# Sum of Squares Method for Sensor Network Localization 

Jiawang Nie *

February 22, 2019


#### Abstract

We formulate the sensor network localization problem as finding the global minimizer of a quartic polynomial. Then sum of squares (SOS) relaxations can be applied to solve it. However, the general SOS relaxations are too expensive to implement for large problems. Exploiting the special features of this polynomial, we propose a new structured SOS relaxation, and discuss its various properties. When distances are given exactly, this SOS relaxation often returns true sensor locations. At each step of interior point methods solving this SOS relaxation, the complexity is $\mathcal{O}\left(n^{3}\right)$, where $n$ is the number of sensors. When the distances have small perturbations, we show that the sensor locations given by this SOS relaxation are accurate within a constant factor of the perturbation error under some technical assumptions. The performance of this SOS relaxation is tested on some randomly generated problems.


Key words: Sensor network localization, graph realization, distance geometry, polynomials, semidefinite program (SDP), sum of squares (SOS), error bound.

## 1 Introduction

An important problem in communication and information theory that has been paid much attention recently is sensor network localization. The basic description of this problem is as follows. For a sequence of unknown vectors (also called sensors) $x_{1}, x_{2}, \cdots, x_{n}$ in Euclidean space $\mathbb{R}^{d}(d=1,2, \cdots)$, we need to find their coordinates such that the distances (not necessarily all) between these sensors and the distances (not necessarily all) to other fixed sensors $a_{1}, \cdots, a_{m}$ (also called anchors) are equal to some given numbers. To be more specific, let $\mathcal{A}=\left\{(i, j) \in[n] \times[n]:\left\|x_{i}-x_{j}\right\|_{2}=d_{i j}\right\}$, and $\mathcal{B}=\left\{(i, k) \in[n] \times[m]:\left\|x_{i}-a_{k}\right\|_{2}=e_{i k}\right\}$, where $d_{i j}, e_{i k}$ are given distances and $[n]=\{1,2, \cdots, n\}$. Then the problem of sensor network localization is to find vectors $\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ such that $\left\|x_{i}-x_{j}\right\|_{2}=d_{i j}$ for every $(i, j) \in \mathcal{A}$ and $\left\|x_{i}-a_{k}\right\|_{2}=e_{i k}$ for every $(i, k) \in \mathcal{B}$. Denote by $G(\mathcal{A})$ the graph whose nodes are $[n]$ and whose edge set is $\mathcal{A}$. For simplicity of notation, let $D=\left(d_{i j}, e_{i k}\right)_{(i, j) \in \mathcal{A},(i, k) \in \mathcal{B}}$ be the given distance data.

Sensor network localization is also known as the graph realization or distance geometry [7. Given a graph $G=(V, E)$, graph realization is to assign each vertex a vector such that the distances between these vectors are equal to the given numbers associated to the edges. The distance geometry problem is to find atom positions of a molecule so that the distances between some atom pairs equal some given numbers. Distance geometry problems arise in the determination of protein structure.

[^0]The sensor locations can be determined by solving the polynomial system

$$
\begin{aligned}
\left\|x_{i}-x_{j}\right\|_{2}^{2}=d_{i j}^{2}, \quad(i, j) \in \mathcal{A} \\
\left\|x_{i}-a_{k}\right\|_{2}^{2}=e_{i k}^{2}, \quad(i, k) \in \mathcal{B} .
\end{aligned}
$$

For a small number of sensors, it might be possible to compute sensor locations by solving these equations. We refer to [29] for methods solving polynomial equations. However, solving polynomial system can be very expensive when there are a lot of sensors. Furthermore, this polynomial system may be inconsistent if the distances $d_{i j}$ or $e_{i k}$ have errors which often occur in practice. In general, the sensor network localization problem is NP-hard ( $1, ~ 18, ~ 26])$, even for the simplest case $d=1$.

Sensor network localization can also be formulated in term of global optimization. Obviously, $x_{1}, \cdots, x_{n}$ are true sensor locations if and only if the optimal value of problem

$$
\begin{equation*}
\min _{x, \cdots, x_{n} \in \mathbb{R}^{d}} \sum_{(i, j) \in \mathcal{A}}\left|\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right|+\sum_{(i, k) \in \mathcal{B}}\left|\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right| \tag{1.1}
\end{equation*}
$$

is zero. So the localization problem is equivalent to finding the global minimizer of this problem. This optimization problem is nonsmooth, nonconvex, and it is also NP-hard to find global solutions, since it is equivalent to sensor network localization problem. So approximation methods are of great interests. Recently, semidefinite programming (SDP) and second-order cone programming (SOCP) relaxations are proposed to solve this problem approximately.

The basic idea of SDP relaxation is to think of the quadratic terms as new variables and add one linear matrix inequality (LMI) they satisfy. Let $X=\left[x_{1}, x_{2}, \cdots, x_{n}\right] \in \mathbb{R}^{d \times n}$ be the matrix variable of sensor locations. Notice the identity:

$$
\left\|x_{i}-x_{j}\right\|_{2}^{2}=h_{i j}^{T} X^{T} X h_{i j}
$$

where $h_{i j}=e_{i}-e_{j}$. Here $e_{i}$ is the $i$-th standard unit vector in $\mathbb{R}^{n}$. By introducing a new variable $Y$, the problem (1.1) becomes

$$
\begin{aligned}
\min _{X \in \mathbb{R}^{d \times n}} & \sum_{(i, j) \in \mathcal{A}}\left|h_{i j} Y h_{i j}-d_{i j}^{2}\right|+\sum_{(i, k) \in \mathcal{B}}\left|\left[\begin{array}{c}
e_{i} \\
-a_{k}
\end{array}\right]^{T}\left[\begin{array}{cc}
Y & X^{T} \\
X & I_{d}
\end{array}\right]\left[\begin{array}{c}
e_{i} \\
-a_{k}
\end{array}\right]-e_{i k}^{2}\right| \\
\text { s.t. } & Y=X^{T} X .
\end{aligned}
$$

Here the quadratic constraint $Y=X^{T} X$ is nonconvex. The SDP relaxation replaces this nonconvex equality by the convex inequality $Y \succeq X^{T} X$, which is the same as

$$
\left[\begin{array}{cc}
Y & X^{T} \\
X & I_{d}
\end{array}\right] \succeq 0 .
$$

The SDP relaxation of (1.1) is

$$
\begin{align*}
\min _{\substack{X \in \mathbb{R}^{d \times n} \\
Y \in \mathbb{R}^{d \times d}}} & \sum_{(i, j) \in \mathcal{A}}\left|h_{i j} Y h_{i j}-d_{i j}^{2}\right|+\sum_{(i, k) \in \mathcal{B}}\left|\left[\begin{array}{c}
e_{i} \\
-a_{k}
\end{array}\right]^{T}\left[\begin{array}{cc}
Y & X^{T} \\
X & I_{d}
\end{array}\right]\left[\begin{array}{c}
e_{i} \\
-a_{k}
\end{array}\right]-e_{i k}^{2}\right|  \tag{1.2}\\
\text { s.t. } & {\left[\begin{array}{cc}
Y & X^{T} \\
X & I_{d}
\end{array}\right] \succeq 0 . } \tag{1.3}
\end{align*}
$$

We refer to [8, 9, [15, 27] for more details about SDP relaxation methods.
The SOCP relaxation comes in a similar way. By introducing new variables $t_{i j}$, problem (1.1) becomes

$$
\begin{aligned}
\min _{x_{i}, t_{i j}, t_{i k}} & \sum_{(i, j) \in \mathcal{A}}\left|t_{i j}-d_{i j}^{2}\right|+\sum_{(i, k) \in \mathcal{B}}\left|t_{i k}-e_{i k}^{2}\right| \\
\text { s.t. } & t_{i j}=\left\|x_{i}-x_{j}\right\|_{2}^{2} \\
& t_{i k}=\left\|x_{i}-a_{k}\right\|_{2}^{2} .
\end{aligned}
$$

Replacing nonconvex equalities $t_{i j}=\left\|x_{i}-x_{j}\right\|_{2}^{2}\left(t_{i k}=\left\|x_{i}-a_{k}\right\|_{2}^{2}\right)$ by convex inequalities $t_{i j} \geq\left\|x_{i}-x_{j}\right\|_{2}^{2}\left(t_{i k} \geq\left\|x_{i}-a_{k}\right\|_{2}^{2}\right)$, we get the SOCP relaxation

$$
\begin{align*}
\min _{x_{i}, t_{i j}, t_{i k}} & \sum_{(i, j) \in \mathcal{A}}\left|t_{i j}-d_{i j}^{2}\right|+\sum_{(i, k) \in \mathcal{B}}\left|t_{i k}-e_{i k}^{2}\right|  \tag{1.4}\\
\text { s.t. } & t_{i j} \geq\left\|x_{i}-x_{j}\right\|_{2}^{2}  \tag{1.5}\\
& t_{i k} \geq\left\|x_{i}-a_{k}\right\|_{2}^{2} . \tag{1.6}
\end{align*}
$$

Usually the SOCP relaxation (1.4)-(1.6) is weaker than the SDP relaxation (1.2)-(1.3), but (1.4)-(1.6) is easier to solve. We refer to [11, 30] for work in this area.

One common feature of SDP and SOCP relaxations is that the computed sensor locations are very inaccurate when the solution of the localization problem is not unique, because many numerical schemes for SDP or SOCP like primal-dual interior point methods often return the analytic center of the solution set. The motivation of this paper is to seek other efficient approaches which can find multiple (if possible) locations.

Now let us come back to the equivalent optimization problem (1.1). Obviously, only the global solutions to (1.1) give true sensor locations while local solutions do not. In general, there are no efficient algorithms to find global minimizers for general nonlinear functions. However, if the objective functions are multivariate polynomials, sum of squares (SOS) relaxations can be applied to solve the problem approximately, and in many situations the global minimizers can be found. We propose to apply SOS relaxations to solve the sensor network localization problem.

In problem (1.1), the objective is a sum of absolute values, and hence not a polynomial. So the SOS methods are not applicable. However, if we replace the absolute values by squares, we can get a new optimization problem

$$
\begin{equation*}
f^{*}:=\min _{X \in \mathbb{R}^{d \times n}} f(X):=\sum_{(i, j) \in \mathcal{A}}\left(\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right)^{2}+\sum_{(i, k) \in \mathcal{B}}\left(\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right)^{2} . \tag{1.7}
\end{equation*}
$$

$f(X)$ can be thought of as the squared $L_{2}$ norm of the error for a given guess $X$. The good property is that $f(X)$ is now a polynomial function of degree four. Therefore, the SOS methods are applicable.

However, if we directly apply the general SOS methods, the computation is very expensive and not practical for large problems. The main contribution of this paper is to propose a sparse SOS relaxation based on the special structures of $f(X)$, which is much easier to solve. The properties and implementations of this particular relaxation are discussed. Usually true sensor locations can be returned. When the distances are perturbed, the solutions returned by this SOS relaxation are shown to be accurate within a factor of the perturbation error.

The notations used in this paper are: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{N}$ denotes the set of nonnegative integers; $A \succeq 0(\succ 0)$ means matrix $A$ is symmetric positive semidefinite (definite); $A^{T}$ denotes the transpose of matrix $A ; \mathcal{S}_{+}^{N}$ denotes the cone of symmetric positive semidefinite matrices of length $N$; for a finite set $S,|S|$ denotes its cardinality; $\equiv$ means an identity; for any $\beta=\left(\beta_{1}, \cdots, \beta_{d}\right) \in \mathbb{N}^{d}$ and $x_{i}=\left(x_{1 i}, \cdots, x_{d i}\right) \in \mathbb{R}^{d}, x_{i}^{\beta}=x_{1 i}^{\beta_{1}} \cdots x_{d i}^{\beta_{d}}$; $|\beta|=\beta_{1}+\cdots+\beta_{d} ; \operatorname{supp}(p)$ denotes the support of polynomial $p(x) ;$ for any $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ with each $\alpha_{i} \in \mathbb{N}^{d}$ and $X=\left[x_{1}, \cdots, x_{n}\right] \in \mathbb{R}^{d \times n}, X^{\alpha}=\prod_{1 \leq i \leq n} x_{i}^{\alpha_{i}} ;|\alpha|=\left|\alpha_{1}\right|+\cdots+\left|\alpha_{n}\right| ;$ for any $x \in \mathbb{R}^{n},\|x\|_{2}=\sqrt{\sum_{i=1}^{n} x_{i}^{2}}$; for any $X=\left(X_{i j}\right) \in \mathbb{R}^{d \times n},\|X\|_{F}=\sqrt{\sum_{i j} X_{i j}^{2}}$. For any two given symmetric matrices $W$ and $S, W \bullet S$ denotes their inner product, i.e., $W \bullet S=$ $\sum_{i, j} W_{i j} S_{i j}=\operatorname{trace}(W S)$.

This paper is organized as follows. Section 2 gives a brief overview of SOS relaxations for minimizing polynomials; Section 3 proposes a structured SOS relaxation to minimize the
polynomial (1.7); Section 4 derives an error bound for the proposed SOS relaxation when distances have errors; Section 5 presents some numerical simulations; lastly Section 6 draws some conclusions.

## 2 Sum of squares (SOS) method

Recently, SOS relaxation receives considerable attention in the global optimization of multivariate polynomial functions. In this section, we give a very brief introduction in this area. We refer to [17, 19, 20, 21] for more details.

### 2.1. SOS polynomials

A polynomial $p(z)$ in $z=\left(z_{1}, \cdots, z_{N}\right)$ is said to be SOS if $p(z) \equiv \sum_{i} p_{i}(z)^{2}$ for some polynomials $p_{i}(z)$. Obviously, if $p(z)$ is SOS, then $p(z)$ is nonnegative, i.e., $p(z) \geq 0$ for all $z \in \mathbb{R}^{N}$. For instance, the polynomial

$$
\begin{aligned}
& \left(z_{1}^{4}+z_{2}^{4}+z_{3}^{4}+z_{4}^{4}-4 z_{1} z_{2} z_{3} z_{4}\right) \\
\equiv & \frac{1}{3}\left\{\left(z_{1}^{2}-z_{2}^{2}-z_{4}^{2}+z_{3}^{2}\right)^{2}+\left(z_{1}^{2}+z_{2}^{2}-z_{4}^{2}-z_{3}^{2}\right)^{2}+\left(z_{1}^{2}-z_{2}^{2}-z_{3}^{2}+z_{4}^{2}\right)^{2}+\right. \\
& \left.2\left(z_{1} z_{4}-z_{2} z_{3}\right)^{2}+2\left(z_{1} z_{2}-z_{3} z_{4}\right)^{2}+2\left(z_{1} z_{3}-z_{2} z_{4}\right)^{2}\right\}
\end{aligned}
$$

is SOS. This identity immediately implies that

$$
z_{1}^{4}+z_{2}^{4}+z_{3}^{4}+z_{4}^{4}-4 z_{1} z_{2} z_{3} z_{4} \geq 0, \quad \forall\left(z_{1}, z_{2}, z_{3}, z_{4}\right) \in \mathbb{R}^{4}
$$

which is an arithmetic-geometric mean inequality. However, the nonnegative polynomials are not necessarily SOS. In other words, the set of SOS polynomials (which is a cone) is properly contained in the set of nonnegative polynomials (which is a larger cone). The process of approximating nonnegative polynomials by SOS polynomials is called SOS relaxation.

The advantage of SOS polynomials over nonnegative polynomials is that it is more tractable to check whether a polynomial is SOS. To test whether a polynomial is SOS is equivalent to test the feasibility of some SDP [20, 21, which has efficient numerical methods. To illustrate this, suppose polynomial $p(z)$ has degree $2 \ell$ (SOS polynomials must have even degree). Then $p(z)$ is SOS if and only if [20, 21] there exists a symmetric matrix $W \succeq 0$ such that

$$
p(z) \equiv \mathbf{m}_{\ell}(z)^{T} W \mathbf{m}_{\ell}(z)
$$

where $\mathbf{m}_{\ell}(z)$ is the column vector of monomials up to degree $\ell$. For instance,

$$
\mathbf{m}_{2}\left(z_{1}, z_{2}\right)=\left[1, z_{1}, z_{2}, z_{1}^{2}, z_{1} z_{2}, z_{2}^{2}\right]^{T}
$$

As is well-known, the number of monomials in $z$ with degrees up to $\ell$ is $\binom{N+\ell}{\ell}$. Thus the size of matrix $W$ is $\binom{N+\ell}{\ell}$. This number can be very large. However, for fixed $\ell($ e.g. $\ell=2),\binom{N+\ell}{\ell}$ is polynomial in $N$. On the other hand, it is NP-hard (about $N$ ) to tell whether a polynomial is nonnegative whenever $2 \ell \geq 4$ (even when $\ell$ is fixed) [17].

### 2.2. SOS relaxation in polynomial optimization

Let $g(z)=\sum_{\alpha \in \mathcal{G}} g_{\alpha} z^{\alpha}$ be a polynomial in $z$. Here $\mathcal{G}$ is the support of $g(z)$. Consider the global optimization problem

$$
g^{*}:=\min _{z \in \mathbb{R}^{N}} g(z) .
$$

This problem is NP-hard when $\operatorname{deg}(g) \geq 4$. Recently, SOS relaxation attracts much attention in solving this problem. The standard SOS relaxation is that

$$
\begin{aligned}
g_{\text {sos }}^{*}:=\max _{\gamma} & \gamma \\
& \text { s.t. } \\
& g(z)-\gamma \text { being SOS } .
\end{aligned}
$$

Obviously we have that $g_{\text {sos }}^{*} \leq g^{*}$. In practice, SOS relaxation provides very good approximations, and often gives exact global minimum, i.e., $g_{s o s}^{*}=g^{*}$, even though theoretically there are many more nonnegative polynomials than SOS polynomials [6]. SOS relaxation is said to be exact if $g_{\text {sos }}^{*}=g^{*}$. In terms of SDP, the SOS relaxation can also be written as

$$
\begin{align*}
g_{\text {sos }}^{*}:=\max _{\gamma, W} & \gamma  \tag{2.1}\\
\text { s.t. } & g(z)-\gamma \equiv \mathbf{m}_{\ell}(z)^{T} W \mathbf{m}_{\ell}(z)  \tag{2.2}\\
& W \succeq 0 \tag{2.3}
\end{align*}
$$

where $2 \ell=\operatorname{deg}(g)$. Notice that (2.2) is an identity about $z$. The above program is convex about $(\gamma, W)$. A lower bound $g_{\text {sos }}^{*}$ can be computed by solving the resulting SDP. It can be shown [17] that the dual of (2.1)-(2.3) is

$$
\begin{align*}
g_{\text {mom }}^{*}:=\min _{y} & \sum_{\alpha \in \mathcal{G}} g_{\alpha} y_{\alpha}  \tag{2.4}\\
\text { s.t. } & M_{\ell}(y) \succeq 0  \tag{2.5}\\
& y_{(0, \cdots, 0)}=1 . \tag{2.6}
\end{align*}
$$

Here $M_{\ell}(y)$ is the moment matrix generated by moment vector $y=\left(y_{\alpha}\right)_{\alpha \in \mathcal{G}}$. Each $y_{\alpha}$ is called the $\alpha$-th moment. The rows and columns of moment matrix $M_{\ell}(y)$ are indexed by integer vectors in $\mathbb{N}^{N}$. Each entry of $M_{\ell}(y)$ is defined as $M_{\ell}(y)(\alpha, \beta):=y_{\alpha+\beta}$ for all $|\alpha|,|\beta| \leq \ell$. For instance, when $\ell=2$ and $N=2$, the vector

$$
y=\left[1, y_{1,0}, y_{0,1}, y_{2,0}, y_{1,1}, y_{0,2}, y_{3,0}, y_{2,1}, y_{1,2}, y_{0,3}, y_{4,0}, y_{3,1}, y_{2,2}, y_{1,3}, y_{0,4}\right]
$$

has moment matrix

$$
M_{2}(y)=\left[\begin{array}{cccccc}
1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\
y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\
y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\
y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\
y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\
y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4}
\end{array}\right]
$$

For SOS relaxation (2.1)-(2.3) and its dual (2.4)-(2.6), the strong duality holds [17, i.e., $g_{\text {sos }}^{*}=g_{\text {mom }}^{*}$. Hence $g_{\text {mom }}^{*}$ is also a lower bound for the global minimum $g^{*}$ of $g(z)$.

Now let us show how to extract minimizer(s) from optimal solutions to (2.4)-(2.6). Let $y^{*}$ be one optimal solution. If moment matrix $M_{\ell}\left(y^{*}\right)$ has rank one, then there exists one vector $w$ such that $M_{\ell}\left(y^{*}\right)=w w^{T}$. Normalize $w$ so that $w_{(0, \cdots, 0)}=1$. Set $z^{*}=w(2: N+1)$. Then $M_{\ell}\left(y^{*}\right)=w w^{T}$ immediately implies that $y^{*}=\mathbf{m}_{\ell}\left(z^{*}\right)$, i.e., $y_{\alpha}^{*}=\left(z^{*}\right)^{\alpha}$. So $g_{\text {mom }}^{*}=g\left(z^{*}\right)$. This says that a lower bound of $g(z)$ is attained at one point $z^{*}$. So $z^{*}$ is one global minimizer. When moment matrix $M_{\ell}\left(y^{*}\right)$ has rank more than one, the process described above does not work. However, if $M_{\ell}\left(y^{*}\right)$ satisfies the so-called flat extension condition

$$
\operatorname{rank} M_{k}\left(y^{*}\right)=\operatorname{rank} M_{k+1}\left(y^{*}\right)
$$

for some $0 \leq k \leq \ell-1$, we can extract more than one minimizers (in this case the global solution is not unique). When flat extension condition is met, it can be shown 10 that there exist distinct vectors $u_{1}, \cdots, u_{r}$ such that

$$
M_{k}\left(y^{*}\right)=\lambda_{1} \mathbf{m}_{k}\left(u_{1}\right) \cdot \mathbf{m}_{k}\left(u_{1}\right)^{T}+\cdots+\lambda_{r} \mathbf{m}_{k}\left(u_{r}\right) \cdot \mathbf{m}_{k}\left(u_{r}\right)^{T}
$$

for some $\lambda_{i}>0, \sum_{i=1}^{r} \lambda_{i}=1$. Here $r=\operatorname{rank} M_{k}\left(y^{*}\right)$. The set $\left\{u_{1}, \cdots, u_{r}\right\}$ is called a $r$ atomic representing support of moment matrix $M_{k}\left(y^{*}\right)$. Vectors $u_{1}, \cdots, u_{r}$ can be shown to be global minimizers of polynomial $g(z)$. These minimizers can be obtained numerically by solving some particular eigenvalue problem. We refer to 10 for flat extension conditions in moment problems and 14 for extracting minimizers.

### 2.3. Exploiting sparsity in SOS relaxation

As mentioned in subsection 2.1, the length of matrix $W$ in SOS relaxation is $\binom{N+\ell}{\ell}$ which can be very huge if either $N$ or $\ell$ is large. So the SOS relaxation is expensive to solve when either $N$ or $\ell$ is large. However, if polynomial $g(z)$ is sparse, i.e., its support $\mathcal{G}=\operatorname{supp}(g)$ is small, the size of the resulting SDP can be reduced significantly. Without loss of generality, assume $(0, \cdots, 0) \in \mathcal{G}$. Then $\operatorname{supp}(g)=\operatorname{supp}(g-\gamma)$ for any number $\gamma$.

Suppose $g(z)-\gamma=\sum_{i} \phi_{i}(z)^{2}$ is an SOS decomposition. Then by Theorem 1 in [24]

$$
\operatorname{supp}\left(\phi_{i}\right) \subset \mathcal{G}^{0}:=\left(\text { the convex hull of } \frac{1}{2} \mathcal{G}^{e}\right)
$$

where $\mathcal{G}^{e}=\{\alpha \in \mathcal{G}: \alpha$ is an even integer vector $\}$. The size of set $\mathcal{G}^{0}$ can be furtherly reduced [16, 31. Here we briefly describe the technique proposed by Waki et al. 31.

For polynomial $g(z)$, define its associated graph $G=([N], E)$ such that $(i, j) \in E$ if and only if $z_{i} z_{j}$ appears in some monomial of $g(z)$. 31 proposed to represent $g(z)-\gamma$ as

$$
g(z)-\gamma \equiv \sum_{i=1}^{K} s_{i}(z), \quad \text { each } s_{i}(z) \text { being SOS with } \operatorname{supp}\left(s_{i}\right) \subset C_{i}
$$

where $\left\{C_{1}, C_{2}, \cdots, C_{K}\right\}$ is the set of all maximal cliques of graph $G$ (this is not necessarily true when $g(z)-\gamma$ is SOS). Usually it is difficult to find all the maximal cliques of a general graph $G$. 31 proposed to replace $\left\{C_{1}, C_{2}, \cdots, C_{K}\right\}$ by the set of all maximal cliques of one chordal extension of $G$. We refer to [5] for properties of chordal graphs. For chordal graphs, there are efficient methods for finding all the maximal cliques. Chordal extension essentially uses the sparse symbolic Cholesky factorization. Let $R$ be the correlative sparsity pattern (csp) matrix of $g(z)$, i.e., $R$ is a random symmetric matrix such that $R(i, j)=0$ for all $z_{i} z_{j}(i \neq j)$ not appearing in any monomial of $g(z)$. Using csp matrix, 31 proposed to find a chordal extension $G^{\prime}$ of $G$ whose maximal cliques of $G^{\prime}$ can be found efficiently.

There is much work in exploiting sparsity in SOS relaxations for minimizing polynomials. We refer to 12, 16, 22, 31, and the references therein.

## 3 SOS relaxations for sensor network localization

This section discusses how to apply SOS relaxations to solve problem (1.7). The variable is $X=\left(x_{j i}\right)_{1 \leq j \leq d, 1 \leq i \leq n}$. Here each $x_{i}=\left[x_{1 i}, \cdots, x_{d i}\right]^{T}$ is the $i$-th sensor location to be computed. Since $f(X)$ is a quartic polynomial, SOS relaxations can be applied directly to find global minimizers. The standard SOS relaxation for (1.7) is

$$
\begin{align*}
f_{\text {sos }}^{*}:=\max & \gamma  \tag{3.1}\\
\text { s.t. } & f(X)-\gamma \equiv \mathbf{m}_{2}(X)^{T} W \mathbf{m}_{2}(X),  \tag{3.2}\\
& W \succeq 0 . \tag{3.3}
\end{align*}
$$

Here $\mathbf{m}_{2}(X)$ stands for the vector of all monomials in $X$ (in graded reverse lexicographical order) of degrees up to two. Its dual is

$$
\begin{align*}
f_{\text {mom }}^{*}:=\min _{y} & \sum_{\alpha \in P} f_{\alpha} y_{\alpha}  \tag{3.4}\\
\text { s.t. } & M_{2}(y) \succeq 0  \tag{3.5}\\
& y_{(0, \cdots, 0)}=1 . \tag{3.6}
\end{align*}
$$

By strong duality, we have $f_{\text {sos }}^{*}=f_{\text {mom }}^{*} \leq f^{*}$. The total number of decision variables in problem (1.7) is $d \cdot n$. The size of matrix $W$ and $M_{2}(y)$ is $\binom{n \cdot d+2}{2}=\mathcal{O}\left(n^{2} d^{2}\right)$, which is polynomial in $n$, $d$. To solve (3.1)-(3.6), there are polynomial time algorithms (e.g., interior point methods). On the other hand, the complexity to solve problem (1.7) is NP-hard. So theoretically we cannot expect SOS relaxation (3.1)-(3.3) to solve (1.7) correctly for every instance. But in practice, SOS relaxations usually provide very good approximations.
Theorem 3.1. If the sensor network localization problem admits a solution for the given data $D=\left(d_{i j}, e_{i k}\right)$, then SOS relaxation is exact, i.e., $f_{\text {sos }}^{*}=f^{*}=0$.

Proof. Since the problem admit a solution, there exists $\hat{X}=\left[\hat{x}_{1}, \cdots, \hat{x}_{n}\right] \in \mathbb{R}^{d \times n}$ such that

$$
\left\|\hat{x}_{i}-\hat{x}_{j}\right\|_{2}=d_{i j}, \quad \forall(i, j) \in \mathcal{A}, \quad\left\|\hat{x}_{i}-a_{k}\right\|_{2}=e_{i k}, \quad \forall(i, k) \in \mathcal{B} .
$$

Thus $f(\hat{X})=0$ and then $f^{*} \leq 0$. Since $f(X)$ is already SOS, we have $f_{\text {sos }}^{*} \geq 0$. Therefore $f_{\text {sos }}^{*}=0$ since $f_{\text {sos }}^{*} \leq f^{*}$.
Remark 3.2. When distance data $D$ admit a solution, the above conclusion that $f_{\text {sos }}^{*}=f^{*}=$ 0 is trivial. Then, does our SOS relaxation make any sense? In this case, the obtained lower bound $f_{\text {sos }}^{*}=0$ is not interesting. However, the solution $y^{*}$ to the dual problem (3.4)-(3.6) can help find sensor locations $x_{1}, x_{2}, \cdots, x_{n}$, as will be shown in the following example.

Example 3.3. Consider a simple example studied in [30], with $n=1, d=2, m=2 . \mathcal{A}=$ $\emptyset, \mathcal{B}=\{(1,1),(1,2)\}, d_{11}=d_{12}=2$. The anchors are $( \pm 1,0)$. For this problem, the SDP and SOCP relaxations return the origin $(0,0)$, while the true solutions are obviously $(0, \pm \sqrt{3})$. Now we formulate it as a polynomial optimization and then apply SOS relaxation to solve it. Problem (1.7) now becomes

$$
\min _{x_{11}, x_{21}} p\left(x_{11}, x_{21}\right):=\left(\left(x_{11}+1\right)^{2}+x_{21}^{2}-4\right)^{2}+\left(\left(x_{11}-1\right)^{2}+x_{21}^{2}-4\right)^{2} .
$$

The SOS relaxation (3.1)-(3.3) is

$$
\begin{aligned}
\max & \gamma \\
\text { s.t. } & p\left(x_{11}, x_{21}\right)-\gamma \equiv \mathbf{m}_{2}\left(x_{11}, x_{21}\right)^{T} W \mathbf{m}_{2}\left(x_{11}, x_{21}\right), \\
& W \succeq 0 .
\end{aligned}
$$

Its dual problem is

$$
\begin{gathered}
\min _{y} 18-4 y_{2,0}-12 y_{0,2}+2 y_{4,0}+4 y_{2,2}+2 y_{0,4} \\
{\left[\begin{array}{cccccc}
1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\
y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\
y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\
y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\
y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\
y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4}
\end{array}\right] \succeq 0 .}
\end{gathered}
$$

Primal-dual interior point methods can be used to solve these two SDPs simultaneously. Here we use software $S e D u M i\left[28\right.$ to solve them. The optimal solution is $\gamma^{*}=-2.21 \cdot 10^{-9}$ and

$$
\begin{aligned}
W^{*} & =\left[\begin{array}{rrrrrr}
18.0000 & 0.0000 & -0.0000 & -3.4082 & 0.0000 & -6.0000 \\
0.0000 & 2.8165 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\
-0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\
-3.4082 & 0.0000 & -0.0000 & 2.0000 & 0.0000 & 1.1361 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.7278 & -0.0000 \\
-6.0000 & -0.0000 & -0.0000 & 1.1361 & -0.0000 & 2.0000
\end{array}\right] . \\
M^{*} & =\left[\begin{array}{rrrrrr}
1.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 3.0000 \\
-0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\
0.0000 & -0.0000 & 3.0000 & 0.0000 & -0.0000 & 0.0000 \\
0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\
-0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 \\
3.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 8.9999
\end{array}\right] .
\end{aligned}
$$

The moment matrix $M^{*}$ has rank two, and satisfies the flat extension condition. Using the technique from [14], we can extract two points $(0.0000, \pm 1.7321)$. This example shows that the SOS relaxation not only finds the global minimum of $f(X)$, but also returns two global minimizers.

The softwares Gloptipoly 13 and SOSTOOLS 23 can be applied directly to solve problem (1.7). If we have a small number of sensors, Gloptipoly and SOSTOOLS work very well. However, for practical problems with a lot of sensors, the implementations of Gloptipoly and SOSTOOLS are usually very expensive. The reason is that the size of matrix in (3.1)-(3.3) or (3.4)-(3.6) is the number $\binom{n \cdot d+2}{2}$, which can be very large. For instance, when $n=50$ and $d=2$, this number is bigger than 5000 . So it is not practical to apply directly the general SOS solvers for sensor network localization problem.

However, it will make a big difference if we can exploit the sparsity of polynomial $f(X)$. From (1.7), we can see that $f(X)$ has very particular structures. Notice that $f(X)$ only has a few number of monomials of degree up to four. Therefore, the techniques to exploit sparsity introduced in Section 2.3 can be applied here. The method described in 31 has been implemented in software SparsePOP [32]. It can help solve larger problems.

We should mention that the performance of SparsePOP depends on the distribution of the edges in $\mathcal{A}$. For instance, when $G(\mathcal{A})$ is sequentially connected, i.e.,

$$
\mathcal{A}=\{(1,2),(2,3), \cdots,(n, n-1)\},
$$

SparsePOP can be applied to solve the localization problem for $(d, n)=(2,100)$ in PCs (e.g, a Laptop with 512 MB ), provided there are enough anchors to make the localization unique. On the other hand, if the edge set $\mathcal{A}$ is randomly sparse, the implementation of SparsePOP is still very expensive. SparsePOP randomly generates positive definite csp matrix (see Section 2.3) and then uses it to find a chordal extension by sparse Cholesky factorization. So if the csp matrix of $f(X)$ is banded with small bandwidth, SparsePOP works very well. However, for general unstructured csp matrices, their Cholesky factors are often dense. So when $\mathcal{A}$ is sparse but not banded, the implementation of SparsePOP is still expensive.

What further can we do in applying SOS ? This is the motivation of this section.

### 3.1. A more structured SOS relaxation

In the previous analysis of exploiting the sparsity, we ignored the fact that $f(X)$ is already
given in SOS form:

$$
f(X)=\sum_{(i, j) \in \mathcal{A}}\left\{\left(\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right)^{2}+\frac{1}{\left|S_{i}\right|} \sum_{k:(i, k) \in \mathcal{B}}\left(\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right)^{2}\right\}
$$

where $S_{i}=\{j:(i, j) \in \mathcal{A}\}$. This leads us to represent $f(X)-\gamma$ as

$$
f(X)-\gamma=\sum_{(i, j) \in \mathcal{A}} s_{i j}\left(x_{i}, x_{j}\right)
$$

where $s_{i j}\left(x_{i}, x_{j}\right)$ are SOS polynomials only in variables $\left(x_{i}, x_{j}\right)$, instead of all $x_{1}, \cdots, x_{n}$. This structure can help save the computation significantly. For this SOS representation, the corresponding SOS relaxation has the special form

$$
\begin{align*}
f_{\text {sos }}^{* *}:=\max & \gamma  \tag{3.7}\\
\text { s.t. } & f(X)-\gamma \equiv \sum_{(i, j) \in \mathcal{A}} \mathbf{m}_{2}\left(x_{i}, x_{j}\right)^{T} W_{i j} \mathbf{m}_{2}\left(x_{i}, x_{j}\right)  \tag{3.8}\\
& W_{i j} \succeq 0, \forall(i, j) \in \mathcal{A} . \tag{3.9}
\end{align*}
$$

The size of matrix $W_{i j}$ is $\binom{2 d+2}{2}=(d+1)(2 d+1)$, which is independent of $n$. The total number of decision variables in (3.7)-(3.9) is $\mathcal{O}\left(d^{4}|\mathcal{A}|\right)$. The dual of (3.7)-(3.9) is

$$
\begin{array}{cl}
f_{\text {mom }}^{* *}:=\sum_{\substack{y=\left(\begin{array}{c}
y_{\nu_{i j}(\alpha)}:(i, j) \in \mathcal{A}, \alpha \in \mathbb{N}^{2 d},|\alpha| \leq 4
\end{array}\right)}} \sum_{\substack{(i, j) \in \mathcal{A} \\
\left(\begin{array}{c}
\alpha=\left(\alpha_{1}, \alpha_{2}\right) \\
\left(\alpha_{1}, \alpha_{2}\right) \in \mathbb{N}^{d} \times \mathbb{N}^{d} \\
\left|\alpha_{1}\right|+\left|\alpha_{2}\right| \leq 4 \\
\hline
\end{array}\right.}} \hat{f}_{\alpha}^{i j} y_{\nu_{i j}(\alpha)}+\sum_{\substack{(i, k) \in \mathcal{B}}} \sum_{\substack{\beta \in \mathbb{N}^{d} \\
|\beta| \leq 4}} \tilde{f}_{\beta}^{i k} y_{\eta_{i}(\beta)} \\
\text { s.t. } & M_{2}\left(\varrho_{i j}(y)\right) \succeq 0, \quad(i, j) \in \mathcal{A} \\
& y_{(0, \cdots, 0)=1} \tag{3.12}
\end{array}
$$

where $\hat{f}_{\alpha}^{i j}, \tilde{f}_{\beta}^{i k}$ are the coefficients of polynomials

$$
\left(\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right)^{2}=\sum_{\substack{\alpha=\left(\alpha_{1}, \alpha_{2}\right) \\\left(\alpha_{1}, \alpha_{2}\right) \mathbb{N}^{2} \times \mathbb{N}^{d} \\\left|\alpha_{1}\right|+\left|\alpha_{2}\right| \leq 4}} \hat{f}_{\alpha}^{i j} x_{i}^{\alpha_{1}} x_{j}^{\alpha_{2}}, \quad\left(\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right)^{2}=\sum_{\substack{\beta \in \mathbb{N}^{d} \\|\beta| \leq 4}} \tilde{f}_{\beta}^{i k} x_{i}^{\beta} .
$$

and $\nu_{i j}(\alpha), \eta_{i}(\beta) \in \mathbb{N}^{n d}$ are the index vectors such that

$$
\begin{aligned}
& \nu_{i j}(\alpha)(k)= \begin{cases}0 & k \in[1, d(i-1)] \\
\alpha(k-d i+d) & k \in[d(i-1)+1, d i] \\
0 & k \in[d i+1,(j-1) d] \\
\alpha(k-d j+2 d) & k \in[d(j-1)+1, d j] \\
0 & k \in[d j+1, d n]\end{cases} \\
& \eta_{i}(\beta)(k)= \begin{cases}0 & k \in[1, d(i-1)] \\
\beta(k-d i+d) & k \in[d(i-1)+1, d i] . \\
0 & k \in[d i+1, d n]\end{cases}
\end{aligned}
$$

Here $\varrho_{i j}: \mathbb{N}^{n d} \rightarrow \mathbb{N}^{2 d}$ is the projection map defined such that

$$
\varrho_{i j}(y)(\zeta)=y_{\nu_{i j}(\zeta)}, \forall \zeta \in \mathbb{N}^{2 d}
$$

Theorem 3.4. For SOS relaxation (3.7)-(3.9) and its dual (3.10)-(3.12), the strong duality $f_{\text {sos }}^{* *}=f_{\text {mom }}^{* *}$ holds.

Proof. By standard duality theory for convex program, it suffices to show that (3.11) admits a strict interior point. We choose $y_{\alpha}$ as

$$
y_{\alpha}:=\frac{\int_{\mathbb{R}^{d \times n}} X^{\alpha} \exp \left\{-\|X\|_{F}^{2}\right\} d X}{\int_{\mathbb{R}^{d \times n}} \exp \left\{-\|X\|_{F}^{2}\right\} d X} .
$$

For every nonzero vector $c=\left(c_{\alpha}\right)_{|\alpha| \leq 2, \alpha=\left(\alpha_{1}, \alpha_{2}\right) \in \mathbb{N}^{2 d}}$, we have

$$
c^{T} M_{2}\left(\varrho_{i j}(y)\right) c=\frac{\int_{\mathbb{R}^{d \times n}}\left(\sum_{|\alpha| \leq 2} c_{\alpha} x_{i}^{\alpha_{1}} x_{j}^{\alpha_{2}}\right)^{2} \exp \left\{-\|X\|_{F}^{2}\right\} d X}{\int_{\mathbb{R}^{d \times n}} \exp \left\{-\|X\|_{F}^{2}\right\} d X}>0 .
$$

Then we have that $M_{2}\left(\varrho_{i j}(y)\right) \succ 0$ for every $(i, j) \in \mathcal{A}$, which completes the proof.
Theorem 3.5. If the sensor network localization problem admits a solution for the given data $D=\left(d_{i j}, e_{i k}\right)_{(i, j) \in \mathcal{A},(i, k) \in \mathcal{B}}$, then SOS relaxation (3.7)-(3.9) is exact, i.e., $f_{\text {sos }}^{* *}=f^{*}=0$.

Proof. From Theorem [3.1, we know $f^{*}=0$. Since $f(X)$ is already SOS with the representation of the form (3.2), we have that $f_{\text {sos }}^{* *} \geq 0$. and then $f_{\text {sos }}^{* *}=0$ since $f_{\text {sos }}^{* *} \leq f^{*}$.

Remark 3.6. Similar to Remark 3.2 when distance data $D=\left(d_{i j}, e_{i k}\right)_{(i, j) \in \mathcal{A},(i, k) \in \mathcal{B}}$ admit a solution, the obtained lower bound $f_{s o s}^{* *}=0$ is not interesting. But the solution to the dual problem (3.10)-(3.12) can help find the sensor locations $x_{1}, x_{2}, \cdots, x_{n}$. Let us illustrate this by another example.

Example 3.7. Consider the sensor network problem, as described in Figure 1 with four


Figure 1: $\square$ : anchor locations, $\circ$ : sensor locations.
sensors and four anchors

$$
a_{1}=\left[\begin{array}{l}
1 \\
1
\end{array}\right], a_{2}=\left[\begin{array}{r}
1 \\
-1
\end{array}\right], a_{3}=\left[\begin{array}{l}
-1 \\
-1
\end{array}\right], a_{4}=\left[\begin{array}{r}
-1 \\
1
\end{array}\right] .
$$

The network is as follows

$$
\mathcal{A}=\{(1,2),(1,4),(2,3),(3,4)\}, \mathcal{B}=\{(1,1),(2,2),(3,3),(4,4)\} .
$$

The distances are given by

$$
d_{12}=d_{14}=d_{23}=d_{34}=s=2-\sqrt{2}, e_{11}=e_{22}=e_{33}=e_{44}=1
$$

Let $X=\left[x_{1}, x_{2}, x_{3}, x_{4}\right]$. Then the polynomial given by (1.7) is

$$
\begin{aligned}
f(X)= & \left(\left\|x_{1}-x_{2}\right\|_{2}^{2}-s\right)^{2}+\left(\left\|x_{1}-x_{4}\right\|_{2}^{2}-s\right)^{2}+ \\
& \left(\left\|x_{2}-x_{3}\right\|_{2}^{2}-s\right)^{2}+\left(\left\|x_{3}-x_{4}\right\|_{2}^{2}-s\right)^{2}+ \\
& \left(\left\|x_{1}-a_{1}\right\|_{2}^{2}-1\right)^{2}+\left(\left\|x_{2}-a_{2}\right\|_{2}^{2}-1\right)^{2}+ \\
& \left(\left\|x_{3}-a_{3}\right\|_{2}^{2}-1\right)^{2}+\left(\left\|x_{4}-a_{4}\right\|_{2}^{2}-1\right)^{2} .
\end{aligned}
$$

For this problem, its SOS relaxation (3.7)-(3.8) is

$$
\begin{gathered}
\max \quad \gamma \\
\text { s.t. } \quad f(X)-\gamma \equiv \\
\mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]\right)^{T} \cdot W_{12} \cdot \mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]\right)+\mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{1} \\
x_{4}
\end{array}\right]\right)^{T} \cdot W_{14} \cdot \mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{1} \\
x_{4}
\end{array}\right]\right)+ \\
\mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{2} \\
x_{3}
\end{array}\right]\right)^{T} \cdot W_{23} \cdot \mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{2} \\
x_{3}
\end{array}\right]\right)+\mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{3} \\
x_{4}
\end{array}\right]\right)^{T} \cdot W_{34} \cdot \mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{3} \\
x_{4}
\end{array}\right]\right) \\
\\
W_{12}, W_{14}, W_{23}, W_{34} \in \mathcal{S}_{+}^{15} .
\end{gathered}
$$

The dual problem (3.7)-(3.8) can be written down similarly. For example, the LMI for pair $(1,2)$ looks like

$$
\left[\begin{array}{cc}
Y_{11} & Y_{12} \\
Y_{12}^{T} & Y_{22}
\end{array}\right] \succeq 0
$$

where

$$
Y_{11}=\left[\begin{array}{lllll}
y_{00000} & y_{10000} & y_{01000} & y_{00100} & y_{00010} \\
y_{10000} & y_{20000} & y_{11000} & y_{10100} & y_{10010} \\
y_{01000} & y_{11000} & y_{02000} & y_{01100} & y_{01010} \\
y_{00100} & y_{10100} & y_{01100} & y_{00200} & y_{00110} \\
y_{00010} & y_{10010} & y_{01010} & y_{00110} & y_{00020}
\end{array}\right]
$$

$Y_{12}$ is
$\left[\begin{array}{llllllllll}y_{20000} & y_{11000} & y_{10100} & y_{10010} & y_{02000} & y_{01100} & y_{01010} & y_{00200} & y_{00110} & y_{00020} \\ y_{30000} & y_{21000} & y_{20100} & y_{20010} & y_{12000} & y_{11100} & y_{11010} & y_{10200} & y_{10110} & y_{10020} \\ y_{21000} & y_{12000} & y_{11100} & y_{11010} & y_{03000} & y_{02100} & y_{02010} & y_{01200} & y_{01110} & y_{01020} \\ y_{20100} & y_{11100} & y_{10200} & y_{10110} & y_{02100} & y_{01200} & y_{01110} & y_{00300} & y_{00210} & y_{00120} \\ y_{20010} & y_{11010} & y_{10110} & y_{10020} & y_{02010} & y_{01110} & y_{01020} & y_{00210} & y_{00120} & y_{00030}\end{array}\right]$
and $Y_{22}$ is
$\left[\begin{array}{lllllllllll}y_{40000} & y_{31000} & y_{30100} & y_{30010} & y_{22000} & y_{21100} & y_{21010} & y_{20200} & y_{20110} & y_{20020} \\ y_{31000} & y_{22000} & y_{21100} & y_{21010} & y_{13000} & y_{12100} & y_{12010} & y_{11200} & y_{11110} & y_{11020} \\ y_{30100} & y_{21100} & y_{20200} & y_{20110} & y_{12100} & y_{11200} & y_{11110} & y_{10300} & y_{10210} & y_{10120} \\ y_{30010} & y_{21010} & y_{20110} & y_{20020} & y_{12010} & y_{11110} & y_{11020} & y_{10210} & y_{10120} & y_{10030} \\ y_{22000} & y_{13000} & y_{12100} & y_{12010} & y_{04000} & y_{03100} & y_{03010} & y_{02200} & y_{02110} & y_{02020} \\ y_{21100} & y_{12100} & y_{11200} & y_{11110} & y_{03100} & y_{02200} & y_{02110} & y_{01300} & y_{02110} & y_{01120} \\ y_{21010} & y_{12010} & y_{11110} & y_{11020} & y_{03010} & y_{02110} & y_{02020} & y_{01210} & y_{01120} & y_{01030} \\ y_{20200} & y_{11200} & y_{10300} & y_{10210} & y_{02200} & y_{01300} & y_{01210} & y_{00400} & y_{00310} & y_{00220} \\ y_{20110} & y_{11110} & y_{10210} & y_{10120} & y_{02110} & y_{01210} & y_{01120} & y_{00310} & y_{00220} & y_{00130} \\ y_{20020} & y_{11020} & y_{10120} & y_{10030} & y_{02020} & y_{01120} & y_{01030} & y_{00220} & y_{00130} & y_{00040}\end{array}\right]$.

In the above indices, $\mathbf{0}=0000$. Solving SOS relaxation (3.7)-(3.9), we get lower bound $f_{\text {sos }}^{*}=-1.7719 \cdot 10^{-9}$. Solving the dual (3.10)-(3.12), we get

$$
\begin{array}{rr}
y_{10000}^{*}= & 0.2929, \quad y_{01000}^{*}=r \\
y_{01000}^{*}= & -0.2929, \quad y_{00100}^{*}=-0.2929, \quad y_{00100}^{*}=0.2929, \quad y_{00010}^{*}=-0.2929, \\
y_{00010}^{*}=-0.2929, \quad y_{00001}^{*}=0.2929 .
\end{array}
$$

The moment matrices $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right.$ have rank one. So the minimizer can be extracted easily, which is

$$
x_{1}^{*}=\left[\begin{array}{l}
0.2929 \\
0.2929
\end{array}\right], x_{2}^{*}=\left[\begin{array}{r}
0.2929 \\
-0.2929
\end{array}\right], x_{3}^{*}=\left[\begin{array}{l}
-0.2929 \\
-0.2929
\end{array}\right], x_{4}^{*}=\left[\begin{array}{r}
-0.2929 \\
0.2929
\end{array}\right] .
$$

They are exactly the true locations $\left( \pm\left(1-\frac{\sqrt{2}}{2}\right), \pm\left(1-\frac{\sqrt{2}}{2}\right)\right)$ (we ignore rounding errors).

### 3.2. The algorithm and complexity

Now we discuss how to extract minimizer(s) $X^{*}=\left[x_{1}^{*}, \cdots, x_{n}^{*}\right]$ from the SOS relaxation (3.7)-(3.8). For each $1 \leq i \leq n$, the $i$-th sensor location $x_{i}^{*}$ can be extracted from the moment matrix $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ if it satisfies the flat extension condition, where $\varrho_{i}$ is the projection map defined such that

$$
\varrho_{i}(y)(\zeta)=y_{\eta_{i}(\zeta)}, \quad \forall \zeta \in \mathbb{N}^{d}
$$

Let $\mathcal{V}_{i}$ be the set of all the vectors which can be extracted from the moment matrix $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$. So $x_{i}^{*} \in \mathcal{V}_{i}$. If $\mathcal{V}_{i}$ is a singleton, then $x_{i}^{*}$ has a unique choice.

The situation will be subtle if some $\mathcal{V}_{i}$ has cardinality more than one. Suppose for some $(i, j) \in \mathcal{A}$ we have $\left|\mathcal{V}_{i}\right|>1$ and $\left|\mathcal{V}_{j}\right|>1$. Can $x_{i}^{*}\left(x_{j}^{*}\right)$ be arbitrary from the set $\mathcal{V}_{i}\left(\mathcal{V}_{j}\right)$ ? The answer to is usually no. Let us suppose there are two sensors $x_{1}$ and $x_{2}$ and the distance data $D$ is given such that the only two possible locations are

$$
x_{1}^{*}=\left[\begin{array}{c}
1 \\
-1
\end{array}\right], x_{2}^{*}=\left[\begin{array}{c}
-1 \\
1
\end{array}\right], \quad \text { or } \quad x_{1}^{*}=\left[\begin{array}{c}
-1 \\
1
\end{array}\right], x_{2}^{*}=\left[\begin{array}{c}
1 \\
-1
\end{array}\right] .
$$

Then we can see

$$
\mathcal{V}_{1}=\mathcal{V}_{2}=\left\{\left[\begin{array}{c}
1 \\
-1
\end{array}\right],\left[\begin{array}{c}
-1 \\
1
\end{array}\right]\right\} .
$$

But obviously we cannot choose $x_{i}^{*}\left(x_{j}^{*}\right)$ such that

$$
x_{1}^{*}=\left[\begin{array}{c}
1 \\
-1
\end{array}\right], x_{2}^{*}=\left[\begin{array}{c}
1 \\
-1
\end{array}\right], \quad \text { or } \quad x_{1}^{*}=\left[\begin{array}{c}
-1 \\
1
\end{array}\right], x_{2}^{*}=\left[\begin{array}{c}
-1 \\
1
\end{array}\right] .
$$

Now what is the rule for choosing $x_{i}^{*}\left(x_{j}^{*}\right)$ from $\mathcal{V}_{i}\left(\mathcal{V}_{j}\right)$ ? So far we have not yet used the information of moment matrix $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$. If $M_{2}\left(\varrho_{i j}(y)\right)$ also satisfies the flat extension condition, we can extract a pair of sensor locations $\left(x_{i}^{*}, x_{j}^{*}\right)$ from $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$. Let $\mathcal{X}_{i j}$ be set of all such pairs that can be extracted from $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$. Now we are wondering whether $\mathcal{V}_{i}$ and $\mathcal{X}_{i j}$ are consistent, i.e., does $\left(x_{i}^{*}, x_{j}^{*}\right) \in \mathcal{X}_{i j}$ imply that $x_{i}^{*} \in \mathcal{V}_{i}, x_{j}^{*} \in \mathcal{V}_{j}$ ? This induces the following theorem.

Theorem 3.8. Let $y^{*}$ be one optimal solution to (3.10)-(3.12). Suppose all $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)((i, j) \in$ $\mathcal{A})$ satisfy the flat extension condition. Let $\mathcal{V}_{i}, \mathcal{X}_{i j}$ be defined as above. Then for any $\left(x_{i}^{*}, x_{j}^{*}\right) \in$ $\mathcal{X}_{i j}$ we must have $x_{i}^{*} \in \mathcal{V}_{i}, x_{j}^{*} \in \mathcal{V}_{j}$.

Proof. Let $\mathcal{X}_{i j}=\left\{\left(x_{i}^{(1)}, x_{j}^{(1)}\right), \cdots,\left(x_{i}^{(r)}, x_{j}^{(r)}\right)\right\}$ be one $r$-atomic representing support of moment matrix $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$. Then we have decomposition

$$
M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)=\sum_{\ell=1}^{r} \lambda_{\ell} \mathbf{m}_{2}\left(\left[\begin{array}{c}
x_{i}^{(\ell)} \\
x_{j}^{(\ell)}
\end{array}\right]\right) \mathbf{m}_{2}\left(\left[\begin{array}{c}
x_{i}^{(\ell)} \\
x_{j}^{(\ell)}
\end{array}\right]\right)^{T}
$$

for some $\lambda_{1}, \cdots, \lambda_{r}>0, \sum_{\ell=1}^{r} \lambda_{\ell}=1$. Notice that $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ is a submatrix of $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$, which immediately implies

$$
M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)=\sum_{\ell=1}^{r} \lambda_{\ell} \mathbf{m}_{2}\left(x_{i}^{(\ell)}\right) \mathbf{m}_{2}\left(x_{i}^{(\ell)}\right)^{T} .
$$

This means that $\left.\left\{x_{i}^{(1)}, \cdots, x_{i}^{(r)}\right)\right\}$ is one $r$-atomic representing support of moment matrix $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ (some $x_{i}^{(\ell)}$ might be same). By definition of $\mathcal{V}_{i}$, we have $\left\{x_{i}^{(1)}, \cdots, x_{i}^{(r)}\right\} \subset \mathcal{V}_{i}$. Similarly we have $\left\{x_{j}^{(1)}, \cdots, x_{j}^{(r)}\right\} \subset \mathcal{V}_{j}$.
Theorem 3.9. Let $y^{*}$ be one optimal solution to (3.10)-(3.12). Assume all $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)((i, j) \in$ $\mathcal{A}$ ) satisfy the flat extension condition. Then any $X^{*}=\left[x_{1}^{*}, \cdots, x_{n}^{*}\right]$ such that $\left(x_{i}^{*}, x_{j}^{*}\right) \in$ $\mathcal{X}_{i j}((i, j) \in \mathcal{A})$ is a global solution to 1.7).

Proof. Fix $X^{*}=\left[x_{1}^{*}, \cdots, x_{n}^{*}\right]$ with $\left(x_{i}^{*}, x_{j}^{*}\right) \in \mathcal{X}_{i j}((i, j) \in \mathcal{A})$. Since moment matrix $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ satisfies flat extension condition, we have decomposition

$$
\left.M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)=\lambda_{i j} \mathbf{m}_{2}\left(\begin{array}{c}
x_{i}^{*} \\
x_{j}^{*}
\end{array}\right]\right) \mathbf{m}_{2}\left(\left[\begin{array}{c}
x_{i}^{*} \\
x_{j}^{*}
\end{array}\right]\right)^{T}+\hat{M}_{i j}
$$

where $1 \geq \lambda_{i j}>0$ and $\hat{M}_{i j} \succeq 0$. Now let $\lambda=\min _{(i, j) \in \mathcal{A}} \lambda_{i j}>0$ and

$$
M_{i j}=\left(\lambda_{i j}-\lambda\right) \mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{i}^{*} \\
x_{j}^{*}
\end{array}\right]\right) \mathbf{m}_{2}\left(\left[\begin{array}{l}
x_{i}^{*} \\
x_{j}^{*}
\end{array}\right]\right)^{T}+\hat{M}_{i j} \succeq 0 .
$$

Notice that $\hat{M}_{i j}$ and $M_{i j}$ are also moment matrices. Without loss of generality, we can assume $\lambda<1$, since otherwise each $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ has rank one and $X^{*}$ is obviously a global minimizer. Define vector $\hat{y}=\left(\hat{y}_{\nu_{i j}(\alpha)}:(i, j) \in \mathcal{A},|\alpha| \leq 4, \alpha \in \mathbb{R}^{2 d}\right)$ as follows

$$
\hat{y}_{\nu_{i j}(\alpha)}=\left(x_{i}^{*}\right)^{\alpha_{1}}\left(x_{j}^{*}\right)^{\alpha_{2}} .
$$

Let $\tilde{y}=\left(\tilde{y}_{\nu_{i j}(\alpha)}:(i, j) \in \mathcal{A},|\alpha| \leq 4, \alpha \in \mathbb{R}^{2 d}\right)$ be such that $y^{*}=\lambda \hat{y}+(1-\lambda) \tilde{y}$. Then we have

$$
M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)=\lambda M_{2}\left(\varrho_{i j}(\hat{y})\right)+(1-\lambda) M_{2}\left(\varrho_{i j}(\tilde{y})\right)
$$

and $(1-\lambda) M_{2}\left(\varrho_{i j}(\tilde{y})\right)=M_{i j}$. Obviously vector $\tilde{y}$ is feasible for (3.11)-(3.12) since

$$
M_{2}\left(\varrho_{i j}(\tilde{y})\right)=\frac{1}{1-\lambda} M_{i j} \succeq 0 .
$$

Let $\mathcal{L}(y)$ be the linear objective function defined in (3.10). Since $y^{*}$ is optimal, we have $\mathcal{L}\left(y^{*}\right) \leq \mathcal{L}(\hat{y})$ and $\mathcal{L}\left(y^{*}\right) \leq \mathcal{L}(\tilde{y})$. By linearity of $\mathcal{L}$, it holds

$$
\mathcal{L}\left(y^{*}\right)=\lambda \mathcal{L}(\hat{y})+(1-\lambda) \mathcal{L}(\tilde{y})=f_{\text {mom }}^{* *}
$$

Therefore, $\mathcal{L}(\hat{y})=f_{\text {mom }}^{* *}$ since $0<\lambda<1$. On the other hand, by definition of $\hat{y}$, we get $f\left(X^{*}\right)=\mathcal{L}(\hat{y})=f_{\text {mom }}^{* *}$, which implies that $X^{*}$ is a global minimizer of (1.7).

Remark 3.10. In the above theorems about $\mathcal{V}_{i}$ and $\mathcal{X}_{i j}$, we need the assumption that the flat extension condition holds. It poses some restrictions on the ranks of moment matrices and their submatrices. What is the geometric meaning behind this condition? Actually, flat extension is a definition frequently used in the area of truncated moment problem. Consider a finite sequence of moments $\gamma=\left(\gamma_{\alpha}\right)_{\alpha \in \mathbb{N}^{n},|\alpha| \leq 2 s}$ ( $s$ is an integer). The flat extension condition says that there exists an integer $0 \leq k<s$ such that

$$
\operatorname{rank} M_{k}(\gamma)=\operatorname{rank} M_{k+1}(\gamma)
$$

If this condition holds, then $\gamma$ has a finite atomic representing measure, i.e.,

$$
M_{s}(\gamma)=\lambda_{1} \mathbf{m}_{s}\left(u_{1}\right) \mathbf{m}_{s}\left(u_{1}\right)^{T}+\cdots+\lambda_{r} \mathbf{m}_{s}\left(u_{r}\right) \mathbf{m}_{s}\left(u_{r}\right)^{T}
$$

where $\lambda_{i}>0, u_{i}$ are distinct, and $r=\operatorname{rank} M_{k}(\gamma)$. In other words, flat extension condition guarantees a finite atomic representing measure whose support has cardinality $r$. But the
converse might not be true, that is, it is possible that flat extension condition fails and $\gamma$ has a representing measure whose support is infinite or finite but greater than $r$. When flat extension condition fails, the rank of $M_{s}(\gamma)$ must be greater than the rank of $M_{s-1}(\gamma)$. Then the truncated moment sequence $\gamma$ might have a finite atomic representing measure but with support cardinality greater than $r$, or might have an infinite representing measure, or it might not have a representing measure. We refer to [10] for more details about the meanings and roles of flat extension condition in truncated moment problem. Now we come back to the sensor network localization problem. Consider moment matrix $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$. If it satisfies the flat extension condition, then there exists $0 \leq k \leq 1$ such that

$$
\operatorname{rank} M_{k}\left(\varrho_{i}\left(y^{*}\right)\right)=\operatorname{rank} M_{k+1}\left(\varrho_{i}\left(y^{*}\right)\right)
$$

We know rank $M_{0}\left(\varrho_{i}\left(y^{*}\right)\right)=1$ and $\operatorname{rank} M_{1}\left(\varrho_{i}\left(y^{*}\right)\right) \leq d+1$, since the size of $M_{1}\left(\varrho_{i}\left(y^{*}\right)\right)$ is $d+1$. The flat extension condition guarantees that $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ has a finite atomic representing measure with support cardinality $\operatorname{rank} M_{1}\left(\varrho_{i}\left(y^{*}\right)\right)$. When this condition fails, it is because either $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ does not have a representing measure, or $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ has an infinite representing measure, or $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ has a finite representing measure whose support has cardinality greater than $\operatorname{rank} M_{1}\left(\varrho_{i}\left(y^{*}\right)\right)$. Thus, if sensor $x_{i}$ has more than $d+1$ locations, flat extension condition might fail and then we are not able to extract $x_{i}$ by applying the technique in [14. When sensor $x_{i}$ has at most $d+1$ possible locations, it is also possible that $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ fails the flat extension condition. It is not clear about the corresponding geometric meaning behind this phenomenon. Notice that sensor network localization problem is NP-hard even if the localization solution is unique, and sparse SOS relaxation (3.7)-(3.9) and its dual (3.10)-(3.12) are solvable in polynomial time. So even if sensor $x_{i}$ has a unique location, we cannot expect that $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ satisfies the flat extension condition, unless NP=P. A similar analysis can be applied to moment matrix $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$. If the sensor pair $\left(x_{i}, x_{j}\right)$ has more than $2 d+1$ locations, the flat extension condition might fail. Similarly, even when ( $x_{i}, x_{j}$ ) has at most $2 d+1$ possible locations this condition might still fail. It is not clear about the geometric meaning lying behind. This is an important and interesting future work.

Now we present the algorithm to solve sensor network localization as follows.
Algorithm 3.11 (Sensor Network Localization via Structured SOS Relaxation).
Input: $d, n, D=\left(d_{i j}, e_{i k}\right)_{(i, j) \in \mathcal{A},((i, j) \in \mathcal{B}}$
Output: $\mathcal{V}_{1}, \mathcal{V}_{2}, \cdots, \mathcal{V}_{n}$ and $\mathcal{X}_{i j}((i, j) \in \mathcal{A})$
Begin
Step 1: Solve the SDP problem (3.10)-(3.12). Get optimal solution $y^{*}$.
Step 2: For each $1 \leq i \leq n$, find the set $\mathcal{V}_{i}$ of vectors $x_{i}^{*}$ from $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$.
Step 3: For every $(i, j) \in \mathcal{A}$ with $\left|\mathcal{V}_{i}\right|+\left|\mathcal{V}_{j}\right|>2$, find the set $\mathcal{X}_{i j}$.

## End

Remark 3.12. The Algorithm 3.11 works when all the moment matrices $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ and $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ satisfy the flat extension condition. If some $M_{2}\left(\varrho_{i}\left(y^{*}\right)\right)$ or $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ does not satisfy the flat extension condition, it is usually because some sensor locations cannot be determined (see Remark 3.10). In such situations, we can apply a tiny perturbation to the coefficients in (3.10), and then simply set $x_{j i}^{*}=y_{\eta_{i}\left(e_{j}\right)}^{*}\left(e_{j}\right.$ is the $j$-th unit vector in $\left.\mathbb{R}^{d}\right)$, which often gives a good approximation for the true location.

Now let us give the complexity for solving sparse SOS relaxation (3.7)-(3.9) and its dual (3.10)-(3.12). Notice that the total number of dual variables is

$$
n\binom{d+4}{4}+|\mathcal{A}|\left\{\binom{2 d+4}{4}-2\binom{d+4}{4}\right\} .
$$

In practice $d$ is usually 1,2 or 3 . So we can assume $d$ is a constant. Thus the total number of dual variables is $\mathcal{O}(n+|\mathcal{A}|)$. If we use interior point methods (e.g., dual-scaling algorithm [2, 3, 4]) to solve this problem, the complexity at each step would be $\mathcal{O}\left((n+|\mathcal{A}|)^{3}\right)$. However, if we exploit the special structures of this sparse SOS relaxation, this complexity bound can be further reduced to $\mathcal{O}\left(n^{3}\right)$.

Notice that in moment matrix $M_{2}\left(\varrho_{i j}(y)\right)$ each entry is a single moment of the form $y_{\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right)}$ with $\alpha_{k} \in \mathbb{N}^{d}$ and $\left|\alpha_{i}\right|+\left|\alpha_{j}\right| \leq 4$, where only $\alpha_{i}, \alpha_{j}$ are nonzero and all other $\alpha_{k}(k \neq i, j)$ are zero vectors. Fix the pair $(i, j) \in \mathcal{A}$. Then the moment $y_{\left(0 \cdots 0 \alpha_{i}, 0 \cdots 0 \alpha_{j} 0 \cdots 0\right)}$ with $\alpha_{i} \neq 0, \alpha_{j} \neq 0$ will not appear in any other moment matrix $M_{2}\left(\varrho_{i j}(y)\right)((k, \ell) \neq(i, j))$. The number of all such moments in $M_{2}\left(\varrho_{i j}(y)\right)$ is

$$
\sigma(d) \stackrel{d e f i n e}{=}\binom{2 d+4}{4}-2\binom{d+4}{4}=\frac{1}{12}(d+1)(d+2)\left(7 d^{2}+9 d-6\right) .
$$

Order the edge pairs in $\mathcal{A}$ in the way such that

$$
\mathcal{A}=\left\{\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right), \cdots,\left(i_{|\mathcal{A}|}, j_{|\mathcal{A}|}\right)\right\} .
$$

Then we relabel the moments $y_{\alpha}$ as follows. Let the moments which only appear in $M_{2}\left(\varrho_{i_{1} j_{1}}(y)\right)$ be $z_{1}, z_{2}, \cdots, z_{\sigma(d)}$, let the moments which only appear in $M_{2}\left(\varrho_{i_{2} j_{2}}(y)\right)$ be $z_{\sigma(d)+1}, \cdots, z_{2 \sigma(d)}$, and similarly for subsequent pairs $\left(i_{k}, j_{k}\right)(k=3, \cdots,|\mathcal{A}|)$. The moments which only appear in $M_{2}\left(\varrho_{i_{|\mathcal{A}|} j_{|\mathcal{A}|}}(y)\right)$ are $z_{(|\mathcal{A}|-1) \sigma(d)+1}, \cdots, z_{|\mathcal{A}| \sigma(d)}$. Now we consider the moment of the form $y_{\left(0 \cdots 0 \alpha_{i} 0 \cdots 0\right)}$, i.e., only the $i$-th subvector of length $d$ in the index of $y_{\alpha}$ is nonzero. The moment $y_{\left(0 \cdots 0 \alpha_{i} 0 \cdots 0\right)}$ will appear in $M_{2}\left(\varrho_{k \ell}(y)\right)$ whenever $k=i$ or $\ell=i$. For each fixed $i$, the number of such moments is $\binom{d+4}{4}$. So the total number of such moments for one sensor is $\omega(d):=\binom{d+4}{4}$. Now we let the moments of the form $y_{\left(\alpha_{1} 0 \cdots 0\right)}$ be $z_{|\mathcal{A}| \sigma(d)+1}, \cdots, z_{|\mathcal{A}| \sigma(d)+\omega(d)}$, let the moments of the form $y_{\left(0 \cdots 0 \alpha_{2} 0 \cdots 0\right)}$ be $z_{|\mathcal{A}| \sigma(d)+\omega(d)+1}, \cdots, z_{|\mathcal{A}| \sigma(d)+2 \omega(d)}$, and similar for all subsequent $k \geq 3$. Lastly, we get a new labeling for the dual variables $y_{\alpha}$ in (3.10)-(3.12), which are

$$
z_{1}, \cdots, z_{|\mathcal{A}| \sigma(d)}, \cdots, z_{|\mathcal{A}| \sigma(d)+n \omega(d)} .
$$

Let $N=|\mathcal{A}| \sigma(d)+n \omega(d)$. Therefore the sparse SOS relaxation (3.7)-(3.9) and its dual (3.10)(3.12) can be written in the standard SDP form

$$
\begin{array}{llllr}
\min & C \bullet W & \text { s.t. } & A(W)=b, & W \succeq 0, \\
\max & b^{T} z & \text { s.t. } & A^{*} z+S=C, & S \succeq 0 . \tag{3.14}
\end{array}
$$

In (3.13)-(3.14), $A^{*} z=\sum_{k=1}^{N} z_{k} A_{k}$ and $A(W)_{k}=A_{k} \bullet W(k=1, \cdots, N) . A_{k}, C$ are constant matrices and $b$ is a constant vector. The definition of $A_{k}, C, b$ is explicit from (3.10)-(3.12). Notice that $A_{k}, C, W, S$ are block diagonal matrices of length $|\mathcal{A}|\binom{2 d+2}{2}$, with each block having size $\binom{2 d+2}{2} . W$ is the primal variable and $S, z$ are dual variables.

Now we apply the dual-scaling algorithm 2, 3,4 to solve (3.13)-(3.14). Applying Newton's method to $A(W)=b, A^{*} z+S=C$, and $W=\nu S^{-1}$ generates the system

$$
\begin{aligned}
A(W+\Delta W) & =b \\
A^{*} \Delta z+\Delta S & =0 \\
\nu S^{-1} \Delta S S^{-1}+\Delta W & =\nu S^{-1}-X .
\end{aligned}
$$

Once $\Delta z$ is known, $\Delta S$ and $\Delta W$ can be given explicitly. $\Delta z$ satisfies the linear system

$$
\nu\left[\begin{array}{ccc}
A_{1} \bullet S^{-1} A_{1} S^{-1} & \cdots & A_{1} \bullet S^{-1} A_{N} S^{-1}  \tag{3.15}\\
\vdots & \ddots & \vdots \\
A_{N} \bullet S^{-1} A_{1} S^{-1} & \cdots & A_{N} \bullet S^{-1} A_{N} S^{-1}
\end{array}\right] \Delta z=b-\nu A\left(S^{-1}\right)
$$

From the labeling of $z$ and block structure of $A_{k}$, we can see that $A_{k}$ and $A_{\ell}$ do not have common nonzero blocks whenever $z_{k}$ and $z_{\ell}(k, \ell \leq|\mathcal{A}| \sigma(d))$ come from different moment matrices $M_{2}\left(\varrho_{i j}(y)\right)$. Since $S$ is also a block matrix like $A_{k}, A_{\ell}$, it holds that $A_{k} \bullet S^{-1} A_{\ell} S^{-1}=0$ whenever $z_{k}$ and $z_{\ell}(k, \ell \leq|\mathcal{A}| \sigma(d))$ come from different moment matrices. Therefore the coefficient matrix in (3.15) has the sparsity pattern

$$
\left[\begin{array}{cccc}
{\left[\begin{array}{cccc}
F_{1} & 0 & 0 & 0 \\
0 & F_{2} & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & F_{\mid \mathcal{A |}}
\end{array}\right]} & \\
& G
\end{array}\right] .
$$

Each block matrix $F_{k}$ has constant size $\sigma(d)$. Matrix $G$ is of dimension $|\mathcal{A}| \sigma(d) \times n \omega(d)$, and $H$ is of dimension $n \omega(d) \times n \omega(d)$. If we use block LU factorization to solve linear system (3.15), it can be done in $\mathcal{O}\left(n^{3}+n|\mathcal{A}|\right)$ operations. Since $S$ is block diagonal, computing $S^{-1}$ costs $\mathcal{O}\left(|\mathcal{A}|+n^{3}\right)$ operations. Obviously, the cardinality of $\mathcal{A}$ is at most $\frac{1}{2} n(n-1)$. So the complexity of dual-scaling algorithm for one step is $\mathcal{O}\left(n^{3}\right)$.

Similarly, we can also get the complexity bound for SDP relaxation (1.2)-(1.3). But the resulting linear system in interior point methods (e.g., dual-scaling algorithm) for (1.2)-(1.3) usually does not have the sparsity pattern as (3.15) does. The complexity for (1.2)-(1.3) at each step is usually $\mathcal{O}\left((n+|\mathcal{A}|)^{3}\right)$. When $|\mathcal{A}|=\mathcal{O}(n)$, the complexities are almost the same. However, if $|\mathcal{A}|=\mathcal{O}\left(n^{1+\tau}\right)$ with $0<\tau \leq 1$, the complexity for sparse SOS relaxation is better. In this case, sparse SOS relaxation is faster to solve for large problems.

## 4 Error bound for the perturbation

In practice, the distance data $D=\left(d_{i j}, e_{i k}\right)_{(i, j) \in \mathcal{A}), e_{i k}((i, j) \in \mathcal{B}}$ may not be given exactly and often have errors due to the inaccuracy in measurements. We now consider the case that $D$ is perturbed by random noises $\epsilon_{i j}$ and $\epsilon_{i k}$, i.e.,

$$
\begin{aligned}
& d_{i j}=\hat{d}_{i j}\left(1+\epsilon_{i j}\right),(i, j) \in \mathcal{A} \\
& e_{i k}=\hat{e}_{i k}\left(1+\epsilon_{i k}\right),(i, k) \in \mathcal{B} .
\end{aligned}
$$

Here $\hat{d}_{i j}$ are true distance between sensor $x_{i}$ and $x_{j}$, and $\hat{e}_{i k}$ are true distance between sensor $x_{i}$ and anchor $a_{k}$. Throughout this section, we make two assumptions.
Assumption 4.1. The distance data $\hat{D}=\left(\hat{d}_{i j}, \hat{e}_{i k}\right)_{(i, j) \in \mathcal{A}(i, k) \in \mathcal{B}}$ is localizable, i.e., there exists $\hat{X}=\left[\hat{x}_{1}, \cdots, \hat{x}_{n}\right]$ such that $\left\|\hat{x}_{i}-\hat{x}_{j}\right\|_{2}=\hat{d}_{i j},\left\|\hat{x}_{i}-a_{k}\right\|_{2}=\hat{e}_{i k}$.

Assumption 4.2. The graph $G(\mathcal{A})$ is connected.
If $G(\mathcal{A})$ is not connected, then $G(\mathcal{A})$ can be decomposed into several connected components. On each component, we can solve an independent sensor network localization problem.

For the convenience of notations, let $\varepsilon=\left(\epsilon_{i j}, \epsilon_{i k}\right)_{i j \in \mathcal{A},(i, k) \in \mathcal{B}}$ be the perturbation and $\delta$ be the maximum error

$$
\delta=\|\varepsilon\|_{\infty}=\max \left(\max _{(i, j) \in \mathcal{A}}\left|\epsilon_{i j}\right|, \max _{(i, k) \in \mathcal{B}}\left|\epsilon_{i k}\right|\right) .
$$

Without loss of generality, assume $\varepsilon=\varepsilon(t)=t \tau$ for some constant vector

$$
\tau=\left(\tau_{i j}, \tau_{i k}\right)_{(i, j) \in \mathcal{A},(i, k) \in \mathcal{B}}
$$

with $\|\tau\|_{\infty}=1$. Here $t$ is a parameter belonging to interval $[0, \delta]$.

Let $y^{*}$ be one optimal solution to (3.10)-(3.12) with distance data $D=\left(d_{i j}, e_{i k}\right)_{(i, j) \in \mathcal{A},(i, k) \in \mathcal{B}}$, $X^{*}=\left[x_{1}^{*}, \cdots, x_{n}^{*}\right]$ be sensor locations extracted from moment matrix $M_{2}\left(y^{*}\right), \hat{y}$ be one optimal solution to (3.10)-(3.12) for distance data $\hat{D}=\left(\hat{d}_{i j}, \hat{e}_{i k}\right)_{(i, j) \in \mathcal{A}(i, k) \in \mathcal{B}}$ and $\hat{X}=$ $\left[\hat{x}_{1}, \cdots, \hat{x}_{n}\right]$ be true sensor locations. The goal of this section is to estimate the error $\left\|x_{i}^{*}-\hat{x}_{i}\right\|(1 \leq i \leq n)$.

Notice that the coefficients $\hat{f}_{\alpha}^{i j}, \tilde{f}_{\beta}^{i k}$ in (3.10) are linear functions with respect to vector

$$
\left(d_{i j}^{2}, d_{i j}^{4}, e_{i k}^{2}, e_{i k}^{4}\right)_{(i, j) \in \mathcal{A},(i, k) \in \mathcal{B}},
$$

and hence they are also functions of $t$. Actually they are univariate polynomials in $t$ of degree four. Denote the objective function in (3.10) by $\mathcal{L}(y, \varepsilon(t))$. We can see that $\mathcal{L}(y, \varepsilon(t))$ is linear with respect to $y$, but nonlinear (quartic polynomial) in $\varepsilon(t)$. In terms of parameter $t$, problem (3.4)-(3.6) becomes

$$
\begin{array}{rl}
f^{*}(\varepsilon(t)):=\min _{y} & \mathcal{L}(y, \varepsilon(t)) \\
\text { s.t. } & M_{2}\left(\varrho_{i j}(y)\right) \succeq 0, \forall(i, j) \in \mathcal{A} \\
& y_{(0, \cdots, 0)}=1 . \tag{4.3}
\end{array}
$$

Denote by $y(t)$ the optimal solution of the above. Then $y(0)=\hat{y}, y(\delta)=y^{*}$.
Assumption 4.3. For each $t \in[0, \delta]$, the solution $y(t)$ to problem 4.1)-4.3) is unique, and all the moment matrices $M_{2}\left(\varrho_{i j}(y(t))\right)$ have rank one.

Remark 4.4. If some moment matrix $M_{2}\left(\varrho_{i j}(y(t))\right)$ has rank more than one, it is usually because either the SOS relaxation (3.7)-(3.9) fails $\left(f_{\text {sos }}^{* *}<f^{*}\right.$, which is very rare from our computational experience), or there is more than one global solution to (1.7). In the latter case, the sensor locations cannot be uniquely determined from the given distance data $D$. In such situations, the solution set is not a singleton and its error analysis is more complicated. So we do not discuss that case here.

Under Assumption4.3 we can extract a unique minimizer $X(t)=\left[x_{1}(t), \cdots, x_{n}(t)\right]$ from the moment matrix $M_{2}\left(\varrho_{i j}(y(t))\right)$. Then we have $X(0)=\hat{X}, X(\delta)=X^{*}$.
Lemma 4.5. As $t \rightarrow 0$, it holds that $\|X(t)-\hat{X}\|_{2} \rightarrow 0$.
Proof. First, we prove that $X(t)$ must be bounded when $t \rightarrow 0$. Otherwise, there exists a sequence $\left\{t_{k}\right\}$ converging to zero such that $\left\|X\left(t_{k}\right)\right\|_{2} \rightarrow \infty$. Under Assumption 4.3 we know $X(t)$ is the unique global minimizer for polynomial $f(X)$. So we have

$$
f(X(t)) \leq f(0)=\sum_{(i, j) \in \mathcal{A}} d_{i j}^{4}+\sum_{(i, k) \in \mathcal{B}} e_{i k}^{4} \leq(1+\delta)^{4}\left\{\sum_{(i, j) \in \mathcal{A}} \hat{d}_{i j}^{4}+\sum_{(i, k) \in \mathcal{B}} \hat{e}_{i k}^{4}\right\}
$$

which implies that $\left\{f\left(X\left(t_{k}\right)\right)\right\}$ is bounded. Let $\digamma^{0}=\{i \in[n]: \exists \ell(i, \ell) \in \mathcal{B}\}$. If for some $i \in \digamma^{0}, x_{i}\left(t_{k}\right)$ goes to infinity as $t_{k} \rightarrow 0$, then

$$
f\left(X\left(t_{k}\right)\right) \geq\left(\left\|x_{i}\left(t_{k}\right)-a_{r}\right\|_{2}^{2}-e_{i, \ell}^{2}\right)^{2} \rightarrow \infty
$$

which is impossible. So for every $i \in \digamma^{0}$, the sequence $\left\{x_{i}\left(t_{k}\right)\right\}_{k=1}^{\infty}$ is bounded. Now let

$$
\digamma^{1}=\digamma^{0} \cup\left\{1 \leq i \leq n: \exists j \in \digamma^{0} \text { s.t. }(i, j) \in \mathcal{A} \text { or }(j, i) \in \mathcal{A}\right\} .
$$

Since the graph $G(\mathcal{A})$ is connected, $\digamma^{1}$ contains $\digamma^{0}$ properly if $\digamma^{0} \neq[n]$. If for some $i \in \digamma^{1} \backslash \digamma^{0}$, $x_{i}\left(t_{k}\right)$ goes to infinity, then there exists $j \in \digamma^{0}$

$$
f\left(X\left(t_{k}\right)\right) \geq\left(\left\|x_{i}\left(t_{k}\right)-x_{j}\left(t_{k}\right)\right\|_{2}^{2}-d_{i j}^{2}\right)^{2} \rightarrow \infty
$$

which is also impossible since $\left\{x_{j}\left(t_{k}\right)\right\}$ is bounded. So for every $i \in \digamma^{1},\left\{x_{i}\left(t_{k}\right)\right\}$ is bounded. Repeat this process, we can get a chain of strictly increasing subsets of [ $n$ ]

$$
\digamma^{0} \subsetneq \digamma^{1} \subsetneq \digamma^{2} \subsetneq \digamma^{3} \subsetneq \cdots
$$

such that for every $i \in \digamma^{\ell}\left\{x_{i}\left(t_{k}\right)\right\}$ is bounded. Since each $\digamma^{i}$ is a subset of finite set $[n]$, there must exist an integer $r$ such that $\digamma^{r}=[n]$. Thus for every $i \in[n], x_{i}\left(t_{k}\right)$ is bounded, which contradicts that $X\left(t_{k}\right) \rightarrow \infty$.

Now we show that $\|X(t)-\hat{X}\|_{2} \rightarrow 0$. Otherwise, there exists a sequence $\left\{t_{k}\right\} \rightarrow 0$ such that $X\left(t_{k}\right) \rightarrow \tilde{X} \neq \hat{X}$. Since $X(t)$ is bounded when $t \rightarrow 0$, we know that the minimizers of $f(X, \varepsilon(t))$ must be contained in a big ball $B(0, R)$ for $R$ large enough. So $f^{*}(\varepsilon(t))=$ $\min \{f(X, \varepsilon(t)): X \in B(0, R)\}$. By continuity, we know $f^{*}\left(\varepsilon\left(t_{k}\right)\right) \rightarrow f^{*}(\varepsilon(0))$. By Assumption 4.1 the sensor network localization is localizable for $\varepsilon(0)=0$. So $f(0)=0=f(\hat{X})$. Since $X(t) \rightarrow \tilde{X}$, we also get $f(\tilde{X})=0$. This implies that problem (4.1) - 4.3) has two distinct optimal solutions, which is not possible by Assumption 4.3

Now we are able to derive the error bound.
Theorem 4.6. For problem (4.1)-(4.3), assume the strong second order sufficient conditions hold, and the optimal solution set of the linearized (with respect to $\varepsilon(t)$ ) problem is nonempty (see Section 4.1 in [33]). Under Assumption 4.3, then it holds:
(i) $\left\|X^{*}-\hat{X}\right\|_{F} \leq L \cdot \delta$ for some constant $L$;
(ii) $f_{\text {mom }}^{* *} \leq \delta^{2}(1+\delta)^{2}\left\{\sum_{(i, j) \in \mathcal{A}} \hat{d}_{i j}^{4}+\sum_{(i, k) \in \mathcal{B}} \hat{e}_{i k}^{4}\right\}$.

Proof. (i) We prove by applying Theorem 4.1.12 in [33]. Consider $(y, \varepsilon)$ as $(x, u)$ in Theorem 4.1.12 in [33. We just check the assumptions there hold. First, by Lemma 4.5 we know that $X(t) \rightarrow \hat{X}$. Second, by the proof of Theorem 3.4, there exists a vector $\tilde{y}$ such that $M_{2}\left(\varrho_{i j}(\tilde{y})\right) \succ 0$ for every $(i, j) \in \mathcal{A}$. So the Robinson constraint qualification holds at $\hat{y}$ for $\varepsilon=0$. Thirdly, other two assumptions of Theorem 4.1.12 in 33 are also satisfied at this theorem. Thus we have $\|y(t)-\hat{y}\|_{2}=\mathcal{O}(t)$.

By Assumption 4.3 the minimizers $X(t)$ can be obtained directly from vector $y(t)$ (see the trick introduced in Section 2.3). Actually $x_{i}(t)$ are the entries of $y(t)$ with indices corresponding to monomials of degree one. So we have $\|X(t)-\hat{X}\|_{F} \leq\|y(t)-\hat{y}\|_{2}=\mathcal{O}(t)$. Let $t=\delta$. Then $\left\|X^{*}-\hat{X}\right\|_{F}=\mathcal{O}(\delta)$, and hence our claim ( $i$ ) holds.
(ii) Since $y^{*}$ is optimal, we have that

$$
f_{\text {mom }}^{* *} \leq f(\hat{X})=\sum_{(i, j) \in \mathcal{A}}\left(\left\|\hat{x}_{i}-\hat{x}_{j}\right\|_{2}^{2}-d_{i j}^{2}\right)^{2}+\sum_{(i, k) \in \mathcal{B}}\left(\left\|\hat{x}_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right)^{2}
$$

Then it holds

$$
\begin{aligned}
f_{\text {mom }}^{*} & \leq \sum_{(i, j) \in \mathcal{A}} \hat{d}_{i j}^{4}\left(\left(1+\epsilon_{i j}\right)^{2}-1\right)^{2}+\sum_{(i, k) \in \mathcal{B}} \hat{e}_{i k}^{4}\left(\left(1+\epsilon_{i k}\right)^{2}-1\right)^{2} \\
& \leq \delta^{2}(1+\delta)^{2}\left\{\sum_{(i, j) \in \mathcal{A}} \hat{d}_{i j}^{4}+\sum_{(i, k) \in \mathcal{B}} \hat{e}_{i k}^{4}\right\}
\end{aligned}
$$

which justifies our claim (ii).
Theorem 4.6 shows that the perturbed solution is accurate within a factor of perturbation error occurring in the distance data $D$, under some technical assumptions. This is observed in Example 5.3

## 5 Some numerical simulations

In this section, we show some numerical implementations in solving sensor network localization via SOS methods. As we discussed in Section 3, the general SOS solvers cannot be applied to solve large problem. So we use the sparse SOS relaxation (3.7)-(3.12) introduced in Section 3.1.

We randomly generate test problems which are similar to those given in 9, and then test the performance of SOS relaxation (3.7)-(3.12). Here 500 sensors $x_{1}^{*}, \cdots, x_{500}^{*}(n=500)$ are randomly generated from the unit square $[-0.5,0.5] \times[-0.5,0.5]$.. Anchors $\left(a_{1}, a_{2}, a_{3}, a_{4}, m=\right.$ 4) are chosen to be the four points $( \pm 0.45, \pm 0.45)$. We apply the sparse SOS relaxation (3.7)-(3.12) to find sensor locations. The computed locations are denoted by $\hat{x}_{1}, \cdots, \hat{x}_{n}$. The accuracy of the computed locations is be measured by the Root Mean Square Distance (RMSD) which is defined as

$$
\mathrm{RMSD}=\left(\frac{1}{n} \sum_{i=1}^{n}\left\|\hat{x}_{i}-x_{i}^{*}\right\|_{2}^{2}\right)^{\frac{1}{2}}
$$

We use RMSD and the consumed CPU time to test the performance. SOS relaxation (3.7)(3.12) will be solved by the sparse SDP solver $S e D u M i ~[28$. The computations are all implemented on a Linux machine with 0.98 GB memory and 1.46 GHz CPU. For these randomly generated problems, the SDP relaxation (1.2) -1.3 cannot be implemented due to the shortage of memory. But SOS relaxation (3.7)-(3.12) can be solved efficiently.

Example 5.1. Randomly generate 500 sensors $x_{1}^{*}, \cdots, x_{500}^{*}$ from the unit square $[-0.5,0.5] \times$ $[-0.5,0.5]$. The edge set $\mathcal{A}$ is chosen as follows. Initially set $\mathcal{A}=\emptyset$. Then for each $i$ from 1 to 500, compute the set $I_{i}=\left\{j \in[500]:,\left\|x_{i}^{*}-x_{j}^{*}\right\|_{2} \leq 0.3, j \geq i\right\}$; if $\left|I_{i}\right| \geq 10$, let $A_{i}$ the subset of $I_{i}$ consisting of the 10 smallest integers; otherwise, let $A_{i}=I_{i}$; then let $\mathcal{A}=\mathcal{A} \cup\{(i, j)$ : $\left.j \in A_{i}\right\}$. The edge set $\mathcal{B}$ is chosen such that $\mathcal{B}=\left\{(i, k) \in[n] \times[m]:\left\|x_{i}^{*}-a_{k}\right\|_{2} \leq 0.3\right\}$, i.e., every anchor is connected to all the sensors that are within distance 0.3 . For every $(i, j) \in \mathcal{A}$ and $(i, k) \in \mathcal{B}$, let the distances be

$$
d_{i j}=\left\|x_{i}^{*}-x_{j}^{*}\right\|_{2}, \quad e_{i k}=\left\|x_{i}^{*}-a_{k}\right\|_{2} .
$$

There are no errors in the distances. The computed results are plotted in Figure 2 The true sensor locations (denoted by circles) and the computed locations (denoted by stars) are connected by solid lines.

From Figure 2 we find that all the stars are located inside circles, which implies that SOS relaxation provides high quality locations. In this example, all the moment matrices $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ in (3.10)-(3.12) have numerical rank one. So every $\mathcal{V}_{i}$ has cardinality one. By Theorem 3.9 $X^{*}=\left[x_{1}^{*}, \cdots, x_{60}^{*}\right]$ with $x_{i}^{*} \in \mathcal{V}_{i}$ is the global minimizer of (1.7) and hence a solution to sensor network localization. The RMSD for SOS relaxation (3.7)-(3.12) is $2.9 \cdot 10^{-6}$ (the computed locations will be exact if we ignore rounding errors involved in floating point operations). Computing the coefficients for SOS relaxation (3.7)-(3.12) and the preprocessing of SeDuMi take about 4045 CPU seconds ( 1.12 hours). The interior point method in SeDuMi consumes about 1079 CPU seconds ( 18 minutes). We generate this random examples 20 times. Every time the RMSD is in the order $O\left(10^{-6}\right)$ and the CPU time consumed by SOS relaxation (3.7)-(3.12) is almost the same.

In Example 5.1 the set $\mathcal{A}$ and $\mathcal{B}$ are sufficient to determine the sensor location uniquely. However, if $\mathcal{A}$ and $\mathcal{B}$ do not contain enough edges, some senors may be uniquely localizable, but the others might not be. In such situations, the SOS relaxation (3.7)-(3.12) is able to find the correct locations for those sensors that are unique localizable, and give estimates for the other sensors that are not uniquely localizable. Let us see another example.


Figure 2: 500 sensors, sufficient edges, SOS relaxation (3.7)-(3.12)

Example 5.2. We generate random test problems almost in the same way as in the Example 5.1 except the following: if $\left|I_{i}\right| \geq 3$, let $A_{i}$ the subset of $I_{i}$ consisting of the 3 smallest integers; otherwise, let $A_{i}=I_{i}$. Then the number of edges might not be sufficient to determine the sensor locations. Assume there are no distance errors. The computed results are plotted in Figure 3 The true sensor locations (denoted by circles) and the computed locations (denoted by stars) are connected by solid lines.

From Figure 3 we can see that most stars are located inside circles, while a few are outside circles. The RMSD for SOS relaxation (3.7)-(3.12) is 0.0118 . For most edges $(i, j)$, the moment matrices $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ satisfy the flat extension condition. Only 22 moment matrices $M_{2}\left(\varrho_{i j}\left(y^{*}\right)\right)$ does not satisfy this condition. So we find that only 22 sensor locations are not correct (with error greater than $10^{-2}$ ), and all the others are correct (with accuracy $10^{-4}$ ). The consumed computational time is now less than the previous example. Computing the coefficients for SOS relaxation (3.7)-(3.12) and the preprocessing of SeDuMi take about 315 CPU seconds ( 5.25 minutes). The interior point method in SeDuMi consumes about 718 CPU seconds (11.9 minutes). From Section 3.2, we know the SOS relaxation has complexity $\mathcal{O}\left(n^{3}+n|\mathcal{A}|\right)$. This example has a much smaller set $\mathcal{A}$ than the previous one does. So it takes much less processing time to initialize $S e D u M i$.

The above two examples show that the SOS relaxation (3.7)-(3.12) can solve large scale sensor network localization problem very efficiently and accurately. Now we conclude this section by one example demonstrating the relationship between distance error and sensor location error.


Figure 3: 500 sensors, insufficient edges, SOS relaxation (3.7)-(3.12)

Example 5.3. Choose $\mathcal{A}, \mathcal{B}$ similarly as in Example 5.1. This usually makes the sensor network location problem have a unique solution. Then perturb the distances as follows

$$
\begin{aligned}
d_{i j} & =\left\|x_{i}^{*}-x_{j}^{*}\right\|_{2}(1+\epsilon \cdot \text { randn }) \\
e_{i k} & =\left\|x_{i}^{*}-a_{k}\right\|_{2}(1+\epsilon \cdot \text { randn })
\end{aligned}
$$

where $\epsilon \in\left\{10^{-4}, 5 \cdot 10^{-4}, 10^{-3}, 5 \cdot 10^{-3}, 10^{-2}, 5 \cdot 10^{-2}, 0.1,0.2,0.3\right\}$ and randn $\in \mathcal{N}(0,1)$. The computed errors RMSD are in the following table:

| $\epsilon$ | 0.0001 | 0.0005 | 0.0010 | 0.0050 | 0.0100 | 0.0500 | 0.1000 | 0.2000 | 0.3000 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| RMSD | $3.1 \cdot 10^{-5}$ | $1.8 \cdot 10^{-4}$ | $2.8 \cdot 10^{-4}$ | 0.0017 | 0.0031 | 0.0195 | 0.0420 | 0.0799 | 0.1852 |

The plot of $R M S D \backslash \varepsilon$ versus distance error $\epsilon$ is in Figure 4

We can see that the sensor location error RMSD has the same magnitude as the distance error $\epsilon$, which is consistent with the error analysis in Section 4.

## 6 Conclusions

In this paper, we formulate the sensor network localization problem as finding the global minimizer(s) of a quartic polynomial. Exploiting the special features of this polynomial, we propose a sparse SOS relaxation. The properties of this SOS relaxation are discussed. Under some technical assumptions, we show that the computed sensor locations are accurate within


Figure 4: The plot of RMSD/ $\epsilon$ versus accuracy parameter $\epsilon$.
a factor of the perturbation error. Our numerical simulations show that this SOS relaxation can solve large localization problem efficiently and accurately.

One natural concern is the reliability of SOS relaxation (3.7)-(3.12). By Theorem 3.5 we know the SOS relaxation is exact whenever the given distance data admits a solution. Furthermore, under the flat extension condition, by Theorem 3.9 we can also find true sensor locations (more than one solution can be returned if the localization is not unique). If the given distances have errors, (1.7) can be considered as a least squares problem. However, as we have discussed in Section 3, our SOS relaxation is not guaranteed to find true sensor locations for every instance, since the problem is NP-hard. To increase our confidence of the SOS relaxation, one may replace the polynomial in (1.7) by the randomly weighted polynomial

$$
\sum_{(i, j) \in \mathcal{A}} \xi_{i j}\left(\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right)^{2}+\sum_{(i, k) \in \mathcal{B}} \varsigma_{i k}\left(\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right)^{2}
$$

where $\xi_{i j}, \varsigma_{i k}$ are random positive numbers.
One important and interesting future work is to study the geometric meaning of flat extension condition. The perturbation error analysis when the localization solution is not unique is also interesting.

Acknowledgement. The author would like to thank Pratik Biswas, Jim Demmel, Laurent El Ghaoui, Bernd Sturmfels, Paul Tseng and Yinyu Ye for helpful discussions.

## References

[1] J. Aspnes, D. Goldberg and Y.R. Yang. On the computational complexity of sensor network localization. Lecture Notes in Computer Science (3121), Springer-Verlag, 2004, pp.32-44.
[2] S. J. Benson, Y. Ye, and X. Zhang. Solving large-scale sparse semidefinite programs for combinatorial optimization. SIAM Journal on Optimization, 10 (2000), pp. 443-461.
[3] S. J. Benson and Y.Ye. DSDP3: Dual scaling algorithm for general positive semidefinite programming. Tech. Report ANL/MCS-P851-1000, Mathematics and Computer Science Division, Argonne National Laboratory, Feb. 2001.
[4] S. J. Benson and Y. Ye. DSDP5: A software package implementing the dual-scaling algorithm for semidefinite programming. Tech. Report ANL/MCS-TM-255, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, June 2002.
[5] J. Blair and B. Peyton. An introduction to chordal graphs and lieque trees. In J. George, J. Gilbert, and J. Liu, editors, graph theory and sparse matrix computations, pages 1-30, Springer Verlag, 1993.
[6] G. Blekherman. Volumes of nonnegative polynomials, sums of squares, and powers of linear forms, preprint, arXiv:math.AG/0402158
[7] L. Blumenthal. Theory and applications of distance geometry, Chelsea Publishing Company, Bronx, New York, 1970.
[8] P. Biswas and Y. Ye. Semidefinite programming for ad hoc wireless sensor network localization. Proc. 3rd IPSN 46-54, 2004.
[9] P. Biswas, T.C. Liang, K.C. Toh, T.C. Wang and Y. Ye. Semidefinite Programming Approaches for Sensor Network Localization with Noisy Distance Measurements. IEEE Transactions on Automation Science and Engineering 3:4 (2006) 360-371.
[10] R.E. Curto and L.A. Fialkow. The truncated complex K-moment problem. Trans. Amer. Math. Soc. 352 (2000) 2825-2855.
[11] L. Doherty, K. S. J. Pister and L. El Ghaoui. Convex Position Estimation in Wireless Sensor Networks. Proc. 20th IEEE Infocom, Vol. 3 (2001), 1655-1663.
[12] K. Gatermann and P. Parrilo. Symmetry groups, semidefinite programs, and sums of squares. Journal of Pure and Appl. Algebra, Vol. 192, No. 1-3, pp. 95-128, 2004.
[13] D. Henrion and J. Lasserre. GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi. ACM Trans. Math. Soft., 29:165-194, 2003.
[14] D. Henrion and J. Lasserre. Detecting global optimality and extracting solutions in GloptiPoly. In Positive Polynomials in Control, D. Henrion and A. Garulli, eds., Lecture Notes on Control and Information Sciences, Springer Verlag, 2005.
[15] N. Krislock, V. Piccialli and H. Wolkowicz. Robust semidefinite programming approaches for sensor network localization with anchors. CORR 2006-12, May. 2006. http://orion.uwaterloo.ca/~hwolkowi/.
[16] M. Kojima, S. Kim and H. Waki. Sparsity in Sums of Squares of Polynomials. Mathematical Programming Vol. 103 (1) 45-62.
[17] J. Lasserre. Global optimization with polynomials and the problem of moments. SIAM Journal on Optimization, 11 (2001), No. 3, 796-817.
[18] J. Moré and Z. Wu. Global continuation for distance geometry problems, SIAM Journal on Optimization, 7(1997), 814-836.
[19] J. Nie, J. Demmel and B. Sturmfels. Minimizing Polynomials via Sum of Squares over the Gradient Ideal. Mathematical Programming, Series A, Vol. 106 (2006), No. 3, 587-606.
[20] P. Parrilo and B. Sturmfels. Minimizing polynomial functions, Proceedings of the DIMACS Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science (March 2001), (eds. S. Basu and L. GonzalezVega), American Mathematical Society, 2003, pp. 83-100.
[21] P. Parrilo. Semidefinite Programming relaxations for semialgebraic problems. Mathematical Programming, Ser. B 96 (2003), No. 2, 293-320.
[22] P. Parrilo. Exploiting structure in sum of squares programs. Proceedings for the $42 n d$ IEEE Conference on Decision and Control, Maui, Hawaii, 2003.
[23] S. Prajna, A. Papachristodoulou and P. Parrilo. SOSTOOLS User's Guide. Website: http://www.mit.edu/~parrilo/SOSTOOLS/
[24] B. Reznick. Extremal psd forms with few terms. Duke Math. J., 45, 363-374 (1978).
[25] B. Reznick. Some concrete aspects of Hilbert's $17^{\text {th }}$ problem. In Contemporary Mathematics, volume 253, pages 251-272. American Mathematical Society, 2000.
[26] J. Saxe. Embeddability of weighted graphs in k-space is strongly NP-hard, in Proc. 17th Allerton Conference in Communications, Control, and Computing, Monticello, IL, 1979, 480-489.
[27] A. Man-cho So and Y. Ye. The theory of semidefinite programming for sensor network localization. To appear in Math. Prog.. Website: http://www.stanford.edu/~yyye
[28] J.F. Sturm. SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, Optimization Methods and Software, 11\&12(1999)625-653.
[29] B. Sturmfels. Solving systems of polynomial equations. Amer.Math.Soc., CBMS regional conferences series, No. 97, providence, Rhode Island, 2002.
[30] P. Tseng. Second-order cone programming relaxation of sensor network localization, August, 2005, to appear in SIAM Journal on Optimization.
[31] H. Waki, S. Kim, M. Kojima and M. Muramatsu. Sums of Squares and Semidefinite Programming Relaxations for Polynomial Optimization Problems with Structured Sparsity. SIAM Journal on Optimization, Vol. 17 (1) 218-242 (2006).
[32] H. Waki, S. Kim, M. Kojima and M. Muramatsu. SparsePOP:a sparse semidefinite programming relaxation of polynomial optimization problems. http://www.is.titech.ac.jp/~kojima.
[33] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. Handbook of semidefinite programming. Kluwer's Publisher, 2000.


[^0]:    *Institute for Mathematics and its Applications (IMA), University of Minnesota, 400 Lind Hall, 207 Church SE, Minneapolis, MN 55455. Email: njw@ima.umn.edu

