertwiners and not necessarily just unitary equivalences. This will allow for a wider range of operations that occur in physics as will be explained later.

Physics 2.9. We think of a C^* -algebra \mathcal{A} as the algebra of observables of a physical system.² An example to relate to is the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space \mathcal{H} . However, the main point of this abstract perspective is to place the emphasis on the observables rather than the Hilbert space of vectors or the particular realization of an abstract observable as an operator. For example, we can think of angular momentum being defined in different ways on different Hilbert spaces (or even classically on phase space), but when we think of angular momentum, we do not think of which Hilbert space it acts on—we just think angular momentum!

Furthermore, we do not measure vectors in a Hilbert space. What we measure are expectation values. This is precisely the meaning of a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ as defined above. A state assigns an expectation value to each physical observable. That is what a physical state is: a list of expectation values for all our observables (satisfying reasonable postulates). For instance, if a is self-adjoint, then $\omega(a)$ is the expectation value of a and $\omega(a^2) - (\omega(a))^2$ is the variance. Therefore, the definition of state includes not only expectation values of observables, but also their moments.

Of course, technically thinking of observables as an algebra is an idealization because observables (as described by the working physicist) are not always bounded operators and therefore they do not form an algebra in the strict sense [6]. We will ignore this issue and consider observables that correspond to bounded operators.

The above definitions of $\mathcal{S}(\mathcal{A})$ and $\mathbf{Rep}(\mathcal{A})$ extend to functors.

Proposition 2.10. *The assignment sending a C^* -algebra \mathcal{A} to $\mathcal{S}(\mathcal{A})$ and sending a morphism $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ of C^* -algebras to $\mathcal{S}(\mathcal{A}') \xleftarrow{\mathcal{S}(f)} \mathcal{S}(\mathcal{A})$, where $\mathcal{S}(f)$ is defined by*

$$(2.11) \quad \mathcal{S}(\mathcal{A}) \ni \omega \mapsto \omega \circ f \in \mathcal{S}(\mathcal{A}'),$$

²Actually, \mathcal{A} contains un-observable operators because it contains elements that are not self-adjoint. Examples include creation and annihilation operators. In fact, it contains observables that are self-adjoint but need not be things we can actually measure in a lab (such as momentum to the 8th power). Nevertheless, we call \mathcal{A} the algebra of observables by slight abuse of terminology.

defines a functor³

$$(2.12) \quad \mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}} \xrightarrow{\mathcal{S}} \mathbf{Set},$$

henceforth referred to as the states pre-sheaf.

Proof. First, $\omega \circ f$ is a state on \mathcal{A}' because ω and f are linear,

$$(2.13) \quad \omega(f(1_{\mathcal{A}'})) = \omega(1_{\mathcal{A}}) = 1,$$

and

$$(2.14) \quad \omega(f(a'^*a')) = \omega(f(a')^*f(a')) \geq 0$$

for all $a' \in \mathcal{A}'$. \mathcal{S} is functorial because the identity $\text{id}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ gets sent to the identity and the composition of C^* -algebra maps $\mathcal{A}'' \xrightarrow{f'} \mathcal{A}' \xrightarrow{f} \mathcal{A}$ gets sent to $\mathcal{S}(f \circ f') = \mathcal{S}(f') \circ \mathcal{S}(f)$.⁴ ■

Physics 2.15. The meaning of this functor physically can be understood by considering a special case, which will be used throughout this work. Suppose \mathcal{A}_0 is a subalgebra of physical observables of \mathcal{A} . Let $i : \mathcal{A}_0 \hookrightarrow \mathcal{A}$ be the inclusion map. The functor $\mathcal{S}(i)$ takes a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ that gave expectation values for all observables in \mathcal{A} and it *restricts* that state to only give expectation values for a smaller collection of observables, mathematically described by \mathcal{A}_0 . In thermodynamic or statistical-mechanical terminology, one can imagine \mathcal{A} as describing the algebra of observables for *microstates* and \mathcal{A}_0 as describing the set of observables for some *macrostates*.⁵ In fact, Jaynes used a closely related idea, that is actually more physically reasonable, by assuming that \mathcal{A}_0 is just a subset of \mathcal{A} and develops thermodynamics from it [7]. In this process of restricting to a subalgebra, one therefore loses some information about the state—we only know fewer of its expectation values. Note that focusing on subalgebras in explaining the physics is not too restrictive since, for example, every (unital) C^* -algebra map of finite-dimensional matrix algebras is unitarily equivalent to one of the form

$$(2.16) \quad A \mapsto \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}$$

for all matrices A in the source of the C^* -algebra map [5]. Here, all the empty positions are filled with 0's. Such maps show up whenever there is a local decomposition of a Hilbert space into tensor products and an observer only has access to one of these components (see Section 6 for more details).

³For any category \mathcal{C} , the *opposite category* \mathcal{C}^{op} has the same objects as \mathcal{C} but a morphism from an object a to an object b in \mathcal{C}^{op} is a morphism from b to a in \mathcal{C} . Also, \mathbf{Set} is the category of sets.

⁴The flipping of the order of morphism composition in the equation $\mathcal{S}(f \circ f') = \mathcal{S}(f') \circ \mathcal{S}(f)$ is why we use op in $\mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}}$. This is sometimes referred to as an anti-homomorphism property or contravariance as opposed to covariance.

⁵I would like to thank V. P. Nair for discussions on these points.

There is a functor $\mathcal{D} : \mathbf{Set} \longrightarrow \mathbf{Cat}$ from the category of sets to the category of categories given by sending a set to the discrete category with only identity morphisms. More explicitly, a set X gets sent to the category $\mathcal{D}(X)$ whose set of objects is X and whose set of morphisms consists only of identities. A function $f : X \longrightarrow Y$ of sets gets sent to the functor $\mathcal{D}(f) : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$ whose value on objects agrees with f . This determines $\mathcal{D}(f)$. Thus, since the composition of functors is a functor, this gives a functor

$$(2.17) \quad \mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}} \xrightarrow{\mathcal{S}} \mathbf{Set} \xrightarrow{\mathcal{D}} \mathbf{Cat},$$

which we denote by **States** and call it the states pre-stack. The categorically-minded reader will immediately point out that **Cat** is actually a 2-category, and we will indeed use this fact in a crucial way when we describe the GNS construction. But for now, let us put this aside.

Proposition 2.18. *The assignment⁶*

$$(2.19) \quad \begin{aligned} \mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}} &\xrightarrow{\mathbf{Rep}} \mathbf{Cat} \\ \mathcal{A} &\mapsto \mathbf{Rep}(\mathcal{A}) \\ \left(\mathcal{A}' \xrightarrow{f} \mathcal{A}\right) &\mapsto \left(\mathbf{Rep}(\mathcal{A}') \xleftarrow{\mathbf{Rep}(f)} \mathbf{Rep}(\mathcal{A})\right), \end{aligned}$$

is a functor. Here $\mathbf{Rep}(f)$, sometimes written as f^* , is the functor defined by sending a representation $(\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}), \mathcal{H})$ to the representation $(\pi \circ f : \mathcal{A}' \longrightarrow \mathcal{B}(\mathcal{H}), \mathcal{H})$ and by sending an intertwiner $(\pi, \mathcal{H}) \xrightarrow{L} (\rho, \mathcal{V})$ to the intertwiner $(\pi \circ f, \mathcal{H}) \xrightarrow{L} (\rho \circ f, \mathcal{V})$.⁷ **Rep** is also called the representation pre-stack.

Proof. Let us first make sure $\mathbf{Rep}(f)$ itself is indeed a functor. For L to be an intertwiner in $\mathbf{Rep}(\mathcal{A}')$, it must be that

$$(2.20) \quad L \circ \pi(f(a')) = \rho(f(a')) \circ L$$

for all $a' \in \mathcal{A}'$. This is true because $f(a') \in \mathcal{A}$ and L is an intertwiner in $\mathbf{Rep}(\mathcal{A})$. It is not difficult to see that $\text{id}_{\mathcal{A}}$ gets sent to $\text{id}_{\mathbf{Rep}(\mathcal{A})}$ and the composition of $\mathcal{A}'' \xrightarrow{f'} \mathcal{A}' \xrightarrow{f} \mathcal{A}$ gets sent to $\mathbf{Rep}(f') \circ \mathbf{Rep}(f)$. \blacksquare

Physics 2.21. The meaning of the functor (2.19) is as follows. With each abstract algebra of observables, there is a collection of Hilbert spaces on which we can realize these observables. This collection is not just a *set*⁸ but a *category* because there are intertwiners between representations. Intertwiners are ubiquitous in physics. Every tensor operator is an intertwiner. For instance, the angular momentum for particles in three-dimensional space is a *vector* of operators. This vector of operators is precisely an

⁶In the second line of (2.19), the assignment on objects is described. In the third line, the assignment on morphisms is specified. We will often use this notation to specify functors.

⁷The same notation L is used because it is the same operator $L : \mathcal{H} \longrightarrow \mathcal{V}$ at the level of Hilbert spaces.

⁸Technically, it is not even a set in the strict sense, but that is not the point we are trying to make.

intertwiner [8]. Other examples of intertwiners are unitary equivalences of representations. These are (some of the) symmetries of quantum mechanics. For instance, different observers might associate a slightly different Hilbert space to a collection of observables. In particular, the observables themselves might be expressed differently. The position and momentum representations of basic quantum mechanics provide one example. The unitary map defined by the Fourier transform is an intertwiner (a unitary equivalence) of representations. The category of representations conveniently packages all of these structures together in a single mathematical entity.

3. THE GNS CONSTRUCTION: FROM OBSERVABLES AND STATES TO HILBERT SPACES

We will split the GNS construction into three parts. First, we will describe the construction as is common in the literature. Then we will describe something that is less commonly illustrated, and is described nicely for physicists in [9], which is what the GNS construction gives for C^* -algebra morphisms (and not necessarily just C^* -algebra isomorphisms). The GNS construction was first introduced by Segal in [2] and we will utilize many of the facts proved in that work. At the end of this section, we state our first result, Theorem 3.26, which says that the GNS construction is an oplax-natural transformation (see Definition A.1 in the Appendix) between the functors introduced in the previous section.

Construction 3.1. Let $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ be a state on a unital C^* -algebra \mathcal{A} . Then the function

$$(3.2) \quad \begin{aligned} \mathcal{A} \times \mathcal{A} &\longrightarrow \mathbb{C} \\ (b, a) &\mapsto \omega(b^*a) \end{aligned}$$

is a map that is conjugate-linear in its first variable and linear in its second. Furthermore, it satisfies⁹

$$(3.3) \quad \omega(b^*a) = \overline{\omega(a^*b)} \quad \forall a, b \in \mathcal{A}$$

and¹⁰

$$(3.4) \quad |\omega(b^*a)|^2 \leq \omega(b^*b)\omega(a^*a) \quad \forall a, b \in \mathcal{A}.$$

⁹Proof: By assumption $\omega((\alpha a + \beta b)^*(\alpha a + \beta b)) \geq 0$ for all $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{A}$, which in particular implies that $\omega((\alpha a + \beta b)^*(\alpha a + \beta b))$ is real. Equating this expression with its conjugate gives $\overline{\alpha\beta}\omega(a^*b) + \alpha\overline{\beta}\omega(b^*a) = \alpha\overline{\beta}\omega(a^*b) + \overline{\alpha}\beta\omega(b^*a)$. Setting $\alpha = \sqrt{-1}$ and $\beta = 1$ gives $-\omega(a^*b) + \omega(b^*a) = \overline{\omega(a^*b)} - \omega(b^*a)$ while setting $\alpha = 1$ and $\beta = 1$ gives $\omega(a^*b) + \omega(b^*a) = \overline{\omega(a^*b)} + \omega(b^*a)$. Adding these two gives $2\omega(b^*a) = 2\overline{\omega(a^*b)}$, which proves the claim.

¹⁰Proof (this is more or less a standard proof of the Cauchy-Schwarz inequality): This splits up into two cases. First, if $\omega(b^*a) = 0$, then the claim is true. In the other case, suppose that $\omega(b^*a) \neq 0$. As in the previous footnote, consider the inequality $\omega((\alpha a + \beta b)^*(\alpha a + \beta b)) \geq 0$ valid for all $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{A}$. Choose $\alpha = \frac{|\omega(b^*a)|}{\omega(b^*b)}\sqrt{\omega(b^*b)}$ and $\beta = -\sqrt{\omega(a^*a)}$. Then, $\omega((\alpha a + \beta b)^*(\alpha a + \beta b)) = 2\omega(b^*b)\omega(a^*a) - 2|\omega(b^*a)|\sqrt{\omega(b^*b)\omega(a^*a)}$ using (3.3) along the way to cancel some terms. Rearranging and canceling the factor of 2 gives $|\omega(b^*a)|\sqrt{\omega(b^*b)\omega(a^*a)} \leq \omega(b^*b)\omega(a^*a)$. Squaring both sides and canceling the common terms proves the claim.

Define the set of null vectors by

$$(3.5) \quad \mathcal{N}_\omega := \{x \in \mathcal{A} : \omega(x^*x) = 0\}.$$

From (3.4), it follows that

$$(3.6) \quad |\omega(a^*x)|^2 \leq \omega(a^*a) \underbrace{\omega(x^*x)}_0 = 0 \implies \omega(a^*x) = 0 \quad \forall a \in \mathcal{A}, x \in \mathcal{N}_\omega.$$

Using this fact,

$$(3.7) \quad \omega((x+y)^*(x+y)) = \omega(x^*x) + \omega(y^*y) + \omega(x^*y) + \omega(y^*x) = 0 \quad \forall x, y \in \mathcal{N}_\omega$$

and

$$(3.8) \quad \omega((ax)^*(ax)) = \omega(x^*a^*ax) = \omega((a^*ax)^*x) = 0 \quad \forall a \in \mathcal{A}, x \in \mathcal{N}_\omega,$$

which together show that \mathcal{N}_ω is a left ideal inside \mathcal{A} , meaning that \mathcal{N}_ω is an additive subgroup of \mathcal{A} and $ax \in \mathcal{N}_\omega$ whenever $a \in \mathcal{A}$ and $x \in \mathcal{N}_\omega$. Furthermore, note that (3.3) and (3.6) imply

$$(3.9) \quad \omega(x^*a) = 0 \quad \forall x \in \mathcal{N}_\omega, a \in \mathcal{A}.$$

Now, write the equivalence class of $a \in \mathcal{A}$ in the quotient vector space $\mathcal{A}/\mathcal{N}_\omega$ as $[a]$. The function (3.2) descends to a well-defined inner product

$$(3.10) \quad \begin{aligned} &\mathcal{A}/\mathcal{N}_\omega \times \mathcal{A}/\mathcal{N}_\omega \xrightarrow{\langle \cdot, \cdot \rangle_\omega} \mathbb{C} \\ &([b], [a]) \mapsto \omega(b^*a) \end{aligned}$$

on $\mathcal{A}/\mathcal{N}_\omega$ by choosing representatives of the equivalence classes. $\langle \cdot, \cdot \rangle_\omega$ is well-defined because for any other representatives b' and a' of $[b]$ and $[a]$, respectively, so that $b - b', a - a' \in \mathcal{N}_\omega$,

$$(3.11) \quad \begin{aligned} \omega(b'^*a') &= \omega\left((b - (b - b'))^*(a - (a - a'))\right) \\ &= \omega(b^*a) - \omega\left((b - b')^*a\right) - \omega\left(b^*(a - a')\right) + \omega\left((b - b')^*(a - a')\right) \\ &= \omega(b^*a) - 0 - 0 + 0 \quad \text{by (3.6) and (3.9)} \\ &= \omega(b^*a). \end{aligned}$$

$\langle \cdot, \cdot \rangle_\omega$ is positive definite by definition of $\mathcal{A}/\mathcal{N}_\omega$ and because ω is a state. Complete $\mathcal{A}/\mathcal{N}_\omega$ with respect to the norm $\|\cdot\|_\omega$ induced by $\langle \cdot, \cdot \rangle_\omega$ and denote this Hilbert space by

$$(3.12) \quad \mathcal{H}_\omega := \overline{\mathcal{A}/\mathcal{N}_\omega}.$$

There is a natural action π_ω of \mathcal{A} on $\mathcal{A}/\mathcal{N}_\omega$ given by¹¹

$$(3.13) \quad \pi_\omega(a)[b] := [ab]$$

¹¹This is well-defined because \mathcal{N}_ω is a left ideal in \mathcal{A} by (3.7) and (3.8).

for all $a \in \mathcal{A}$ and $[b] \in \mathcal{A}/\mathcal{N}_\omega$. $\pi_\omega(a)$ is a bounded operator on $\mathcal{A}/\mathcal{N}_\omega$ for all $a \in \mathcal{A}$ because¹²

$$(3.14) \quad \|\pi_\omega(a)[b]\|_\omega^2 = \|[ab]\|_\omega^2 = \omega((ab)^*ab) = \omega(b^*a^*ab) \leq \|a^*a\|\omega(b^*b) = \|a^*a\|\|[b]\|_\omega^2$$

for all $[b] \in \mathcal{A}/\mathcal{N}_\omega$. Therefore, $\pi_\omega(a)$ extends uniquely to a bounded operator on \mathcal{H}_ω . It is immediate from (3.13) that $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ is a unital algebra homomorphism. It is a map of C^* -algebras because

$$(3.15) \quad \begin{aligned} \langle [c], \pi_\omega(a^*)[b] \rangle_\omega &= \langle [c], [a^*b] \rangle_\omega = \omega(c^*a^*b) = \omega((ac)^*b) \\ &= \langle [ac], [b] \rangle_\omega = \langle \pi_\omega(a)[c], [b] \rangle_\omega \quad \forall a \in \mathcal{A}, [b], [c] \in \mathcal{A}/\mathcal{N}_\omega, \end{aligned}$$

which shows that $\pi_\omega(a^*) = \pi_\omega(a)^*$. Thus, associated to every state $\omega : \mathcal{A} \rightarrow \mathbb{C}$, we have constructed a representation $(\pi_\omega, \mathcal{H}_\omega)$ of \mathcal{A} . We denote this assignment by $\mathbf{GNS}_\mathcal{A} : \mathbf{States}(\mathcal{A}) \rightarrow \mathbf{Rep}(\mathcal{A})$, i.e. $\mathbf{GNS}_\mathcal{A}(\omega) := (\pi_\omega, \mathcal{H}_\omega)$. It is automatically a functor because $\mathbf{States}(\mathcal{A})$ has no non-trivial morphisms. This construction is called the GNS construction for \mathcal{A} .

Physics 3.16. As we discussed earlier, a state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is a list of expectation values of all the observables of interest described by \mathcal{A} . As a particular example, consider again the case where $\mathcal{A} = \mathcal{B}(\mathcal{H})$ for a Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle$. Then, there is actually a one-to-one correspondence between states $\omega : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ satisfying a certain condition¹³ and density matrices, i.e. bounded linear operators $\rho \in \mathcal{B}(\mathcal{H})$ that are self-adjoint and $\text{tr}(\rho) = 1$ (see Proposition 19.8 and Theorem 19.9 of [6]). The correspondence is obtained by the map that sends a density matrix ρ to the state ω_ρ defined by $\omega_\rho(a) := \text{tr}(\rho a)$ for all $a \in \mathcal{A}$. Therefore, we will *think* of an abstract state $\omega : \mathcal{A} \rightarrow \mathbb{C}$ as being equivalent to a density matrix.¹⁴ This example will help us interpret the GNS construction physically. The meaning of the function $(b, a) \mapsto \omega(b^*a)$ for two observables a and b in \mathcal{A} is less mysterious if we focus on the case $b = a$ and think of a and a^* as annihilation and creation operators, respectively. Then a^*a is the number operator and $\omega(a^*a)$ is the expectation value of the particle number for the state ω .

The meaning of the null-space \mathcal{N}_ω can be interpreted as the set of observables that annihilate the state ω for all observable purposes. If we go back to the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and the special case of $\rho = P_\psi$ (written as $|\psi\rangle\langle\psi|$ in Dirac notation), the projection operator onto the subspace spanned by a unit vector $\psi \in \mathcal{H}$, then for an observable $x \in \mathcal{A}$ to be in \mathcal{N}_ω would mean that $\text{tr}(P_\psi x^*x) = \langle x\psi, x\psi \rangle = 0$, which, since $\langle \cdot, \cdot \rangle$ is an inner product, would mean that $x\psi = 0$, i.e. x annihilates ψ . Now, consider two observables $a, b \in \mathcal{A}$ such that $b - a \in \mathcal{N}_\omega$. This means that $(b - a)\psi = 0$, i.e. $b\psi = a\psi$,

¹²In the second last step, we have used the fact that $\omega(y^*xy) \leq \|x\|\omega(y^*y)$ for all $x, y \in \mathcal{A}$ (see Proposition 2.1.5. part (ii) of [4] for a proof).

¹³If $\dim \mathcal{H} < \infty$, no such additional condition is necessary. However, in the case $\dim \mathcal{H} = \infty$, one needs stronger, but reasonable, continuity assumptions on the state.

¹⁴In some cases of interest, such a density matrix need not exist. This occurs for instance in the Unruh effect whereupon restricting the algebra of observables to a Rindler observer does not lead to a density matrix, but rather an abstract state satisfying the KMS condition (see Section 5.1 of Wald [10]).

which means that the observables b and a cannot be distinguished by the state $\langle \psi, \cdot \psi \rangle$ associated to the vector ψ . This argument extends to mixtures as well. The simple case of a density matrix of the form $\rho = \lambda P_\psi + (1 - \lambda)P_\phi$, with $\psi, \phi \in \mathcal{H}$ unit vectors, P_ψ, P_ϕ their associated projections, and $\lambda \in (0, 1)$ illustrates the general case. Let ω_ρ be the associated state $\omega_\rho = \text{tr}(\rho \cdot)$. Then $x \in \mathcal{N}_{\omega_\rho}$ implies $\lambda \langle x\psi, x\psi \rangle + (1 - \lambda) \langle x\phi, x\phi \rangle = 0$. Since all terms are non-negative, this requires $\langle x\psi, x\psi \rangle = 0$ and $\langle x\phi, x\phi \rangle = 0$ individually. Hence, one concludes that $x \in \mathcal{N}_{\omega_\rho}$ implies that x annihilates *all* the vectors comprising the mixture. Similarly, if $b - a \in \mathcal{N}_{\omega_\rho}$, then one concludes $a\psi = b\psi$ and $a\phi = b\phi$ so that a and b are indistinguishable observables with respect to *both* the vectors comprising the mixture. Therefore, to summarize, if we fix a state ω on an algebra of observables \mathcal{A} , it may be that with respect to that particular state, there are some observables that are indistinguishable in terms of their expectation values. That is why we consider the quotient $\mathcal{A}/\mathcal{N}_\omega$ where we have identified these equivalent observables. Therefore, the GNS construction tells us that the associated Hilbert space is just equivalence classes of observables of \mathcal{A} distinguished by the state ω .

Construction 3.17. Let $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ be a morphism of C^* -algebras and let $\omega : \mathcal{A} \rightarrow \mathbb{C}$ be a state on \mathcal{A} . Then, as discussed in Proposition 2.10, $\omega \circ f : \mathcal{A}' \rightarrow \mathbb{C}$ is a state on \mathcal{A}' . By applying the previous construction, we get two representations $\pi_{\omega \circ f} : \mathcal{A}' \rightarrow \mathcal{B}(\mathcal{H}_{\omega \circ f})$ and $\pi_\omega : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ with $\pi_{\omega \circ f}$ a representation of \mathcal{A}' and π_ω a representation of \mathcal{A} . There is a canonical map $L_f : \mathcal{A}'/\mathcal{N}_{\omega \circ f} \rightarrow \mathcal{A}/\mathcal{N}_\omega$ obtained from the diagram¹⁵

$$(3.18) \quad \begin{array}{ccc} \mathcal{A}' & \xrightarrow{f} & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{A}'/\mathcal{N}_{\omega \circ f} & \xrightarrow{L_f} & \mathcal{A}/\mathcal{N}_\omega \end{array}$$

given by

$$(3.19) \quad L_f([a']) := [f(a')]$$

for all $[a'] \in \mathcal{A}'/\mathcal{N}_{\omega \circ f}$. This is well-defined because for any $x' \in \mathcal{N}_{\omega \circ f}$,¹⁶

$$(3.20) \quad \begin{array}{ccc} & (\omega \circ f)(x'^* x') & \\ \text{since } x' \in \mathcal{N}_{\omega \circ f} \swarrow & & \searrow \text{by Def'n 2.3} \\ 0 & & \omega(f(x')^* f(x')) \end{array},$$

i.e. $f(x') \in \mathcal{N}_\omega$. A similar calculation shows that

$$(3.21) \quad \|L_f([a'])\|_\omega^2 = \| [f(a')] \|_\omega^2 = \omega(f(a')^* f(a')) = (\omega \circ f)(a'^* a') = \|[a']\|_{\omega \circ f}^2$$

¹⁵The double arrow \twoheadrightarrow signifies a surjection.

¹⁶Diagrams such as (3.20) are read from top to bottom in either clockwise or counterclockwise order to replicate the argument in the order in which it was originally conceived.

for all $[a'] \in \mathcal{A}'/\mathcal{N}_{\omega \circ f}$ so that L_f is an injective bounded linear map and therefore extends uniquely to an injective bounded linear map $L_f : \mathcal{H}_{\omega \circ f} \longrightarrow \mathcal{H}_\omega$, which is denoted by the same letter. Furthermore, the map L_f is an intertwiner $(\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}) \longrightarrow (\pi_\omega \circ f, \mathcal{H}_\omega)$ of representations of \mathcal{A}' , which means that the diagram

$$(3.22) \quad \begin{array}{ccc} \mathcal{H}_{\omega \circ f} & \xrightarrow{L_f} & \mathcal{H}_\omega \\ \pi_{\omega \circ f}(a') \downarrow & & \downarrow \pi_\omega(f(a')) \\ \mathcal{H}_{\omega \circ f} & \xrightarrow{L_f} & \mathcal{H}_\omega \end{array}$$

commutes for all $a' \in \mathcal{A}'$. This is true because for any $[b'] \in \mathcal{A}'/\mathcal{N}_{\omega \circ f}$,

$$(3.23) \quad \begin{aligned} L_f\left(\pi_{\omega \circ f}(a')([b'])\right) &= L_f([a'b']) && \text{by (3.13) for } \pi_{\omega \circ f} \\ &= [f(a'b')] && \text{by (3.19)} \\ &= [f(a')f(b')] && \text{by Def'n 2.3} \\ &= \pi_\omega(f(a'))([f(b')]) && \text{by (3.13) for } \pi_\omega \\ &= \pi_\omega(f(a'))(L_f([b'])) && \text{by (3.19)} \end{aligned}$$

for all $a' \in \mathcal{A}'$. By continuity, L_f is an intertwiner on all of $\mathcal{H}_{\omega \circ f}$. The assignment sending a state $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ and a morphism $f : \mathcal{A}' \longrightarrow \mathcal{A}$ of C^* -algebras to the intertwiner $L_f : (\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}) \longrightarrow (\pi_\omega \circ f, \mathcal{H}_\omega)$ of representations of \mathcal{A}' therefore defines a natural transformation¹⁷

$$(3.24) \quad \begin{array}{ccc} \mathbf{States}(\mathcal{A}) & \xrightarrow{\mathbf{GNS}_{\mathcal{A}}} & \mathbf{Rep}(\mathcal{A}) \\ \mathbf{States}(f) \downarrow & \nearrow \mathbf{GNS}_f & \downarrow \mathbf{Rep}(f) \\ \mathbf{States}(\mathcal{A}') & \xrightarrow{\mathbf{GNS}_{\mathcal{A}'}} & \mathbf{Rep}(\mathcal{A}') \end{array}$$

associated to every morphism $f : \mathcal{A}' \longrightarrow \mathcal{A}$ of C^* -algebras. We denote the intertwiner L_f by $\mathbf{GNS}_f(\omega)$ to explicitly indicate what it depends on.

Physics 3.25. Let us go back to the case $i : \mathcal{A}_0 \hookrightarrow \mathcal{A}$ of restricting ourselves to a subalgebra of observables and let ω be a state on \mathcal{A} . Let $\omega_0 := \omega \circ i$ be the state pulled back to \mathcal{A}_0 . Since \mathcal{A}_0 is a subalgebra of \mathcal{A} , there are fewer experiments we can perform on the state ω_0 . Although $\mathcal{N}_{\omega_0} \subseteq \mathcal{N}_\omega$, which was proven in (3.20), the fact that $L_i : \mathcal{A}_0/\mathcal{N}_{\omega_0} \longrightarrow \mathcal{A}/\mathcal{N}_\omega$ is injective says that the equivalence classes of distinguishable observables for the state ω_0 (cf. Physics 3.16) are also distinguishable by ω , but not necessarily conversely. In this sense, there are fewer distinguishable observables for ω_0

¹⁷In the present situation, the definition of the natural transformation \mathbf{GNS}_f reduces to an assignment on objects of $\mathbf{States}(\mathcal{A})$ to morphisms of $\mathbf{Rep}(\mathcal{A}')$ because $\mathbf{States}(\mathcal{A})$ is a discrete category.

than there are for ω . This is consistent with the perspective that \mathcal{A}_0 describes macrostate observables while \mathcal{A} describes microstate observables (cf. Physics 2.15). Since the intertwiner provides a subspace $L_i(\mathcal{H}_{\omega_0})$ of the Hilbert space \mathcal{H}_{ω} , the act of restricting our view to a subalgebra corresponds to restricting to a subspace of our Hilbert space.¹⁸

Theorem 3.26. *The assignments¹⁹*

$$(3.27) \quad \begin{aligned} \mathbf{C}^*\text{-}\mathbf{Alg}_0^{\text{op}} &\xrightarrow{\text{GNS}} \mathbf{Cat}_1 \\ \mathcal{A} &\mapsto \left(\mathbf{States}(\mathcal{A}) \xrightarrow{\text{GNS}_{\mathcal{A}}} \mathbf{Rep}(\mathcal{A}) \right) \end{aligned}$$

from Construction 3.1 and

$$(3.28) \quad \begin{aligned} \mathbf{C}^*\text{-}\mathbf{Alg}_1^{\text{op}} &\xrightarrow{\text{GNS}} \mathbf{Cat}_2 \\ (\mathcal{A}' \xrightarrow{f} \mathcal{A}) &\mapsto \left(\text{GNS}_{\mathcal{A}'} \circ \mathbf{States}(f) \xrightarrow{\text{GNS}_f} \mathbf{Rep}(f) \circ \text{GNS}_{\mathcal{A}} \right) \end{aligned}$$

from Construction 3.17 define an oplax-natural transformation²⁰

$$(3.29) \quad \begin{array}{ccc} & \text{States} & \\ & \Downarrow \text{GNS} & \\ \mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}} & & \mathbf{Cat} \\ & \Uparrow \text{Rep} & \end{array}$$

Proof. There are only two things to check because $\mathbf{C}^*\text{-}\mathbf{Alg}$ has no nontrivial 2-morphisms (see Definition A.1). First, the GNS construction applied to the identity morphism $\text{id}_{\mathcal{A}}$ for a C^* -algebra \mathcal{A} gives $\text{GNS}_{\text{id}_{\mathcal{A}}}$ which is precisely the identity natural transformation $\text{GNS}_{\mathcal{A}} \circ \mathbf{States}(\text{id}_{\mathcal{A}}) = \text{GNS}_{\mathcal{A}} \Rightarrow \text{GNS}_{\mathcal{A}} = \mathbf{Rep}(\text{id}_{\mathcal{A}}) \circ \text{GNS}_{\mathcal{A}}$. Second, associated to a pair of composable morphisms

$$(3.30) \quad \mathcal{A}'' \xrightarrow{f'} \mathcal{A}' \xrightarrow{f} \mathcal{A},$$

there are two diagrams one obtains. On the one hand, applying the GNS construction to the composition $f \circ f'$ gives $\text{GNS}_{f \circ f'}$. On the other hand, applying GNS to each f' and f and then composing gives another natural transformation. These two results look

¹⁸This phrasing is a bit misleading, however, since *every* C^* -algebra morphism $f : \mathcal{A}' \longrightarrow \mathcal{A}$ will lead to L_f being injective regardless of whether or not f is injective since our argument did not depend on this. Nevertheless, for psychological reasons and simplicity for interpretation, we will always use inclusions for explaining the physics.

¹⁹Given a 2-category (or a category) \mathcal{C} , the objects, 1-morphisms, and 2-morphisms are denoted by \mathcal{C}_0 , \mathcal{C}_1 , and \mathcal{C}_2 , respectively.

²⁰We are viewing \mathbf{Cat} as a strict 2-category whose 2-morphisms are natural transformations. By also viewing $\mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}}$ as a 2-category (all of whose 2-morphisms are identities), we can view \mathbf{States} and \mathbf{Rep} as (strict) 2-functors. Because GNS_f is not invertible, which is usually required in the definition of a pseudo-natural transformation, we use the more general notion of oplax-natural transformation described in Definition A.1 of the Appendix.

like

$$(3.31) \quad \begin{array}{ccc} \text{States}(\mathcal{A}) & \xrightarrow{\text{GNS}_{\mathcal{A}}} & \text{Rep}(\mathcal{A}) \\ \downarrow \text{States}(f \circ f') & \nearrow \text{GNS}_{f \circ f'} & \downarrow \text{Rep}(f \circ f') \\ \text{States}(\mathcal{A}'') & \xrightarrow{\text{GNS}_{\mathcal{A}''}} & \text{Rep}(\mathcal{A}'') \end{array} \quad \& \quad \begin{array}{ccc} \text{States}(\mathcal{A}) & \xrightarrow{\text{GNS}_{\mathcal{A}}} & \text{Rep}(\mathcal{A}) \\ \downarrow \text{States}(f) & \nearrow \text{GNS}_f & \downarrow \text{Rep}(f) \\ \text{States}(\mathcal{A}') & \xrightarrow{\text{GNS}_{\mathcal{A}'}} & \text{Rep}(\mathcal{A}') \\ \downarrow \text{States}(f') & \nearrow \text{GNS}_{f'} & \downarrow \text{Rep}(f') \\ \text{States}(\mathcal{A}'') & \xrightarrow{\text{GNS}_{\mathcal{A}''}} & \text{Rep}(\mathcal{A}'') \end{array}$$

respectively. The second condition that **GNS** be an oplax-natural transformation is that the compositions in these two diagrams are equal. This follows from the commutativity of the individual squares and triangles in the diagram

$$(3.32) \quad \begin{array}{ccccc} & & f \circ f' & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{A}'' & \xrightarrow{f'} & \mathcal{A}' & \xrightarrow{f} & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}/\mathcal{N}_{\omega \circ f \circ f'} & \xrightarrow{L_{f'}} & \mathcal{A}/\mathcal{N}_{\omega \circ f} & \xrightarrow{L_f} & \mathcal{A}/\mathcal{N}_{\omega} \\ & \xrightarrow{L_{f \circ f'}} & & & \end{array}$$

for any state ω on \mathcal{A} . By continuity, this equality extends to the completions. \blacksquare

Physics 3.33. Oplax-naturality means the following if we restrict our attention to a subalgebra and then restrict to yet another subalgebra, as in

$$(3.34) \quad \mathcal{A}_1 \xrightarrow{j} \mathcal{A}_0 \xrightarrow{i} \mathcal{A}.$$

Equality of the two diagrams in (3.31) means that constructing the physical subspace $\mathcal{H}_{\omega \circ i \circ j}$ of \mathcal{H}_{ω} of quantum configurations for the state ω with respect to the subalgebra \mathcal{A}_1 is the same subspace obtained from first restricting to \mathcal{A}_0 and then to \mathcal{A}_1 , i.e.

$$(3.35) \quad \begin{array}{ccc} & \mathcal{H}_{\omega \circ i} & \\ L_j \nearrow & & \searrow L_i \\ \mathcal{H}_{\omega \circ i \circ j} & \xrightarrow{L_{i \circ j}} & \mathcal{H}_{\omega} \end{array}$$

commutes, where we have used the notation from Construction 3.17.

Remark 3.36. **GNS** being an oplax-natural transformation provides the correct categorical structured needed to reflect the functoriality of the GNS construction as can be seen by the equality of the diagrams in (3.31). In current terminology [11], Theorem 3.26 shows that the GNS construction is not only a functor for a fixed C^* -algebra \mathcal{A} ,

but it is also a morphism of pre-stacks over the category of all C^* -algebras. Note that it is *not* a morphism of pre-sheaves of categories because the outer diagram in (3.24) does *not* commute (a condition that is required to have a morphism of pre-sheaves). Instead, a natural transformation (which is a 2-morphism in **Cat**) is needed to compensate for the lack of commutativity, and this is why 2-categories play a crucial role in the GNS construction.

4. SOME COMMENTS ON THE CATEGORY OF STATES

One would like to think of **States**(\mathcal{A}) as a category of states with non-trivial morphisms. Namely, a morphism from $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ to $\mu : \mathcal{A} \longrightarrow \mathbb{C}$ consists of a C^* -algebra morphism $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ such that the diagram

$$(4.1) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \\ \omega \searrow & & \swarrow \mu \\ & \mathbb{C} & \end{array}$$

commutes, i.e. $\mu \circ \phi = \omega$. Let us call this closely related category *states*(\mathcal{A}). While one can define a functor *states*(\mathcal{A}) \longrightarrow **Rep**(\mathcal{A}) as a special case of **GNS** $_{\mathcal{A}}$ on objects and **GNS** $_{\phi}$ on morphisms, this is too restrictive and not what we want in general because mappings of *different* algebras show up in many applications. Recall from Physics 2.15, that a C^* -algebra map $\mathcal{A}_0 \longrightarrow \mathcal{A}$ is supposed to be thought of as using macrostate observables described by \mathcal{A}_0 instead of microstate observables described by \mathcal{A} . To incorporate this, we would therefore still want to think of the different categories of states as a pre-sheaf of categories on the category of C^* -algebras, i.e. a functor *states* : **C**^{*}-**Alg**^{op} \longrightarrow **Cat**. For a morphism $f : \mathcal{A}' \longrightarrow \mathcal{A}$ this should get mapped to a functor *states*(f) : *states*(\mathcal{A}) \longrightarrow *states*(\mathcal{A}'). How should this functor be defined? This agrees with **States**(f) at the level of objects. However, for a morphism $\phi : \omega \longrightarrow \mu$ of states in \mathcal{A} , all we have is the collection of morphisms

$$(4.2) \quad \begin{array}{ccc} \mathcal{A}' & & \mathcal{A}' \\ f \searrow & & \swarrow f \\ \mathcal{A} & \xrightarrow{\phi} & \mathcal{A} \\ \omega \searrow & & \swarrow \mu \\ & \mathbb{C} & \end{array} .$$

From these data, we are supposed to produce a map of C^* -algebras $\phi' : \mathcal{A}' \longrightarrow \mathcal{A}'$ such that the diagram

$$(4.3) \quad \begin{array}{ccc} \mathcal{A}' & \xrightarrow{\phi'} & \mathcal{A}' \\ \omega \circ f \searrow & & \swarrow \mu \circ f \\ & \mathbb{C} & \end{array}$$

commutes. One can show that the only such maps $f : \mathcal{A}' \longrightarrow \mathcal{A}$ of C^* -algebras for which we can do this in a functorial manner are C^* -algebra *isomorphisms*. Since we specifically do not want this for physical reasons, we use the discrete category **States**(\mathcal{A}) instead of the more reasonable, yet naive, category *states*(\mathcal{A}).

5. A RIGHT ADJOINT TO THE GNS CONSTRUCTION

Besides producing a representation $(\pi_\omega, \mathcal{H}_\omega)$ of \mathcal{A} given a state ω on \mathcal{A} , the GNS construction also produces a cyclic vector in \mathcal{H}_ω . This fact will let us construct a sort of inverse to the GNS construction provided that we include this extra datum in the definition of the oplax-natural transformation **GNS**.

Definition 5.1. A cyclic vector Ω for a representation π of a C^* -algebra \mathcal{A} on a Hilbert space \mathcal{H} is a normalized (i.e. unit) vector $\Omega \in \mathcal{H}$ such that

$$(5.2) \quad \{\pi(a)\Omega : a \in \mathcal{A}\}$$

is a dense subset in \mathcal{H} (with respect to the norm induced by the inner product on \mathcal{H}). A representation (π, \mathcal{H}) of \mathcal{A} together with a cyclic vector Ω is called a cyclic representation and is written as a triple $(\pi, \mathcal{H}, \Omega)$. A representation (π, \mathcal{H}) of \mathcal{A} together with a normalized vector (not necessarily cyclic) is called a pointed representation.

The reason for demanding normalized vectors in the above definition is so that they produce states, as will be explained shortly.

Physics 5.3. When \mathcal{A} is the algebra of observables for a quantum field theory (in a particular phase), the vacuum vector is typically a cyclic vector—any particle content state is obtained by creation operators on the ground state. When a representation is irreducible, every non-zero vector is cyclic—by using annihilation operators, one can get to the ground state. One should take this comment with a grain of salt due to domain and distribution issues (see Folland [12]). A finite-dimensional and completely rigorous example occurs in the theory of spin by use of ladder operators.

Definition 5.4. Let **Rep**[•](\mathcal{A}) be the category of pointed representations of \mathcal{A} . Namely, an object of **Rep**[•](\mathcal{A}) is a pointed representation $(\pi, \mathcal{H}, \Omega)$ of \mathcal{A} and a morphism from $(\pi, \mathcal{H}, \Omega)$ to $(\pi', \mathcal{H}', \Omega')$ is an intertwiner $L : \mathcal{H} \longrightarrow \mathcal{H}'$ of representations of \mathcal{A} such that

$$(5.5) \quad L(\Omega) = \Omega' \quad \& \quad L^*L = \text{id}_{\mathcal{H}}.$$

Let **Rep**[◊](\mathcal{A}) be the sub-category of **Rep**[•](\mathcal{A}) of cyclic representations of \mathcal{A} .

Proof. Some things must be checked so that the above definition is in fact valid. For instance, let

$$(5.6) \quad (\pi, \mathcal{H}, \Omega) \xrightarrow{L} (\pi', \mathcal{H}', \Omega') \xrightarrow{L'} (\pi'', \mathcal{H}'', \Omega'')$$

be a pair of composable morphisms. Then the composition $L'L$ satisfies

$$(5.7) \quad (L'L)^*(L'L) = L^*L'^*L'L = L^*L = \text{id}_{\mathcal{H}}.$$

Associativity follows from associativity of composition of functions. The other axioms of a category all hold. \blacksquare

Remark 5.8. $\mathbf{Rep}^\circ(\mathcal{A})$ is a full subcategory of $\mathbf{Rep}^\bullet(\mathcal{A})$ because a vector being cyclic is a property and not additional structure.

The following lemma will be useful for proving that certain linear maps are isometries.

Lemma 5.9. *Let \mathcal{H} and \mathcal{H}' be Hilbert spaces and let $L : \mathcal{H} \longrightarrow \mathcal{H}'$ be a bounded linear map. The following conditions on L are equivalent.*

- (a) $L^*L = \text{id}_{\mathcal{H}}$.
- (b) $\|L\psi\|' = \|\psi\|$ for all $\psi \in \mathcal{H}$.
- (c) $\langle L\psi, L\phi \rangle' = \langle \psi, \phi \rangle$ for all $\psi, \phi \in \mathcal{H}$.

In this notation, a prime superscript has been used to distinguish the norm and inner product on \mathcal{H}' from that of \mathcal{H} .

Remark 5.10. The condition $L^*L = \text{id}_{\mathcal{H}}$ implies that L is injective. We do *not* require L to be unitary, which would impose $LL^* = \text{id}_{\mathcal{H}'}$ as well. However, note that if $L : (\pi, \mathcal{H}, \Omega) \longrightarrow (\pi', \mathcal{H}', \Omega')$ is a morphism of *cyclic* representations, then L sends a dense subset of \mathcal{H} to a dense subset of \mathcal{H}' because $L(\Omega) = \Omega'$. Therefore, in this case, L is unitary. Also note that the number of morphisms from a cyclic representation to a pointed representation is quite small: there is either one or none at all.

Construction 5.11. Let $(\pi, \mathcal{H}, \Omega)$ be a pointed representation of a C^* -algebra \mathcal{A} . The vector Ω defines a state ω_Ω on $\mathcal{B}(\mathcal{H})$ by the formula

$$(5.12) \quad \mathcal{B}(\mathcal{H}) \ni B \mapsto \omega_\Omega(B) := \langle \Omega, B\Omega \rangle.$$

We often refer to ω_Ω as a vector state. Pulling this state back to \mathcal{A} along π defines a state $\omega_\Omega \circ \pi : \mathcal{A} \longrightarrow \mathbb{C}$ on \mathcal{A} . $\omega_\Omega \circ \pi$ is sometimes denoted by $\mathbf{rest}_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$ for “restriction.”

Remark 5.13. If the vector Ω were not required to be normalized, but merely nonzero, then one could still define a state by the assignment

$$(5.14) \quad \mathcal{B}(\mathcal{H}) \ni B \mapsto \frac{\langle \Omega, B\Omega \rangle}{\langle \Omega, \Omega \rangle}.$$

Nevertheless, such an assignment would spoil other desirable properties that will be discussed shortly (see Remark 5.54).

Lemma 5.15. *Let $L : (\pi, \mathcal{H}, \Omega) \longrightarrow (\pi', \mathcal{H}', \Omega')$ be a morphism of pointed representations of \mathcal{A} . Then,*

$$(5.16) \quad \omega_{\Omega'} \circ \pi' = \omega_{\Omega} \circ \pi,$$

i.e. the two states $\mathbf{rest}_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$ and $\mathbf{rest}_{\mathcal{A}}((\pi', \mathcal{H}', \Omega'))$ are equal.

Proof. For any $a \in \mathcal{A}$,

$$(5.17) \quad \begin{aligned} \langle \Omega', \pi'(a)\Omega' \rangle &= \langle L(\Omega), \pi'(a)L(\Omega) \rangle && \text{by (5.4)} \\ &= \langle L(\Omega), L\pi(a)\Omega \rangle && \text{by (2.7)} \\ &= \langle \Omega, L^*L\pi(a)\Omega \rangle && \text{by Def'n of } L^* \\ &= \langle \Omega, \pi(a)\Omega \rangle && \text{by (5.4).} \end{aligned}$$

■

Physics 5.18. Imagine a context in which we begin with a representation of \mathcal{A} on a Hilbert space \mathcal{H} and a vacuum vector $\Omega \in \mathcal{H}$. Given a subalgebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$, the construction $\mathbf{rest}_{\mathcal{A}}$ restricts the vacuum state $\omega_{\Omega} := \langle \Omega, \cdot \Omega \rangle$ on $\mathcal{B}(\mathcal{H})$ to a state on this subalgebra. This is useful if we can only make measurements of certain observables. For instance, a Rindler observer has a restricted algebra of observables so that restricting a Minkowski vacuum state to their algebra results in a thermal state, a phenomenon known as the Unruh effect [10]. If we change our representation in such a way that the two are still related by an intertwiner satisfying (5.5), then we get the *same* state. Note that it is not required that L be unitary—an isometry preserving the unit vector suffices. $\mathbf{rest}_{\mathcal{A}}$ is also a natural construction from the physics perspective since every unit vector in \mathcal{H} gives a state on any C^* -subalgebra of $\mathcal{B}(\mathcal{H})$. What is not obvious is that there is a canonical way to go back—the purpose of this section is to make this statement precise and prove that the GNS construction achieves this.

Proposition 5.19. *Let \mathcal{A} be a C^* -algebra. The assignment*

$$(5.20) \quad \begin{aligned} \mathbf{Rep}^{\bullet}(\mathcal{A})_0 &\ni (\pi, \mathcal{H}, \Omega) \mapsto \omega_{\Omega} \circ \pi \in \mathbf{States}(\mathcal{A})_0 \\ \mathbf{Rep}^{\bullet}(\mathcal{A})_1 &\ni ((\pi, \mathcal{H}, \Omega) \xrightarrow{L} (\pi', \mathcal{H}', \Omega')) \mapsto \text{id}_{\omega_{\Omega} \circ \pi} \in \mathbf{States}(\mathcal{A})_1 \end{aligned}$$

from Construction 5.11 defines a functor $\mathbf{rest}_{\mathcal{A}} : \mathbf{Rep}^{\bullet}(\mathcal{A}) \longrightarrow \mathbf{States}(\mathcal{A})$.

Proof. This follows directly from Construction 5.11 and Lemma 5.15. ■

Construction 5.21. Let $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ be a morphism of C^* -algebras. The induced functor $\mathbf{Rep}(f) : \mathbf{Rep}(\mathcal{A}) \longrightarrow \mathbf{Rep}(\mathcal{A}')$ extends to a functor $\mathbf{Rep}^{\bullet}(f) : \mathbf{Rep}^{\bullet}(\mathcal{A}) \longrightarrow \mathbf{Rep}^{\bullet}(\mathcal{A}')$ as follows. Let $(\pi, \mathcal{H}, \Omega)$ be a pointed representation of \mathcal{A} . Then this gets sent to $(\pi \circ f, \mathcal{H}, \Omega)$. Note that even if $(\pi, \mathcal{H}, \Omega)$ is a cyclic representation, $(\pi \circ f, \mathcal{H}, \Omega)$ is *not* necessarily a cyclic representation of \mathcal{A}' since

$$(5.22) \quad \{ \pi(f(a'))\Omega : a' \in \mathcal{A}' \}$$

is not necessarily dense in \mathcal{H} . Nevertheless, $(\pi \circ f, \mathcal{H}, \Omega)$ is a pointed representation. A morphism of pointed representations of \mathcal{A} gets sent to a morphism of pointed representations of \mathcal{A}' under the functor $\mathbf{Rep}^\bullet(f)$ using the same intertwiner. In fact, the diagram

$$(5.23) \quad \begin{array}{ccc} \mathbf{Rep}^\bullet(\mathcal{A}) & \xrightarrow{\mathbf{rest}_\mathcal{A}} & \mathbf{States}(\mathcal{A}) \\ \mathbf{Rep}^\bullet(f) \downarrow & & \downarrow \mathbf{States}(f) \\ \mathbf{Rep}^\bullet(\mathcal{A}') & \xrightarrow{\mathbf{rest}_{\mathcal{A}'}} & \mathbf{States}(\mathcal{A}') \end{array}$$

commutes.

This proves the following fact.

Proposition 5.24. *\mathbf{rest} , as defined in Construction 5.11, is a natural transformation²¹*

$$(5.25) \quad \begin{array}{ccc} & \mathbf{States} & \\ \curvearrowright & \uparrow \mathbf{rest} & \curvearrowleft \\ \mathbf{C}^*\text{-}\mathbf{Alg}^{\mathrm{op}} & & \mathbf{Cat} \\ & \downarrow \mathbf{Rep}^\bullet & \\ & \mathbf{Rep}^\bullet & \end{array}$$

Physics 5.26. Following the earlier examples of a subalgebra $\mathcal{A}_0 \hookrightarrow \mathcal{A}$, one interpretation of the functor $\mathbf{Rep}^\bullet(\mathcal{A}_0 \hookrightarrow \mathcal{A}) : \mathbf{Rep}^\bullet(\mathcal{A}) \longrightarrow \mathbf{Rep}^\bullet(\mathcal{A}_0)$ can be given as follows. Let $(\pi, \mathcal{H}, \Omega)$ be a cyclic representation of \mathcal{A} with Ω viewed as a vacuum vector for some quantum field theory. View \mathcal{A}_0 as a subalgebra of \mathcal{A} corresponding to low energy observables. Low energy observables may be unable to produce states of the parent theory whose rest mass is beyond the energy scale of the low energy observer. This corresponds to the fact that the vector Ω might no longer be cyclic with respect to the pullback representation to the algebra \mathcal{A}_0 (cf. Physics 5.3).

Commutativity of the diagram (5.23), and hence naturality of \mathbf{rest} , says that the state the low energy observer sees is independent of the order in which they disregard certain information. The observer can either forget the representation first and then focus on the low energy observables or first focus on the induced low energy representation and then forget the representation. The induced states are the same.

We will now modify the GNS construction to include the construction of a cyclic vector. Due to the similarity of this construction and that of Constructions 3.1 and 3.17, we will skip many details and only focus on the new ones.

Construction 5.27. For every C^* -algebra \mathcal{A} , define a functor

$$(5.28) \quad \mathbf{GNS}_\mathcal{A}^\bullet : \mathbf{States}(\mathcal{A}) \longrightarrow \mathbf{Rep}^\bullet(\mathcal{A})$$

²¹This is special case of a pseudo-natural transformation since $\mathbf{rest}_f = \mathrm{id}$ in (5.23).

by the following assignment. To a state $\omega : \mathcal{A} \longrightarrow \mathbb{C}$, assign the cyclic representation²² $\mathbf{GNS}_{\mathcal{A}}^{\bullet}(\omega) := (\pi_{\omega}, \mathcal{H}_{\omega}, [1_{\mathcal{A}}])$. Because $\mathbf{States}(\mathcal{A})$ has no non-trivial morphisms, this defines a functor. Furthermore, the image of this functor actually lands in the subcategory $\mathbf{Rep}^{\circ}(\mathcal{A})$ [2]. To every morphism $\mathcal{A}' \xrightarrow{f} \mathcal{A}$ of unital C^* -algebras, define a natural transformation

$$(5.29) \quad \begin{array}{ccc} \mathbf{States}(\mathcal{A}) & \xrightarrow{\mathbf{GNS}_{\mathcal{A}}^{\bullet}} & \mathbf{Rep}^{\bullet}(\mathcal{A}) \\ \downarrow \mathbf{States}(f) & \nearrow \mathbf{GNS}_f^{\bullet} & \downarrow \mathbf{Rep}^{\bullet}(f) \\ \mathbf{States}(\mathcal{A}') & \xrightarrow{\mathbf{GNS}_{\mathcal{A}'}^{\bullet}} & \mathbf{Rep}^{\bullet}(\mathcal{A}') \end{array}$$

as follows. To every state $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ on \mathcal{A} define the morphism

$$(5.30) \quad (\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}, [1_{\mathcal{A}'}]) \xrightarrow{\mathbf{GNS}_f^{\bullet}(\omega)} (\pi_{\omega} \circ f, \mathcal{H}_{\omega}, [1_{\mathcal{A}}])$$

of pointed representations to be exactly the same as L_f in (3.19) and simply note that a property of this linear map is that

$$(5.31) \quad L_f([1_{\mathcal{A}'}]) = [f(1_{\mathcal{A}'}]) = [1_{\mathcal{A}}]$$

since f is a morphism of unital C^* -algebras. $L_f^* L_f = \text{id}_{\mathcal{H}_{\omega \circ f}}$ follows from the calculation (3.21) and Lemma 5.9.

Remark 5.32. Note that although $(\pi_{\omega}, \mathcal{H}_{\omega}, [1_{\mathcal{A}}])$ and $(\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}, [1_{\mathcal{A}'}])$ are cyclic representations of \mathcal{A} and \mathcal{A}' , respectively, the pointed representation $(\pi_{\omega} \circ f, \mathcal{H}_{\omega}, [1_{\mathcal{A}}])$ of \mathcal{A}' obtained by pullback along f is *not* necessarily cyclic. This is why the target of the GNS functor was chosen to be the category of pointed representations instead of cyclic representations. This is analogous to the fact that the restriction of an irreducible representation to a subalgebra need not be irreducible.

Theorem 5.33. *The assignments*

$$(5.34) \quad \begin{array}{l} \mathbf{C}^*\text{-Alg}_0^{\text{op}} \xrightarrow{\mathbf{GNS}^{\bullet}} \mathbf{Cat}_1 \\ \mathcal{A} \mapsto \left(\mathbf{States}(\mathcal{A}) \xrightarrow{\mathbf{GNS}_{\mathcal{A}}^{\bullet}} \mathbf{Rep}^{\bullet}(\mathcal{A}) \right) \end{array}$$

and

$$(5.35) \quad \begin{array}{l} \mathbf{C}^*\text{-Alg}_1^{\text{op}} \xrightarrow{\mathbf{GNS}^{\bullet}} \mathbf{Cat}_2 \\ (\mathcal{A}' \xrightarrow{f} \mathcal{A}) \mapsto \left(\mathbf{GNS}_{\mathcal{A}'}^{\bullet} \circ \mathbf{States}(f) \xrightarrow{\mathbf{GNS}_f^{\bullet}} \mathbf{Rep}^{\bullet}(f) \circ \mathbf{GNS}_{\mathcal{A}}^{\bullet} \right) \end{array}$$

²² $[1_{\mathcal{A}}]$ is a cyclic vector because $\{\pi_{\omega}(a)[1_{\mathcal{A}}] = [a] : a \in \mathcal{A}\} =: \mathcal{A}/\mathcal{N}_{\omega}$ is dense in \mathcal{H}_{ω} by definition.

from Construction 5.27 define an oplax-natural transformation

$$(5.36) \quad \begin{array}{ccc} & \text{States} & \\ \text{C}^*\text{-Alg}^{\text{op}} & \xrightarrow{\text{GNS}^\bullet} & \text{Cat} \\ & \text{Rep}^\bullet & \end{array}$$

Proof. The proof is not much different than what it was for **GNS** in Theorem 3.26. ■

Physics 5.37. The weak form of naturality for **GNS**[•] guarantees that the way in which the Hilbert spaces and their cyclic vectors fit into the larger space is independent of how the restrictions to successive subalgebras is grouped. For instance, one could imagine a sequence of course-grained subalgebras. This naturality on one pair of subalgebras guarantees consistency for all such tuples of subalgebras when applying the GNS construction.

There is one last construction we must confront. This involves relating the composition of oplax-natural transformations **rest** and **GNS**[•] with the identity natural transformation.

Lemma 5.38. *The vertical composition*

$$(5.39) \quad \begin{array}{ccc} & \text{States} & \\ \text{C}^*\text{-Alg}^{\text{op}} & \xrightarrow{\text{GNS}^\bullet} \text{Rep}^\bullet \xrightarrow{\text{rest}} & \text{Cat} \\ & \text{States} & \end{array}$$

of oplax-natural transformations is equal to the identity natural transformation.

Proof. Let \mathcal{A} be a C^* -algebra. The composition acting on a state $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ gives

$$(5.40) \quad \begin{aligned} \text{States}(\mathcal{A}) &\xrightarrow{\text{GNS}^\bullet_{\mathcal{A}}} \text{Rep}^\bullet(\mathcal{A}) \xrightarrow{\text{rest}_{\mathcal{A}}} \text{States}(\mathcal{A}) \\ \omega &\longmapsto (\pi_\omega, \mathcal{H}_\omega, [1_{\mathcal{A}}]) \mapsto \langle [1_{\mathcal{A}}], \pi_\omega(\cdot)[1_{\mathcal{A}}] \rangle_\omega. \end{aligned}$$

$\langle [1_{\mathcal{A}}], \pi_\omega(\cdot)[1_{\mathcal{A}}] \rangle_\omega$ agrees with ω as a state on \mathcal{A} because

$$(5.41) \quad \langle [1_{\mathcal{A}}], \pi_\omega(a)[1_{\mathcal{A}}] \rangle_\omega = \langle [1_{\mathcal{A}}], [a] \rangle_\omega = \omega(1_{\mathcal{A}}^* a) = \omega(a) \quad \forall a \in \mathcal{A}.$$

There are no non-trivial morphisms in **States**(\mathcal{A}) so the composition is the identity functor. To every morphism $f : \mathcal{A}' \longrightarrow \mathcal{A}$ of C^* -algebras, the composition of natural

transformations

$$(5.42) \quad \begin{array}{ccccc} \mathbf{States}(\mathcal{A}) & \xrightarrow{\mathbf{GNS}_{\mathcal{A}}^{\bullet}} & \mathbf{Rep}^{\bullet}(\mathcal{A}) & \xrightarrow{\mathbf{rest}_{\mathcal{A}}} & \mathbf{States}(\mathcal{A}) \\ \downarrow \mathbf{States}(f) & \nearrow \mathbf{GNS}_f^{\bullet} & \downarrow \mathbf{Rep}^{\bullet}(f) & \nearrow \text{id}=\mathbf{rest}_f & \downarrow \mathbf{States}(f) \\ \mathbf{States}(\mathcal{A}') & \xrightarrow{\mathbf{GNS}_{\mathcal{A}'}^{\bullet}} & \mathbf{Rep}^{\bullet}(\mathcal{A}') & \xrightarrow{\mathbf{rest}_{\mathcal{A}'}} & \mathbf{States}(\mathcal{A}') \end{array}$$

must equal the identity natural transformation. This follows immediately from the fact that the compositions on the top and bottom of this diagram are identities by (5.41) and since $\mathbf{States}(\mathcal{A}')$ has no non-trivial morphisms. \blacksquare

Physics 5.43. The interpretation of this is immediate when viewing \mathbf{GNS}^{\bullet} and \mathbf{rest} as processes/constructions. If you start with a state, construct the GNS representation, and then forget the representation, you get back your original state. In other words, there is no loss of information.

However, the composition in the order

$$(5.44) \quad \begin{array}{ccc} & \mathbf{Rep}^{\bullet} & \\ \text{rest} \downarrow & & \uparrow \mathbf{GNS}^{\bullet} \\ \mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}} & \xrightarrow{\mathbf{States}} & \mathbf{Cat} \\ \mathbf{GNS}^{\bullet} \downarrow & & \uparrow \mathbf{Rep}^{\bullet} \end{array}$$

is certainly not the identity. In the following, we construct the required modification (see Definition A.10 in the Appendix).

Construction 5.45. Let \mathcal{A} be a unital C^* -algebra and consider the diagram

$$(5.46) \quad \begin{array}{ccc} & \mathbf{States}(\mathcal{A}) & \\ \mathbf{rest}_{\mathcal{A}} \nearrow & & \searrow \mathbf{GNS}_{\mathcal{A}}^{\bullet} \\ \mathbf{Rep}^{\bullet}(\mathcal{A}) & \xrightarrow{\text{id}_{\mathbf{Rep}^{\bullet}(\mathcal{A})}} & \mathbf{Rep}^{\bullet}(\mathcal{A}) \end{array}$$

of functors. Recalling the notation from Constructions 5.11 and 3.1, observe what happens to a pointed representation $(\pi, \mathcal{H}, \Omega)$ of \mathcal{A} along the top two functors

$$(5.47) \quad (\pi, \mathcal{H}, \Omega) \xrightarrow{\mathbf{rest}_{\mathcal{A}}} \langle \Omega, \pi(\cdot)\Omega \rangle =: \omega \xrightarrow{\mathbf{GNS}_{\mathcal{A}}^{\bullet}} (\pi_{\omega}, \mathcal{H}_{\omega}, [1_{\mathcal{A}}])$$

Therefore, we have two pointed representations of \mathcal{A} whose associated states agree, i.e.

$$(5.48) \quad \langle \Omega, \pi(a)\Omega \rangle = \langle [1_{\mathcal{A}}], \pi_{\omega}(a)[1_{\mathcal{A}}] \rangle_{\omega} \quad \text{for all } a \in \mathcal{A}.$$

If $(\pi, \mathcal{H}, \Omega)$ were also a cyclic representation, then it was already known by Segal that any other cyclic representation restricting to the same state is unitarily equivalent to

it [2]. We slightly modify Segal's proof for our construction. Define the linear map

$$(5.49) \quad \begin{array}{ccc} \mathcal{A}/\mathcal{N}_\omega & \xrightarrow{m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))} & \mathcal{H} \\ [a] & \longmapsto & \pi(a)\Omega \end{array} .$$

To see that this is well-defined, let $x \in \mathcal{N}_\omega$ so that

$$(5.50) \quad \begin{array}{ccc} & \omega(x^*x) & \\ \text{\textit{by (5.47)}} \swarrow & & \searrow \\ 0 & \langle \Omega, \pi(x^*x)\Omega \rangle & \\ & \text{\textit{by (5.49)}} \downarrow & \\ & \langle \pi(x)\Omega, \pi(x)\Omega \rangle & \end{array} .$$

Since $\langle \cdot, \cdot \rangle$ is an inner product, this holds if and only if $\pi(x)\Omega = 0$, proving well-definedness. Because²³

$$(5.51) \quad \|m_{\mathcal{A}}([a])\|^2 = \langle m_{\mathcal{A}}([a]), m_{\mathcal{A}}([a]) \rangle = \langle \pi(a)\Omega, \pi(a)\Omega \rangle = \omega(a^*a) = \|[a]\|_\omega^2$$

for all $[a] \in \mathcal{A}/\mathcal{N}_\omega$, $m_{\mathcal{A}}$ is bounded and extends uniquely to a bounded linear map $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega)) : \mathcal{H}_\omega \longrightarrow \mathcal{H}$. By Lemma 5.9 and (5.51), this extension is also an isometry.

We now show that $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega)) : (\pi_\omega, \mathcal{H}_\omega, [1_{\mathcal{A}}]) \longrightarrow (\pi, \mathcal{H}, \Omega)$ is an intertwiner, which means that the diagram

$$(5.52) \quad \begin{array}{ccc} \mathcal{H}_\omega & \xrightarrow{m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))} & \mathcal{H} \\ \pi_\omega(a) \downarrow & & \downarrow \pi(a) \\ \mathcal{H}_\omega & \xrightarrow{m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))} & \mathcal{H} \end{array}$$

commutes for all $a \in \mathcal{A}$. Following the image of an element $[b] \in \mathcal{A}/\mathcal{N}_\omega$ along both of these morphisms gives

$$(5.53) \quad \begin{array}{ccc} m_{\mathcal{A}}(\pi_\omega(a)[b]) & & \pi(a)m_{\mathcal{A}}([b]) \\ \text{\textit{by (3.13)}} \downarrow & & \downarrow \text{\textit{by (5.49)}} \\ m_{\mathcal{A}}([ab]) & & \pi(a)\pi(b)\Omega \\ \text{\textit{by (5.49)}} \downarrow & & \downarrow \\ & \pi(ab)\Omega & \end{array}$$

proving that the diagram (5.52) indeed commutes (upon extension to the completion). Hence, $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega)) : (\pi_\omega, \mathcal{H}_\omega, [1_{\mathcal{A}}]) \longrightarrow (\pi, \mathcal{H}, \Omega)$ is a morphism in $\mathbf{Rep}^\bullet(\mathcal{A})$.

²³We occasionally write $m_{\mathcal{A}}([a])$ instead of $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))([a])$ because the notation would be too difficult to read otherwise. Since our representation $(\pi, \mathcal{H}, \Omega)$ is fixed for now, this should cause no confusion.

Note that if $(\pi, \mathcal{H}, \Omega)$ is cyclic, then Remark 5.10 shows that $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$ is a unitary equivalence.

Remark 5.54. If, in the definition of a pointed representation, we use arbitrary vectors instead of normalized ones and we define $\mathbf{rest}_{\mathcal{A}}(\pi, \mathcal{H}, \Omega)$ to be the state

$$(5.55) \quad \frac{\langle \Omega, \pi(\cdot)\Omega \rangle}{\langle \Omega, \Omega \rangle},$$

then the map (5.49) cannot be an isometry unless $\langle \Omega, \Omega \rangle = 1$.

Physics 5.56. The map $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$ tells us that if we start with an arbitrary representation (π, \mathcal{H}) of the algebra of observables \mathcal{A} together with a normalized vector $\Omega \in \mathcal{H}$ (the representation need not be irreducible and the vector need not be cyclic), if we forget about our Hilbert space, and remember only the algebra of observables and our state, then we might not be able to recover our exact Hilbert space back, but we can get close. The best we can do from the GNS construction is to get a new Hilbert space that embeds into the Hilbert space we started with via $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$. Furthermore, in this subspace, the vector we started with becomes cyclic with respect to the algebra of observables. In other words, we lose some information, namely the vectors orthogonal to this subspace, but we keep many of the essential features of our initial state and our algebra of observables.

Lemma 5.57. m from Construction 5.45 defines a modification (cf. Definition A.10)

$$(5.58) \quad \begin{array}{ccc} & \mathbf{States} & \\ \mathbf{rest} \nearrow & \Downarrow m & \searrow \mathbf{GNS}^{\bullet} \\ \mathbf{Rep}^{\bullet} & \xrightarrow{\mathbf{id}_{\mathbf{Rep}^{\bullet}}} & \mathbf{Rep}^{\bullet} \end{array}$$

via the assignment

$$(5.59) \quad \mathbf{C}^*\text{-Alg}_0 \ni \mathcal{A} \longmapsto \begin{array}{ccc} & \mathbf{States}(\mathcal{A}) & \\ \mathbf{rest}_{\mathcal{A}} \nearrow & \Downarrow m_{\mathcal{A}} & \searrow \mathbf{GNS}_{\mathcal{A}}^{\bullet} \\ \mathbf{Rep}^{\bullet}(\mathcal{A}) & \xrightarrow{\mathbf{id}_{\mathbf{Rep}^{\bullet}(\mathcal{A})}} & \mathbf{Rep}^{\bullet}(\mathcal{A}) \end{array}.$$

Furthermore, for each C^* -algebra \mathcal{A} , $m_{\mathcal{A}}$ restricts to a well-defined and vertically invertible natural transformation on the subcategory $\mathbf{Rep}^{\circ}(\mathcal{A})$.

Proof. In order for m to be a modification, for every morphism $f : \mathcal{A}' \rightarrow \mathcal{A}$ of C^* -algebras, the following equality must hold (see equation (A.14))

$$(5.60) \quad \begin{array}{ccc} \text{Rep}^\bullet(\mathcal{A}) & \xrightarrow{\text{id}_{\text{Rep}^\bullet(\mathcal{A})}} & \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{Rep}^\bullet(f) & \nearrow \text{GNS}^\bullet_{\mathcal{A}} & \downarrow \text{Rep}^\bullet(f) \\ \text{Rep}^\bullet(\mathcal{A}') & \xrightarrow{\text{id}_{\text{Rep}^\bullet(\mathcal{A}')}} & \text{Rep}^\bullet(\mathcal{A}') \\ \downarrow \text{rest}_{\mathcal{A}'} & \nearrow \text{GNS}^\bullet_{\mathcal{A}'} & \downarrow \text{rest}_{\mathcal{A}'} \\ \text{States}(\mathcal{A}') & \xrightarrow{\text{id}_{\text{States}(\mathcal{A}')}} & \text{States}(\mathcal{A}') \end{array} = \begin{array}{ccc} \text{Rep}^\bullet(\mathcal{A}) & \xrightarrow{\text{id}_{\text{Rep}^\bullet(\mathcal{A})}} & \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{Rep}^\bullet(f) & \nearrow \text{id}_{\text{Rep}^\bullet(f)} & \downarrow \text{Rep}^\bullet(f) \\ \text{Rep}^\bullet(\mathcal{A}') & \xrightarrow{\text{id}_{\text{Rep}^\bullet(\mathcal{A}')}} & \text{Rep}^\bullet(\mathcal{A}') \\ \downarrow \text{rest}_{\mathcal{A}'} & \nearrow \text{id}_{\text{rest}_{\mathcal{A}'}} & \downarrow \text{rest}_{\mathcal{A}'} \\ \text{States}(\mathcal{A}') & \xrightarrow{\text{id}_{\text{States}(\mathcal{A}')}} & \text{States}(\mathcal{A}') \end{array}$$

i.e. for every object $(\pi, \mathcal{H}, \Omega)$ of $\text{Rep}^\bullet(\mathcal{A})$ with $\omega := \langle \Omega, \pi(\cdot)\Omega \rangle$, the diagram

$$(5.61) \quad \begin{array}{ccc} & (\pi_\omega \circ f, \mathcal{H}_\omega, [1_{\mathcal{A}}]) & \\ \text{GNS}^\bullet_{f(\omega)} \nearrow & & \searrow f^*(m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))) \\ (\pi_{\omega \circ f}, \mathcal{H}_{\omega \circ f}, [1_{\mathcal{A}'}]) & \xrightarrow{m_{\mathcal{A}'}((\pi \circ f, \mathcal{H}, \Omega))} & (\pi \circ f, \mathcal{H}, \Omega) \end{array}$$

of intertwiners of pointed representations of \mathcal{A}' must commute. The image of a vector $[a'] \in \mathcal{A}'/\mathcal{N}_{\omega \circ f}$ under the top two linear maps is $\pi(f(a'))\Omega$ while the image under the bottom map is $(\pi \circ f)(a')\Omega$. These are equal elements in \mathcal{H} . Because the maps agree on a dense subspace, the diagram (5.61) commutes. Finally, when $m_{\mathcal{A}}$ is restricted to the subcategory $\text{Rep}^\circ(\mathcal{A})$, it was shown at the end of Construction (5.45) that it is unitary on objects of $\text{Rep}^\circ(\mathcal{A})$ and therefore a vertically invertible natural transformation. ■

Physics 5.62. Commutativity of (5.61), i.e. m being a modification, encodes the fact that the way in which our GNS Hilbert spaces sit inside the original space agrees with respect to the smaller collection of observables in the case that one is restricting to a subalgebra.

Everything we have done up to this point leads to the following theorem encompassing the GNS construction. To state it, we introduce the functor 2-category (see Definition A.28).

Definition 5.63. Let $\text{Fun}(\mathbf{C}^*\text{-Alg}^{\text{op}}, \mathbf{Cat})$ be the 2-category whose objects are functors from $\mathbf{C}^*\text{-Alg}^{\text{op}}$ to \mathbf{Cat} , 1-morphisms are oplax-natural transformations, and 2-morphisms are modifications. Compositions and identities are defined as in ordinary 2-category theory (see Appendix A for definitions).

Theorem 5.64. The oplax-natural transformation $\text{GNS}^\bullet : \text{States} \Rightarrow \text{Rep}^\bullet$ is left adjoint to rest . In fact, the quadruple $(\text{GNS}^\bullet, \text{rest}, \text{id}, m)$ is an adjunction in $\text{Fun}(\mathbf{C}^*\text{-Alg}^{\text{op}}, \mathbf{Cat})$.

Proof. The only thing left to check are the zig-zag identities from Lemma A.29. Using the notation from that lemma, $F := \text{States}$, $G := \text{Rep}^\bullet$, $\sigma := \text{GNS}^\bullet$, $\rho := \text{rest}$, $\eta := \text{id}$,

and $\epsilon := m$. By Remark A.32, it suffices to prove

$$(5.65) \quad \begin{array}{c} \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{rest}_\mathcal{A} \\ \text{States}(\mathcal{A}) \xrightarrow{m_\mathcal{A}} \text{id}_{\text{Rep}^\bullet(\mathcal{A})} \\ \downarrow \text{GNS}_\mathcal{A}^\bullet \\ \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{rest}_\mathcal{A} \\ \text{States}(\mathcal{A}) \end{array} \xrightarrow{\text{id}_{\text{States}(\mathcal{A})}} \begin{array}{c} \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{rest}_\mathcal{A} \\ \text{States}(\mathcal{A}) \end{array} = \begin{array}{c} \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{rest}_\mathcal{A} \\ \text{States}(\mathcal{A}) \end{array}$$

and

$$(5.66) \quad \begin{array}{c} \text{States}(\mathcal{A}) \\ \downarrow \text{GNS}_\mathcal{A}^\bullet \\ \text{Rep}^\bullet(\mathcal{A}) \\ \downarrow \text{rest}_\mathcal{A} \\ \text{States}(\mathcal{A}) \xrightarrow{m_\mathcal{A}} \text{id}_{\text{Rep}^\bullet(\mathcal{A})} \\ \downarrow \text{GNS}_\mathcal{A}^\bullet \\ \text{Rep}^\bullet(\mathcal{A}) \end{array} \xrightarrow{\text{id}_{\text{States}(\mathcal{A})}} \begin{array}{c} \text{States}(\mathcal{A}) \\ \downarrow \text{GNS}_\mathcal{A}^\bullet \\ \text{Rep}^\bullet(\mathcal{A}) \end{array} = \begin{array}{c} \text{States}(\mathcal{A}) \\ \downarrow \text{GNS}_\mathcal{A}^\bullet \\ \text{Rep}^\bullet(\mathcal{A}) \end{array}$$

for each object \mathcal{A} of $\mathbf{C}^*\text{-Alg}^{\text{op}}$. Fortunately, these identities are essentially tautologous. For (5.65), since $\mathbf{States}(\mathcal{A})$ has no non-trivial morphisms, the equality holds. For (5.66), it suffices to check what happens to a state ω . Under the composition in (5.66), ω gets sent to

$$(5.67) \quad \omega \mapsto (\pi_\omega, \mathcal{H}_\omega, [1_\mathcal{A}]) \mapsto \langle [1_\mathcal{A}], \pi_\omega(\cdot)[1_\mathcal{A}] \rangle = \omega \mapsto (\pi_\omega, \mathcal{H}_\omega, [1_\mathcal{A}]),$$

which is exactly the same representation as in the second step. Finally, $m_\mathcal{A}((\pi_\omega, \mathcal{H}_\omega, [1_\mathcal{A}]))$ is the identity intertwiner because it sends $[a] \in \mathcal{A}/\mathcal{N}_\omega$ to $\pi(a)[1_\mathcal{A}]$ by (5.49) but $\pi(a)[1_\mathcal{A}] = [a]$ by (3.13). \blacksquare

In particular, by Remark A.32, this theorem implies the following.

Corollary 5.68. *For every C^* -algebra \mathcal{A} , the quadruple $(\mathbf{GNS}_\mathcal{A}^\bullet, \text{rest}_\mathcal{A}, \text{id}, m_\mathcal{A})$ is an adjunction (in the usual sense).*

This means that for every state $\omega \in \mathbf{States}(\mathcal{A})_0$ and pointed representation $(\pi, \mathcal{H}, \Omega) \in \mathbf{Rep}^\bullet(\mathcal{A})_0$, there is a natural bijection of morphisms²⁴

$$(5.69) \quad \mathbf{Rep}^\bullet(\mathcal{A})\left(\mathbf{GNS}_\mathcal{A}^\bullet(\omega), (\pi, \mathcal{H}, \Omega)\right) \cong \mathbf{States}(\mathcal{A})\left(\omega, \text{rest}_\mathcal{A}(\pi, \mathcal{H}, \Omega)\right),$$

²⁴This is how one remembers that \mathbf{GNS}^\bullet is left adjoint to rest .

which illustrates in what sense the GNS construction $\mathbf{GNS}_{\mathcal{A}}^{\bullet}(\omega)$ is optimal: for every other choice of representation $(\pi, \mathcal{H}, \Omega)$ on which to realize the state ω as the restriction of a vector state, there is always a (unique) isometric intertwiner from the GNS Hilbert space to \mathcal{H} . In particular, the GNS Hilbert space is the *smallest* space on which one can represent a state as the restriction of a vector state. If ω and $\mathbf{rest}_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$ do not agree on \mathcal{A} , this result also says that there is *no* intertwiner between the pointed representations $\mathbf{GNS}_{\mathcal{A}}^{\bullet}(\omega)$ and $(\pi, \mathcal{H}, \Omega)$ (since $\mathbf{States}(\mathcal{A})$ is a discrete category). This can change if we have an isomorphism of our algebra.

A special case of the adjunction $(\mathbf{GNS}_{\mathcal{A}}^{\bullet}, \mathbf{rest}_{\mathcal{A}}, \mathrm{id}, m_{\mathcal{A}})$ occurs when restricted to the category of cyclic representations of \mathcal{A} . In this case, it is an adjoint equivalence (meaning, an equivalence of categories). In other words, in the cyclic case, the categories $\mathbf{Rep}^{\odot}(\mathcal{A})$ and $\mathbf{States}(\mathcal{A})$ are equivalent and the restriction functor exhibits this equivalence with a canonical inverse given by the GNS construction. In particular, this reproduces the well-known result [2] that there is a one-to-one correspondence between isomorphism classes of cyclic representations of \mathcal{A} and states on \mathcal{A} .

Our results can be summarized by saying that we can now provide a *definition* instead of a *construction* that produces, in a functorial manner, cyclic representations from states on C^* -algebras.

Definition 5.70. The GNS construction is the left adjoint of \mathbf{rest} .

This means that the GNS construction is characterized by the following data:

- i) a function $\mathcal{G}_{\mathcal{A}} : \mathcal{S}(\mathcal{A}) \longrightarrow \mathbf{Rep}^{\bullet}(\mathcal{A})_0$ for each C^* -algebra \mathcal{A} sending states on \mathcal{A} to pointed representations of \mathcal{A} ,
- ii) a function $\mathcal{G}_f : \mathcal{S}(\mathcal{A}) \longrightarrow \mathbf{Rep}^{\bullet}(\mathcal{A}')_1$ for each morphism $f : \mathcal{A}' \longrightarrow \mathcal{A}$ of C^* -algebras sending states on \mathcal{A} to isometric intertwiners of pointed representations of \mathcal{A}' , and
- iii) a function $m_{\mathcal{A}} : \mathbf{Rep}^{\bullet}(\mathcal{A})_0 \longrightarrow \mathbf{Rep}^{\bullet}(\mathcal{A})_1$ for each C^* -algebra \mathcal{A}

subject to the following conditions:

- (a) $\mathcal{G}_f(\omega)$ is an isometric intertwiner of representations of \mathcal{A}' from $\mathcal{G}_{\mathcal{A}'}(\omega \circ f)$ to $f^* \mathcal{G}_{\mathcal{A}}(\omega)$ for all states $\omega \in \mathcal{S}(\mathcal{A})$,
- (b) $m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))$ is a isometric intertwiner from $\mathcal{G}_{\mathcal{A}}(\mathbf{rest}_{\mathcal{A}}((\pi, \mathcal{H}, \Omega)))$ to $(\pi, \mathcal{H}, \Omega)$ for all pointed representations $(\pi, \mathcal{H}, \Omega) \in \mathbf{Rep}^{\bullet}(\mathcal{A})_0$,
- (c) $\mathcal{G}_{\mathrm{id}_{\mathcal{A}}} = i_{\mathcal{A}} \circ \mathcal{G}_{\mathcal{A}}$, where $i_{\mathcal{A}} : \mathbf{Rep}^{\bullet}(\mathcal{A})_0 \longrightarrow \mathbf{Rep}^{\bullet}(\mathcal{A})_1$ is the map that assigns the identity intertwiner to each pointed representation,
- (d) $\mathcal{G}_{f \circ f'}(\omega) = f'^*(\mathcal{G}_f(\omega)) \circ \mathcal{G}_{f'}(\omega \circ f)$ for all composable pairs of C^* -algebra morphisms $\mathcal{A}'' \xrightarrow{f'} \mathcal{A}' \xrightarrow{f} \mathcal{A}$ and states $\omega \in \mathcal{S}(\mathcal{A})$,
- (e) $f^* m_{\mathcal{A}}((\pi, \mathcal{H}, \Omega)) \circ \mathcal{G}_f(\mathbf{rest}_{\mathcal{A}}((\pi, \mathcal{H}, \Omega))) = m_{\mathcal{A}'}((\pi \circ f, \mathcal{H}, \Omega))$ for all pointed representations $(\pi, \mathcal{H}, \Omega) \in \mathbf{Rep}^{\bullet}(\mathcal{A})_0$,
- (f) $\mathbf{rest}_{\mathcal{A}} \circ \mathcal{G}_{\mathcal{A}} = \mathrm{id}_{\mathcal{S}(\mathcal{A})}$, and
- (g) $m_{\mathcal{A}}(\mathcal{G}_{\mathcal{A}}(\omega)) = \mathrm{id}_{\mathcal{G}_{\mathcal{A}}(\omega)}$ for all states $\omega \in \mathcal{S}(\mathcal{A})$.

²⁵Here, f^* is the pullback of a representation along the map f .

The reader should compare this short definition to Constructions 3.1, 3.17, and 5.27, which are the usual definitions of the GNS construction. Many of these properties are well-known, though less emphasis is placed on allowing non-unitary intertwiners, obfuscating the underlying categorical perspective outlined here. We have established such a categorical framework for the GNS constructing viewing it as a morphism/process in an appropriate category, namely $\mathbf{Fun}(\mathbf{C}^*\text{-}\mathbf{Alg}^{\text{op}}, \mathbf{Cat})$, and we have illustrated that the universal properties of the GNS construction are encoded in this morphism being left adjoint to the simple morphism/process that views normalized vectors as states.

6. EXAMPLES

The authors of [9] include several examples, and we will go through the simplest ones to illustrate the meaning of our constructions and theorems.

Example 6.1. Let $\mathcal{A} := \mathcal{B}(\mathbb{C}^2)$, 2×2 matrices with complex coefficients. This is the algebra of observables for a spin- $\frac{1}{2}$ system, i.e. a qubit. Label an orthonormal basis by $\{\uparrow, \downarrow\}$ —this basis refers to the spin of a particle along a particular axis. Let \mathcal{A} act on \mathbb{C}^2 by the identity representation, meaning that the representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathbb{C}^2)$ is just the identity map. Let $\omega_{\uparrow} : \mathcal{B}(\mathbb{C}^2) \rightarrow \mathbb{C}$ be the state corresponding to a pure state with spin up, i.e. ²⁶ $\omega_{\uparrow}(a) := \langle \uparrow | a | \uparrow \rangle$ for all $a \in \mathcal{B}(\mathcal{H})$. Applying the restriction functor $\mathbf{rest}_{\mathcal{A}}$ to the pointed representation $(\pi, \mathbb{C}^2, |\uparrow\rangle)$ gives ω_{\uparrow} . Next, apply the GNS construction $\mathbf{GNS}_{\mathcal{A}}^{\bullet}$ to the state ω_{\uparrow} . As a vector space, $\mathcal{B}(\mathbb{C}^2)$ is four-dimensional with a basis given by

$$(6.2) \quad e_{\uparrow\uparrow} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{\uparrow\downarrow} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{\downarrow\uparrow} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \& \quad e_{\downarrow\downarrow} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The expectation values for these operators are given by

$$(6.3) \quad \omega_{\uparrow}(e_{\uparrow\uparrow}) = 1, \quad \omega_{\uparrow}(e_{\uparrow\downarrow}) = 0, \quad \omega_{\uparrow}(e_{\downarrow\uparrow}) = 0, \quad \& \quad \omega_{\uparrow}(e_{\downarrow\downarrow}) = 0.$$

Notice that ²⁷ $e_{\uparrow\downarrow}^{\dagger} e_{\uparrow\downarrow} = e_{\downarrow\downarrow}$ and $e_{\downarrow\downarrow}^{\dagger} e_{\downarrow\downarrow} = e_{\downarrow\downarrow}$ so that $\omega_{\uparrow}(e_{\uparrow\downarrow}^{\dagger} e_{\uparrow\downarrow}) = 0$ and $\omega_{\uparrow}(e_{\downarrow\downarrow}^{\dagger} e_{\downarrow\downarrow}) = 0$. In fact,

$$(6.4) \quad \mathcal{N}_{\omega_{\uparrow}} = \text{span}\{e_{\uparrow\downarrow}, e_{\downarrow\downarrow}\}.$$

Then, ²⁸ $\mathcal{H}_{\omega_{\uparrow}} := \mathcal{B}(\mathbb{C}^2) / \mathcal{N}_{\omega_{\uparrow}}$ consists of equivalence classes of matrices

$$(6.5) \quad a = \begin{pmatrix} a_{\uparrow\uparrow} & a_{\uparrow\downarrow} \\ a_{\downarrow\uparrow} & a_{\downarrow\downarrow} \end{pmatrix},$$

where $a_{ij} \in \mathbb{C}$ with $i, j \in \{\uparrow, \downarrow\}$, and $a \sim b$ if and only if

$$(6.6) \quad b - a = \begin{pmatrix} 0 & b_{\uparrow\downarrow} - a_{\uparrow\downarrow} \\ 0 & b_{\downarrow\downarrow} - a_{\downarrow\downarrow} \end{pmatrix}.$$

²⁶We are using Dirac bra-ket notation for the examples.

²⁷To avoid confusion with the physics literature, for the purposes of this section, we will use \dagger to denote the adjoint instead of $*$.

²⁸No completion is necessary here since the vector spaces are finite-dimensional.

Note that the inner product on $\mathcal{H}_{\omega_\uparrow}$ in this case is given by

$$(6.7) \quad \mathcal{H}_{\omega_\uparrow} \times \mathcal{H}_{\omega_\uparrow} \ni ([a], [b]) \mapsto \omega_\uparrow(a^\dagger b) = \overline{a_{\uparrow\uparrow}} b_{\uparrow\uparrow} + \overline{a_{\downarrow\uparrow}} b_{\downarrow\uparrow},$$

where the overline denotes complex conjugation. The associated cyclic representation from the GNS construction applied to the state ω_\uparrow is $(\pi_{\omega_\uparrow}, \mathcal{H}_{\omega_\uparrow}, [\mathbf{1}])$, where $\mathbf{1}$ is the 2×2 identity matrix and $\pi_{\omega_\uparrow}(a)([b]) = [ab]$ is obtained from ordinary matrix multiplication. The intertwiner $m_{\mathcal{A}}$ from (5.49) applied to our representation $(\pi, \mathbb{C}^2, |\uparrow\rangle)$ is the map $[a] \mapsto a|\uparrow\rangle$. Since our representation was cyclic to begin with, this intertwiner is unitary. This map compares our original Hilbert space representation to the one obtained from the GNS construction in a canonical way.

Example 6.8. Let $\mathcal{A} := \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2)$ and let \mathcal{A} act on $\mathbb{C}^2 \otimes \mathbb{C}^2$ via

$$(6.9) \quad \mathcal{A} \xrightarrow{\pi} \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2) \\ a \otimes b \mapsto \left(\mathbb{C}^2 \otimes \mathbb{C}^2 \ni |\psi\rangle \otimes |\phi\rangle \xrightarrow{\pi(a \otimes b)} a|\psi\rangle \otimes b|\phi\rangle \right)$$

on elementary tensors and appropriately extended. Let ω be the state

$$(6.10) \quad \mathcal{A} \ni a \otimes b \mapsto \omega(a \otimes b) := \langle \Psi | \pi(a \otimes b) | \Psi \rangle$$

on \mathcal{A} obtained by pulling back the vector state associated to the unit vector

$$(6.11) \quad |\Psi\rangle := \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$

where $|\uparrow\downarrow\rangle$ is short for $|\uparrow\rangle \otimes |\downarrow\rangle$. Let the inner product on $\mathbb{C}^2 \otimes \mathbb{C}^2$ be the usual Euclidean inner product. Using the same notation as in the previous example,

$$(6.12) \quad \begin{aligned} \omega(a \otimes b) &= \frac{1}{2} \left(\langle \uparrow\downarrow | - \langle \downarrow\uparrow | \right) \pi(a \otimes b) \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \\ &= \frac{1}{2} \left(\langle \uparrow | \otimes \langle \downarrow | - \langle \downarrow | \otimes \langle \uparrow | \right) \left(a|\uparrow\rangle \otimes b|\downarrow\rangle - a|\downarrow\rangle \otimes b|\uparrow\rangle \right) \\ &= \frac{1}{2} \left(a_{\uparrow\uparrow} b_{\downarrow\downarrow} - a_{\uparrow\downarrow} b_{\downarrow\uparrow} - a_{\downarrow\uparrow} b_{\uparrow\downarrow} + a_{\downarrow\downarrow} b_{\uparrow\uparrow} \right). \end{aligned}$$

It is not necessary for us to calculate \mathcal{H}_ω explicitly but we use the isomorphism π in (6.9) to identify its elements as equivalence classes $[a \otimes b]$ of elements in $\mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2)$. Because $|\Psi\rangle$ is cyclic, Lemma 5.57 shows that the map

$$(6.13) \quad \begin{aligned} (\pi_\omega, \mathcal{H}_\omega, [\mathbf{1}]) &\xrightarrow{m_{\mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2)}} (\pi, \mathbb{C}^2 \otimes \mathbb{C}^2, |\Psi\rangle) \\ [a \otimes b] &\longmapsto \pi(a \otimes b) |\Psi\rangle \end{aligned}$$

extended linearly is a unitary intertwiner. Now, let $i_1 : \mathcal{B}(\mathbb{C}^2) \longrightarrow \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2)$ be the C^* -algebra map defined by

$$(6.14) \quad i_1(a) := a \otimes \mathbf{1}.$$

Physically, such a map corresponds to an observer \mathcal{O}_1 only being able to make measurements on the observables $\mathcal{B}(\mathbb{C}^2)$ corresponding to a single particle. It is convenient to denote the first \mathbb{C}^2 by \mathcal{H}_1 and the second by \mathcal{H}_2 . This situation occurs, for instance, in

an EPR-like experiment, where a particle decomposes into two particles whose spins are correlated in a way described by the vector (6.11). The two particles fly off in opposite directions and observers far away are waiting to measure the spin.

$$\mathcal{O}_1 \xleftarrow{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)} \mathcal{O}_2$$

Observer \mathcal{O}_1 cannot measure the observables $\mathcal{B}(\mathcal{H}_2)$ and vice versa. Therefore, the state that \mathcal{O}_1 sees is given by the restriction

$$(6.15) \quad \omega_1 := \omega \circ i_1 : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathbb{C}.$$

This state corresponds to the density matrix

$$(6.16) \quad \rho_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

on \mathcal{H}_1 using our ordered basis $(|\uparrow\rangle, |\downarrow\rangle)$. What is the GNS construction applied to the state ω_1 and how is it related to the original pointed representation $(\pi, \mathcal{H}_1 \otimes \mathcal{H}_2, |\Psi\rangle)$? Let $a \in \mathcal{B}(\mathcal{H}_1)$. Then

$$(6.17) \quad (a^\dagger a)_{ik} = \sum_{j \in \{\uparrow, \downarrow\}} (a^\dagger)_{ij} a_{jk} = \sum_{j \in \{\uparrow, \downarrow\}} \overline{a_{ji}} a_{jk}$$

implies

$$(6.18) \quad \begin{aligned} \omega_1(a^\dagger a) &= \text{tr}(\rho_1 a^\dagger a) = \frac{1}{2} \left(\langle \uparrow | a^\dagger a | \uparrow \rangle + \langle \downarrow | a^\dagger a | \downarrow \rangle \right) \\ &= \frac{1}{2} \sum_{j \in \{\uparrow, \downarrow\}} \left(|a_{j\uparrow}|^2 + |a_{j\downarrow}|^2 \right) = \frac{1}{2} \sum_{j, k \in \{\uparrow, \downarrow\}} |a_{jk}|^2 \end{aligned}$$

so that $\omega_1(a^\dagger a) = 0$ if and only if $a = 0$. Therefore, $\mathcal{N}_{\omega_1} = \{0\}$ and hence $\mathcal{H}_{\omega_1} = \mathcal{B}(\mathcal{H}_1)$ as a vector space. The inner product on \mathcal{H}_{ω_1} is given by

$$(6.19) \quad \mathcal{H}_{\omega_1} \times \mathcal{H}_{\omega_1} \ni (a, b) \mapsto \omega_1(a^\dagger b) = \frac{1}{2} \text{tr}(a^\dagger b),$$

which is half the Hilbert-Schmidt (Frobenius) inner product. Furthermore, the associated GNS representation π_{ω_1} acts as

$$(6.20) \quad \pi_{\omega_1}(a)b = ab = \sum_{i, j, k \in \{\uparrow, \downarrow\}} a_{ij} b_{jk} e_{ik},$$

where the e_{ik} are as in (6.2). The induced map $L_{i_1} : \mathcal{H}_{\omega_1} \longrightarrow \mathcal{H}_\omega$ corresponding to (3.18) is given by

$$(6.21) \quad \begin{aligned} \mathcal{B}(\mathcal{H}_1) &\equiv \mathcal{H}_{\omega_1} \xrightarrow{L_{i_1} = \text{GNS}_{i_1}^\bullet(\omega)} \mathcal{H}_\omega \\ a &\longmapsto [a \otimes \mathbb{1}] \end{aligned}$$

Using this with the intertwiner $m_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$ from (6.13), gives a canonical intertwiner of $\mathcal{B}(\mathcal{H}_1)$ -representations to our original Hilbert space

$$(6.22) \quad \begin{array}{ccccc} \mathcal{H}_{\omega_1} & \xrightarrow{\text{GNS}_{i_1}^\bullet(\omega)} & \mathcal{H}_\omega & \xrightarrow{i_1^*(m_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)})} & \mathcal{H}_1 \otimes \mathcal{H}_2 \\ a & \longmapsto & [a \otimes \mathbb{1}] & \longmapsto & \pi(a \otimes \mathbb{1})|\Psi\rangle \end{array} .$$

This canonical map is the top arrow in the diagram (5.61). This exhibits our Hilbert space \mathcal{H}_{ω_1} , which was the Hilbert space from the GNS construction associated to the EPR density matrix ρ_1 for observer \mathcal{O}_1 , as a vector subspace of our original Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ for the entangled EPR vector $|\Psi\rangle$. Note that the map (6.22) is surjective because

$$(6.23) \quad \pi(a \otimes \mathbb{1})|\Psi\rangle = \frac{1}{\sqrt{2}} \left(a_{\uparrow\uparrow}|\uparrow\downarrow\rangle + a_{\downarrow\uparrow}|\downarrow\downarrow\rangle - a_{\uparrow\downarrow}|\uparrow\uparrow\rangle - a_{\downarrow\downarrow}|\downarrow\uparrow\rangle \right).$$

It is also an isometry by Lemma 5.9 because

$$(6.24) \quad \|\pi(a \otimes \mathbb{1})|\Psi\rangle\|^2 = \langle \Psi | \pi(a^\dagger \otimes \mathbb{1}) \pi(a \otimes \mathbb{1}) | \Psi \rangle = \frac{1}{2} \sum_{j,k \in \{\uparrow, \downarrow\}} |a_{jk}|^2 = \frac{1}{2} \text{tr}(a^\dagger a) = \|a\|_{\omega_1}^2$$

for all $a \in \mathcal{H}_{\omega_1}$. Hence, (6.22) is a unitary intertwiner.

APPENDIX A. 2-CATEGORICAL PRELIMINARIES

In the GNS construction, we use oplax-natural transformations, which are different from the pseudo-natural transformations that appear in the early literature on 2-categories [13]. Fortunately, the difference is minor. For completeness, we include this definition along with the notion of modifications [14].

Definition A.1. Let \mathcal{C} and \mathcal{D} be two (strict)²⁹ 2-categories and let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two 2-functors. An oplax-natural transformation ρ from F to G , written as $\rho : F \rightrightarrows G$, consists of

- i) a function $\rho : \mathcal{C}_0 \longrightarrow \mathcal{D}_1$ assigning a 1-morphism in \mathcal{D} to an object x in \mathcal{C} in the following manner

$$(A.2) \quad \begin{array}{ccc} x & \xrightarrow{\rho} & \begin{array}{c} F(x) \\ \downarrow \rho(x) \\ G(x) \end{array} \end{array}$$

²⁹A definition exists for weak 2-categories and weak 2-functors but such a definition is not needed here.

- ii) and a function $\rho : C_1 \longrightarrow D_2$ assigning a 2-morphism³⁰ in \mathcal{D} to every 1-morphism $y \xleftarrow{\alpha} x$ in \mathcal{C} in the following manner

$$(A.3) \quad y \xleftarrow{\alpha} x \quad \xrightarrow{\rho} \quad \begin{array}{ccc} F(y) & \xleftarrow{F(\alpha)} & F(x) \\ \rho(y) \downarrow & \searrow \rho(\alpha) & \downarrow \rho(x) \\ G(y) & \xleftarrow{G(\alpha)} & G(x) \end{array} .$$

These data must satisfy the following conditions:

- (a) For every object x in \mathcal{C} ,

$$(A.4) \quad \rho(\text{id}_x) = \text{id}_{\rho(x)} .$$

- (b) For every pair $(z \xleftarrow{\alpha} y, y \xleftarrow{\beta} x)$ of composable 1-morphisms in \mathcal{C} , the diagram³¹

$$(A.5) \quad \begin{array}{ccc} \rho(z) \circ F(\alpha) \circ F(\beta) & \xrightarrow{\rho(\alpha) \circ \text{id}_{F(\beta)}} & G(\alpha) \circ \rho(y) \circ F(\beta) \\ \text{id} \parallel & & \parallel \text{id}_{G(\alpha) \circ \rho(\beta)} \\ \rho(z) \circ F(\alpha\beta) & & \\ \rho(\alpha\beta) \Downarrow & & \\ G(\alpha\beta) \circ \rho(x) & \xlongequal{\text{id}} & G(\alpha) \circ G(\beta) \circ \rho(x) \end{array}$$

commutes, i.e.

$$(A.6) \quad \begin{array}{ccccc} F(z) & \xleftarrow{F(\alpha)} & F(y) & \xleftarrow{F(\beta)} & F(x) \\ \rho(z) \downarrow & \searrow \rho(\alpha) & \downarrow \rho(y) & \searrow \rho(\beta) & \downarrow \rho(x) \\ G(z) & \xleftarrow{G(\alpha)} & G(y) & \xleftarrow{G(\beta)} & G(x) \end{array} = \begin{array}{ccc} F(z) & \xleftarrow{F(\alpha\beta)} & F(x) \\ \rho(z) \downarrow & \searrow \rho(\alpha\beta) & \downarrow \rho(x) \\ G(z) & \xleftarrow{G(\alpha\beta)} & G(x) \end{array} .$$

- (c) For every 2-morphism

$$(A.7) \quad \begin{array}{ccc} & \alpha & \\ y & \begin{array}{c} \Downarrow \Sigma \\ \Downarrow \end{array} & x \\ & \gamma & \end{array} ,$$

³⁰For a pseudo-natural transformation, one requires this 2-morphism to be vertically invertible, motivated by the fact that equations should replace isomorphisms upon categorification [15]. In cases where invertibility is not imposed, one has two possibilities depending on the direction of this 2-morphism. A lax-natural transformation (see Definition 7.5.2 of [14]) uses the opposite direction, which is why we use the prefix oplax.

³¹Horizontal composition of 2-morphisms is written using \circ .

the diagram

$$(A.8) \quad \begin{array}{ccc} G(\alpha) \circ \rho(x) & \xleftarrow{\rho(\alpha)} & \rho(y) \circ F(\alpha) \\ \downarrow G(\Sigma) \circ \text{id}_{\rho(x)} & & \downarrow \text{id}_{\rho(y)} \circ F(\Sigma) \\ G(\gamma) \circ \rho(x) & \xleftarrow{\rho(\gamma)} & \rho(y) \circ F(\gamma) \end{array}$$

commutes, i.e.

$$(A.9) \quad \begin{array}{ccc} \begin{array}{ccc} & F(\alpha) & \\ \swarrow & \Downarrow & \searrow \\ F(y) & F(\Sigma) & F(x) \\ \downarrow \rho(y) & \downarrow F(\gamma) & \downarrow \rho(x) \\ G(y) & \rho(\gamma) & G(x) \\ & \Downarrow & \\ & G(\gamma) & \end{array} & = & \begin{array}{ccc} & F(\alpha) & \\ \swarrow & \Downarrow & \searrow \\ F(y) & \rho(\alpha) & F(x) \\ \downarrow \rho(y) & \downarrow G(\alpha) & \downarrow \rho(x) \\ G(y) & G(\Sigma) & G(x) \\ & \Downarrow & \\ & G(\gamma) & \end{array} \end{array}$$

The definition of a modification does not change if one uses oplax-natural transformations instead of pseudo-natural transformations.

Definition A.10. Let \mathcal{C} and \mathcal{D} be two 2-categories, $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two 2-functors, and $\rho, \sigma : F \Rightarrow G$ be two oplax-natural transformations. A modification m from σ to ρ , written as $m : \sigma \Rightarrow \rho$ and drawn as

$$(A.11) \quad \begin{array}{ccc} & F & \\ \swarrow & \Downarrow & \searrow \\ \mathcal{D} & \rho & \mathcal{C} \\ \swarrow & \Downarrow & \searrow \\ & G & \end{array} \quad m : \sigma \Rightarrow \rho$$

consists of a function $m : \mathcal{C}_0 \longrightarrow \mathcal{D}_2$ assigning a 2-morphism in \mathcal{D} to an object x in \mathcal{C} in the following manner

$$(A.12) \quad x \xrightarrow{m} \begin{array}{c} F(x) \\ \swarrow \quad \searrow \\ \rho(x) \quad \xleftarrow{m(x)} \quad \sigma(x) \\ \searrow \quad \swarrow \\ G(x) \end{array}$$

This assignment must satisfy the condition that for every 1-morphism $y \xleftarrow{\alpha} x$, the diagram

$$(A.13) \quad \begin{array}{ccc} G(\alpha) \circ \sigma(x) & \xleftarrow{\sigma(\alpha)} & \sigma(y) \circ F(\alpha) \\ \downarrow \text{id}_{G(\alpha)} \circ m(x) & & \downarrow m(y) \circ \text{id}_{F(\alpha)} \\ G(\alpha) \circ \rho(x) & \xleftarrow{\rho(\alpha)} & \rho(y) \circ F(\alpha) \end{array}$$

commutes, i.e.

$$(A.14) \quad \begin{array}{ccc} F(y) & \xleftarrow{F(\alpha)} & F(x) \\ \sigma(y) \swarrow \xrightarrow{m(y)} \rho(y) & \searrow \rho(\alpha) & \searrow \rho(x) \\ G(y) & \xleftarrow{G(\alpha)} & G(x) \end{array} = \begin{array}{ccc} F(y) & \xleftarrow{F(\alpha)} & F(x) \\ \sigma(y) \swarrow \xrightarrow{\sigma(\alpha)} \sigma(x) & \searrow m(x) & \searrow \rho(x) \\ G(y) & \xleftarrow{G(\alpha)} & G(x) \end{array} .$$

Compositions of oplax-natural transformations and modifications are not changed as a result of these alterations to the usual definitions. In particular, the vertical composition of oplax-natural transformations is denoted using vertical concatenation as in

$$(A.15) \quad \begin{array}{ccc} & F & \\ \curvearrowright & \Downarrow \begin{smallmatrix} \rho \\ \sigma \end{smallmatrix} & \curvearrowleft \\ \mathcal{D} & & \mathcal{C} \\ \curvearrowleft & \Uparrow H & \curvearrowright \end{array} := \begin{array}{ccc} & F & \\ \curvearrowright & \Downarrow \begin{smallmatrix} \rho \\ G \\ \sigma \end{smallmatrix} & \curvearrowleft \\ \mathcal{D} & & \mathcal{C} \\ \curvearrowleft & \Uparrow H & \curvearrowright \end{array}$$

and is defined by the assignments

$$(A.16) \quad x \xrightarrow{\begin{smallmatrix} \rho \\ \sigma \end{smallmatrix}} \begin{array}{c} F(x) \\ \downarrow \rho(x) \\ G(x) \\ \downarrow \sigma(x) \\ H(x) \end{array}$$

for each object x in \mathcal{C} and

$$(A.17) \quad y \xleftarrow{\alpha} x \xrightarrow{\rho} \begin{array}{ccc} F(y) & \xleftarrow{F(\alpha)} & F(x) \\ \rho(y) \downarrow & \searrow \rho(\alpha) & \downarrow \rho(x) \\ G(y) & \xleftarrow{G(\alpha)} & G(x) \\ \sigma(y) \downarrow & \searrow \sigma(\alpha) & \downarrow \sigma(x) \\ H(y) & \xleftarrow{H(\alpha)} & H(x) \end{array}$$

for each morphism $y \xleftarrow{\alpha} x$ in \mathcal{C} . Similarly, the vertical composition of modifications is denoted using vertical concatenation as in

$$(A.18) \quad \begin{array}{ccc} & F & \\ \curvearrowright & \Downarrow \begin{smallmatrix} \rho \\ \lambda \end{smallmatrix} \begin{smallmatrix} m \\ n \end{smallmatrix} \Downarrow \begin{smallmatrix} \sigma \\ \tau \end{smallmatrix} & \curvearrowleft \\ \mathcal{D} & & \mathcal{C} \\ \curvearrowleft & \Uparrow H & \curvearrowright \end{array} := \begin{array}{ccc} & F & \\ \curvearrowright & \Downarrow \begin{smallmatrix} \rho \\ \lambda \end{smallmatrix} \begin{smallmatrix} m \\ G \\ n \end{smallmatrix} \Downarrow \begin{smallmatrix} \sigma \\ \tau \end{smallmatrix} & \curvearrowleft \\ \mathcal{D} & & \mathcal{C} \\ \curvearrowleft & \Uparrow H & \curvearrowright \end{array}$$

and is defined by the assignment

$$(A.19) \quad x \xrightarrow{\quad m_n \quad} \begin{array}{c} F(x) \\ \rho(x) \leftarrow \begin{array}{c} \xleftarrow{m(x)} \\ \xrightarrow{\quad} \end{array} \rightarrow \sigma(x) \\ G(x) \\ \lambda(x) \leftarrow \begin{array}{c} \xleftarrow{n(x)} \\ \xrightarrow{\quad} \end{array} \rightarrow \tau(x) \\ H(x) \end{array}$$

for each object x in \mathcal{C} .

Definition A.20. Let \mathcal{C} be a (strict) 2-category. An adjunction in \mathcal{C} consists of a pair of objects x, y in \mathcal{C} , a pair of morphisms

$$(A.21) \quad x \xrightleftharpoons[g]{f} y$$

and a pair of 2-morphisms

$$(A.22) \quad \begin{array}{c} x \xrightarrow{\text{id}_x} x \\ \searrow f \quad \Downarrow \eta \quad \nearrow g \\ y \end{array} \quad \& \quad \begin{array}{c} x \\ \nearrow g \quad \Downarrow \epsilon \quad \searrow f \\ y \xrightarrow{\text{id}_y} y \end{array}$$

satisfying

$$(A.23) \quad \begin{array}{c} \text{id}_x \\ \curvearrowright \quad \Downarrow \eta \quad \curvearrowleft \\ x \xrightarrow{f} y \xrightarrow{g} x \xrightarrow{f} y \\ \quad \quad \quad \Downarrow \epsilon \quad \quad \quad \\ \quad \quad \quad \text{id}_y \end{array} = \begin{array}{c} f \\ \curvearrowright \quad \Downarrow \text{id}_f \quad \curvearrowleft \\ x \xrightarrow{\text{id}_f} y \end{array}$$

and

$$(A.24) \quad \begin{array}{c} \text{id}_x \\ \curvearrowright \quad \Downarrow \eta \quad \curvearrowleft \\ y \xrightarrow{g} x \xrightarrow{f} y \xrightarrow{g} x \\ \quad \quad \quad \Downarrow \epsilon \quad \quad \quad \\ \quad \quad \quad \text{id}_y \end{array} = \begin{array}{c} g \\ \curvearrowright \quad \Downarrow \text{id}_g \quad \curvearrowleft \\ y \xrightarrow{\text{id}_g} x \end{array} .$$

Conditions (A.23) and (A.24) are known as the zig-zag identities. An adjunction as above is typically written as a quadruple (f, g, η, ϵ) and we say f is left adjoint to g and write $f \dashv g$.

A left adjoint is unique in the following sense.

Lemma A.25. *Let \mathcal{C} be a (strict) 2-category and let x and y be two objects in \mathcal{C} . Let $x \xleftarrow{g} y$ be a 1-morphism and let (f, g, η, ϵ) and $(f', g, \eta', \epsilon')$ be adjunctions in which f and f' are both left adjoint to g . Then there exists a vertically invertible 2-morphism $\sigma : f \Rightarrow f'$ such that*

$$(A.26) \quad \begin{array}{ccc} x & \xrightarrow{\text{id}_x} & x \\ & \searrow f & \downarrow \eta \\ & \swarrow f' & \downarrow \eta' \\ y & & y \end{array} \quad = \quad \begin{array}{ccc} x & \xrightarrow{\text{id}_x} & x \\ & \searrow f' & \downarrow \eta' \\ & \swarrow f & \downarrow \eta \\ y & & y \end{array}$$

and

$$(A.27) \quad \begin{array}{ccc} & x & \\ g \nearrow & \downarrow \epsilon' & \searrow f' \\ y & \xrightarrow{\text{id}_y} & y \end{array} \quad = \quad \begin{array}{ccc} & x & \\ g \nearrow & \downarrow \epsilon & \searrow f \\ y & \xrightarrow{\text{id}_y} & y \end{array} .$$

In this paper, we focus on an adjunction in a particular 2-category obtained from functors between 2-categories.

Definition A.28. Let \mathcal{C} and \mathcal{D} be two (strict) 2-categories. Let $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ be the 2-category whose objects are (strict) functors from \mathcal{C} to \mathcal{D} , 1-morphisms are oplax-natural transformations, and 2-morphisms are modifications.

We spell out what it means to have an adjunction in this 2-category explicitly.

Lemma A.29. *Let \mathcal{C} and \mathcal{D} be two (strict) 2-categories and let $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ be the functor 2-category described in Definition (A.28). An adjunction in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ consists of two (strict) functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, two oplax-natural transformations $\sigma : F \Rightarrow G$ and $\rho : G \Rightarrow F$, and two modifications $\eta : \text{id}_F \Rightarrow \rho_\sigma$ and $\epsilon : \rho_\sigma \Rightarrow \text{id}_G$ such that the diagrams*

$$(A.30) \quad \begin{array}{ccc} & \rho_\sigma & \\ \epsilon_\rho \nearrow & \downarrow \rho_\sigma & \searrow \text{id}_\rho \\ \text{id}_G \xleftarrow{\rho} & \text{id}_\rho & \xrightarrow{\rho} \text{id}_F \end{array} \quad \& \quad \begin{array}{ccc} & \sigma_\rho & \\ \eta_\sigma \nearrow & \downarrow \sigma_\rho & \searrow \text{id}_\sigma \\ \text{id}_F \xrightarrow{\sigma} & \text{id}_\sigma & \xrightarrow{\sigma} \text{id}_G \end{array}$$

both commute, i.e.

$$(A.31) \quad \begin{array}{c} G \\ \downarrow \rho \\ F \xRightarrow{\epsilon} \text{id}_G \\ \downarrow \sigma \\ G \xRightarrow{\eta} \text{id}_F \\ \downarrow \rho \\ F \end{array} = \begin{array}{c} G \\ \xRightarrow{\text{id}_\rho} \rho \\ F \end{array} \quad \& \quad \begin{array}{c} F \\ \downarrow \sigma \\ G \xRightarrow{\eta} \text{id}_F \\ \downarrow \rho \\ F \xRightarrow{\epsilon} \text{id}_G \\ \downarrow \sigma \\ G \end{array} = \begin{array}{c} F \\ \xRightarrow{\text{id}_\sigma} \sigma \\ G \end{array} ,$$

respectively.

Remark A.32. Because the zig-zag identities only involve the equality of modifications, and since the datum of a modification consists only of an assignment of 2-morphisms in \mathcal{D} to objects of \mathcal{C} , they can be re-expressed as

$$(A.33) \quad \begin{array}{c} G(x) \\ \downarrow \rho(x) \\ F(x) \xRightarrow{\epsilon(x)} \text{id}_{G(x)} \\ \downarrow \sigma(x) \\ G(x) \xRightarrow{\eta(x)} \text{id}_{F(x)} \\ \downarrow \rho(x) \\ F(x) \end{array} = \begin{array}{c} G(x) \\ \xRightarrow{\text{id}_{\rho(x)}} \rho(x) \\ F(x) \end{array}$$

and

$$(A.34) \quad \begin{array}{c} F(x) \\ \downarrow \sigma(x) \\ G(x) \xRightarrow{\eta(x)} \text{id}_{F(x)} \\ \downarrow \rho(x) \\ F(x) \xRightarrow{\epsilon(x)} \text{id}_{G(x)} \\ \downarrow \sigma(x) \\ G(x) \end{array} = \begin{array}{c} F(x) \\ \xRightarrow{\text{id}_{\sigma(x)}} \sigma(x) \\ G(x) \end{array}$$

for every object x of \mathcal{C} , i.e. for every object x in \mathcal{C} , the quadruple $(\sigma(x), \rho(x), \eta(x), \epsilon(x))$ is an adjunction.

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INDEX OF NOTATION

Notation	Name/description	Location	Page
\mathcal{A}	unital C^* -algebra	Def'n 2.2	3
$\mathbf{C}^*\text{-Alg}$	category of unital C^* -algebras	Def'n 2.4	3
ω	a state (on some C^* -algebra)	Def'n 2.5	3
$\mathcal{S}(\mathcal{A})$	set of states on \mathcal{A}	Def'n 2.5	3
$\mathbf{Rep}(\mathcal{A})$	category of representations of \mathcal{A}	Def'n 2.6	4
\mathcal{H}	Hilbert space	Def'n 2.6	4
$\mathcal{B}(\mathcal{H})$	bounded linear operators on \mathcal{H}	Def'n 2.6	4
\mathcal{S}	states pre-sheaf	Prop 2.10	4
States	states pre-stack	Eqn (2.17)	6
Rep	representation pre-stack	Eqn (2.19)	6
\mathcal{N}_ω	null-space associated to ω	Eqn (3.5)	8
$[a]$	typical element of $\mathcal{A}/\mathcal{N}_\omega$	Eqn (3.10)	8
\mathcal{H}_ω	Hilbert space associated to ω via GNS	Eqn (3.12)	8
π_ω	representation associated to ω via GNS	Eqn (3.13)	8
$\mathbf{GNS}_\mathcal{A}$	GNS construction for \mathcal{A}	Con 3.1	7–8
\mathbf{GNS}_f	GNS construction for $\mathcal{A}' \xrightarrow{f} \mathcal{A}$	Con 3.17	10–11
GNS	the GNS construction	Thm 3.26	12
Ω	unit vector (occasionally cyclic)	Def'n 5.1	15
$(\pi, \mathcal{H}, \Omega)$	pointed (or cyclic) representation	Def'n 5.1	15
$\mathbf{Rep}^\bullet(\mathcal{A})$	category of pointed representations of \mathcal{A}	Def'n 5.4	15
$\mathbf{Rep}^\circ(\mathcal{A})$	category of cyclic representations of \mathcal{A}	Def'n 5.4	15
$\mathbf{rest}_\mathcal{A}$	restriction to states on \mathcal{A} functor	Con 5.11	16
ω_Ω	vector state $\langle \Omega, \cdot \Omega \rangle$	Eqn 5.12	16
\mathbf{Rep}^\bullet	pointed representation pre-stack	Con 5.21	17
\mathbf{rest}_f	restriction to states for $\mathcal{A}' \xrightarrow{f} \mathcal{A}$	Con 5.21	17

rest	restriction natural transformation	Prop 5.24	18
GNS[•]	pointed GNS construction	Thm 5.33	19
<i>m</i>	GNS modification	Lem 5.57	23

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