A minimum effort optimal control problem for the wave equation

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Abstract A minimum effort optimal control problem for the undamped wave equation is considered which involves L^∞–control costs. Since the problem is non-differentiable a regularized problem is introduced. Uniqueness of the solution of the regularized problem is proven and the convergence of the regularized solutions is analyzed. Further, a semi-smooth Newton method is formulated to solve the regularized problems and its superlinear convergence is shown. Thereby special attention has to be paid to the well-posedness of the Newton iteration. Numerical examples confirm the theoretical results.

1 Introduction

In this paper a minimum effort problem for the wave equation is considered. Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, \ldots, 4\}$, be a bounded domain, $T > 0$, $J = (0, T)$,

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$Q = J \times \Omega$, and $\Sigma = J \times \partial\Omega$. We consider the following problem

$$\begin{cases} \min_{(y,u)\in X\times U} & \frac{1}{2}\left\|\mathbf{C}_{\omega_o}y - z\right\|^2_{L^2(Q)} + \frac{\alpha}{2}\left\|u\right\|^2_{L^\infty(Q)}, \quad \text{s.t.} \\ & \mathbf{A}y = \mathbf{B}_{\omega_c}u \quad \text{in } Q, \\ & y(0) = y_0 \quad \text{in } \Omega, \\ & \mathbf{C}y = 0 \quad \text{on } \Sigma \end{cases} \qquad (P_1)$$

for given state space X, control space U, initial point y_0, parameter $\alpha > 0$ and desired state $z \in L^2(Q)$. $\mathbf{A}$ denotes the wave operator, $\mathbf{B}_{\omega_c}$ the control operator, $\mathbf{C}_{\omega_o}\colon X \to L^2(Q)$ an observation operator, where $\omega_o, \omega_c \subset \Omega$ describe the area of observation and control, and $\mathbf{C}\colon X \to L^2(Q)$ denotes a boundary operator. A detailed formulation in a functional analytic setting is given in the following section.

The interpretation of the cost functional in (P_1) can be described as minimizing the tracking error by means of a control which is pointwise as small as possible. The appearance of the L^∞–control costs leads to nondifferentiability. The analytic and efficient numerical treatment of this nonsmooth problem by a semi-smooth Newton method stands in the focus of this work. We prove superlinear convergence of this iterative method and present numerical examples.

Numerical methods for minimum effort problems in the context of ordinary differential equation are developed in publications, see, e.g., Neustadt [13], and Ito and Kunisch [9] and the references given there. In the context of partial differential equations there exist only few results, see the publications on elliptic equations by Grund and Rösch [5] and Clason, Ito, and Kunisch [1].

The literature for numerical methods for optimal control of the wave equation is significantly less rich than for that for equations of parabolic type. Let us mention some selected contributions for the wave equation. In [6] Gugat treats state constrained optimal control problems by penalty techniques. Gugat and Leugering [7] analyze bang-bang properties for L^∞–norm minimal control problems for exact and approximate controllability problems and give numerical results. Time optimal control problems are considered by Kunisch and Wachsmuth [11] and semi-smooth Newton methods for control constrained optimal control problems with L^2–control costs in Kröner, Kunisch, and Vexler [10]. Gerdts, Greif, and Pesch in [4] present numerical results driving a string to rest and give further relevant references. A detailed analysis of discretization issues for controllability problems related to the wave equation is contained in the work by Zuazua, see e.g. [16] and Ervedoza and Zuazua [2].

We will present an equivalent formulation of the minimal effort problem (P_1) with a state equation having a bilinear structure and controls satisfying pointwise constraints, i.e. we move the difficulty of nondifferentiability of the control costs in the cost functional to additional constraints. To solve the problem we apply a semi-smooth Newton method.

Special attention has to be paid to the well-posedness of the iteration of the semi-smooth Newton scheme. The lack of smoothing properties of the

equations (when compared to elliptic equations) requires careful choice of the function space setting to achieve superlinear convergent algorithms; in particular, the problem is not reformulated in the reduced form as in [1] and the well-definedness of the Newton updates has to be shown.

The restriction to dimensions $d \leq 4$ is due to the Sobolev embedding theorem which is needed in Lemma 5.1 to verify Newton differentiability.

The paper is organized as follows. In Section 2 we make some preliminary remarks, in Section 3 we formulate the minimal effort problem, in Section 4 we present a regularized problem, in Section 5 we formulate the semi-smooth Newton method, in Section 6 we discretize the problem, and in Section 7 we present numerical examples.

2 Preliminaries

In this paper we use the usual notations for Lebesgue and Sobolev spaces. Further, we set

$$Y^1 = H_0^1(\Omega) \times L^2(\Omega), \qquad P^1 = L^2(\Omega) \times H_0^1(\Omega),$$
$$Y^0 = L^2(\Omega) \times H^{-1}(\Omega), \qquad P^0 = H^{-1}(\Omega) \times L^2(\Omega).$$

There holds the following relation between these spaces

$$(Y^1)^* = P^0, \quad (Y^0)^* = P^1,$$

where * indicates the dual space. For a Banach space Y we set

$$L^2(Y) = L^2(J, Y),$$
$$H^1(Y) = H^1(J, Y).$$

Further, we introduce the following operators

$$\mathbf{A} \colon X = L^2(Y^1) \cap H^1(Y^0) \longrightarrow Y = L^2(Y^0) \oplus \left(H^1(P^0)\right)^*,$$
$$\mathbf{A}^* \colon Y^* = L^2(P^1) \cap H^1(P^0) \longrightarrow X^* = L^2(P^0) \oplus \left(H^1(Y^0)\right)^*$$

with

$$\mathbf{A} = \begin{pmatrix} \partial_t & -\mathrm{id} \\ -\Delta & \partial_t \end{pmatrix}, \quad \mathbf{A}^* = \begin{pmatrix} -\partial_t & -\Delta \\ -\mathrm{id} & -\partial_t \end{pmatrix} \tag{2.1}$$

for the Laplacian $(-\Delta) \colon H_0^1(\Omega) \to H^{-1}(\Omega)$ and identity map $\mathrm{id} \colon L^2(Q) \to L^2(Q)$. For $(y, p) = (y_1, y_2, p_1, p_2) \in X \times Y^*$ there holds the relation

$$\langle \mathbf{A}y, p \rangle_{Y, Y^*} + (y(0), p(0))_{L^2(\Omega)} = \langle y, \mathbf{A}^*p \rangle_{X, X^*} + (y(T), p(T))_{L^2(\Omega)}, \tag{2.2}$$

since

$$\left\langle \begin{pmatrix} \partial_t y_1 - y_2 \\ \partial_t y_2 - \Delta y_1 \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\rangle_{Y, Y^*} + (y(0), p(0))_{L^2(\Omega)}$$
$$= \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} -\partial_t p_1 - \Delta p_2 \\ -\partial_t p_2 - p_1 \end{pmatrix} \right\rangle_{X, X^*} + (y(T), p(T))_{L^2(\Omega)}. \tag{2.3}$$

We introduce the observation and control operator

$$\mathbf{C}_{\omega_o} = \big(\chi_{\omega_o} \,\mathrm{id},\, 0\big) : L_2 \to L_2, \tag{2.4}$$

$$\mathbf{B}_{\omega_c} = \big(0,\, \chi_{\omega_c} \,\mathrm{id}\big) : L_2 \to L_2 \tag{2.5}$$

with the characteristic functions χ_{ω_o} and χ_{ω_c} of $J \times \omega_o$ and $J \times \omega_c$ for given nonempty open subsets $\omega_o, \omega_c \subset \Omega$. Here we used the notation $L_2 = L^2(L^2(\Omega) \times L^2(\Omega))$. Further, we define the boundary operators

$$\mathbf{C} = \boldsymbol{C}_\Omega,$$

$$\mathbf{B} = \boldsymbol{B}_\Omega.$$

For the inner product in $L^2(Q)$ we write

$$(\cdot, \cdot) = (\cdot, \cdot)_{L^2(Q)}.$$

Throughout this paper $\mathcal{C} > 0$ denotes a generic constant.

3 The minimum effort problem for the wave equation

In this section we present the minimum effort problem (P_1) in detail and formulate an equivalent problem in which we move the difficulty of the nondifferentiability of the control costs to additional control constraints. Furthermore we derive the optimality system for the latter problem.

To make the minimum effort problem (P_1) precise we choose $U = L^\infty(Q)$, $y_0 \in Y^1$ and the operators $\mathbf{A}, \mathbf{B}_{\omega_c} \mathbf{C}_{\omega_o}, \mathbf{C}$ and $\mathbf{B}$ as defined in the previous section.

Problem (P_1) can be formulated equivalently as

$$\begin{cases} \displaystyle\min_{(y,u,c)\in X\times U\times \mathbb{R}_0^+} \quad J(y,c) = \frac{1}{2}\left\|\mathbf{C}_{\omega_o}y - z\right\|_{L^2(Q)}^2 + \frac{\alpha}{2}c^2, \quad \text{s.t.} \\[2mm] \qquad\qquad \mathbf{A}y = \mathbf{B}_{\omega_c}u \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \qquad \text{in } \Omega, \\ \qquad\qquad \mathbf{C}y = 0 \qquad \text{on } \Sigma, \\ \qquad\qquad \|u\|_U \le c. \end{cases} \tag{P_2}$$

Except for the case $c = 0$ problem (P_2) is equivalent to problem (P_3) given by

$$\begin{cases} \displaystyle\min_{(y,u,c)\in X\times U\times \mathbb{R}_0^+} \quad J(y,c) = \frac{1}{2}\left\|\mathbf{C}_{\omega_o}y - z\right\|_{L^2(Q)}^2 + \frac{\alpha}{2}c^2, \quad \text{s.t.} \\[2mm] \qquad\qquad \mathbf{A}y = c\mathbf{B}_{\omega_c}u \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \qquad \text{in } \Omega, \\ \qquad\qquad \mathbf{C}y = 0 \qquad \text{on } \Sigma, \\ \qquad\qquad \|u\|_U \le 1. \end{cases} \tag{P_3}$$

By standard arguments the existence of a solution $(y^*, u^*, c^*) \in X \times U \times \mathbb{R}_0^+$ of problem (P_3) can be verified. The reformulation of the problem has the advantage that the domain of the control space does not depend on the parameter c.

Remark 3.1 For $c = 0$ any control u with $\|u\|_U \leq 1$ is a minimizer of (P_3). To avoid this case we assume that

$$J(y^*, c^*) < \frac{1}{2} \|z\|_{L^2(Q)}^2 \tag{3.1}$$

for a solution (y^*, u^*, c^*). If $c = 0$ problem (P_3) reduces to

$$\begin{cases} \min_{(y,u) \in X \times U} & J(y) = \frac{1}{2} \|\mathbf{C}_{\omega_o} y - z\|_{L^2(Q)}^2, \quad \text{s.t.} \\ & \mathbf{A}y = 0 \quad \text{in } Q, \\ & y(0) = y_0 \quad \text{in } \Omega, \\ & \mathbf{C}y = 0 \quad \text{on } \Sigma, \\ & \|u\|_U \leq 1. \end{cases}$$

Thus, y is determined by the equation and u can be chosen arbitrarily as far as the pointwise constraints are satisfied. If (3.1) holds, we have

$$\frac{1}{2} \|\mathbf{C}_{\omega_o} y - z\|_{L^2(Q)}^2 + \frac{\alpha}{2} c^2 < \frac{1}{2} \|z\|_{L^2(Q)}^2$$

and with $c = 0$ this leads to the contradiction

$$\frac{1}{2} \|z\|_{L^2(Q)}^2 < \frac{1}{2} \|z\|_{L^2(Q)}^2.$$

By standard techniques the optimality system can be derived.

Lemma 3.1 *The optimality system for problem (P_3) is given by*

$$\begin{cases} \mathbf{A}^* p + \mathbf{C}_{\omega_o}^* \mathbf{C}_{\omega_o} y - \mathbf{C}_{\omega_o}^* z = 0, \quad p(T) = 0, \quad \mathbf{B}p|_\Sigma = 0, \\ \quad (-\mathbf{B}_{\omega_c}^* p, \delta u - u) \geq 0 \quad \text{for all } \delta u \text{ with } \|\delta u\|_{L^\infty(Q)} \leq 1, \\ \quad \alpha c - (u, \mathbf{B}_{\omega_c}^* p) = 0, \\ \quad \mathbf{A}y - c\mathbf{B}_{\omega_c} u = 0, \quad y(0) = y_0, \quad \mathbf{C}y|_\Sigma = 0 \end{cases} \tag{3.2}$$

with $p \in Y^*$.

From the pointwise inspection of the second relation in the optimality system (3.2) we obtain for $(t, x) \in J \times \Omega$

$$u(t, x) = \begin{cases} 1 & \text{if } \mathbf{B}_{\omega_c}^* p(t, x) > 0, \\ -1 & \text{if } \mathbf{B}_{\omega_c}^* p(t, x) < 0, \\ s \in [-1, 1] & \text{if } \mathbf{B}_{\omega_c}^* p(t, x) = 0 \end{cases} \tag{3.3}$$

or equivalently $u \in \mathrm{Sgn}(\mathbf{B}_{\omega_c}^* p)$ with

$$\mathrm{Sgn}(s) = \begin{cases} 1 & \text{if } s > 0, \\ -1 & \text{if } s < 0, \\ [-1,1] & \text{if } s = 0. \end{cases}$$

Eliminating the control we obtain the reduced system

$$\begin{cases} \mathbf{A}^* p + \mathbf{C}_{\omega_o}^* \mathbf{C}_{\omega_o} y - \mathbf{C}_{\omega_o}^* z = 0, \quad p(T) = 0, \quad \mathbf{B}p|_\Sigma = 0, \\ \quad \alpha c - \left\| \mathbf{B}_{\omega_c}^* p \right\|_{L^1(Q)} = 0, \\ \quad \mathbf{A}y - c\mathbf{B}_{\omega_c} \mathrm{sgn}(\mathbf{B}_{\omega_c}^* p) \ni 0, \quad y(0) = y_0, \quad \mathbf{C}y|_\Sigma = 0. \end{cases} \tag{3.4}$$

Under certain conditions the solution of (P_3) is unique.

Lemma 3.2 *For $c \neq 0$ and $\omega_c \subset \omega_o$ the solution of problem (P_3) is unique if we set the control to zero on $Q \setminus (J \times \omega_c)$.*

Remark 3.2 The value of the control on the domain $Q \setminus (J \times \omega_c)$ has no influence on the solution of the control problem as far as $\|u\|_{L^\infty(Q \setminus (J \times \omega_c))} \leq \|u\|_{L^\infty(J \times \omega_c)}$. To obtain uniqueness we set $u \equiv 0$ on $Q \setminus (J \times \omega_c)$.

Proof of Lemma 3.2 Because of the equivalence of (P_3) and (P_1) for $c \neq 0$ it suffices to prove uniqueness for the latter one. Let $S \colon U \to L^2(Q)$ be the control-to-state mapping for the state equation given in (P_1). Further, let (y, u) be a solution of (P_1) with the cost given by

$$F(u) := \frac{1}{2} \left\| \mathbf{C}_{\omega_o} S(u) - z \right\|_{L^2(Q)}^2 + \frac{\alpha}{2} \left\| u \right\|_{L^\infty(Q)}^2.$$

For $\omega_o = \Omega$ the map $\mathbf{C}_{\omega_o} S$ is injective and we have strict convexity of

$$\frac{1}{2} \left\| \mathbf{C}_{\omega_o} S(u) - z \right\|_{L^2(Q)}^2.$$

Further, the L^∞-norm is convex, so F is strictly convex and we obtain uniqueness.

Uniqueness in the more general case $\omega_c \subset \omega_o$ is proved as follows. Let

$$\begin{aligned} \mathbf{A}y_1 &= \mathbf{B}_{\omega_c} u_1 \text{ in } Q, \quad y_1(0) = y_0 \text{ in } \Omega, \quad \mathbf{C}y_1 = 0 \text{ on } \Sigma, \\ \mathbf{A}y_2 &= \mathbf{B}_{\omega_c} u_2 \text{ in } Q, \quad y_2(0) = y_0 \text{ in } \Omega, \quad \mathbf{C}y_2 = 0 \text{ on } \Sigma \end{aligned}$$

for $(y_i, u_i) \in X \times U$, $i = 1, 2$, with

$$\mathbf{C}_{\omega_o} y_1 = \mathbf{C}_{\omega_o} y_2. \tag{3.5}$$

This implies, that $y_1 - y_2 = 0$ on $J \times \omega_o$. Hence, $\mathbf{A}(y_1 - y_2) = 0$ on $J \times \omega_o$ and thus, $u_1 = u_2$ on $J \times \omega_c$. Consequently, we derive $y_1 = y_2$ on Q. Thus $\mathbf{C}_{\omega_o} S$ is injective and we are in the situation as above.

Remark 3.3 For general $\omega_o \subset \Omega$ and $\omega_c \subset \Omega$ we cannot expect uniqueness due to the finite speed of propagation. Consider a one dimensional domain $\Omega = (0, L)$, $L > 0$, with $\omega_o = (0, \varepsilon)$, $\varepsilon > 0$ small, and $\omega_c = \Omega$. Let (y, u) be a corresponding solution of (P_1) with $y \neq 0$ in an open subset of $Q \setminus (J \times \omega_o)$. Then there exists an open set J in $Q \setminus (J \times \omega_o)$ in which the adjoint state p does not vanish. Thus we have $u \neq 0$ on J. Let $\hat{y}$ be the solution of

$$A\hat{y} = \mathbf{B}_{\omega_c} g \text{ in } Q, \quad \hat{y}(0) = 0 \text{ in } \Omega, \quad \hat{y} = 0 \text{ on } \Sigma$$

with

$$g = \begin{cases} -\operatorname{sgn}(u)\eta \text{ in } B_\delta, \\ 0 \qquad\qquad \text{else} \end{cases}$$

for $\delta, \eta > 0$ and $B_\delta \subset J$ and the usual *sgn*–function. Here, B_δ denotes a ball with radius δ with respect to the topology of Q. Then $\|u + g\|_{L^\infty(Q)} = \|u\|_{L^\infty(Q)}$ and $\mathbf{C}_{\omega_o}(\hat{y} + y) = \mathbf{C}_{\omega_o} y$ for $\delta, \eta > 0$ sufficiently small. Thus we obtain a second solution $(u + g, \hat{y})$.

4 The regularized minimum effort problem

The optimality system in (3.4) is not (in a generalized sense) differentiable. Therefore we consider a regularized minimum effort problem given by

$$\begin{cases} \min_{(y,u,c) \in X \times U \times \mathbb{R}_0^+} \quad J_\beta(y, u, c) = \frac{1}{2} \|\mathbf{C}_{\omega_o} y - z\|_{L^2(Q)}^2 + \frac{\beta c}{2} \|u\|_{L^2(Q)}^2 + \frac{\alpha}{2} c^2, \\ \qquad\qquad \text{s.t.} \\ \qquad\qquad \mathbf{A}y = c\mathbf{B}_{\omega_c} u \quad \text{in } Q, \\ \qquad\qquad\quad y(0) = y_0 \qquad \text{in } \Omega, \\ \qquad\qquad\quad\; \mathbf{C}y = 0 \qquad \text{on } \Sigma, \\ \qquad\quad \|u\|_{L^\infty(Q)} \leq 1 \end{cases}$$

$$(P_{\text{reg}})$$

for $y_0 \in Y^1$, parameters $\alpha, \beta > 0$, and $z \in L^2(Q)$.

The existence of a solution follows by standard arguments which we denote by $(y_\beta, u_\beta, c_\beta)$.

Remark 4.1 The regularization term scales linearly with the parameter c. Alternative regularizations, where the penalty term is constant or quadratic in c, are discussed in [1].

Remark 4.2 Let $(y_\beta, u_\beta, c_\beta)$ be a solution of (P_{reg}). To exclude the case $c_\beta = 0$ for β sufficiently small we assume $z \not\equiv 0$ and

$$J(y^*, c^*) < \frac{1}{2} \|z\|_{L^2(Q)}^2. \tag{4.1}$$

If $c_\beta = 0$ we have

$$
\begin{aligned}
\frac{1}{2} \|z\|_{L^2(Q)}^2 &= \frac{1}{2} \|\mathbf{C}_{\omega_o} y_\beta - z\|_{L^2(Q)}^2 + \frac{\beta c_\beta}{2} \|u_\beta\|_{L^2(Q)}^2 + \frac{\alpha}{2} c_\beta^2 \\
&\leq \frac{1}{2} \|\mathbf{C}_{\omega_o} y^* - z\|_{L^2(Q)}^2 + \frac{\beta c^*}{2} \|u^*\|_{L^2(Q)}^2 + \frac{\alpha}{2} (c^*)^2 \\
&\leq J(y^*, c^*) + \frac{\beta (c^*)^2}{2} \operatorname{meas}(Q) \\
&\leq \frac{1}{2} \|z\|_{L^2(Q)}^2 + J(y^*, c^*) - \|z\|_{L^2(Q)}^2 + \frac{\beta (c^*)^2}{2} \operatorname{meas}(Q)
\end{aligned}
$$

which is a contradiction to (4.1) for all $\beta > 0$ sufficiently small.

The optimality system for the regularized problem is given by

$$
\begin{cases}
(\beta u_\beta - \mathbf{B}_{\omega_c}^* p_\beta, u - u_\beta) \geq 0 & \text{for all } u \text{ with } \|u\|_{L^\infty(Q)} \leq 1, \\
\mathbf{A}^* p_\beta + \mathbf{C}_{\omega_o}^* (\mathbf{C}_{\omega_o} y_\beta - z) = 0, \quad p_\beta(T) = 0, \quad \mathbf{B} p_\beta|_\Sigma = 0, \\
\alpha c_\beta + \dfrac{\beta}{2} \|u_\beta\|_{L^2(Q)}^2 - (u_\beta, \mathbf{B}_{\omega_c}^* p_\beta) = 0, \\
\mathbf{A} y_\beta - c_\beta \mathbf{B}_{\omega_c} u_\beta = 0, \quad y_\beta(0) = y_0, \quad \mathbf{C} y_\beta|_\Sigma = 0
\end{cases}
$$

with $p_\beta \in Y^*$.

By pointwise inspection of the first relation we have

$$
u_\beta = \operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p_\beta) = \begin{cases}
1 & \mathbf{B}_{\omega_c}^* p_\beta(t,x) > \beta, \\
-1 & \mathbf{B}_{\omega_c}^* p_\beta(t,x) < -\beta, \\
\frac{1}{\beta} \mathbf{B}_{\omega_c}^* p_\beta(t,x) & |\mathbf{B}_{\omega_c}^* p(t,x)| \leq \beta.
\end{cases} \tag{4.2}
$$

Before we prove uniqueness of a solution of (P_{reg}) we recall the following well-known property. Let $N = \{0\} \times L^2(Q)$. Then we can introduce the inverse operator

$$
\mathbf{A}^{-1} \colon N \to L_2, \quad f \mapsto y,
$$

where $y \in X$, $y(0) = 0$, is the unique solution of

$$
\langle \mathbf{A} y, \varphi \rangle_{Y, Y^*} = (f, \varphi)_{L_2} \quad \forall \varphi \in Y^*.
$$

By a priori estimates, see, e.g., Lions and Magenes [12, p. 265], we obtain that $\mathbf{A}^{-1}$ is a bounded linear operator. Consequently, there exists a well-defined dual operator $(\mathbf{A}^{-1})^* \colon L_2 \to N$ satisfying

$$
((\mathbf{A}^{-1})^* w, v)_{L_2} = (w, \mathbf{A}^{-1} v)_{L_2} \tag{4.3}
$$

for $w \in L_2$ and $v \in N$.

Using this property we can guarantee uniqueness of a solution of the regularized problem under certain conditions. The uniqueness is not obvious because of the bilinear structure of the state equation.

Lemma 4.1 *Let $(y_\beta, u_\beta, c_\beta)$ be a solution of (P_{reg}). Then y_β and u_β are uniquely determined by c_β. Conversely, c_β and y_β are uniquely determined by u_β. Further, for $\alpha > 0$ sufficiently large there exists a unique solution of problem (P_{reg}).*

Proof To prove uniqueness we use a Taylor expansion argument as in [1, Appendix A]. To utilize (4.3) we need to transform (P_{reg}) into a problem with homogeneous initial condition. For this purpose let $\bar{y} \in X$ be the solution of

$$\begin{cases} \mathbf{A}\bar{y} = 0 & \text{in } Q, \\ \bar{y}(0) = y_0 & \text{in } \Omega, \\ \mathbf{C}\bar{y} = 0 & \text{on } \Sigma. \end{cases}$$

We set $\bar{z} = -\mathbf{C}_{\omega_o}\bar{y} + z$ and introduce problem (P_{hom}) given by

$$\begin{cases} \displaystyle\min_{(y,u,c)\in X\times U\times\mathbb{R}_0^+} \quad J_\beta(y,u,c) = \frac{1}{2}\left\|\mathbf{C}_{\omega_o}y - \bar{z}\right\|_{L^2(Q)}^2 + \frac{\beta c}{2}\|u\|_{L^2(Q)}^2 + \frac{\alpha}{2}c^2, \\ \qquad\text{s.t.} \\ \qquad\qquad \mathbf{A}y = c\mathbf{B}_{\omega_c}u \quad \text{in } Q, \\ \qquad\qquad y(0) = 0 \qquad\;\; \text{in } \Omega, \\ \qquad\qquad \mathbf{C}y = 0 \qquad\;\; \text{on } \Sigma, \\ \qquad\qquad \|u\|_U \leq 1. \end{cases}$$

$$(P_{\mathrm{hom}})$$

The control problems (P_{reg}) and (P_{hom}) are equivalent. Thus, without restriction of generality we can assume that the initial state y_0 is zero.

We define the reduced cost

$$F(u,c) = \frac{1}{2}\left\|\mathbf{C}_{\omega_o}\mathbf{A}^{-1}(c\mathbf{B}_{\omega_c}u) - z\right\|_{L^2(Q)}^2 + \frac{\beta c}{2}\|u\|_{L^2(Q)}^2 + \frac{\alpha}{2}c^2.$$

To shorten notations we set $\mathbf{M} = \mathbf{C}_{\omega_o}\mathbf{A}^{-1}\mathbf{B}_{\omega_c}$, i.e.

$$\mathbf{M}\colon L^2(Q) \xrightarrow{\;\mathbf{B}_{\omega_c}\;} N \xrightarrow{\;\mathbf{A}^{-1}\;} L_2 \xrightarrow{\;\mathbf{C}_{\omega_o}\;} L^2(Q).$$

Since $\mathbf{M}$ is a linear, bounded operator and using (4.3) we derive the optimality conditions

$$c_\beta(\beta u_\beta - \mathbf{M}^* z + c_\beta \mathbf{M}^* \mathbf{M} u_\beta, u - u_\beta) \geq 0 \quad \text{for all } \|u\|_{L^\infty(Q)} \leq 1, \qquad (4.4)$$

$$\alpha c_\beta + \frac{\beta}{2}\|u_\beta\|_{L^2(Q)}^2 - (u_\beta, \mathbf{M}^* z) + c_\beta\|\mathbf{M}u_\beta\|_{L^2(Q)}^2 = 0. \qquad (4.5)$$

The partial derivatives of F at (u_β, c_β) are given by

$$F_{uu} = c_\beta^2 \mathbf{M}^* \mathbf{M} + \beta c_\beta \,\mathrm{id},$$

$$F_{cc} = \|\mathbf{M} u_\beta\|_{L^2(Q)}^2 + \alpha,$$

$$F_{uc} = 2 c_\beta \mathbf{M}^* \mathbf{M} u_\beta - \mathbf{M}^* z + \beta u_\beta,$$

$$F_{uuc} = 2 c_\beta \mathbf{M}^* \mathbf{M} + \beta \,\mathrm{id},$$

$$F_{ccu} = 2 \mathbf{M}^* \mathbf{M} u_\beta,$$

$$F_{uucc} = 2 \mathbf{M}^* \mathbf{M}.$$

Let (u, c) be an admissible pair. Then we set

$$\hat{u} = u - u_\beta, \quad \hat{c} = c - c_\beta.$$

The Taylor expansion is given as follows, we use the fact, that $F_c(u_\beta, c_\beta) = 0$, see (4.5), and that the derivatives commute

$$\begin{aligned}
F(u, c) - F(u_\beta, c_\beta) = {}& c_\beta (\beta u_\beta - \mathbf{M}^* z + c_\beta \mathbf{M}^* \mathbf{M} u_\beta, \hat{u}) \\
& + \frac{c_\beta^2}{2} \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 + \frac{\beta c_\beta}{2} \|\hat{u}\|_{L^2(Q)}^2 \\
& + \frac{1}{2} \left(\|\mathbf{M} u_\beta\|_{L^2(Q)}^2 + \alpha \right) \hat{c}^2 \\
& + (2 c_\beta \mathbf{M}^* \mathbf{M} u_\beta - \mathbf{M}^* z + \beta u_\beta, \hat{u}) \hat{c} \\
& + \frac{3}{6} \left(2 c_\beta \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 \hat{c} + \beta \|\hat{u}\|_{L^2(Q)}^2 \hat{c} \right) \\
& + (\mathbf{M} u_\beta, \mathbf{M} \hat{u}) \hat{c}^2 \\
& + \frac{6}{24} 2 \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 \hat{c}^2.
\end{aligned}$$

Using twice (4.4) we further have

$$\begin{aligned}
F(u, c) - F(u_\beta, c_\beta) \geq {}& \frac{c_\beta^2}{2} \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 + \frac{\beta}{2} (c_\beta + \hat{c}) \|\hat{u}\|_{L^2(Q)}^2 + \\
& + \frac{1}{2} \left(\|\mathbf{M} u_\beta\|_{L^2(Q)}^2 + \alpha \right) \hat{c}^2 + c_\beta (\mathbf{M} u_\beta, \mathbf{M} \hat{u}) \hat{c} \\
& + c_\beta \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 \hat{c} \\
& + (\mathbf{M} u_\beta, \mathbf{M} \hat{u}) \hat{c}^2 + \frac{1}{2} \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 \hat{c}^2.
\end{aligned}$$

With

$$(\mathbf{M} u_\beta, \mathbf{M} \hat{u}) = (\sqrt{\eta} \mathbf{M} u_\beta, (\sqrt{\eta})^{-1} \mathbf{M} \hat{u}) \geq -\frac{\eta}{2} \|\mathbf{M} u_\beta\|_{L^2(Q)}^2 - \frac{1}{2\eta} \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2$$

we obtain

$$\begin{aligned}
F(u, c) - F(u_\beta, c_\beta) \geq {}& \frac{c_\beta^2}{2} (1 - \frac{1}{\eta}) \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 + \frac{\beta}{2} c \|\hat{u}\|_{L^2(Q)}^2 \\
& + \frac{1}{2} (\alpha - \eta \|\mathbf{M} u_\beta\|_{L^2(Q)}^2) \hat{c}^2 + c_\beta \|\mathbf{M} \hat{u}\|_{L^2(Q)}^2 \hat{c}.
\end{aligned} \tag{4.6}$$

Set $K := \sup \{ \|\mathbf{M}u\|^2_{L^2(Q)} \mid \|u\|_{L^\infty(Q)} \leq 1 \}$ and choose $\eta = 1$. For $\alpha > K^2$ we have

$$F(u,c) - F(u_\beta, c_\beta) \geq \frac{\beta}{2} c \|\hat{u}\|^2_{L^2(Q)} + \frac{1}{2}(\alpha - K^2)\hat{c}^2 + c_\beta \|\mathbf{M}\hat{u}\|^2_{L^2(Q)} \hat{c} \geq 0. \tag{4.7}$$

Let (u_β, c_β) and $(u_{\beta'}, c_{\beta'})$ be two solutions. Then we obtain from (4.7), that $(u_\beta, c_\beta) = (u_{\beta'}, c_{\beta'})$.

From (4.6) we see, that for $c_\beta = c_{\beta'}$ also $u_\beta = u_{\beta'}$ for any $\eta \geq 1$ and for $u_\beta = u_{\beta'}$ we have $c_\beta = c_{\beta'}$ for $\eta > 0$ sufficiently small. These last two statements do not require any assumption on α.

In the following we analyze convergence of the the solution of (P_{reg}) for $\beta \to 0$ and proceed as in [1].

Lemma 4.2 *For $\beta > 0$ let $(y_\beta, u_\beta, c_\beta)$ denote a solution of (P_{reg}). Further let (P_3) have a solution (y^*, u^*, c^*) which we associate with $\beta = 0$ and also denote by (y_0, u_0, c_0). Then for any $0 \leq \beta \leq \beta'$ we have*

$$c_{\beta'} \|u_{\beta'}\|^2_{L^2(Q)} \leq c_\beta \|u_\beta\|^2_{L^2(Q)}, \tag{4.8}$$

$$J(y_\beta, c_\beta) \leq J(y_{\beta'}, c_{\beta'}), \tag{4.9}$$

$$J(y_\beta, c_\beta) + \frac{\beta c_\beta}{2} \|u_\beta\|^2_{L^2(Q)} \leq J(y^*, c^*) + \frac{\beta c^*}{2} \|u^*\|^2_{L^2(Q)}. \tag{4.10}$$

Proof We recall the proof from [1] and apply it to the time-dependent case. Since $(y_\beta, u_\beta, c_\beta)$ is a solution of (P_{reg}) and (y_0, u_0, c_0) a solution of (P_3), respectively, we have for $0 \leq \beta \leq \beta'$ that

$$J(y_\beta, c_\beta) + \frac{\beta c_\beta}{2} \|u_\beta\|^2_{L^2(Q)} \leq J(y_{\beta'}, u_{\beta'}) + \frac{\beta c_{\beta'}}{2} \|u_{\beta'}\|^2_{L^2(Q)}.$$

Thus, further

$$J(y_\beta, c_\beta) + \frac{\beta c_\beta}{2} \|u_\beta\|^2_{L^2(Q)} + \frac{(\beta' - \beta)c_{\beta'}}{2} \|u_{\beta'}\|^2_{L^2(Q)}$$

$$\leq J(y_{\beta'}, c_{\beta'}) + \frac{\beta' c_{\beta'}}{2} \|u_{\beta'}\|^2_{L^2(Q)} \tag{4.11}$$

$$\leq J(y_\beta, c_\beta) + \frac{\beta' c_\beta}{2} \|u_\beta\|^2_{L^2(Q)}.$$

From the outer inequality we have $(\beta' - \beta)(c_{\beta'} \|u_{\beta'}\|^2_{L^2(Q)} - c_\beta \|u_\beta\|^2_{L^2(Q)}) \leq 0$ implying the first assertion. From (4.11) we derive

$$J(y_\beta, c_\beta) - J(y_{\beta'}, c_{\beta'}) \leq \beta(c_{\beta'} \|u_{\beta'}\|^2_{L^2(Q)} - c_\beta \|u_\beta\|^2_{L^2(Q)}) \tag{4.12}$$

and the right hand side is smaller than or equal to zero by the previous result and thus (4.9) follows. Assertion (4.10) follows from the last inequality in (4.11) by setting $\beta' = \beta$ and $\beta = 0$.

After this preparation we prove convergence of minimizers of (P_{reg}) following [1].

Theorem 4.1 *Let (P_3) have a unique solution. Then any selection of solutions*
$\left\{ (y_\beta, u_\beta, c_\beta) \right\}_{\beta > 0}$ of P_β are bounded in $X \times L^\infty(Q) \times \mathbb{R}_0^+$ and converges weak* to the solution of (P_3) for $\beta \to 0$. It converges strongly in $X \times L^q(Q) \times \mathbb{R}_0^+$ for $q \in [1, \infty)$.

Proof The point $(\hat{y}, \hat{u}, \hat{c}) = (y_0, 0, 0)$ is feasible for the constraints. Thus we have

$$\|\mathbf{C}_{\omega_o} y_\beta - z\|_{L^2(Q)}^2 + \beta c_\beta \|u_\beta\|_{L^2(Q)}^2 + \alpha c_\beta^2 \leq \|\mathbf{C}_{\omega_o} y_0 - z\|_{L^2(Q)}^2$$

and consequently, the boundedness of c_β follows. The controls u_β are bounded by the constant 1 in $L^\infty(Q)$ and hence, y_β is bounded in X.

Therefore, there exists $(\bar{y}, \bar{u}, \bar{c}) \in X \times L^\infty(Q) \times \mathbb{R}_0^+$ such that for a subsequence there holds

$$(y_\beta, u_\beta, c_\beta) \rightharpoonup^* (\bar{y}, \bar{u}, \bar{c})$$

in $X \times L^\infty(Q) \times \mathbb{R}$. By passing to the limit in the equation we obtain that $(\bar{y}, \bar{u}, \bar{c})$ is a solution of

$$\begin{cases} \mathbf{A}y = c\mathbf{B}_{\omega_c} u & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ \mathbf{C}y = 0 & \text{on } \Sigma. \end{cases}$$

Since the L^∞–norm is weak* lower semicontinuous, we have $\|\bar{u}\|_{L^\infty(Q)} \leq 1$. Further, by the weak lower semicontinuity of $J_\beta \colon L^2(Q) \times L^2(Q) \times \mathbb{R}_0^+ \to \mathbb{R}$ we derive that $(\bar{y}, \bar{u}, \bar{c})$ is a solution of (P_3). Uniqueness of the solution of (P_3) implies that $(\bar{y}, \bar{u}, \bar{c}) = (y^*, u^*, c^*)$. Thus we have proved weak* convergence.

For strong convergence insert the weak limit (u^*, c^*) in (4.8) with $\beta = 0$ and obtain for all $\beta' > 0$ from the lower semicontinuity of the norm that

$$c_{\beta'} \|u_{\beta'}\|_{L^2(Q)}^2 \leq c^* \|u^*\|_{L^2(Q)}^2 \leq \liminf_{\beta' \to 0} c_{\beta'} \|u_{\beta'}\|_{L^2(Q)}^2.$$

This implies

$$\limsup_{\beta' \to 0} \|u_{\beta'}\|_{L^2(Q)}^2 \leq \|u^*\|_{L^2(Q)}^2 \leq \liminf_{\beta' \to 0} \|u_{\beta'}\|_{L^2(Q)}^2$$

and consequently, strong convergence in $L^2(Q)$. Using

$$\|u_{\beta'} - u^*\|_{L^p(Q)} \leq \|u_{\beta'} - u^*\|_{L^2(Q)} \|u_{\beta'} - u^*\|_{L^\infty(Q)}$$

we obtain strong convergence of $u_{\beta'} \to u^*$ in every $L^q(Q)$, $q \in [1, \infty)$. Furthermore, this implies strong convergence of the corresponding state $y_{\beta'}$.

Remark 4.3 In case of non-unique solvability of (P_3) the assertion of Theorem 4.1 remains true if we consider the convergence of subsequences to a solution of (P_3).

From the strong convergence of u_β we can derive a convergence rate for the error in the cost functional.

Corollary 4.1 *Let (P_3) have a unique solution (y^*, u^*, c^*). Then there holds*

$$J(y_\beta, c_\beta) - J(y^*, c^*) = o(\beta)$$

for $\beta \to 0$.

Proof From (4.12) we have

$$0 \leq J(y_\beta, c_\beta) - J(y^*, u^*) \leq \beta(c_\beta \|u_\beta\|_{L^2(Q)} - c^* \|u^*\|_{L^2(Q)})$$

which proves the assertion.

From know on we will assume, that problem (P_reg) has a unique solution.

5 Semi-smooth Newton method

In this section we formulate the semi-smooth Newton method and prove its superlinear convergence. To keep notations simple we omit the index β for the solution of the regularized problem. Using

$$\|p\|_{L^1_\beta(Q)} = \int_Q |p(t,x)|_\beta dx dt, \quad |p(t,x)|_\beta = \begin{cases} p(t,x) - \frac{\beta}{2} & \text{if } p(t,x) > \beta, \\ -p(t,x) - \frac{\beta}{2} & \text{if } p(t,x) < -\beta, \\ \frac{1}{2\beta}p(t,x)^2 & \text{if } |p(t,x)| \leq \beta \end{cases}$$

we reformulate the optimality system for the regularized problem. We eliminate the control u using (4.2) and obtain

$$\mathbf{A}^* p + \mathbf{C}^*_{\omega_o}(\mathbf{C}_{\omega_o} y - z) = 0, \quad p(T) = 0, \quad \mathbf{B}p|_\Sigma = 0, \tag{5.1}$$

$$\alpha c - \left\|\mathbf{B}^*_{\omega_c} p\right\|_{L^1_\beta(Q)} = 0, \tag{5.2}$$

$$\mathbf{A}y - c\mathbf{B}_{\omega_c} \text{sgn}_\beta(\mathbf{B}^*_{\omega_c} p) = 0, \quad y(0) = y_0 \quad \mathbf{C}y|_\Sigma = 0. \tag{5.3}$$

To write the system equivalently as an operator equation we set

$$W = X \times Y_0^* \times \mathbb{R}^+,$$
$$Z = X^* \times \mathbb{R} \times Y \times Y^1.$$

For convenience of the reader we recall that $Y^1 = H^1_0(\Omega) \times L^2(\Omega)$, $Y^0 = L^2(\Omega) \times H^{-1}(\Omega)$, $X = L^2(Y^1) \cap H^1(Y^0)$, $P^1 = (Y^0)^*$, $P^0 = (Y^1)^*$, $Y_0^* =$

$\left\{\, p \in L^2(P^1) \cap H^1(P^0) \mid p(T) = 0 \,\right\}$, and $Y = L^2(Y^0) \oplus \left(H^1(P^0) \right)^*$. Then, we can define the operator $\mathbf{T}$ by

$$\mathbf{T} \colon W \longrightarrow Z, \quad \mathbf{T}(x) = \mathbf{T}(y, p, c) = \begin{pmatrix} \mathbf{A}^* p + \mathbf{C}^*_{\omega_o} \mathbf{C}_{\omega_o} y - \mathbf{C}^*_{\omega_o} z \\ \alpha c - \left\| \mathbf{B}^*_{\omega_c} p \right\|_{L^1_\beta(Q)} \\ \mathbf{A} y - c \mathbf{B}_{\omega_c} \operatorname{sgn}_\beta(\mathbf{B}^*_{\omega_c} p) \\ y(0) - y_0 \end{pmatrix} \qquad (5.4)$$

and obtain (5.1)–(5.3) equivalently as

$$\mathbf{T}(x) = 0 \qquad (5.5)$$

for $x \in W$.

To formulate the semi-smooth Newton method we need Newton differentiability of the operator $\mathbf{T}$. Let

$$W^{\mathbb{R}} = X \times Y_0^* \times \mathbb{R}.$$

Lemma 5.1 *The operator $\mathbf{T}$ is Newton differentiable, i.e. for all $x \in W$ and $h \in W^{\mathbb{R}}$ there holds*

$$\| \mathbf{T}(x + h) - \mathbf{T}(x) - \mathbf{T}'(x + h) h \|_Z = o(h) \quad \text{for } \| h \|_{W^{\mathbb{R}}} \to 0. \qquad (5.6)$$

Proof The operator

$$\max \colon L^p(Q) \to L^q(Q), \quad p > q \geq 1$$

is Newton differentiable with derivative

$$(D_N \max(0, v - \beta) h)(t, x) = \begin{cases} h(t, x), & v(t, x) > \beta, \\ 0, & v(t, x) \leq \beta \end{cases}$$

for $v, h \in L^p(Q)$, $\beta \in \mathbb{R}$, and $(t, x) \in Q$, see Ito and Kunisch [8, Example 8.14]. For the min operator an analog Newton derivative can be obtained. Since

$$\operatorname{sgn}_\beta(v) = \frac{1}{\beta} (v - \max(0, v - \beta) - \min(0, v + \beta))$$

we obtain for the operator

$$\operatorname{sgn}_\beta \colon L^p(Q) \to L^q(Q), \quad p > q \geq 1,$$

the Newton derivative

$$\left(D_N \operatorname{sgn}_\beta(p) h \right)(t, x) = \begin{cases} 0, & |p(t, x)| > \beta, \\ \frac{1}{\beta} h(t, x), & |p(t, x)| \leq \beta \end{cases}$$

for $v, h \in L^p(Q)$, $\beta \in \mathbb{R}^+$, and $(t, x) \in Q$.

The mapping $w \colon \mathbb{R} \to \mathbb{R}$, $s \mapsto |w(s)|_\beta$ defines a differentiable Nemytskii operator from $L^p(Q)$ to $L^2(Q)$ for $p \geq 4$ according to Tröltzsch [15, Chapter 4.3.3]. Thus, the mapping

$$\| \cdot \|_{L^1_\beta(Q)} \colon L^p(Q) \to \mathbb{R}, \quad p \geq 4$$

is Newton differentiable with Newton derivative

$$D_N(\|v\|_{L^1_\beta(Q)})h = (\text{sgn}_\beta(v), h)$$

for $v, h \in L^p(Q)$, see Clason, Ito, and Kunisch [1].

Further, since $\mathbf{B}^*_{\omega_c}p \in C(H^1(\Omega)) \hookrightarrow L^q(Q)$ for $q = \frac{2d}{d-2}$ the mappings

$$
\begin{aligned}
p &\mapsto \mathbf{B}^*_{\omega_c}p \mapsto \left\|\mathbf{B}^*_{\omega_c}p\right\|_{L^1_\beta(Q)}, & Y^* &\to L^4(Q) \to \mathbb{R}, \\
p &\mapsto \mathbf{B}^*_{\omega_c}p \mapsto c\mathbf{B}_{\omega_c}\text{sgn}_\beta(\mathbf{B}^*_{\omega_c}p), & Y^* &\to L^4(Q) \to L^2(Q) \to X^*
\end{aligned}
\tag{5.7}
$$

for $d \leq 4$ are Newton differentiable. Consequently, we obtain the assertion.

To formulate the semi-smooth Newton method we consider

$$\mathbf{T}' \colon W \to L(W^{\mathbb{R}}, Z) \tag{5.8}$$

with

$$
\mathbf{T}'(x)(\delta y, \delta p, \delta c) = \begin{pmatrix}
\mathbf{A}^*\delta p + \mathbf{C}^*_{\omega_o}\mathbf{C}_{\omega_o}\delta y \\
\alpha\delta c - (\text{sgn}_\beta(\mathbf{B}^*_{\omega_c}p), \mathbf{B}^*_{\omega_c}\delta p) \\
\mathbf{A}\delta y - \delta c\mathbf{B}_{\omega_c}\text{sgn}_\beta(\mathbf{B}^*_{\omega_c}p) - \frac{c}{\beta}\mathbf{B}_{\omega_c}\mathbf{B}^*_{\omega_c}\delta p\chi_{I_p} \\
\delta y(0)
\end{pmatrix}
\tag{5.9}
$$

for $x = (y, p, c) \in W$ and $(\delta y, \delta p, \delta c) \in W^{\mathbb{R}}$. Here χ_{I_p} denotes the characteristic function for the set

$$I_p = \left\{ (t, x) \in Q \mid |\mathbf{B}^*_{\omega_c}p(t, x)| \leq \beta \right\}. \tag{5.10}$$

The operator $\mathbf{T}'(x)$ is invertible on its image as we see in the next lemma. The proof is presented in the appendix.

Lemma 5.2 *For $x \in W$ the operator*

$$\mathbf{T}'(x) \colon W^{\mathbb{R}} \to \text{Im}(\mathbf{T}'(x)) \subset Z$$

is bijective and we can define

$$\mathbf{T}'(x)^{-1} \colon \text{Im}(\mathbf{T}'(x)) \to W^{\mathbb{R}}.$$

Furthermore, there holds the following estimate

$$\left\|\mathbf{T}'(x)^{-1}(z)\right\|_{W^{\mathbb{R}}} \leq \mathcal{C}\|z\|_Z \tag{5.11}$$

for $z \in \text{Im}(\mathbf{T}'(x)) \cap Z_1$ uniformly in $x \in W$, where

$$Z_1 = S \times \mathbb{R} \times M \times \{0\} \subset Z$$

and $S = \left\{ (\chi_{\omega_o}v, 0) \mid v \in L^2(L^2(\Omega)) \right\}$, $M = \left\{ (0, v) \mid v \in L^2(H^{-1}(\Omega)) \right\}$.

Directly applying the Newton method to equation (5.5) leads to the iteration

$$\mathbf{T}'(x^k)(\delta x) = -\mathbf{T}(x^k), \tag{5.12}$$

$$x^{k+1} = \delta x + x^k, x^0 \in W, \tag{5.13}$$

where in every Newton step the following system

$$\mathbf{A}^* \delta p + \mathbf{C}^*_{\omega_o} \mathbf{C}_{\omega_o} \delta y = -\mathbf{A}^* p^k - \mathbf{C}^*_{\omega_o} \mathbf{C}_{\omega_o} y^k + \mathbf{C}^*_{\omega_o} z, \tag{5.14}$$

$$\delta p(T) = 0,$$
$$\mathbf{B} \delta p|_\Sigma = 0,$$
$$\alpha \delta c - (\mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c} p^k), \mathbf{B}^*_{\omega_c} \delta p) = -\alpha c^k + \left\| \mathbf{B}^*_{\omega_c} p^k \right\|_{L^1_\beta(Q)},$$

$$\mathbf{A} \delta y - \delta c \mathbf{B}_{\omega_c} \mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c} p^k) - \frac{c^k}{\beta} \mathbf{B}_{\omega_c} \mathbf{B}^*_{\omega_c} \delta p \chi_{I_k} = -\mathbf{A} y^k + c^k \mathbf{B}_{\omega_c} \mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c} p^k), \tag{5.15}$$

$$\delta y(0) = -y^k(0) + y_0, \tag{5.16}$$
$$\mathbf{C} \delta y|_\Sigma = 0$$

with $I_k = I_{p^k}$ has to be solved. To simplify the system we reformulate it equivalently as follows

$$\mathbf{A}^* p^{k+1} + \mathbf{C}^*_{\omega_o} \mathbf{C}_{\omega_o} y^{k+1} = \mathbf{C}^*_{\omega_o} z, \tag{5.17}$$

$$p^{k+1}(T) = 0, \tag{5.18}$$

$$\mathbf{B} p^{k+1}|_\Sigma = 0, \tag{5.19}$$

$$\alpha c^{k+1} - (\mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c} p^k), \mathbf{B}^*_{\omega_c} p^{k+1}) =$$
$$- (\mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c} p^k), \mathbf{B}^*_{\omega_c} p^k) + \left\| \mathbf{B}^*_{\omega_c} p^k \right\|_{L^1_\beta(Q)}, \tag{5.20}$$

$$\mathbf{A} y^{k+1} - c^{k+1} \mathbf{B}_{\omega_c} \mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c} p^k) - \frac{c^k}{\beta} \mathbf{B}_{\omega_c} \mathbf{B}^*_{\omega_c} p^{k+1} \chi_{I_k} = -\frac{c^k}{\beta} \mathbf{B}_{\omega_c} \mathbf{B}^*_{\omega_c} p^k \chi_{I_k}, \tag{5.21}$$

$$y^{k+1}(0) = y_0, \tag{5.22}$$

$$\mathbf{C} y|_\Sigma = 0, \tag{5.23}$$

$$\delta y = y^{k+1} - y^k \tag{5.24}$$

for $k \in \mathbb{N}_0$. The iterates p^{k+1} are solutions of wave equations with right hand side in $L^2(Q) \times \{0\}$ and the iterates y^{k+1} are solutions of wave equations with right hand side in $\{0\} \times L^2(Q)$ for all $k \geq 0$ and initialization $(y^0, p^0, c^0) \in W$.

Under certain conditions the well-definedness of the Newton iteration can be shown.

Lemma 5.3 *For $x^0 \in W$ the Newton iterates x^k satisfy*

$$x^k \in W, \quad \mathbf{T}(x^k) \in \mathrm{Im}(\mathbf{T}'(x^k))$$

for $k \in \mathbb{N}_0$ if $c^k > 0$.

Remark 5.1 Let x^* be the solution of (P_{reg}). In Theorem 5.1 we will show that for β and $\left\| x^0 - x^* \right\|_{W^{\mathbb{R}}}$ sufficiently small the iterates c^k remain positive.

Proof of Lemma 5.3 For given iterate $x^k \in W$ we consider the control problem

$$
\begin{cases}
\displaystyle\min_{(y,u,c)\in X\times U\times\mathbb{R}_0^+} \quad J(c,u,y) = \frac{1}{2}\left\|\mathbf{C}_{\omega_o}y - z\right\|^2_{L^2(Q)} + \frac{\beta c^k}{2}\left\|u\right\|^2_{L^2(Q)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \dfrac{\alpha}{2}|c - z_1|^2, \\[2mm]
\text{s.t.} \\
\quad \mathbf{A}y - c\mathbf{B}_{\omega_c}\mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c}p^k) - c^k\mathbf{B}_{\omega_c}u\chi_{I_k} = z_2 \quad \text{in } Q, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad y(0) = y_0 \quad \text{in } \Omega, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbf{C}y|_\Sigma = 0 \quad \text{on } \Sigma
\end{cases}
\tag{5.25}
$$

with $I_k = I_{p^k}$, $z_2 \in L^2(\{0\} \times L^2(\Omega))$, $z_1 \in \mathbb{R}$, $y_0 \in Y^1$, $x^k = (y^k, p^k, c^k)$ and α, β, y_0 as in (5.17)–(5.23). This problem has a unique solution (y,u,c).

The optimality system of (5.25) is given by (5.17)–(5.23) if we choose

$$z_1 = -\frac{1}{\alpha}\left((\mathrm{sgn}_\beta(\mathbf{B}^*_{\omega_c}p^k),\mathbf{B}^*_{\omega_c}p^k) - \left\|\mathbf{B}^*_{\omega_c}p^k\right\|_{L^1_\beta(Q)}\right), \tag{5.26}$$

$$z_2 = -\frac{c^k}{\beta}\mathbf{B}_{\omega_c}\mathbf{B}^*_{\omega_c}p^k\chi_{I_k}. \tag{5.27}$$

From (5.17)–(5.19) we derive that $p \in Y_0^*$. This implies $x^k \in W$ for all Newton iterates. Since (5.12)–(5.13) and (5.17)–(5.24) are equivalent, the second assertion follows, when setting $\delta y = y - y^k$.

To apply (5.11) we need $\mathbf{T}(x^k) \in Z_1$. For $k \geq 1$ this follows immediately from (5.17)–(5.23). To obtain $\mathbf{T}(x^0) \in Z_1$, we choose $x^0 = (y^0, p^0, c^0) \in W$, such that

$$y^0(0) = y_0, \tag{5.28}$$

$$\partial_t y^0_1 = y^0_2, \tag{5.29}$$

$$\mathbf{A}^* p^0 + \mathbf{C}^*_{\omega_o}\mathbf{C}_{\omega_o}y^0 - \mathbf{C}^*_{\omega_o}z = 0. \tag{5.30}$$

To prove superlinear convergence of the Newton method we need the following estimate.

Lemma 5.4 *Let $x^* \in W$ be the solution of (P_{reg}) and let $x^0 \in W$ satisfy (5.28)–(5.30). Then the Newton iterates satisfy*

$$x^{k+1} - x^* = -\mathbf{T}'(x^k)^{-1} \left(\mathbf{T}(x^k) - \mathbf{T}(x^*) - \mathbf{T}'(x^k)(x^k - x^*) \right) \qquad (5.31)$$

and there holds the following estimate

$$\left\| x^{k+1} - x^* \right\|_{W^{\mathbb{R}}} \leq \mathcal{C} \left\| \mathbf{T}(x^k) - \mathbf{T}(x^*) - \mathbf{T}'(x^k)(x^k - x^*) \right\|_Z \qquad (5.32)$$

for $k \in \mathbb{N}_0$ if $c^k > 0$ with $x^k = (y^k, p^k, c^k)$.

Proof There holds $\mathbf{T}(x^*) = 0$ and $\mathbf{T}'(x^k)(x^k - x^*) \in \mathrm{Im}(\mathbf{T}'(x^k))$, and according to Lemma 5.3 we have $\mathbf{T}(x^k) \in \mathrm{Im}(\mathbf{T}'(x^k))$. Consequently,

$$\mathbf{T}(x^*) - \mathbf{T}(x^k) - \mathbf{T}'(x^k)(x^k - x^*) \in \mathrm{Im}(\mathbf{T}'(x^k))$$

for all $k \in \mathbb{N}_0$. Further, we derive from (5.17)–(5.24) and (5.28)–(5.30) that for $k \in \mathbb{N}_0$

$$\mathbf{T}(x^*) - \mathbf{T}(x^k) - \mathbf{T}'(x^k)(x^k - x^*) \in \mathrm{Im}(\mathbf{T}'(x^k)) \cap Z_1.$$

Thus, the assertion follows with Lemma 5.2.

The superlinear convergence of the Newton method is shown in the next main theorem.

Theorem 5.1 *Let $x^* = (y^*, p^*, c^*)$ be the solution of (P_{reg}) with $z \not\equiv 0$ and β sufficiently small, such that $c^* > 0$ (cf. Remark 4.2). Further let $x^0 \in W$ satisfy (5.28)–(5.30) and let $\left\| x^0 - x^* \right\|_{W^{\mathbb{R}}}$ be sufficiently small. Then the iterates $x^k = (y^k, p^k, c^k) \in W$ of the semi-smooth Newton method (5.12)-(5.13) are well defined and they satisfy*

$$\left\| x^{k+1} - x^* \right\|_{W^{\mathbb{R}}} \leq o(\left\| x^k - x^* \right\|_{W^{\mathbb{R}}}) \qquad (5.33)$$

for $\left\| x^k - x^* \right\|_{W^{\mathbb{R}}} \to 0$.

Proof From the estimates (5.6) and (5.32) and the positiveness of $c^* > 0$ we conclude that $c^k > 0$ for all $k \in \mathbb{N}$ if $\left\| x^0 - x^* \right\|_{W^{\mathbb{R}}}$ is sufficiently small. Thus by Lemma 5.3 all iterates are in W.

Estimate (5.33) follows from Lemma 5.1, Lemma 5.4 and [8, Proof of Theorem 8.16].

To realize the semi-smooth Newton method we introduce the active sets

$$A_k^+ = \{ (t, x) \in Q \mid \mathbf{B}_{\omega_c}^* p^k(t, x) > \beta \},$$
$$A_k^- = \{ (t, x) \in Q \mid \mathbf{B}_{\omega_c}^* p^k(t, x) < -\beta \}$$

for iterates $p^k \in Y^*$. With $I_k = I_{p^k}$ (cf. the definition in (5.10)) we have $Q = I_k \cup A_k^+ \cup A_k^-$. The Newton method is realized as presented in Algorithm 5.1.

Remark 5.2 The solution of system (5.17)–(5.24) in Step 7 of Algorithm 5.1 can be found by solving the control problem (5.25) if we assume that the scalar c is always positive.

Algorithm 5.1 Semi-smooth Newton algorithm with path-following

---

1: Choose $n = 0$, $\underline{y}_0 = (\underline{y}_{01}, \underline{y}_{02}) \in X$ satisfying (5.28) and (5.29), $\underline{c}_0 \in \mathbb{R}^+$,
    $q \in (0, 1)$, $tol, tol_\beta, \beta_0 \in \mathbb{R}^+$, and $\bar{n} \in \mathbb{N}$.
2: For given $\underline{y}_0$ solve the adjoint equation (5.1) and obtain $\underline{p}_0 \in Y^*$.
3: **repeat**
4:     Set $k = 0$ and $(y^0, p^0, c^0) = (\underline{y}_n, \underline{p}_n, \underline{c}_n)$.
5:     **repeat**
6:        Compute the active and inactive sets A_k^+, A_k^-, and I_k:

$$A_k^+ = \{\, (t, x) \in Q \mid \mathbf{B}_{\omega_c}^* p^k(t, x) > \beta \,\},$$
$$I_k = \{\, (t, x) \in Q \mid |\mathbf{B}_{\omega_c}^* p^k(t, x)| \leq \beta \,\},$$
$$A_k^- = \{\, (t, x) \in Q \mid \mathbf{B}_{\omega_c}^* p^k(t, x) < -\beta \,\}.$$

7:        Solve for $x^k = (y^k, p^k, c^k)$ system (5.17)–(5.24) and obtain

$$x^{k+1} = (y^{k+1}, p^{k+1}, c^{k+1}).$$

8:        Set $k = k + 1$.
9:     **until** $\left\| x^k - x^{k-1} \right\|_{W^{\mathbb{R}}} < tol$.
10:     Set $(\underline{y}_{n+1}, \underline{p}_{n+1}, \underline{c}_{n+1}) = x^k$.
11:     Compute $u^{k+1} = \mathrm{sgn}_\beta(\mathbf{B}_{\omega_c}^*(p^{k+1}))$.
12:     Set $\beta_{n+1} = q\beta_n$.
13:     Set $n = n + 1$.
14: **until** $\beta_{n+1} < tol_\beta$ or $n > \bar{n}$.

---

6 Discretization

To realize Algorithm 5.1 numerically we present the discretization of problem (5.25) for data given by (5.26)–(5.27).

For the discretization of the state equation we apply a continuous Galerkin method following Kröner, Kunisch, and Vexler [10]. For temporal discretization we apply a Petrov–Galerkin method with continuous piecewise linear ansatz functions and discontinuous (in time) piecewise constant test functions. For the spatial discretization we use conforming linear finite elements. Let

$$\bar{J} = \{0\} \cup J_1 \cup \cdots \cup J_M$$

be a partition of the time interval $\bar{J} = [0, T]$ with subintervals $J_m = (t_{m-1}, t_m]$ of size k_m and time points

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T.$$

We define the time discretization parameter k as a piecewise constant function by setting $k|_{J_m} = k_m$ for $m = 1, \ldots, M$. Further, for

$$0 = l_0 < l_1 < \cdots < l_{N-1} < l_N = L$$

let

$$\mathcal{T}_h = L_1 \cup \cdots \cup L_N$$

be a partition of the space interval $\Omega = (0, L)$ with subintervals $L_n = (l_{n-1}, l_n)$ of size h_n and $h = \max_{n=1,\ldots,M} h_n$. We construct on the mesh $\mathcal{T}_h$ a conforming finite element space V_h in a standard way by setting

$$V_h = \left\{ v \in H_0^1(\Omega) \mid v|_{L_n} \in \mathcal{P}^1(L_n) \right\}.$$

Then the discrete ansatz and test space are given by

$$X_{kh} = \left\{ v \in C(\bar{J}, L^2(\Omega)) \mid v|_{J_m} \in \mathcal{P}_1(J_m, V_h) \right\},$$
$$\widetilde{X}_{kh} = \left\{ v \in L^2(J, H_0^1(\Omega)) \mid v|_{J_m} \in \mathcal{P}_0(J_m, V_h) \text{ and } v(0) \in L^2(\Omega) \right\},$$

where $\mathcal{P}_r(J_m, V_h)$ denotes the space of all polynomials of degree lower or equal $r = 0, 1$ defined on J_m with values in V_h. For the discretization of the control space we set

$$U_{kh} = \widetilde{X}_{kh}.$$

In the following we present the discrete optimality system for (5.25) assuming that the iterates c^k are positive. With the notation

$$(\cdot, \cdot)_{J_m} := \int_{J_m} (\cdot, \cdot)_{L^2(\Omega)} dt$$

for the Newton iterates $c^k \in \mathbb{R}$, $u_{kh}^k \in U_{kh}$, $y_{kh}^k = (y_1^k, y_2^k) \in X_{kh} \times X_{kh}$, and $p_{kh}^k = (p_1^k, p_2^k) \in \widetilde{X}_{kh} \times \widetilde{X}_{kh}$, $k \in \mathbb{N}$, the adjoint equation is given by

$$-\sum_{m=0}^{M-1} (\psi^1(t_m), p_1^{k+1}(t_{m+1}) - p_1^{k+1}(t_m))_{L^2(\Omega)} + (\nabla\psi^1, \nabla p_2^{k+1})$$
$$+ (\psi^1(t_M), p_1^{k+1}(t_M))_{L^2(\Omega)} = -(\psi^1, \chi_{\omega_o}(y_1^{k+1} - z)) \quad \forall \psi^1 \in X_{kh}, \quad (6.1)$$

$$-\sum_{m=0}^{M-1} (\psi^2(t_m), p_2^{k+1}(t_{m+1}) - p_2^{k+1}(t_m))_{L^2(\Omega)} - (\psi^2, p_1^{k+1})$$
$$+ (\psi^2(t_M), p_2^k(t_M))_{L^2(\Omega)} = 0 \quad \forall \psi^2 \in X_{kh}, \quad (6.2)$$

the optimality conditions by

$$\alpha c^{k+1} - (\text{sgn}_\beta(\chi_{\omega_c} p_2^k), \chi_{\omega_c} p_2^{k+1}) = -(\text{sgn}_\beta(\chi_{\omega_c} p_2^k), \chi_{\omega_c} p_2^k) + \left\| \chi_{\omega_c} p_2^k \right\|_{L_\beta^1(Q)},$$
$$(6.3)$$

$$(\beta u_{kh}^{k+1}, \tau u) = (\chi_{I_{kh}} \chi_{\omega_c} p_2^{k+1}, \tau u) \quad \forall \tau u \in U_{kh}, \quad (6.4)$$

for $I_{kh} = I_{p_{kh}}$ and the state equation by

$$\sum_{m=1}^{M} (\partial_t y_1^{k+1}, \xi^1)_{J_m} - (y_2^{k+1}, \xi^1) + (y_1^{k+1}(0) - y_{0,1}, \xi^1(0))_{L^2(\Omega)} = 0 \quad \forall \xi^1 \in \widetilde{X}_{kh},$$
$$(6.5)$$

$$\sum_{m=1}^{M} (\partial_t y_2^{k+1}, \xi^2)_{J_m} + (\nabla y_1^{k+1}, \nabla \xi^2) + (y_2^{k+1}(0) - y_{0,2}, \xi^2(0))_{L^2(\Omega)}$$
$$- c^{k+1}(\text{sgn}_\beta(\chi_{\omega_c} p_2^k), \xi^2) - c^k(\chi_{\omega_c} u_{kh}^{k+1} \chi_{I_{kh}}, \xi^2)$$
$$= -c^k(\text{sgn}_\beta(\chi_{\omega_c} p_2^k), \xi^2) \quad \forall \xi^2 \in \widetilde{X}_{kh} \quad (6.6)$$

with $y_0 = (y_{0,1}, y_{0,2})$.

When evaluating the time integrals by a trapedoizal rule the time stepping scheme for the state equation results in a Crank Nicolson scheme.

To solve the system (6.1)–(6.6) we introduce the control-to-state operator for the discrete state equation (6.5)–(6.6)

$$S_{kh}^k : U_{kh} \times \mathbb{R} \to L^2(Q), \quad (u_{kh}, c) \mapsto y_1$$

and the discrete reduced cost functional

$$j_{kh}^k : U_{kh} \times \mathbb{R} \to \mathbb{R}_0^+,$$
$$j_{kh}^k(u_{kh}, c) = \frac{1}{2} \|\chi_{\omega_o} S_{kh}(u_{kh}, c)\|_{L^2(Q)}^2 + \frac{\beta c}{2} \|u_{kh}\|_U^2 + \frac{\alpha}{2} |c - z_1|^2,$$

with

$$z_1 = -\frac{1}{\alpha} \left((\text{sgn}_\beta(\chi_{\omega_c} p_2^k), \chi_{\omega_c} p_2^k) - \|\chi_{\omega_c} p_2^k\|_{L_\beta^1(Q)} \right),$$

where $p_{kh}^k = (p_1^k, p_2^k)$ results from the previous iterate. Then the solution of the system is given as a solution of the reduced problem

$$\min j_{kh}^k(u_{kh}, c), \quad (u_{kh}, c) \in U_{kh} \times \mathbb{R}.$$

The necessary optimality condition is given by

$$(j_{kh}^k)'(u_{kh}, c)(\delta u, \delta c) = 0 \quad \forall (\delta u, \delta c) \in U_{kh} \times \mathbb{R}.$$

We solve this reduced problem by a classical Newton method, i.e. the Newton update $(\tau u, \tau c) \in U_{kh} \times \mathbb{R}$ is given by

$$(j_{kh}^k)''(u_{kh}, c)(\tau u, \tau c, \delta u, \delta c) = -(j_{kh}^k)'(u_{kh}, c)(\delta u, \delta c) \quad \forall (\delta u, \delta c) \in U_{kh} \times \mathbb{R}. \quad (6.7)$$

The explicit representations of the derivatives of the reduced cost functional are given in the Appendix 8.2.

7 Numerical examples

In this section we present numerical examples confirming the theoretical results from above. In the first three examples we consider the convergence behaviour of the Newton iteration in the inner loop of Algorithm 5.1, i.e. we consider the case with path iteration number $\bar{n} = 0$. Further, we present an example in which we consider the algorithm with $\bar{n}$ large and analyze the behaviour for $\beta \to 0$.

The computations are done by using MATLAB, for the plot in Figure 7.3 the optimization library RoDoBo [14] was used.

Example 7.1 Let the data be given as follows

$$z(t,x) = -\sin(2\pi x), \quad \alpha = 10^{-2}, \quad \beta = 10^{-3}, \quad y_0(x) = (x(1-x),0), \quad \bar{n} = 0$$

for $x \in \Omega = (0,1)$ and $T = 1$. The control and observation area is given by

$$\omega_o = (0,1), \quad \omega_c = (0,1).$$

As an initial point for the algorithm we choose

$$y^0(t,x) = (x(1-x),0), \quad c^0 = 10$$

satisfying (5.28) and (5.29). We discretize our problem as presented in the previous section on uniform meshes with $N = 256$ and $M = 255$.

Table 7.1: Error of the Newton iterates

k	c	e_c^k	$e_c^k/e_c^{(k-1)}$	e_y^k	$e_y^k/e_y^{(k-1)}$	e_p^k	$e_p^k/e_p^{(k-1)}$	am	ap
0	10.0	8.49	-	9.63	-	0.74	-	42025	17450
1	1.214853	2.95e-01	3.48e-02	8.30e+00	8.63e-01	2.32e-01	3.12e-01	32298	25740
2	1.393382	1.17e-01	3.95e-01	7.51e-01	9.05e-02	3.12e-02	1.34e-01	22186	33754
3	1.489242	2.08e-02	1.78e-01	3.14e-01	4.18e-01	8.52e-03	2.73e-01	26101	31073
4	1.509587	4.43e-04	2.13e-02	3.28e-02	1.04e-01	1.04e-03	1.22e-01	24874	31490
5	1.510022	8.19e-06	1.85e-02	1.48e-03	4.52e-02	4.40e-05	4.22e-02	25120	31450
6	1.510031	9.14e-08	1.12e-02	5.53e-06	3.73e-03	9.35e-08	2.13e-03	25133	31448
7	1.510031	1.62e-10	1.77e-03	3.70e-10	6.69e-05	6.97e-12	7.45e-05	25133	31448
8	1.510031	0.00	0.00	0.00	0.00	0.00	0.00	25133	31448

In Table 7.1 we see the errors in the scalar $e_c^k = |c^k - c^*|$, in the state $e_y^k = \left\| y^k - y^* \right\|_X$ and in the adjoint state $e_p^k = \left\| p^k - p^* \right\|_{L^2(P^1) \cap L^2(P^0)}$ in every Newton iteration k. For the exact solution (y^*, p^*, c^*) we choose the 8th iterate. We do not consider the full norm of Y_0^* for the adjoint state, since we discretize the adjoint state by piecewise constants in time. By am we denote the number of mesh points in set A^- and by ap the number of mesh points in set A^+. As the stopping criterion for the Newton iteration we choose $tol = 10^{-9}$. If we go beyond this tolerance the residuums in the conjugate gradient method to solve the Newton equation (6.7) reach the machine accuracy. The behaviour of the errors presented in Table 7.1 indicate superlinear convergence.

Example 7.2 In this example we keep the data as above except for

$$\omega_o = (0, 1), \quad \omega_c = (0, 1/3),$$

i.e. the control domain is a subset of the domain of observation, cf. Lemma 3.2. In Table 7.2 the behaviour of the errors of the Newton iterates are shown. For the exact solution we choose the 6th Newton iterate and as in the previous

Table 7.2: Error of the Newton iterates

k	c	e_c^k	$e_k^c/e_{(k-1)}^c$	e_y^k	$e_k^y/e_{(k-1)}^y$	e_p^k	$e_k^p/e_{(k-1)}^p$	am	ap
0	10.0	9.62	-	11.64	-	8.67e-01	-	19923	0
1	0.346298	2.88e-02	3.00e-03	4.43e-01	3.80e-02	1.71e-02	1.97e-02	15559	2782
2	0.374087	1.06e-03	3.67e-02	2.81e-02	6.35e-02	1.21e-03	7.10e-02	15892	1699
3	0.375135	9.41e-06	8.89e-03	2.13e-04	7.59e-03	8.98e-06	7.40e-03	15875	1784
4	0.375145	4.24e-09	4.51e-04	5.15e-08	2.41e-04	1.64e-09	1.83e-04	15875	1785
5	0.375145	3.36e-13	7.93e-05	1.31e-12	2.54e-05	4.86e-14	2.96e-05	15875	1785
6	0.375145	0.00	0.00	0.00	0.00	0.00	0.00	15875	1785

example the iterates converge superlinearly.

Example 7.3 In this example we choose the data as above except for

$$\omega_o = (1/2, 1), \quad \omega_c = (0, 1/3),$$

i.e. $\omega_c \not\subset \omega_o$. Further, we set $tol = 10^{-7}$ for the reason already mentioned in Example 7.1. The behaviour of the errors of the Newton iterates is presented in Table 7.3. As the exact solution we take the 4th iterate and again we obtain

Table 7.3: Error of the Newton iterates

k	c	e_c^k	e_c^k/e_c^{k-1}	e_y^k	e_y^k/e_y^k	e_p^k	e_p^k/e_p^{k-1}	am	ap
0	10.0	9.45	-	10.67	-	0.41	-	0	10075
1	0.491219	1.28e-02	1.35e-03	3.91e+00	3.67e-01	1.81e-02	4.36e-02	0	11444
2	0.503892	1.57e-04	1.22e-02	1.07e-02	2.73e-03	2.63e-05	1.45e-03	0	11927
3	0.504049	1.10e-08	7.01e-05	2.01e-08	1.89e-06	4.69e-10	1.78e-05	0	11927
4	0.504049	0.00	0.00	0.00	0.00	0.00	0.00	0	11927

superlinear convergence.

We note that in these three examples above am and ap are identified before we stop. In fact not only the cardinality of the sets A^- and A^+ stagnates but the sets themselves are identified.

Example 7.4 In this example we apply a simple path-following strategy by choosing in every iteration the new regularization parameter by the rule

$$\beta_{n+1} = q\beta_n, \quad n \in \mathbb{N}_0,$$

with some given $q \in (0, 1)$ and $\beta_0 > 0$.

We choose

$$z = -1, \quad \alpha = 10^{-2}, \quad \beta_0 = 10^{-1}, \quad y_0(x) = (\sin(2\pi x), -4\sin(2\pi x)), \quad \bar{n} = 6 \tag{7.1}$$

for $x \in \Omega = (0,1)$ and $T = 1$. Further we set $q = 0.2$ and $\omega_c = \omega_o = \Omega$. We solve the problem on a spatial and temporal mesh with $N = 100$ and $M = 127$. For initialization we choose

$$y^0(t, x) = ((t-1)^4 \sin(2\pi x), 4(t-1)^3 \sin(2\pi x)), \quad c^0 = 1 \tag{7.2}$$

for $(t, x) \in Q$. The results are presented in Table 7.4. For decreasing β the corresponding values of the cost functional and the behaviour of the error $e_{\beta_n}^J = J(u_{\beta_n}, c_{\beta_n}) - J(u^*, c^*)$ is shown. For the exact solution (u^*, c^*) we take $(u_{\beta_6}, c_{\beta_6})$. Further the number of Newton steps ns is presented.

Table 7.4: Error in the cost functional

n	β_n	$J(u_{\beta_n}, c_{\beta_n})$	$J(u_{\beta_n}, c_{\beta_n}) - J(u^*, c^*)$	$e_{\beta_n}^J / e_{\beta_{n-1}}^J$	c_{β_n}	am	ap	ns
0	1.0	0.25171	3.77e-02	-	2.94061	2590	0	3
1	2.0e-02	0.22346	9.47e-03	0.25	3.70135	6833	0	3
2	4.0e-03	0.21623	2.24e-03	0.24	3.96424	8720	722	3
3	8.0e-04	0.21444	4.53e-04	0.20	4.04065	9702	1563	2
4	1.6e-04	0.21407	8.57e-05	0.18	4.05790	10164	1915	2
5	3.2e-05	0.21401	1.47e-05	0.17	4.06136	10371	2054	2
6	6.4e-06	0.21400	0.00	0.00	4.06209	10520	2120	1

The values of the cost functional decrease which confirms the theoretical result in (4.9). Further, the behaviour of the errors indicates superlinear convergence for $\beta \to 0$, which confirms the result of Corollary 4.1. The number of active points in A^- is larger than in A^+ which we expect for the given desired state.

For β smaller than presented in Table 7.4 the number of active and inactive nodes remains constant up to 3 switching nodes, however one looses the superlinear convergence. The number of Newton steps decreases which relies on the fact that the iteration is nested. For a non nested iteration (i.e. $(\underline{y}_n, \underline{p}_n, \underline{c}_n) = (\underline{y}_0, \underline{p}_0, \underline{c}_0)$ for all n) the number of Newton steps is increasing for smaller β, see Table 7.5.

Table 7.5: Newton steps for decreasing β

β	1.0e-00	2.0e-02	4.0e-03	8.0e-04	1.6e-04	3.2e-05	6.4e-06
ns	3	5	5	5	6	7	12

In Figure 7.1 we compare for time horizon $T = 2$ the state of the regularized control problem for data given in (7.1), (7.2) and $\beta = 4 \cdot 10^{-3}$ with the solution of the state equation for $u \equiv 0$. The plots show the behaviour of the state with
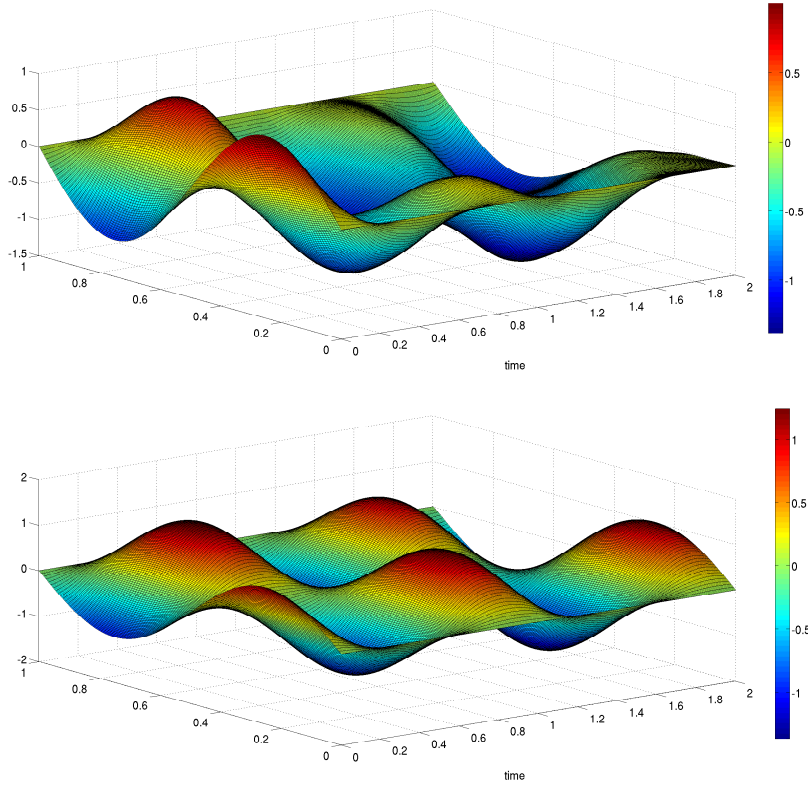
Fig. 7.1: State for the controlled and uncontrolled ($u \equiv 0$) problem

respect to time. The first plot indicates that the state tries to reach the desired state $z = -1$ different from the second (uncontrolled) one.

If we go beyond the time horizon $T = 2$ the tracking of the desired state by the optimal state of the regularized problem further improves.

In Figure 7.2 we see the corresponding optimal control of the regularized problem which is nearly of bang-bang type.

Figure 7.3 shows the optimal state for problem (P_1) when replacing the L^∞– by L^2–control costs with α given as in (7.1). The tracking of the desired state is nearly the same as in case of the regularized problem presented in Figure 7.1. But we see that in some parts the deflection in positive direction is less than for the regularized problem. This reflects our expectation, since the L^2–control space is larger than the L^∞–space and thus allows a better approximation of the desired state.

We also tested the case with $\omega_o = [0, 1/3]$ and $\omega_c = [2/3, 1]$ and observed similar numerical behaviour for different initializations $\underline{y}_0$ and $\underline{c}_0$.
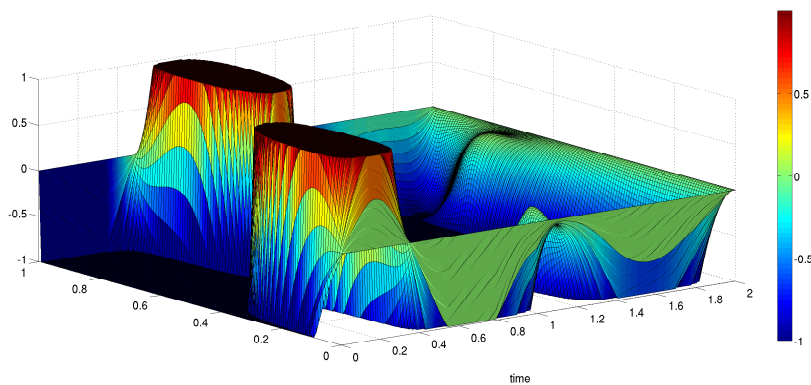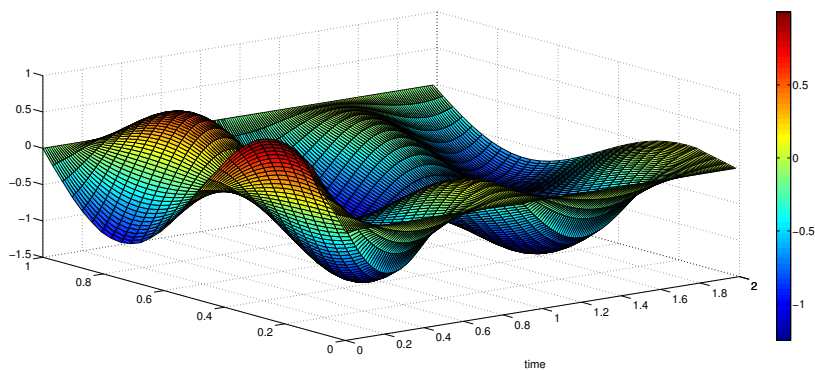
Fig. 7.2: Optimal control

Fig. 7.3: Optimal state for L^2–control costs

8 Appendix

8.1 Proof of Lemma 5.2

In the first step we show the bijectivity of the map $\mathbf{T}'(x)\colon W^{\mathbb{R}} \to \mathrm{Im}(\mathbf{T}'(x))$ for $x \in W$. The surjectivity is obvious. To verify injectivity we proceed as follows. Assume $\mathbf{T}'(x)v = \mathbf{T}'(x)w$ for given $v, w \in W^{\mathbb{R}}$. Then $v - w$ is the

solution of the following optimal control problem

$$
\begin{cases}
\displaystyle \min_{(\delta y, \delta u, \delta c) \in X \times U \times \mathbb{R}} \quad J(\delta y, \delta u, \delta c) = \frac{1}{2} \left\| \mathbf{C}_{\omega_o} \delta y - z_0 \right\|_{L^2(Q)}^2 + \frac{\beta c}{2} \left\| \delta u \right\|_{L^2(Q)}^2 \\
\hspace{6cm} + \frac{\alpha}{2} |\delta c|^2, \quad \text{s.t.} \\
\mathbf{A} \delta y - \delta c\, \mathbf{B}_{\omega_c} \operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p) - c \mathbf{B}_{\omega_c} \delta u \chi_{I_p} = 0 \quad \text{in } Q, \\
\hspace{5cm} \delta y(0) = 0 \quad \text{in } \Omega, \\
\hspace{5cm} \mathbf{C} \delta y|_\Sigma = 0 \quad \text{on } \Sigma.
\end{cases}
\tag{8.1}
$$

with $x = (y, p, c)$ and $z_0 \equiv 0$. Existence of a unique solution follows by considering the reduced functional $j(\delta u, \delta c) = J(\delta c, \delta u, \delta y(\delta u, \delta c))$, where δy is the solution to the constraining partial differential equation as a function of $(\delta u, \delta c)$. The solution is necessarily zero.

In the second step we prove the estimate (5.11). Let $x = (y, p, c) \in W$, $\delta x = (\delta y, \delta p, \delta c) \in W^{\mathbb{R}}$ and $z = (z_0, z_1, z_2, 0) \in \operatorname{Im}(\mathbf{T}'(x)) \cap Z_1$. Then the equation $\mathbf{T}'(x)^{-1}(z) = \delta x$ is equivalent to the following system

$$
\mathbf{A}^* \delta p + \mathbf{C}_{\omega_o}^* \mathbf{C}_{\omega_o} \delta y = \mathbf{C}_{\omega_o}^* z_0,
\tag{8.2}
$$

$$
\delta p(T) = 0,
$$

$$
\mathbf{B} \delta p|_\Sigma = 0,
$$

$$
\alpha \delta c - (\operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p), \mathbf{B}_{\omega_c}^* \delta p) = z_1,
\tag{8.3}
$$

$$
\mathbf{A} \delta y - \delta c \mathbf{B}_{\omega_c} \operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p) - \frac{c}{\beta} \mathbf{B}_{\omega_c} \mathbf{B}_{\omega_c}^* \delta p \chi_{I_p} = z_2,
\tag{8.4}
$$

$$
\delta y(0) = 0,
$$

$$
\mathbf{C} \delta y|_\Sigma = 0.
$$

Multiplying (8.2) with δy and (8.4) with $-\delta p$ and adding both equations we obtain

$$
\left\| \mathbf{C}_{\omega_o} \delta y \right\|_{L^2(Q)}^2 + \delta c (\operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p), \mathbf{B}_{\omega_c} \delta p) + \frac{c}{\beta} \left\| \mathbf{B}_{\omega_c}^* \delta p \chi_{I_p} \right\|_{L^2(Q)}^2
$$
$$
= -\langle z_2, \delta p \rangle_{Y,Y^*} + (\mathbf{C}_{\omega_o}^* z_0, \delta y). \quad (8.5)
$$

Here we used (2.2) and that $\delta y(0) = 0$ and $\delta p(T) = 0$.

By multiplying (8.3) with $-\frac{1}{\alpha}(\operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p), \mathbf{B}_{\omega_c}^* \delta p)$ and adding it to (8.5) we have

$$
\left\| \mathbf{C}_{\omega_o} \delta y \right\|_{L^2(Q)}^2 + \frac{1}{\alpha} (\operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p), \mathbf{B}_{\omega_c} \delta p)^2 \le \left\| z_2 \right\|_Y \left\| \delta p \right\|_{Y^*}
$$
$$
+ \left\| \mathbf{C}_{\omega_o}^* z_0 \right\|_{L^2(Q)} \left\| \mathbf{C}_{\omega_o} \delta y \right\|_{L^2(Q)} + \frac{1}{\alpha} |(\operatorname{sgn}_\beta(\mathbf{B}_{\omega_c}^* p), \mathbf{B}_{\omega_c}^* \delta p)| \, |z_1|.
$$

From the priori estimate in [12, p. 265] we have

$$
\left\| \delta p \right\|_{Y^*} \le \mathcal{C} \left\| \mathbf{C}_{\omega_o}^* (\mathbf{C}_{\omega_o} \delta y - z_0) \right\|_{L_2}
\tag{8.6}
$$

with $Y^* = L^2(P^1) \cap H^1(P^0)$. Using Young's inequality we further derive

$$\|\mathbf{C}_{\omega_o}\delta y\|^2_{L^2(Q)} \leq \mathcal{C}\,\|z_2\|^2_Y + \frac{1}{4}\|\mathbf{C}_{\omega_o}\delta y\|^2_{L^2(Q)} + \mathcal{C}\,\|\mathbf{C}^*_{\omega_o}z_0\|^2_{L_2}$$
$$+ \|\mathbf{C}^*_{\omega_o}z_0\|^2_{L_2} + \frac{1}{4}\|\mathbf{C}_{\omega_o}\delta y\|^2_{L^2(Q)} + \frac{1}{4}\|\mathbf{C}_{\omega_o}\delta y\|^2_{L^2(Q)} + \mathcal{C}|z_1|^2$$

and hence,

$$\|\mathbf{C}_{\omega_o}\delta y\|_{L^2(Q)} \leq \mathcal{C}\left(\|z_0\|_{L_2} + |z_1| + \|z_2\|_Y\right). \tag{8.7}$$

This implies

$$\|\delta p\|_{Y^*} \leq \mathcal{C}\,\|z\|_Z \tag{8.8}$$

and together with (8.3)

$$|\delta c| \leq \mathcal{C}\,\|z\|_Z. \tag{8.9}$$

Finally, from (8.4), (8.8), (8.9)

$$\|\delta y\|_X \leq \mathcal{C}\,\|z\|_Z.$$

8.2 Tangent and additional adjoint equations

Let $\delta y_{kh} = (\delta y_1^{k+1}, \delta y_2^{k+1})$ be the solution of the tangent equation

$$\sum_{m=1}^M (\partial_t \delta y_1^{k+1}, \xi^1)_{J_m} - (\delta y_2^{k+1}, \xi^1) + (\delta y_1^{k+1}(0), \xi^1(0))_{L^2(\Omega)} = 0 \quad \forall \xi^1 \in \widetilde{X}_{kh},$$

$$\sum_{m=1}^M (\partial_t \delta y_2^{k+1}, \xi^2)_{J_m} + (\nabla \delta y_1^{k+1}, \nabla \xi^2) + (\delta y_2^{k+1}(0), \xi^2(0))_{L^2(\Omega)}$$
$$- \delta c^{k+1}(\operatorname{sgn}_\beta(\chi_{\omega_c}p_2^k), \xi^2) - c^k(\chi_{\omega_c}\delta u_{kh}^{k+1}\chi_{I_{kh}}, \xi^2) = 0 \quad \forall \xi^2 \in \widetilde{X}_{kh},$$

and $\delta p_{kh} = (\delta p_1^{k+1}, \delta p_2^{k+1})$ of the additional adjoint

$$-\sum_{m=0}^{M-1}(\psi^1(t_m), \delta p_1^{k+1}(t_{m+1}) - \delta p_1^{k+1}(t_m))_{L^2(\Omega)} + (\nabla \psi^1, \nabla \delta p_2^{k+1})$$
$$+ (\psi^1(t_M), \delta p_1^{k+1}(t_M))_{L^2(\Omega)} = -(\psi^1, \chi_{\omega_o}\delta y_1^{k+1}) \quad \forall \psi^1 \in X_{kh},$$

$$-\sum_{m=0}^{M-1}(\psi^2(t_m), \delta p_2^{k+1}(t_{m+1}) - \delta p_2^{k+1}(t_m))_{L^2(\Omega)} - (\psi^2, \delta p_1^{k+1})$$
$$+ (\psi^2(t_M), \delta p_2^k(t_M))_{L^2(\Omega)} = 0 \quad \forall \psi^2 \in X_{kh},$$

then the first and second derivative of j_{kh} at a point $(u_{kh}, c) \in U_{kh} \times \mathbb{R}$ are given by

$$(j_{kh}^k)'(u_{kh}, c)(\delta u, \delta c) = \beta c^k(u_{kh}, \delta u) + \alpha(c + \frac{1}{\alpha}(\mathrm{sgn}_\beta(\chi_{\omega_c}p_2^k), \chi_{\omega_c}p_2^k))\delta c$$
$$- \left\| \chi_{\omega_c}p_2^k \right\|_{L^1_\beta(Q)} \delta c - \delta c(\mathrm{sgn}_\beta(\chi_{\omega_c}p_2^k), p_2^{k+1}) - c^k(\chi_{\omega_c}\delta u \chi_{I_{kh}}, p_2^{k+1})$$

and

$$(j_{kh}^k)''(u_{kh}, c)(\tau u, \tau c, \delta u, \delta c) = \beta c^k(\tau u, \delta u) + \alpha \delta c \cdot \tau c - \delta c(\mathrm{sgn}_\beta(\chi_{\omega_c}p_2^k), \delta p_2^{k+1})$$
$$- c^k(\chi_{\omega_c}\tau u \chi_{I_{kh}}, \delta p_1^{k+1})$$

for $\delta u, \tau u \in U_{kh}$ and $\delta c, \tau c \in \mathbb{R}$.

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