FINITE ELEMENT ERROR ESTIMATES FOR AN OPTIMAL CONTROL PROBLEM GOVERNED BY THE BURGERS EQUATION

PEDRO MERINO

ABSTRACT. We derive a-priori error estimates for the finite-element approximation of a distributed optimal control problem governed by the steady one-dimensional Burgers equation with pointwise box constraints on the control. Here the approximation of the state and the control is done by using piecewise linear functions. With this choice, an L^2 superlinear order of convergence for the control is obtained; moreover, under a further assumption on the regularity structure of the optimal control this error estimate can be improved to $h^{3/2}$. The theoretical findings are tested experimentally by means of numerical examples.

1. INTRODUCTION

We consider the finite element approximation of the following optimal control problem of the steady one-dimensional Burgers equation with pointwise control constraints:

$$(\mathbf{P}) \begin{cases} \min J(y,u) = \dfrac{1}{2}\|y - y_d\|_{L^2(0,1)}^2 + \dfrac{\lambda}{2}\|u\|_{L^2(0,1)}^2 \\ \text{subject to:} \\ -\nu y'' + yy' = Bu \quad \text{in } (0,1), \\ y(0) = y(1) = 0, \\ \alpha \le u(x) \le \beta, \quad \text{a.e. in } (0,1). \end{cases}$$

The Burgers equation is a well-known one dimensional model for turbulence and its control has been studied by several authors c.f. [1], [2],[7]. Our aim in this paper consists in deriving a-priori error estimates for the optimal control problem in the L^2-norm.

Finite element approximations for control constrained control problems in fluid mechanics have been previously considered in [8] and [5] for piecewise constant controls. In particular, in the last, the authors report an error order of h^2 if the control space is not discretized, whereas an order of h is obtained for the piecewise constant discretization. It is natural to expect that these error estimates also holds in the case of the Burgers equation using the theory developed in [5]. However, if the control space is discretized by piecewise linear functions, results were only obtained for the semilinear case in [3] and in [11] for the linear–quadratic case. Since the optimal control is Lipschitz continuous, its approximations by piecewise linear functions seems to be a natural choice, which in addition piecewise linear functions have less degrees of freedom than piecewise constant functions. Here we aim to perform this task by combining the arguments in [5] and [3] to obtain a superlinear error of convergence for the L^2–norm estimate of the control. In addition,

by considering a stronger assumption on the structure of the optimal control and relying on the one–dimensional setting of our problem, we are able to improve the order of the error to $h^{3/2}$.

The paper is organized as follows: first we briefly comment the properties the optimal control problem and its conditions for optimality, next we refer to the finite element method approximation of the Burgers equation and the corresponding error estimates . Next, we discuss the approximation of the optimal control problem by piecewise linear functions by establishing a superlinear order of convergence for the optimal control. We finish the theory by showing that the superlinear error of convergence can be improved under certain assumptions on the regularity of the optimal control. Finally, we discuss some numerical experiments to confront our theoretical findings.

2. The control problem

We consider the discretization analysis for the following optimal control problem, governed by Burgers equation:

$$
(\mathbf{P}) \begin{cases} \min_{(y,u)\in H_0^1(0,1)\times U_{ad}} J(y,u) = \frac{1}{2}\|y - y_d\|^2 + \frac{\lambda}{2}\|u\|^2 \\ \text{subject to:} \\ -\nu y'' + yy' = Bu \quad \text{in } (0,1), \\ y(0) = y(1) = 0. \end{cases}
$$

Here, U_{ad} is the set of admissible controls defined by $U_{ad} = \{u \in L^2(0,1) : \alpha \leq u \leq \beta\}$ with constants α and β satisfying $\alpha < \beta$. $\lambda > 0$ is the usual Tychonoff parameter. We shall denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the norm and the scalar product in $L^2(0,1)$, respectively. $B(x) = \mathcal{X}_\omega(x)$ is the indicator function defined in an open subinterval $\omega \subset \Omega := (0,1)$, whereas ν denotes the viscosity parameter which is assumed that satisfies (6). For different spaces, the open ball centered in u with radius $r > 0$ will be denoted by $B(u,r)$ if there is no risk of confusion.

2.1. **The state equation equation.** The steady Burgers equation is given by

$$
-\nu y'' + yy' = f \quad \text{in } (0,1), \tag{3a}
$$

$$
y(0) = y(1) = 0. \tag{3b}
$$

The weak formulation of the homogeneous Dirichlet problem for the Burgers equation is as follows: given $f \in L^2(0,1)$, find $y \in H_0^1(0,1)$ such that

$$
a(y,\varphi) + b(y,y,\varphi) = (f,\varphi), \quad \forall \varphi \in H_0^1(0,1), \tag{4}
$$

where: $a : H_0^1(0,1) \times H_0^1(0,1) \to \mathbb{R}$ is the continuous, bilinear and symetric form defined by

$$
a(\phi,\varphi) = \nu \int_0^1 \phi'\varphi' dx,
$$

and $b : (H_0^1(0,1))^3 \to \mathbb{R}$ stands for the continuous trilinear form defined by

$$
b(\phi,\varphi,\psi) = \frac{1}{3} \int_0^1 [(\phi\varphi)'\psi + \phi\varphi'\psi] dx.
$$

The trilinear b enjoys the following important properties c.f. [13]

$$|b(\phi,\varphi,\psi)| \leq \|\phi\|_{H_0^1(0,1)}\|\varphi\|_{H_0^1(0,1)}\|\psi\|_{H_0^1(0,1)}, \quad \forall(\phi,\varphi,\psi) \in (H_0^1(0,1))^3, \qquad (5a)$$

$$b(\phi,\varphi,\psi) + b(\phi,\psi,\varphi) = 0, \quad \forall(\phi,\varphi,\psi) \in (H_0^1(0,1))^3, \qquad (5b)$$

$$b(\phi,\varphi,\varphi) = 0, \quad \forall(\phi,\varphi) \in (H_0^1(0,1))^2. \qquad (5c)$$

It is well known, cf. [13, Theorem 2.10] that if the condition

$$\|f\| < \nu^2. \qquad (6)$$

holds, the Burgers equation (4) has a unique solution depending on the right hand side. Indeed, for every $f \in L^2(0,1)$, there exists $y \in H_0^1(0,1)$ which satisfies (4) and fulfill the relation $\|y\|_{H_0^1(0,1)} \leq \frac{1}{\nu}\|f\|$. In addition, by taking the nonlinearity to the right hand side and relying on elliptic regularity results, it can be shown that y belongs to $H^{2+m}(0,1)$ for every integer $m \geq 0$, provided that $f \in H^m(0,1)$. In the following, we link $f \in L^2(0,1)$ to its associated state $y \in H_0^1(0,1)$ as the solution of (4) and we will indicate this explicitly by writing $y = y(f)$ to emphasize that the state y is generated by the right-hand side f. The following property will be useful in the forthcoming sections.

Lemma 1. *Let $(u_k)_{k \in \mathbb{N}}$ be a sequence of functions which converges weakly to $\bar{u}$ in $L^2(0,1)$ satisfying $\|u_k\| < \nu^2$, then the sequence $(y_k)_{k \in \mathbb{N}}$ of the corresponding associated states converges strongly to $\bar{y}$ in $H_0^1(0,1)$.*

Proof. The result is a straightforward consequence of the properties of the solutions of the Burgers equation and the compactness of the usual embeddings. ∎

2.2. **Existence of solution for the optimal control problem.** The arguments for proving existence of an optimal control are standard since U_{ad} is a nonempty, closed and convex set in $L^2(0,1)$.

In the following, $\mathcal{F}_{ad}$ will denote the set feasible pairs for (**P**), that is, those pairs $(y,u) \in H_0^1(0,1) \times U_{ad}$ such that (87b) is satisfied with $f = Bu$. Note that $\mathcal{F}_{ad}$ is nonempty.

Theorem 1. *If the inequality (6) holds then the problem (P) has a solution.*

Remark 1. *Despite the strict convexity of the objective functional and uniqueness of the solution of the state equation, uniqueness of the optimal control can not be guaranteed since $\mathcal{F}_{ad}$ is not necessarily convex.*

3. Optimality conditions

In this section we shall derive first-order necessary and second-order sufficient conditions for local solutions of (**P**), both play an important role in the derivation of error estimates. Therefore, we make precise the notion of local minimum.

Definition 1. *A pair $(\bar{y},\bar{u}) \in \mathcal{F}_{ad}$ will be referred as local optimal pair for (**P**) if there exist positive reals ρ_u and ρ_y such that*

$$J(y,u) \geq J(\bar{y},\bar{u}), \quad \forall(y,u) \in \mathcal{F}_{ad} \cap (B(\bar{y},\rho_y) \times B(\bar{u},\rho_u)).$$

For convenience, we introduce the following operator.

Definition 2. *We define the operator $R : H_0^1(0,1) \times U_{ad} \to H^{-1}(0,1)$ by the relation*

$$\langle R(y,u),\varphi \rangle = a(y,\varphi) + b(y,y,\varphi) - (Bu,\varphi), \quad \forall\varphi \in H^{-1}(0,1),$$

where $\langle \cdot,\cdot \rangle$ denotes the dual pairing between $H^{-1}(0,1)$ and $H_0^1(0,1)$.

Note that $\langle R(y,u), \varphi \rangle = 0$ indicates that y is the weak solution of the state equation (87b) associated to the control u.

In the next lemmas we study the differentiability of the operator R.

Lemma 2. *Let $(w,h) \in L^2(0,1) \times H_0^1(0,1)$. The operator R given in Definition 2 has first and second derivatives given by:*

$$R'(y,u) : H_0^1(0,1) \times L^2(0,1) \rightarrow H^{-1}(0,1),$$
$$\langle R'(y,u)(w,h), \varphi \rangle = a(w,\varphi) + b(w,y,\varphi) + b(y,w,\varphi) - (Bh, \varphi), \quad (7a)$$

and

$$R''(y,u) : (H_0^1(0,1) \times L^2(0,1))^2 \rightarrow H^{-1}(0,1),$$
$$\langle R''(y,u)(w_1,h_1)(w_2,h_2), \varphi \rangle = b(w_1, w_2, \varphi) + b(w_2, w_1, \varphi), \quad (7b)$$

respectively for any (w,h), (w_1,h_1) and (w_2,h_2) in $H_0^1(0,1) \times L^2(0,1)$ accordingly, and for all $\varphi \in H_0^1(0,1)$.

Proof. The result follows from the linear properties of a, b and the scalar product in $L^2(0,1)$. ∎

3.1. **First-order necessary conditions.** The following first-order necessary conditions are derived in the spirit of [14].

Lemma 3. *Let $(\bar{y}, \bar{u}) \in \mathcal{F}_{ad}$ be a local optimal pair for (**P**), then $(\bar{y}, \bar{u})$ is a regular point for (**P**) in the sense of [14].*

Proof. The regular point condition of $(\bar{y}, \bar{u})$ for the problem (**P**) is achieved by noting that for every $f \in H^{-1}(0,1)$ the linear equation

$$R'(\bar{y}, \bar{u})(w,h) = f$$

has a unique solution (w,h) of the form $\theta(y - \bar{y}, u - \bar{u})$, with $(y,u) \in H_0^1(0,1) \times U_{ad}$ and $\theta \geq 0$. ∎

Theorem 2. *Let $(\bar{y}, \bar{u})$ be a solution of (**P**) such that $\|B\bar{u}\| < \nu^2$, then there exists an adjoint state $\bar{p} \in H_0^1(0,1)$ such that the following optimality system is fullfilled:*

$$-\nu\bar{y}'' + \bar{y}\bar{y}' = B\bar{u}, \qquad in\ (0,1), \quad with \quad \bar{y}(0) = \bar{y}(1) = 0, \qquad (8a)$$
$$-\nu\bar{p}'' - \bar{y}\bar{p}' = \bar{y} - y_d, \qquad in\ (0,1), \quad with \quad \bar{p}(0) = \bar{p}(1) = 0, \qquad (8b)$$
$$(B\bar{p} + \lambda\bar{u}, u - \bar{u}) \geq 0, \qquad \forall u \in U_{ad}. \qquad (8c)$$

Moreover, (8c) can be equivalently expressed in terms of the projection operator:

$$\bar{u}(x) = P_{[\alpha,\beta]}\left(-\frac{1}{\lambda}B\bar{p}(x)\right) \qquad (9)$$

Proof. This system obtained by using Lemma 3 and applying the theory in [14] . ∎

Remark 2. *It is worth to point out that extra regularity of the optimal quantities can be deduced using standard elliptic regularity results from [10]. Indeed, by taking $\omega = (0,1)$ and since $y \in H^2(0,1)$ and $y_d \in L^2(0,1)$ we have that $\bar{p} \in H^2(0,1) \hookrightarrow C^{1,\frac{1}{2}}([0,1])$. From the characterization of $\bar{u}$ given by (9) and properties of the projection operator $P_{[\alpha,\beta]}$, it follows that $\bar{u} \in C^{0,1}([0,1])$. By bootstrapping arguments on the state equation, we have that $\bar{y}$ solves Poisson's equation with a right hand side in $C^{0,\frac{1}{2}}([0,1])$ therefore, by elliptic regularity results, we conclude that $\bar{y} \in C^{2,\frac{1}{2}}([0,1])$.*

Furthermore, if y_d is assumed in to be in $C^{2,\frac{1}{2}}([0,1])$ then $\bar{p}$ is also in $C^{2,\frac{1}{2}}([0,1])$. This high regularity of $\bar{p}$, however, can not be transferred to the optimal control because the projection operator.

For our forthcoming analysis, we introduce the *Lagrangian* $\mathcal{L} : H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \to \mathbb{R}$ defined by:

$$\mathcal{L}(y,u,p) = \frac{1}{2}\|y - y_d\| + \frac{\lambda}{2}\|Bu\| - \langle R(y,u), p \rangle,$$

whose corresponding first and second derivatives (with respect to the first and second variable) at $(y,u,p) \in H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1)$, are given by:

$$\mathcal{L}'(y,u,p)(w,h) = (y - y_d, w) + \lambda(u,h) - \langle R'(y,u)(w,h), p \rangle, \tag{10a}$$

$$\text{for all } (w,h) \in H_0^1(0,1) \times L^2(0,1),$$

$$\mathcal{L}''(y,u,p)(w_1,h_1)(w_2,h_2) = (w_1, w_2) + \lambda(h_1, h_2) - \langle R''(y,u)(w_1,h_1)(w_2,h_2), p \rangle, \tag{10b}$$

$$\text{for all } (w_i, h_i) \in H_0^1(0,1) \times L^2(0,1), i = 1,2.$$

These expressions allow us to write down optimality system (8) in terms of the derivatives of $\mathcal{L}$ in the following usual way:

$$\text{(8b) is equivalent to} \quad \frac{\partial \mathcal{L}}{\partial y}(\bar{y}, \bar{u}, \bar{p}) = 0, \quad \text{and} \tag{11a}$$

$$\text{(8c) is equivalent to} \quad \frac{\partial \mathcal{L}}{\partial u}(\bar{y}, \bar{u}, \bar{p})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad}. \tag{11b}$$

3.2. **Second-order sufficient optimality conditions.** The forthcoming analysis of Section 5.1 concerning the approximation of the optimal control problem by the finite element method, requires the formulation of second-order sufficient optimality conditions. By the nature of the nonlinearity of the Burgers equation, it shall be notice that the two norm-discrepancy does not occur in our formulation. In order to establish second-order sufficient conditions we introduce the *critical cone*. For $\tau > 0$, we define the set

$$\Omega_\tau := \{x \in (0,1) : |\bar{p}(x) + \lambda\bar{u}(x)| > \tau\}.$$

The critical cone $C_{\bar{u}}^\tau$ consists of those directions $v \in L^2(0,1)$, such that

$$v(x) = 0 \text{ if } x \in \Omega_\tau, \tag{12a}$$

$$v(x) \geq 0 \text{ if } x \in \Omega\backslash\Omega_\tau \text{ and } \bar{u}(x) = u_a(x), \text{ and} \tag{12b}$$

$$v(x) \leq 0 \text{ if } x \in \Omega\backslash\Omega_\tau \text{ and } \bar{u}(x) = u_b(x). \tag{12c}$$

The next theorem states second-order sufficient conditions for $(\mathbf{P})$. For a better presentation we will use the notation $\mathcal{L}''(\bar{y}, \bar{u}, \bar{p})[y, u]^2 = \mathcal{L}''(\bar{y}, \bar{u}, \bar{p})(y, u)(y, u)$.

Theorem 3. *Let $(\bar{u}, \bar{y}) \in \mathcal{F}_{ad}$ be a feasible pair for $(\mathbf{P})$ satisfying first-order necessary conditions formulated in Theorem 2, with adjoint state $\bar{p} \in H_0^1(0,1)$. In addition, suppose there are $\tau > 0$ and $\delta > 0$, such that the coercivity property*

$$\delta\|h\|^2 \leq (w,w) + \lambda(h,h) - 2(ww', \bar{p}), \tag{13}$$

is satisfied for all $h \in C_{\bar{u}}^\tau$ and all $w \in H_0^1(0,1)$ such that $R'(\bar{y}, \bar{u})(w,h) = 0$, then there exist constants $\sigma > 0$ and $\varepsilon > 0$ such that

$$J(\bar{y}, \bar{u}) + \sigma\|u - \bar{u}\|^2 \leq J(y,u) \tag{14}$$

holds for every $u \in U_{ad} \cap \overline{B(\bar{u}, \varepsilon)}$ *and* $y \in H_0^1(0,1)$ *obeying* $R(y, u) = 0$.

Proof. We argue by contradiction by adapting the the proof of Theorem 4.1 in [9]. Therefore, we assume the existence of a sequence $(u_k)_{k \in \mathbb{N}}$ in U_{ad} converging to $\bar{u}$, such that the sequence $(y_k)_{k \in \mathbb{N}}$ of their associated states converge to $\bar{y}$. Therefore, (y_k, u_k) is an admissible pair which fulfills the relation

$$J(\bar{y}, \bar{u}) + \frac{1}{k} \|u_k - \bar{u}\|^2 > J(y_k, u_k). \tag{15}$$

Let us define the sequence of directions $h_k := \dfrac{u_k - u}{\rho_k}$, and the sequence $w_k := \dfrac{y_k - \bar{y}}{\rho_k}$, with $\rho_k := \|u_k - \bar{u}\|$, for every $k \in \mathbb{N}$. Clearly $(h_k)_{k \in \mathbb{N}}$ is bounded, with $\|h_k\| = 1$ and so $(w_k)_{k \in \mathbb{N}}$ is also bounded in $H_0^1(0,1)$; this implies the existence of subsequences denoted again by $(h_k)_{k \in \mathbb{N}}$ and $(w_k)_{k \in \mathbb{N}}$ respectively, such that $h_k \rightharpoonup h$ in $L^2(0,1)$ and $w_k \rightharpoonup w$ in $H_0^1(0,1)$. Moreover, since h belongs to the closed and convex set (and therefore weakly closed) $C_{\bar{u}}^\tau \cap \overline{B(\bar{u}, 1)}$, it follows that h also belongs to $C_{\bar{u}}^\tau \cap \overline{B(\bar{u}, 1)}$. Let us check that (w, h) satisfies $R'(\bar{y}, \bar{u})(w, h) = 0$. From the definition of R and Lemma 2, the pair (w_k, h) satisfies:

$$a(w_k, \varphi) + b(w_k, y_k, \varphi) + b(\bar{y}, w_k, \varphi) - (Bh, \varphi) = 0 \quad \forall \varphi \in H_0^1(0,1).$$

Taking the limit $k \to \infty$ in (16), by the convergence properties of $(h_k)_{k \in \mathbb{N}}$ and $(w_k)_{k \in \mathbb{N}}$ we see that the pair (w, h) satisfies the linearized equation:

$$R'(\bar{y}, \bar{u})(w, h) = a(w, \varphi) + b(w, \bar{y}, \varphi) + b(\bar{y}, w, \varphi) - (Bh, \varphi) = 0 \quad \forall \varphi \in H_0^1(0,1). \tag{17}$$

By applying the mean value theorem to the Lagrangian in (15), we obtain

$$\rho_k \mathcal{L}'(\xi_k, \zeta_k, \bar{p})(w_k, h) = \mathcal{L}(y_k, u_k, \bar{p}) - \mathcal{L}(\bar{y}, \bar{u}, \bar{p}) = J(\bar{y}, \bar{u}) - J(y_k, u_k) < \frac{1}{k} \|u_k - \bar{u}\|^2, \tag{18}$$

where ξ_k is between y_k and $\bar{y}$, and ζ_k is between w_k and w. Since $h \rightharpoonup h$ in $L^2(0,1)$, Lemma 1 implies that $w_k \to w$ in $L^2(0,1)$ therefore, from (18) we arrive to

$$\mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(w, h) \leq 0. \tag{19}$$

On the other hand, we find that $\mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(w_k, h) \geq 0$ holds for every $k \in \mathbb{N}$ in view of the first-order necessary conditions expressed in (11). After passing to the limit $k \to \infty$ and using the same convergence arguments it follows that

$$\mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(w, h) \geq 0, \tag{20}$$

thus we have that $\mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(w, h) = 0$.

Now, we show that $h = 0$. We recall that if $h = 0$ then $w = 0$ because w is the unique solution of the linearized equation (17). By using the second-order Taylor expansion of the Lagrangian and having in mind (7b) and (10b) we get

$$\rho_k \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(w_k, h) + \frac{\rho_k^2}{2} \mathcal{L}''(\bar{y}, \bar{u}, \bar{p})[w_k, h]^2 = \mathcal{L}(y_k, u_k, \bar{p}) - \mathcal{L}(\bar{y}, \bar{u}, \bar{p}), \tag{21}$$

hence, by using (21) in (15) we estimate

$$\rho_k \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(w_k, h) + \frac{\rho_k^2}{2}((w_k, w_k) + \lambda(h_k, h_k) - 2(w_k w_k', \bar{p})) < \frac{\rho_k^2}{k}. \tag{22}$$

From the definition of w_k and h_k and (11) the first term in the last inequality is nonnegative, therefore we have

$$(w_k, w_k) + \lambda(h_k, h_k) - 2(w_k w_k', \bar{p}) < \frac{2}{k}. \tag{23}$$

Once again, since $h_k \rightharpoonup h$ in $L^2(0,1)$ and $w_k \rightharpoonup w$ in $H_0^1(0,1)$ then

$$(w,w) + \lambda(h,h) - 2(ww', \bar{p}) \leq \liminf_{k\to\infty} (w_k, w_k) + \lambda(h_k, h_k) - 2(w_k w_k', \bar{p}) \leq 0, \tag{24}$$

which together with second-order condition (13) implies that $h = 0$. Finally, by observing that $\|h\| = 1$, we can infer from (23) the final contradiction:

$$\lambda = \liminf_{k\to 0}(w_k, w_k) + \lambda - 2(w_k w_k', \bar{p})$$
$$\leq \liminf_{k\to 0}(w_k, w_k) + \lambda(h_k, h_k) - 2(w_k w_k', \bar{p}) \leq 0.$$

∎

4. Finite element approximation of the Burgers equation

This section is devoted to the approximation of Burgers equation by using the finite element method and the derivation of the corresponding error of convergence. Let n be a positive integer, we define $h := 1/n$ and a uniform mesh on the interval $[0,1]$ denoted by $\mathcal{I}_h$, which consists of n subintervals: $I_i = [x_{i-1}, x_i]$ of $[0,1]$ for $i = 1, \ldots, n$, such that $0 = x_0 < x_1 \ldots < x_n = 1$ and $[0,1] = \cup_{i=1}^n I_i$. We also introduce the finite dimensional space $V_h \subset H_0^1(0,1)$ defined by

$$V_h = \{y_h \in C([0,1]) : y_{h|I_i} \in \mathcal{P}_1, \text{ for } i = 1, \ldots, n, \text{ with } y_h(0) = y_h(1) = 0\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than one. Therefore, we define the discrete Burgers equation in V_h as follows: given $f \in L^2(0,1)$ find $y_h \in V_h$ such that

$$a(y_h, \varphi_h) + b(y_h, y_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h. \tag{25}$$

Theorem 4. *If $f \in U_h$ is such that $\|f\| < \nu^2$ then, equation (25) has a unique solution $y_h \in V_h$ such that*

$$\|y_h\|_{H_0^1(0,1)} \leq \frac{1}{\nu}\|f\|. \tag{26}$$

Proof. The proof is completely analogous to the proof in [13, Theorem 2.10]. ∎

Let us denote by $\Pi_h : C([0,1]) \to V_h$ the usual Lagrange interpolation operator such that for every $z \in H_0^1(0,1)$, the element $\Pi_h z$ is the unique element in V_h which satisfies $\Pi_h z(x_i) = z(x_i)$ for $i = 0, 1, \ldots, n$.

For convenience, we recall a well known result which establishes an estimate for the interpolation error cf. [6].

Lemma 4. *Let nonnegative integers m and k and $p, q \in [1, \infty]$. If the embeddings*

$$W^{k+1,p}(T) \hookrightarrow C^0(T), \text{ and}$$
$$W^{k+1,p}(T) \hookrightarrow W^{m,q}(T)$$

hold, then there exists a constant $C > 0$ independent of h such that the following interpolation error is satisfied

$$\|y - \Pi_T y\|_{W^{m,q}(T)} \leq C h^{n(\frac{1}{q} - \frac{1}{p}) + k + 1 - m} \|y\|_{W^{k+1,p}(T)}, \tag{27}$$

where $\Pi_T y$ is the restriction of $\Pi_h y$ to an element T of the discretization of the domain with dimension n.

Moreover, Lemma 4 implies that

$$\lim_{h\to 0}\frac{1}{h}\|z - \Pi_h z\| = 0, \quad \forall z \in W^{1,p}(0,1), \text{ and } 1 < p. \tag{28}$$

The proof for this result can be found in [4, Lemma 7]. We are interested in the error estimate for the approximation of the solution of the Burgers equation using linear finite elements, to this purpose we convent that C denotes a generic constant which is positive and independent of h.

Lemma 5. *Let $f \in L^2(0,1)$ be such that $\|f\| < \nu^2$ and $y \in H_0^1(0,1)$ such that $R(y,f) = 0$. If y_h denotes the corresponding solution of the discrete equation (25) with right-hand side f; then, the estimate*

$$\|y - y_h\|_{H_0^1(0,1)} \le C\|y - \Pi_h y\|_{H_0^1(0,1)}, \tag{29}$$

is satisfied.

Proof. Since $y \in H_0^1(0,1)$ and $y_h \in V_h$ satisfy equations (4) and (25) respectively, after subtracting both equations we get

$$a(y - y_h, \varphi_h) + b(y,y,\varphi_h) - b(y_h,y_h,\varphi_h) = 0, \forall \varphi_h \in V_h. \tag{30}$$

In particular, if z_h is an arbitrary element in V_h, we choose $\varphi_h = y_h - z_h$ in (30), resulting in

$$a(y_h - z_h, y_h - z_h) = a(y - z_h, y_h - z_h) + b(y,y,y_h - z_h) - b(y_h,y_h,y_h - z_h) \tag{31}$$

Let us estimate the right-hand side of (31). In view of (5b) and (5c) we find that

$$\begin{aligned}
b(y,y,y_h - z_h) &- b(y_h,y_h,y_h - z_h) \\
&= b(y, y - z_h, y_h - z_h) + b(y, z_h, y_h - z_h) - b(y_h, y_h, y_h - z_h) \\
&= b(y, y - z_h, y_h - z_h) + b(y - z_h, y_h, y_h - z_h) - b(y_h - z_h, y_h, y_h - z_h)
\end{aligned}$$

using [13, Lemma 3.4, p.9] and inequality (26) we find out that

$$\begin{aligned}
b(y,&y, y_h - z_h) - b(y_h,y_h,y_h - z_h) \\
&\le (\|y\|_{H_0^1(0,1)} + \|y_h\|_{H_0^1(0,1)})\|y - z_h\|_{H_0^1(0,1)}\|y_h - z_h\|_{H_0^1(0,1)} + \|y_h\|_{H_0^1(0,1)}\|y_h - z_h\|_{H_0^1(0,1)}^2 \\
&\le \frac{2}{\nu}\|f\|\|y - z_h\|_{H_0^1(0,1)}\|y_h - z_h\|_{H_0^1(0,1)} + \frac{1}{\nu}\|f\|\|y_h - z_h\|_{H_0^1(0,1)}^2 \\
&\le 2\nu\|y - z_h\|_{H_0^1(0,1)}\|y_h - z_h\|_{H_0^1(0,1)} + \frac{1}{\nu}\|f\|\|y_h - z_h\|_{H_0^1(0,1)}^2. \tag{32}
\end{aligned}$$

Using (32) in identity (31), the continuity of a and b implies that

$$\nu\|y_h - z_h\|_{H_0^1(0,1)}^2 \le 3\nu\|y - z_h\|_{H_0^1(0,1)}\|y_h - z_h\|_{H_0^1(0,1)} + \frac{1}{\nu}\|f\|\|y_h - z_h\|_{H_0^1(0,1)}^2,$$

from which, we conclude that

$$\nu\left(1 - \frac{\|f\|}{\nu^2}\right)\|y_h - z_h\|_{H_0^1(0,1)}^2 \le 3\nu\|y - z_h\|_{H_0^1(0,1)}\|y_h - z_h\|_{H_0^1(0,1)}. \tag{33}$$

Observe that the coefficient on the left-hand side is a positive number. Taking $z_h = \Pi_h y$ in (33) and using the fact that Π_h is a continuous operator, it follows that

$$\|y_h - \Pi_h y\|_{H_0^1(0,1)} \le C\|y - \Pi_h y\|_{H_0^1(0,1)}. \tag{34}$$

Finally, the last inequality implies the desired estimate as follows

$$\|y - y_h\|_{H_0^1(0,1)} \le \|y - \Pi_h y\|_{H_0^1(0,1)} + \|y_h - \Pi_h y\|_{H_0^1(0,1)} \le C\|y - \Pi_h y\|_{H_0^1(0,1)}. \tag{35}$$

$\blacksquare$

Combining Lemmas 4 and 5 we arrive to the following result.

Theorem 5. *Let $f \in L^2(0,1)$ be such that $\|f\| < \nu^2$ and let $y \in H_0^1(0,1)$ and $y_h \in V_h$ be the solutions of the equations (4) and (25) respectively. Then the estimate*

$$\|y - y_h\|_{H_0^1(0,1)} \leq Ch\|y\|_{H^2(0,1)} \tag{36}$$

is fulfilled.

In the process of deriving error estimates for the finite element approximation of the optimal control problem (**P**), we will need the following estimate in the L^2–norm.

Theorem 6. *Let $f \in L^2(0,1)$ be such that $\nu^2 > \|f\|$ and let $y \in H_0^1(0,1)$ and $y_h \in V_h$ the solutions of the state equations (4) and (25) respectively. Then, the estimate*

$$\|y - y_h\|_{L^2(0,1)} \leq Ch^2\|y\|_{H^2(0,1)} \tag{37}$$

is fulfilled.

Proof. In order to derive the L^2–estimate for the approximation error of the Burgers equation, we introduce the following auxiliary linear problem:

$$\begin{cases} \text{Given } r \in L^2(0,1), \text{ find } z \in H_0^1(0,1) \text{ such that:} \\ a(z,\varphi) + b(y,\varphi,z) + b(\varphi,y,z) = (r,\varphi), \quad \forall \varphi \in H_0^1(0,1), \end{cases}$$

and its finite element approximation:

$$\begin{cases} \text{Given } r \in L^2(0,1), \text{ find } z_h \in V_h \text{ such that:} \\ a(z_h,\varphi_h) + b(y,\varphi_h,z_h) + b(\varphi_h,y,z_h) = (r,\varphi_h), \quad \forall \varphi_h \in V_h. \end{cases}$$

Based on the properties of b, it is clear that equations (38) and (39) fulfill the hypothesis of the Lax-Milgram theorem in their respective formulation spaces. Indeed, from its definition the symmetric bilinear form $\tilde{a}(z,\varphi) := a(z,\varphi) + b(y,\varphi,z) + b(\varphi,y,z)$ is continuous in $H_0^1(0,1)$ from its definition. V-ellipticity follows from (5) and the estimate $\|y\|_{H_0^1(0,1)} \leq \frac{1}{\nu}\|f\|$ as follows:

$$\tilde{a}(\varphi,\varphi) = a(\varphi,\varphi) + b(\varphi,y,\varphi) \geq (\nu - \|y\|_{H_0^1(0,1)})\|\varphi\|_{H_0^1(0,1)}^2$$

$$> \nu(1 - \frac{\|f\|}{\nu^2})\|\varphi\|_{H_0^1(0,1)}^2 > 0. \tag{40}$$

Therefore, there exist unique solutions $z \in H_0^1(0,1) \cap H^2(0,1)$ of (38) and $z_h \in V_h$ of (39), respectively. Moreover, by linearity we can easily check that z and z_h satisfy

$$\|z - z_h\|_{H_0^1(0,1)} \leq Ch\|z\|_{H^2(0,1)}. \tag{41}$$

Now, let us observe that (30) implies that y and y_h fulfill the relation

$$a(y - y_h, z_h) = -b(y, y - y_h, z_h) - b(y - y_h, y_h, z_h). \tag{42}$$

On the other hand, taking $\varphi = y - y_h$ in (38) we have

$$(r, y - y_h) = a(z, y - y_h) + b(y - y_h, y, z) + b(y, y - y_h, z)$$
$$= a(z - z_h, y - y_h) + a(z_h, y - y_h) + b(y - y_h, y, z) + b(y, y - y_h, z),$$

where we replace the identity (42) to attain

$$
\begin{aligned}
(r, y - y_h) =& a(z - z_h, y - y_h) + b(y - y_h, y, z) + b(y, y - y_h, z) \\
& - b(y, y - y_h, z_h) - b(y - y_h, y_h, z_h) \\
=& a(z - z_h, y - y_h) + b(y - y_h, y - y_h, z) + b(y - y_h, y_h, z - z_h) \\
& + b(y, y - y_h, z - z_h),
\end{aligned}
$$

then, by continuity of a and b we estimate

$$
\begin{aligned}
(r, y - y_h) \leq & C \|z - z_h\|_{H_0^1(0,1)} \|y - y_h\|_{H_0^1(0,1)} + \|z\|_{H_0^1(0,1)} \|y - y_h\|_{H_0^1(0,1)}^2 \\
& + (\|y\|_{H_0^1(0,1)} + \|y_h\|_{H_0^1(0,1)}) \|y - y_h\|_{H_0^1(0,1)} \|z - z_h\|_{H_0^1(0,1)},
\end{aligned}
$$

finally, by using (36) and (41) we arrive to

$$
\|y - y_h\| = \sup_{\|r\| \leq 1} (r, y - y_h) \leq C h^2 \|y\|_{H^2(0,1)},
$$

which finishes the proof.   ∎

5. NUMERICAL APPROXIMATION OF THE CONTROL PROBLEM

For convenience, we use the following notation:

- For every control $u \in U_{ad}$ satisfying $\|Bu\| < \nu^2$, $y(u)$ denotes the unique solution of (87b) in $H_0^1(0,1)$.
- For every control $u \in U_{ad}$ satisfying $\|Bu\| < \nu^2$, $y_h(u)$ denotes the unique solution of (43) in V_h.

$p(u)$ and $p_h(u)$ will be used analogously to denote the corresponding adjoint states.

Let us define the set of discrete admissible controls by $U_{ad,h} = U_{ad} \cap V_h$. In addition, the state equation is approximated by the following problem: for a given $u \in U_{ad}$, find $y_h \in V_h$ satisfying

$$
a(y_h, \varphi_h) + b(y_h, y_h, \varphi) = (Bu, \varphi_h), \quad \forall \varphi_h \in V_h. \tag{43}
$$

We are interested in unique local solutions close to the optimal state $\bar{y}$. With respect to this, we have the following preliminar result.

Lemma 6. *Let $(\bar{y}, \bar{u}) \in H_0^1(0,1) \times U_{ad}$ the optimal pair for (**P**) with $\nu^2 > \|B\bar{u}\|$. Then, there exist positive numbers ρ_1 and ρ_2 independent of the mesh parameter h, such that for all $u \in B(\bar{u}, \rho_1)$ there exists a unique $y_h(u) \in V_h \cap B(\bar{y}, \rho_2)$ satisfying equation (43). Moreover, the corresponding discrete state $y_h(u)$ satisfies*

$$
\|y_h(u)\|_{H_0^1(0,1)} \leq \frac{1}{\nu} \|Bu\|.
$$

Proof. If we define $\delta := \nu^2 - \|B\bar{u}\|$ and $\rho_1 := \frac{\delta}{2}$, then for any $u \in B(\bar{u}, \rho_1) \cap U_{ad}$ we have that $\|Bu\| \leq \|Bu - B\bar{u}\| + \nu^2 - \delta \leq \|u - \bar{u}\| + \nu^2 - \delta < \nu^2$. According to Theorem 4 there exists $y_h = y_h(u)$ satisfying equation (43), with the bound $\|y_h\|_{H_0^1(0,1)} \leq \frac{1}{\nu} \|Bu\|$. ρ_2 can be chosen using the estimate

$$
\begin{aligned}
\|\bar{y} - y_h(u)\|_{H_0^1(\Omega)} & \leq \|\bar{y} - y(u)\|_{H_0^1(,1)} + \|y(u) - y_h(u)\|_{H_0^1(0,1)} \\
& \leq \frac{1}{\nu} \|B(\bar{u} - u)\| + C h \|y(u)\|_{H^2(0,1)}.
\end{aligned}
$$

                                                                             ∎

Lemma 7. *Let* $(\bar{y}, \bar{u}) \in H_0^1(0,1) \times U_{ad}$ *an optimal pair for* (**P**) *with* $\nu^2 > \|B\bar{u}\|$. *Consider controls* u *and* v *in the open ball* $B(\bar{u}, \rho_1)$ *from Lemma 6 , then it follows that*

$$\|y(u) - y_h(v)\|_{H_0^1(0,1)} \le C(h\|y(h)\|_{H^2(0,1)} + \|u - v\|) \qquad (44)$$

Proof. The proof is analogous to the proof of Lemma 5. Since $y_h(u)$ and $y_h(v)$ are the solutions of (43) with right-hand side u and v accordingly, we subtract the corresponding equations to satisfy

$$a(y_h(v) - y_h(u), \varphi_h) + b(y_h(v), y_h(v), \varphi_h) - b(y_h(u), y_h(u), \varphi_h) = (B(v - u), \varphi_h), \qquad (45)$$

for all $\varphi_h \in V_h$. In particular, choosing $\varphi_h = y_h(v) - y_h(u)$ in (45) and using (5c) we estimate

$$\begin{aligned}
\nu\|y_h(v) - y_h(u)\|^2_{H_0^1(0,1)} =& b(y_h(u), y_h(u), y_h(v) - y_h(u)) - b(y_h(v), y_h(v), y_h(v) - y_h(u)) \\
& + (B(v - u), y_h(v) - y_h(u)), \\
=& b(y_h(u) - y_h(v), y_h(u), y_h(v) - y_h(u)) + (B(v - u), y_h(v) - y_h(u)) \\
\le& \|y_h(u)\|_{H_0^1(0,1)} \|y(v) - y(u)\|^2_{H_0^1(0,1)} + \|v - u\| \|y(v) - y(u)\|_{H_0^1(0,1)}.
\end{aligned}$$

Taking the first term on the right to the left side, and taking into account Lemma 6 we get

$$(\nu - \frac{1}{\nu}\|Bu\|)\|y_h(u) - y_h(v)\|^2_{H_0^1(0,1)} \le \|u - v\| \|y_h(v) - y_h(u)\|_{H_0^1(0,1)},$$

thus, we get the estimate

$$\|y_h(u) - y_h(v)\|_{H_0^1(0,1)} \le \frac{\nu}{\nu^2 - \|Bu\|} \|u - v\|. \qquad (46)$$

By noticing that $\|y(u) - y_h(v)\|_{H_0^1(0,1)} \le \|y(u) - y_h(u)\|_{H_0^1(0,1)} + \|y_h(u) - y_h(v)\|_{H_0^1(0,1)}$, it is easy to derive (44) using the estimate (46) and the error bound established in Theorem 5. ∎

We are in place to formulate the discrete optimal control problem associated to (**P**). Let us define the discrete admissible set $U_{ad,h} := U_{ad} \cap V_h$. In addition, we shall not consider any source of error on y_d. The discrete optimal control problem is given by:

$$(\mathbf{P_h}) \begin{cases} \min_{(y,u) \in V_h \times U_{ad,h}} J(y, u) = \frac{1}{2}\|y - y_d\|^2 + \frac{\lambda}{2}\|u\|^2 \\ \text{subject to (43).} \end{cases}$$

Remark 3. *The set* $\mathcal{F}_{ad,h} := \{(y, u) \in V_h \times U_{ad,h} : (y, u)\, satisfiying\, (43)\}$ *is not empty. Therefore, existence of a solution of* (**P_h**) *is a direct consequence of the compactness of* $\mathcal{F}_{ad,h}$ *and continuity of* J *in* $\mathcal{F}_{ad,h}$.

The optimality system for a local solution $\bar{u}_h$ of (**P_h**) can be derived analogously to the continuous optimality system; therefore, we state this without proof in the following theorem.

Theorem 7. *Let* $(\bar{y}_h, \bar{u}_h) \in \mathcal{F}_{ad,h}$ *be a local solution of* (**P_h**) *such that* $\|B\bar{u}\| < \nu^2$, *then there exists a discrete adjoint state* $\bar{p}_h \in V_h$ *such that the following optimality*

system is fulfilled:

$$a(\bar{y}_h, \varphi_h) + b(\bar{y}_h, \bar{y}_h, \varphi_h) = (B\bar{u}_h, \varphi_h), \qquad\qquad \forall \varphi_h \in V_h, \qquad (47\text{a})$$

$$a(\bar{p}_h, \varphi_h) - (\bar{y}_h \bar{p}_h', \varphi_h) = (y_d - \bar{y}_h, \varphi_h), \qquad\qquad \forall \varphi_h \in V_h, \qquad (47\text{b})$$

$$(B\bar{p}_h + \lambda \bar{u}_h, u - \bar{u}_h) \geq 0, \qquad\qquad \forall u \in U_{ad,h}. \qquad (47\text{c})$$

Later on, in the derivation of the order of convergence for the optimal control, we shall need this optimality system as well as the following estimate for the adjoint equation.

Theorem 8. *Let $(y(u), u)$ a feasible pair for problem* (**P**)*, with $\|Bu\| < \nu^2$ and let the adjoint state $p(u)$ solution of the following equation:*

$$a(p(u), \varphi) + b(y(u), \varphi, p(u)) + b(\varphi, y(u), p(u)) = (y(u) - y_d, \varphi), \quad \forall \varphi \in H_0^1(0,1). \quad (48)$$

If $p_h(u) \in V_h$ is the solution of the discretized version of equation (48)*; that is:*

$$a(p_h, \varphi_h) + b(y_h(u), \varphi_h, p) + b(\varphi_h, y_h(u), p) = (y_h(u) - y_d, \varphi_h), \quad \forall \varphi_h \in V_h, \quad (49)$$

then there exists a constant C, independent of h such that

$$\|p(u) - p_h(u)\|_{H_0^1(0,1)} \leq Ch, \qquad (50)$$

moreover, the estimate in the L^2-norm holds:

$$\|p(u) - p_h(u)\| \leq Ch^2. \qquad (51)$$

Proof. The proof is analogous to the proof of Theorem 6. To simplify notation we define $p = p(u)$ and $p_h = ph(u)$. Let us take $\varphi_h \in V_h$ as test function in the weak formulation of (48) and (49) respectively, and then substract the resulting equations obtaining

$$\begin{aligned}
a(p_h - p, \varphi_h) = & \, b(y(u), \varphi_h, p) - b(y_h(u), \varphi_h, p_h) \\
& + b(\varphi_h, y(u), p) - b(\varphi_h, y_h(u), p_h) \\
& + (y_h(u) - y(u), \varphi_h),
\end{aligned} \qquad (52)$$

by applying property (5) and choosing $\varphi_h = p_h - \Pi_h p$, from (52) we have that

$$\begin{aligned}
a(p_h - & \Pi_h p, p_h - \Pi_h p) \\
= & \, a(p - \Pi_h p, p_h - \Pi_h p) + b(y(u) - y_h(u), p_h - \Pi_h p, p) \\
& + b(y_h(u), p_h - \Pi_h p, p - p_h) + b(p_h - \Pi_h p, y(u) - y_h(u), p) \\
& + b(p_h - \Pi_h p, y_h(u), p - p_h) + (y(u) - y_h(u), p_h - \Pi_h p) \\
= & \, a(p - \Pi_h p, p_h - \Pi_h p) + b(y(u) - y_h(u), p_h - \Pi_h p, p) \\
& + b(y_h(u), p_h - \Pi_h p, p - \Pi_h p) + b(p_h - \Pi_h p, y(u) - y_h(u), p) \\
& + b(p_h - \Pi_h p, y_h(u), p - \Pi_h p) + b(p_h - \Pi_h p, y_h(u), \Pi_h p - p_h) \\
& + (y(u) - y_h(u), p_h - \Pi_h p).
\end{aligned} \qquad (53)$$

Taking into account the definition of the trilinear form b, the property (5a) and Lemma 6 we estimate

$$\nu\|p_h - \Pi_h p\|_{H_0^1(0,1)}$$
$$\leq (2\|y_h(u)\|_{H_0^1(0,1)} + \nu)\|p - \Pi_h p\|_{H_0^1(0,1)} + \|y_h(u)\|_{H_0^1(0,1)}\|p_h - \Pi_h p\|_{H_0^1(0,1)}$$
$$+ (2\|p\|_{H_0^1(0,1)} + 1)\|y(u) - y_h(u)\|_{H_0^1(0,1)}$$
$$\leq (\frac{2}{\nu}\|Bu\| + \nu)\|p - \Pi_h p\|_{H_0^1(0,1)} + (2\|p\|_{H_0^1(0,1)} + 1)\|y(u) - y_h(u)\|_{H_0^1(0,1)}$$
$$+ \frac{1}{\nu}\|Bu\|_{H_0^1(0,1)}\|p_h - \Pi_h p\|_{H_0^1(0,1)} \tag{54}$$

Taking the last term in (54) to the left-hand side, since $\|Bu\| \leq \nu^2$ and using estimates (36) and Lemma 4 we have that there exist a constant $C > 0$ independent of h such that

$$\|p_h - \Pi_h p\|_{H_0^1(0,1)} \leq C(\|p - \Pi_h p\|_{H_0^1(0,1)} + \|y(u) - y_h(u)\|_{H_0^1(0,1)})$$
$$\leq Ch(\|p\|_{H^2(0,1)} + \|y(u)\|_{H^2(0,1)}).$$

From the last inequality we obtain the desired H_0^1-estimate (50) since

$$\|p - p_h\|_{H_0^1(0,1)} \leq \|p - \Pi_h p\|_{H_0^1(0,1)} + \|\Pi_h p - p_h\|_{H_0^1(0,1)} \leq Ch.$$

for some constant $C > 0$ independent of h. Now, we prove (51). By similar arguments used to derive (37) we consider the auxiliary linear elliptic problem (38), with $y = y(u)$:

$$\begin{cases} \text{Given } r \in L^2(0,1), \text{ find } z \in H_0^1(0,1) \text{ such that:} \\ a(z,\varphi) + b(y(u),z,\varphi) + b(z,y(u),\varphi) = (r,\varphi), \quad \forall \varphi \in H_0^1(0,1), \end{cases} \tag{55}$$

Once again, the bilinear form $\tilde{a}(z,\varphi) := a(z,\varphi) + b(y(u),z,\varphi) + b(z,y(u),\varphi)$ is elliptic. Therefore, (55) has a unique solution $z \in H_0^1(0,1)$ with $\|z\|_{H_0^1(0,1)} \leq \|r\|$. We denote by $z_h \in V_h$ the corresponding finite element approximation (39), which fulfills the error estimate $\|z - z_h\|_{H_0^1(0,1)} \leq Ch$, for some constant $C > 0$ independent of h. After subtracting the auxiliary problem (55) and its discretization choosing $\varphi = p_h - p$ we get

$$(r, p_h - p) = a(z, p - p_h) + b(y(u), z, p_h - p) + b(z, y(u), p_h - p)$$
$$= a(z - z_h, p - p_h) + a(z_h, p - p_h) + b(y(u), z, p_h - p) + b(z, y(u), p_h - p). \tag{56}$$

Choosing $\varphi_h = z_h$ in identity (52) and inserting in (56) leads to

$$(r, p_h - p) = a(z - z_h, p - p_h) + (y_h(u) - y(u), z_h)$$
$$+ b(y(u), z_h, p) - b(y_h(u), z_h, p_h) + b(z_h, y(u), p) - b(z_h, y_h(u), p_h)$$
$$+ b(y(u), z, p_h - p) + b(z, y(u), p_h - p)$$
$$= a(z - z_h, p - p_h) + (y_h(u) - y(u), z_h)$$
$$+ b(y(u), z - z_h, p_h - p) + b(z - z_h, y(u), p_h - p)$$
$$- b(y_h(u) - y(u), z_h, p_h) - b(z_h, y_h(u) - y(u), p_h),$$
$$= a(z - z_h, p - p_h) + (y_h(u) - y(u), z_h)$$
$$+ b(y(u), z - z_h, p_h - p) + b(z - z_h, y(u), p_h - p)$$
$$+ (z_h p_h', y_h(u) - y(u)),$$

since a is continuous in $(H_0^1(0,1))^2$ and b satisfies (5a), from estimates (36) and (37)similarly to Theorem 6 we get the estimate (51). ∎

We are interested in the convergence properties of local solutions of $(\mathbf{P_h})$. Following ideas given in [4], the following convergence properties are established.

Assumption 1. *In order to establish an order of convergence, we will assume the following*

$$\delta_\nu := \sup_{h>0}\{\nu^2 - \|Bu_h\|\} > 0. \tag{57}$$

This uniform bound for $\bar{u}_h$ is needed to have the following result.

Corollary 1. *We notice that in our notation $\bar{p}_h = p_h(\bar{u}_h)$ and $\bar{p} = p(\bar{u})$; consequently, under the assumption that $\delta_\nu := \sup_{h>0}\{\nu^2 - \|Bu_h\|\} > 0$ then Theorem 8 implies that*

$$\begin{aligned}
\|\bar{p}_h - \bar{p}\| &\leq \|p_h(\bar{u}_h) - p(\bar{u}_h)\| + \|p(\bar{u}_h) - \bar{p}(\bar{u})\| \\
&\leq c(h^2 + \|\bar{u} - \bar{u}_h\|),
\end{aligned} \tag{58}$$

for some constant c independent of the size of the mesh h.

Theorem 9. *Let $(\bar{y}_h, \bar{u}_h)$ an optimal pair for $(\mathbf{P_h})$. Then, there is a subsequence $(\bar{u}_h)_{h>0}$ which converges weakly in $L^2(0,1)$ to a limit $\bar{u}$. Moreover, the weak limit $\bar{u}$ is a solution of $(\mathbf{P})$ which satisfies*

$$\lim_{h\to 0} J(\bar{y}_h, \bar{u}_h) = J(\bar{y}, \bar{u}) = \inf(P) \quad and \quad \lim_{h\to 0}\|\bar{u}_h - \bar{u}\| = 0. \tag{59}$$

Proof. Let $(\bar{u}_h)_{h>0}$ be the sequence such that $\bar{u}_h$ is the optimal control for $(\mathbf{P_h})$. Since this sequence is formed by admissible controls for the problem $(\mathbf{P_h})$ then it is also bounded. Thus, we can extract a weak convergent subsequence in $L^2(0,1)$, denoted again by $(\bar{u}_h)_{h>0}$. Let us denote its weak limit by $\bar{u}$ and denote by $\bar{y} := y(\bar{u})$ the associated state. It is clear, in view of Lemma 1, that $\bar{y}_h \to \bar{y}$ in $H_0^1(0,1)$. Noticing that the pair $(y_h(\Pi_h\bar{u}), \Pi_h\bar{u})$ is feasible for $(\mathbf{P_h})$ and by convexity of the objective functional we can conclude that

$$\begin{aligned}
J(\bar{y}, \bar{u}) &\leq \liminf_{h\to 0} J(\bar{y}_h, \bar{u}_h) \\
&\leq \liminf_{h\to 0} J(y_h(\Pi_h\bar{u}), \Pi_h\bar{u}) \\
&\leq \limsup_{h\to 0} J(y_h(\Pi_h\bar{u}), \Pi_h\bar{u}) = J(\bar{y}, \bar{u}) = \inf(\mathbf{P}),
\end{aligned}$$

which together with the fact that $(\bar{y}, \bar{u}) \in \mathcal{F}_{ad}$ imply that $\bar{u}$ is an optimal control for $(\mathbf{P})$. Then, the first identity of (59) follows by applying Mazur's Theorem and convexity of the objective functional. The second identity of (59) is obtained by the following argument

$$\begin{aligned}
\frac{\lambda}{2}\|\bar{u}_h - \bar{u}\|^2 &= \frac{\lambda}{2}\|\bar{u}_h\|^2 - \frac{\lambda}{2}\|\bar{u}\|^2 + \lambda(\bar{u}, \bar{u}_h - \bar{u}) \\
&= J(\bar{y}_h, \bar{u}_h) - J(\bar{y}, \bar{u}) + \frac{1}{2}\|\bar{y} - \bar{y}_h\|^2 + \lambda(\bar{u}, \bar{u}_h - \bar{u}),
\end{aligned} \tag{60}$$

where the last term tends to 0 as $h \to 0$ by the weak convergence of $\bar{u}_h$ to $\bar{u}$ and the first part of (59). ∎

5.1. **Derivation of the order of convergence.** In this section we derive the main result of this paper by adapting the theory developed in [3]. First we recall some auxiliary results. We denote the solution of the adjoint equation by $p(\bar{u}_h)$ satisfying

$$a(p, \varphi) + b(\bar{y}_h, \varphi, p) + b(\varphi, \bar{y}_h, p) = (\bar{y}_h - y_d, \varphi), \tag{61}$$

for all $\varphi \in H_0^1(0, 1)$, where $\bar{y}_h$ is the solution of (47a).

Lemma 8. *Let us assume that there is a constant $c > 0$ independent of h such that $\|\bar{u}_h - \bar{u}\| \geq ch$, then there is an $h_0 > 0$ and $\mu > 0$ such that the estimate*

$$\mu\|\bar{u}_h - \bar{u}\|^2 \leq [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}), \tag{62}$$

is satisfied for all $h < h_0$.

Proof. From the first derivative of the Lagrangian given by (10a), it follows that

$$\begin{aligned}
[\mathcal{L}'(y(\bar{u}_h), &\bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&= [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(y(\bar{u}_h), \bar{u}_h, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&\quad + [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, \bar{p}) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&= [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(y(\bar{u}_h), \bar{u}_h, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&\quad + \|y(\bar{u}_h) - \bar{y}\|^2 + \lambda\|\bar{u}_h - \bar{u}\|^2 - 2b(y(\bar{u}_h) - \bar{y}, y(\bar{u}_h) - \bar{y}, \bar{p}).
\end{aligned} \tag{63}$$

Since $y(\bar{u}_h)$ and $\bar{y} = y(\bar{u})$ satisfy (4) for $f = \bar{u}_h$ and $f = \bar{u}$ respectively, the first term on the right-hand side in (63) satisfies:

$$\begin{aligned}
[\mathcal{L}'(y(\bar{u}_h), &\bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(y(\bar{u}_h), \bar{u}_h, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&= -\langle R'(y(\bar{u}_h), \bar{u}_h)(y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}), p(\bar{u}_h) - \bar{p}\rangle \\
&= -a(y(\bar{u}_h) - \bar{y}, p(\bar{u}_h) - \bar{p}) - b(y(\bar{u}_h) - \bar{y}, y(\bar{u}_h), p(\bar{u}_h) - \bar{p}) - b(y(\bar{u}_h), y(\bar{u}_h) - \bar{y}, p(\bar{u}_h) - \bar{p}) \\
&\quad + (B(\bar{u}_h - \bar{u}), p(\bar{u}_h) - \bar{p}) \\
&= b(y(\bar{u}_h), y(\bar{u}_h), p(\bar{u}_h) - \bar{p}) - (B\bar{u}_h, p(\bar{u}_h) - \bar{p}) - b(\bar{y}, \bar{y}, \bar{p}_h - \bar{p}) + (B\bar{u}, p(\bar{u}_h) - \bar{p}) \\
&\quad - b(y(\bar{u}_h) - \bar{y}, y(\bar{u}_h), p(\bar{u}_h) - \bar{p}) - b(y(\bar{u}_h), y(\bar{u}_h) - \bar{y}, p(\bar{u}_h) - \bar{p}) + (B(\bar{u}_h - \bar{u}), p(\bar{u}_h) - \bar{p}) \\
&= -b(y(\bar{u}_h) - \bar{y}, y(\bar{u}_h) - \bar{y}, p(\bar{u}_h) - \bar{p}),
\end{aligned} \tag{64}$$

by replacing (64) in (63) we observe

$$\begin{aligned}
[\mathcal{L}'(y(\bar{u}_h), &\bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&= \|y(\bar{u}_h) - \bar{y}\|^2 + \lambda\|\bar{u}_h - \bar{u}\|^2 - 2b(y(\bar{u}_h) - \bar{y}, y(\bar{u}_h) - \bar{y}, \bar{p}) \\
&\quad - b(y(\bar{u}_h) - \bar{y}, y(\bar{u}_h) - \bar{y}, p(\bar{u}_h) - \bar{p}).
\end{aligned} \tag{65}$$

Now, let us define the sequences $z_h = \dfrac{y(\bar{u}_h) - \bar{y}}{\|\bar{u}_h - \bar{u}\|}$ and $v_h = \dfrac{\bar{u}_h - \bar{u}}{\|\bar{u}_h - \bar{u}\|}$. By noticing that these sequences are bounded, we can extract a subsequence of h denoted again by h, such that $h \to 0$ and $v_h \rightharpoonup v$ in $L^2(0, 1)$ and $z_h \rightharpoonup z$ in $H_0^1(0, 1)$. We shall proof that (z, v) belongs to the critical cone $C_{\bar{u}}^\tau$. By its definition, v_h fulfills (12b) and (12c), and so does v. In order to check that $v(x) = 0$ whenever $x \in \Omega_\tau$, we argue as in [3]. If $h \to 0$, from Theorem 9, we have that there also exists a subsequence $\bar{u}_h(x) \to \bar{u}(x)$ a.e. $x \in (0, 1)$ We notice that since $\bar{u}_h \to \bar{u}$ the associated adjoint

state $\bar{p}_h \to \bar{p}$, both in in $L^2(0,1)$ which together with $v_h \rightharpoonup v$ imply

$$
\begin{aligned}
(B\bar{p} + \lambda\bar{u}, v) &= \lim_{h\to 0}(B\bar{p}_h + \lambda\bar{u}_h, v_h)\\
&= \lim_{h\to 0}\frac{1}{\|\bar{u}_h - \bar{u}\|}(B\bar{p}_h + \lambda\bar{u}_h, \bar{u}_h - \bar{u})\\
&= \lim_{h\to 0}\frac{1}{\|\bar{u}_h - \bar{u}\|}(B\bar{p}_h + \lambda\bar{u}_h, \bar{u}_h - \Pi_h\bar{u}) + \frac{1}{\|\bar{u}_h - \bar{u}\|}(B\bar{p}_h + \lambda\bar{u}_h, \Pi_h\bar{u} - \bar{u}).
\end{aligned}
$$

By considering (47c) we infer

$$
\begin{aligned}
(B\bar{p} + \lambda\bar{u}, v) &\leq \lim_{h\to 0}\frac{1}{\|\bar{u}_h - \bar{u}\|}(B(\bar{p}_h - \bar{p}) + \lambda(\bar{u}_h - \bar{u}), \Pi_h\bar{u} - \bar{u})\\
&\quad + \lim_{h\to 0}\frac{1}{\|\bar{u}_h - \bar{u}\|}(B\bar{p} + \lambda\bar{u}, \Pi_h\bar{u} - \bar{u})\\
&\leq \lim_{k\to 0}\frac{\|\Pi_h\bar{u} - \bar{u}\|}{\|\bar{u}_h - \bar{u}\|}\left(\|B(\bar{p}_h - \bar{p})\| + \lambda\|\bar{u}_h - \bar{u}\|\right) + \|B\bar{p} + \lambda\bar{u}\|\lim_{h\to 0}\frac{\|\Pi_h\bar{u} - \bar{u}\|}{\|\bar{u}_h - \bar{u}\|}\\
&\leq c\lim_{h\to 0}\frac{\|\Pi_h\bar{u} - \bar{u}\|}{h} = 0,
\end{aligned}
$$

where the last estimation follows from Lemma 4. From the last inequality and the fact that $\tau < |\bar{p}(x) + \lambda\bar{u}(x)|$, we infer that $v(x) = 0$ in Ω_τ and thus we have that $v \in C_{\bar{u}}^\tau$. Furthermore, the pair (z, v) satisfies the linear equation given by $R'(\bar{y}, \bar{u})(z, v) = 0$. To see this, substract $\langle R(y(\bar{u}_h), \varphi\rangle$ and $\langle R(\bar{y}, \bar{u}), \varphi\rangle$ to obtain

$$
a(z_h, \varphi) + b(z_h, y(\bar{u}_h), \varphi) + b(\bar{y}, z_h, \varphi) - (Bv_h, \varphi) = 0 \quad \forall\varphi \in H_0^1(0,1), \qquad (66)
$$

hence, using the convergence properties of z_h and v_h and Lemma 1, after passing to the limit $k \to \infty$, we get $R'(\bar{y}, \bar{u})(z, v) = 0$.

From (65) we have

$$
\begin{aligned}
&[\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u})\\
&= \left(\|z_h\|^2 + \lambda\|v_h\|^2 - 2b(z_h, z_h, \bar{p}) - b(z_h, z_h, \bar{p}(\bar{u}_h) - \bar{p})\right)\|\bar{u}_h - \bar{u}\|^2, \qquad (67)
\end{aligned}
$$

Moreover, since the pair $(z, v) \in C_{\bar{u}}^\tau$ with $R'(\bar{y}, \bar{u})(z, v) = 0$ and $p(\bar{u}_h) \to \bar{p}$ uniformly, by applying second order sufficient conditions (13) we infer that

$$
\begin{aligned}
\liminf_{h\to 0}&\left(\|z_h\|^2 + \lambda\|v_h\|^2 - 2b(z_h, z_h, \bar{p}) - b(z_h, z_h, p(\bar{u}_h) - \bar{p})\right)\\
&\geq \|z\|^2 + \lambda\|v\|^2 - 2b(z, z, \bar{p}) \geq \delta\|v\|^2. \qquad (68)
\end{aligned}
$$

Finally, from (67), (68) and choosing $\mu = \frac{\delta}{2}\|v\|^2$, there exists an h_0 such that for all $h < h_0$ we get

$$
\mu\|\bar{u}_h - \bar{u}\|^2 \leq [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}).
$$

∎

Lemma 9. *It holds that*

$$
\lim_{h\to 0}\frac{1}{h^2}\left|\left((B\bar{p} + \lambda\bar{u}), \Pi_h\bar{u} - \bar{u}\right)\right| = 0. \qquad (69)
$$

Proof. The proof of this Lemma can be found in [3, Lemma 4.4].   ∎

Theorem 10. *Let $\bar{u}$ be a local solution of* (**P**) *satisfying second-order sufficient condition* (13). *If $\bar{u}_h$ is a local solution of* (**P$_h$**) *satisfying Assumption 1 and such that $\lim_{h\to 0} \|\bar{u}_h - \bar{u}\| = 0$ then the following convergence property holds*

$$\lim_{h\to 0} \frac{1}{h}\|\bar{u}_h - \bar{u}\| = 0. \tag{70}$$

Proof. As in [3], the proof is argued by contradiction. Let us assume that (70) is false, therefore there exist a constant $c > 0$ and a subsequence $(\bar{u}_h)_{h>0}$ such that the relation

$$\|\bar{u}_h - \bar{u}\| \geq ch, \tag{71}$$

holds for all $h > 0$ sufficiently small. From Lemma 8 there exists $h_0 > 0$ such that for all $h < h_0$ the estimate (62) holds.

We proceed to estimate the right hand side of (62). Since $\bar{u}$ and $\bar{u}_h$ are local solutions for (**P**) and (**P$_h$**) respectively, then they satisfy the first order necessary conditions given by Theorems 2 and 7. We observe that $\bar{u}_h$ is feasible for (**P**) and $\Pi_h \bar{u}$ is feasible for (**P**). Therefore, taking $u = \bar{u}_h$ in (8c) and $u = \bar{u}$ in (47c) it comes

$$(B\bar{p} + \lambda\bar{u}, \bar{u}_h - \bar{u}) \geq 0, \text{ and} \tag{72}$$

$$(B\bar{p}_h + \lambda\bar{u}_h, \Pi_h\bar{u} - \bar{u}_h) \geq 0, \tag{73}$$

Then, inequalities (72) and (73) imply that

$$
\begin{aligned}
&\mathcal{L}'(\bar{y}_h, \bar{u}_h, \bar{p}_h)(\bar{y}_h - \Pi_h\bar{y}, \bar{u}_h - \Pi_h\bar{u}) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(\bar{y}_h - \bar{y}, \bar{u}_h - \bar{u}) \\
&= (\bar{p}_h, B(\bar{u}_h - \Pi_h\bar{u})) + \lambda(\bar{u}_h, \bar{u}_h - \Pi\bar{u}) - (\bar{p}, B(\bar{u}_h - \bar{u})) - \lambda(\bar{u}, \bar{u}_h - \bar{u}) \\
&= (B\bar{p}_h + \lambda\bar{u}_h, \bar{u}_h - \Pi_h\bar{u}) - (B\bar{p} + \lambda u, \bar{u}_h - u) \leq 0.
\end{aligned}
\tag{74}
$$

With the help of (74) we estimate

$$
\begin{aligned}
&[\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&= \mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h))(y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) - \mathcal{L}'(\bar{y}_h, \bar{u}_h, \bar{p}_h)(\bar{y}_h - \Pi_h\bar{y}, \bar{u}_h - \Pi_h\bar{u}) \\
&\quad + \mathcal{L}'(\bar{y}_h, \bar{u}_h, \bar{p}_h)(\bar{y}_h - \Pi_h\bar{y}, \bar{u}_h - \Pi_h\bar{u}) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})(y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&\leq (y(\bar{u}_h) - y_d, y(\bar{u}_h) - \bar{y}) + \lambda(\bar{u}_h, \bar{u}_h - \bar{u}) - \langle R'(\bar{y}(\bar{u}_h), \bar{u}_h)(y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}), p(\bar{u}_h)\rangle \\
&\quad - (\bar{y}_h - y_d, \bar{y}_h - \Pi_h y) - \lambda(\bar{u}_h, \bar{u}_h - \Pi_h\bar{u}) + \langle R'(\bar{y}_h, \bar{u}_h)(\bar{y}_h - \Pi_h\bar{y}, \bar{u}_h - \Pi_h\bar{u}), \bar{p}_h\rangle
\end{aligned}
$$

since $\bar{p}_h$ and $p(\bar{u}_h)$ satisfy adjoint equations (47b) and (61) we have

$$
\begin{aligned}
&= (Bp(\bar{u}_h), \bar{u}_h - \bar{u}) + \lambda(\bar{u}_h, \bar{u}_h - \bar{u}) - (B\bar{p}_h, \bar{u}_h - \Pi_h\bar{u}) - \lambda(\bar{u}_h, \bar{u}_h - \Pi_h\bar{u}) \\
&= (\lambda\bar{u}_h + B\bar{p}_h, \Pi_h\bar{u} - \bar{u}) + (B(p(\bar{u}_h) - \bar{p}_h), \bar{u}_h - \bar{u}) \\
&= (\lambda(\bar{u}_h - \bar{u}) + B(\bar{p}_h - \bar{p}), \Pi_h\bar{u} - \bar{u}) + (\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u}) + (B(p(\bar{u}_h) - \bar{p}_h), \bar{u}_h - \bar{u})
\end{aligned}
\tag{75}
$$

Now, from our assumption (71) we apply Lemma 8 to (75) we get

$$
\begin{aligned}
\mu\|\bar{u}_h - \bar{u}\|^2 &\leq [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}) \\
&\leq (\lambda(\bar{u}_h - \bar{u}) + B(\bar{p}_h - \bar{p}), \Pi_h\bar{u} - \bar{u}) + (\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u}) \\
&\quad + (B(p(\bar{u}_h) - \bar{p}_h), \bar{u}_h - \bar{u})
\end{aligned}
\tag{76}
$$

According to the estimate (58), from (76) we have

$$
\begin{aligned}
\mu\|\bar{u}_h - \bar{u}\|^2 \leq{}& (\lambda\|\bar{u}_h - \bar{u}\| + \|\bar{p}_h - \bar{p}\|)\|\Pi_h\bar{u} - \bar{u}\| + (\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u}) \\
&+ \|p_h(\bar{u}_h) - \bar{p}_h\|\|\bar{u}_h - \bar{u}\| \\
\leq{}& c_1(\|\bar{u} - \bar{u}_h\| + h^2)\|\Pi_h\bar{u} - \bar{u}\| + (\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u}) + c_2 h^2\|\bar{u}_h - \bar{u}\|
\end{aligned}
\tag{77}
$$

next, dividing the last relation by $h\|\bar{u}_h - \bar{u}\|$, then (71) implies

$$
\mu\|\bar{u}_h - \bar{u}\| \leq c\frac{\|\Pi_h\bar{u} - \bar{u}\|}{h} + \frac{(\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u})}{h^2} + c_2 h,
\tag{78}
$$

for some constant $c > 0$ independent of h. Finally, in view of Lemma 9 and the relation (28) we divide by h and pass to the limit

$$
\mu\lim_{k\to\infty}\frac{1}{h}\|\bar{u}_h - \bar{u}\| \leq \lim_{k\to\infty}\left(c\frac{\|\Pi_h\bar{u} - \bar{u}\|}{h} + \frac{(\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u})}{h^2}\right) = 0.
$$

This contradicts our assumption (71), and therefore the statement of Theorem 10 is true. ∎

5.2. **An improved error estimate.** We make a further error analysis by taking into account a stronger assumption on the estructure of the optimal control which allow us to derive a better interpolation error in the L^1–norm which is crucial to make an improvement in the overall error estimate. The following assumption was proposed by Rösch in [12], and guarantees that $\bar{u}$ is Lipschitz continuous and piecewise of class C^2 on the domain $\Omega = (0, 1)$.

Assumption 2. *There exists a finite number of points $t_k \in [0, 1]$, for $k = 0, \ldots, N$ such that $t_0 = 0$ and $t_N = 1$, such that the optimal control $\bar{u} \in C^2[t_{k-1}, t_k]$ for all $k = 1, \ldots, N$.*

The following interpolation error is a consequence of the last assumption and its proved in [12, Lemma 3].

Lemma 10. *Under Assumption 2 there exists a positive constant c, such that the following bound for the interpolation error*

$$
\|\bar{u} - \Pi_h\bar{u}\| \leq ch^{3/2},
\tag{79}
$$

holds.

Lemma 11. *Under Assumption 2 there exists a positive constant c, such that the following estimate follows:*

$$
|(\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u})| \leq ch^3.
\tag{80}
$$

Proof. The proof is done along the lines of the proof of Lemma 4.4 in [3], in combination with the arguments of Lemma 3 in [12]. In our mesh $\mathcal{I}_h$, we consider the following sets

$$
\begin{aligned}
I_h^+ &= \{i : |\lambda\bar{u}(x) + B\bar{p}(x)| > 0, \forall x \in I_i \in \mathcal{I}_h\}, \text{ and} \\
I_h^0 &= \{i : \exists \xi_i \in I_i, \text{such that } \lambda\bar{u}(\xi_i) + B\bar{p}(\xi_i) = 0\}.
\end{aligned}
$$

Notice that $\lambda\bar{u} + B\bar{p} \in C^{0,1}$ by the regularity of $\bar{u}$ and $\bar{p}$. Moreover, from the variational inequality (8c) we have the characterization

$$\lambda\bar{u}(x) + B\bar{p}(x) \geq 0 \,\text{if}\, \bar{u}(x) = \alpha,$$
$$\lambda\bar{u}(x) + B\bar{p}(x) \leq 0 \,\text{if}\, \bar{u}(x) = \beta,$$
$$\lambda\bar{u}(x) + B\bar{p}(x) = 0 \,\text{if}\, \alpha < \bar{u}(x) < \beta.$$

Therefore, if $i \in I_h^+$ we have that $\lambda\bar{u}(x) + B\bar{p}(x) \neq 0$ for all $x \in I_i$, and thus $\bar{u}(x) = \alpha$ or $\bar{u}(x) = \beta$ accordingly, which in turn implies that $\Pi_h\bar{u}(x) = \bar{u}(x)$ for all $x \in I_i$. From this observation we have:

$$
\begin{aligned}
|(\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u})| &= \left| \int_0^1 (\lambda\bar{u}(x) + B\bar{p}(x))(\Pi_h\bar{u}(x) - \bar{u}(x))\, dx \right| \\
&= \left| \sum_{i=1}^n \int_{I_i} (\lambda\bar{u}(x) + B\bar{p}(x))(\Pi_h\bar{u}(x) - \bar{u}(x))\, dx \right| \\
&= \left| \sum_{i \in I_h^0} \int_{I_i} (\lambda\bar{u}(x) + B\bar{p}(x))(\Pi_h\bar{u}(x) - \bar{u}(x))\, dx \right| \\
&\leq \sum_{i \in I_h^0} \int_{I_i} |\lambda\bar{u}(x) + B\bar{p}(x) - \lambda\bar{u}(\xi_i) - B\bar{p}(\xi_i)|\, |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx.
\end{aligned}
$$
$$(81)$$

Since $\lambda\bar{u} + B\bar{p} \in C^{0,1}$. By denoting its Lipschitz constant by $\tilde{L}$; from (81) we have that

$$
\begin{aligned}
|(\lambda\bar{u} + B\bar{p}, \Pi_h\bar{u} - \bar{u})| &\leq \sum_{i \in I_h^0} \int_{I_i} L|x - \xi_i|\, |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx. \\
&\leq Lh \sum_{i \in I_h^0} \int_{I_i} |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx. \\
&\leq Lh \int_0^1 |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx.
\end{aligned}
$$
$$(82)$$

Now, consider the integrand on the right–hand side of (82). By Assumption 2, we distinguish the intervals $[t_{i-1}, t_i]$ between the class $\mathcal{I}_1$ containing the intervals where $\bar{u} \in C^2[t_{i-1}, t_i]$ and the class $\mathcal{I}_2$ formed by the remaining intervals where $\bar{u}$ is only Lipschitz. From interpolation error for piecewise linear functions in one dimension, we get:

$$
\begin{aligned}
\int_0^1 |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx \\
&= \sum_{I_1} \int_{t_{i-1}}^{t_i} |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx + \sum_{I_2} \int_{t_{i-1}}^{t_i} |\Pi_h\bar{u}(x) - \bar{u}(x)|\, dx \\
&\leq \sum_{I_1} \frac{\|\bar{u}''(\zeta_i)\|}{8} h^2\, h + \sum_{I_2} \frac{L}{4} h\, h\, dx \\
&\leq \sum_{I_1} \max_{1 \leq i \leq N} \frac{\|\bar{u}''(\zeta_i)\|}{8} h^3\, h + (N-1)\frac{L}{2} h^2\, dx
\end{aligned}
$$
$$(83)$$

where $\zeta_i \in (t_{i-1}, t_i)$. Note that by the Assumption 2 the class I_2 contains at most $N - 1$ intervals and that N is independent of h. Therefore, from (82) and (83) we deduce the estimate (80).   ■

Theorem 11. *Let $\bar{u}$ be a local solution of* (**P**) *satisfying second-order sufficient condition* (13). *If $\bar{u}_h$ is a local solution of* (**P$_h$**) *such that $\lim_{h \to 0} \|\bar{u}_h - \bar{u}\| = 0$; under Assumptions 1 and 2 then the following error estimate holds*

$$\|\bar{u}_h - \bar{u}\| \le ch^{3/2} \tag{84}$$

Proof. Analogous to the proof of Lemma 8, we have a pair of sequences $z_h = \dfrac{y(\bar{u}_h) - \bar{y}}{\|\bar{u}_h - \bar{u}\|}$ and $v_h = \dfrac{\bar{u}_h - \bar{u}}{\|\bar{u}_h - \bar{u}\|}$ such that $v_h \rightharpoonup v$ in $L^2(0,1)$ and $z_h \rightharpoonup z$ in $H_0^1(0,1)$ when $h \to 0$. By construction of v_h, it is easy to see that v satisfies the sign condition (12b) and (12c). To verify that the condition (12a) is satisfied by v, we first observe that $\int_{\Omega_\tau} (B\bar{p}(x) + \lambda \bar{u}(x))(\bar{u}_h(x) - \bar{u}(x))\, dx \ge 0$, implying that $\int_{\Omega_\tau} (B\bar{p}(x) + \lambda \bar{u}(x))v\, dx \ge 0$.

On the other hand,

$$\int_{\Omega_\tau} (B\bar{p}(x) + \lambda \bar{u}(x)) v(x)\, dx$$
$$= \frac{1}{\|\bar{u}_h - \bar{u}\|} \lim_{h \to 0} \int_{\Omega_\tau} (B\bar{p}_h(x) + \lambda \bar{u}_h(x))(\bar{u}_h(x) - \bar{u}(x))\, dx$$
$$= \frac{1}{\|\bar{u}_h - \bar{u}\|} \lim_{h \to 0} \left[\int_{\Omega_\tau} (B\bar{p}_h(x) + \lambda \bar{u}_h(x))(\bar{u}_h(x) - \Pi_h \bar{u}(x))\, dx \right.$$
$$\left. + \int_{\Omega_\tau} (B\bar{p}_h(x) + \lambda \bar{u}_h(x))(\Pi_h \bar{u}(x) - \bar{u}(x))\, dx \right].$$

The first integral on the right-hand side of the last identity is less or equal than 0 by the first order necessary optimality conditions (47c); while the second integral is equal to 0 by noticing that the optimal control is active on Ω_τ, i.e. $\bar{u}(x) = \alpha$ or $\bar{u}(x) = \beta$ and thus $\Pi_h \bar{u}(x) - \bar{u}(x) = 0$. Since $|B\bar{p}(x) + \lambda \bar{u}(x)| > \tau > 0$ on Ω_τ, we apply second the order sufficient condition of Theorem 3 and by the repeating the arguments in the proof of Lemma 8, we can deduce the existence of an $h_0 > 0$ and $\mu > 0$, such that

$$\mu \|\bar{u}_h - \bar{u}\|^2 \le [\mathcal{L}'(y(\bar{u}_h), \bar{u}_h, p(\bar{u}_h)) - \mathcal{L}'(\bar{y}, \bar{u}, \bar{p})](y(\bar{u}_h) - \bar{y}, \bar{u}_h - \bar{u}),$$
$$\le c_1(\|\bar{u} - \bar{u}_h\| + h^2)\|\Pi_h \bar{u} - \bar{u}\| + (\lambda \bar{u} + B\bar{p}, \Pi_h \bar{u} - \bar{u}) + c_2 h^2 \|\bar{u}_h - \bar{u}\|. \tag{85}$$

Now, by applying the Young's inequality and taking into account Lemma 10 and Lemma 11, we finally deduce that

$$\frac{\mu}{2} \|\bar{u}_h - \bar{u}\|^2 \le c_2 h^4 + c_3 \|\Pi_h \bar{u} - \bar{u}\|^2 + (\lambda \bar{u} + B\bar{p}, \Pi_h \bar{u} - \bar{u})$$
$$\le c_2 h^4 + c_4 h^3 + ch^3, \tag{86}$$

which implies the error estimate (84)   ■

6. Numerical experiments

For the sake of illustration of our theory, we develop a numerical test where the exact solution of the optimal control problem is known. The optimization problem

is solved by a BFGS method, which stops when the norm of the residual $u_{k+1} - u_k$ is less than the tolerance of $1e-7$. Our example reads as follows

$$(\mathbf{E}) \begin{cases} \min_{(y,u)\in H_0^1(0,1)\times U_{ad}} J(y,u) = \frac{1}{2}\|y - y_d\|^2 + \frac{\lambda}{2}\|u - u_d\|^2 \\ \text{subject to:} \\ -\nu y'' + yy' = u + f \quad \text{in } (0,1), \\ y(0) = y(1) = 0, \end{cases}$$

where:

$$\nu = 0.78,$$
$$u_a = -1, \quad u_b = 1$$
$$y_d(x) = -x(x-1),$$
$$u_d(x) = \begin{cases} 1 & 0 \le x < 1/3, \\ -6x + 3 & 1/3 \le x \le 2/3, \\ -1 & 2/3 < x < 1, \end{cases}$$
$$f(x) = -u_d + x(x-1)(2x-1) + 2\nu.$$

With these choices, problem (E) has the optimal control $\bar{u} = u_d$ with associated optimal state $\bar{y} = y_d$, and adjoint state $\bar{p} = 0$ together satisfying the optimality conditions stablished in Theorem 2. Note that since $\bar{p}$ vanishes, the optimal quantities also satisfy the second order optimality condition (13). Figure 1 shows the computed optimal control $\bar{u}$ and its associated optimal state $\bar{y}$ at $h = 0.0244$.

In the next table we estimate numerically the order of convergence in the L^2- norm (EOC). From the numerical results, it can be observed a quadratic order of convergence for the optimal control values of λ close to 1, but this order is lower if the value of λ decreases. It should be notice that our control satisfies Assumption 2.

h	Error in L^2	EOC
0.0476	0.0111045	-
0.0244	0.0031017	1.91
0.0123	0.0008099	1.94
0.0062	0.0002152	1.95
0.0031	0.0000656	1.94
0.0016	0.0000300	1.87
0.0008	0.0000190	1.71

TABLE 1. Numeric computation of the errors and the order of convergence

The numerical approximations can be observed in the following figure. Note that $\|\bar{u}\| = \sqrt{7}/3$; hence, the condition $\|\bar{u}\| \le \nu^2$ is satisfied.

REFERENCES

[1] John A Burns and Sungkwon Kang. A control problem for burgers' equation with bounded input/output. *Nonlinear Dynamics*, 2(4):235–262, 1991.
[2] John A Burns and Sungkwon Kang. A stabilization problem for burgers? equation with unbounded control and observation. In *Estimation and Control of Distributed Parameter Systems*, pages 51–72. Springer, 1991.
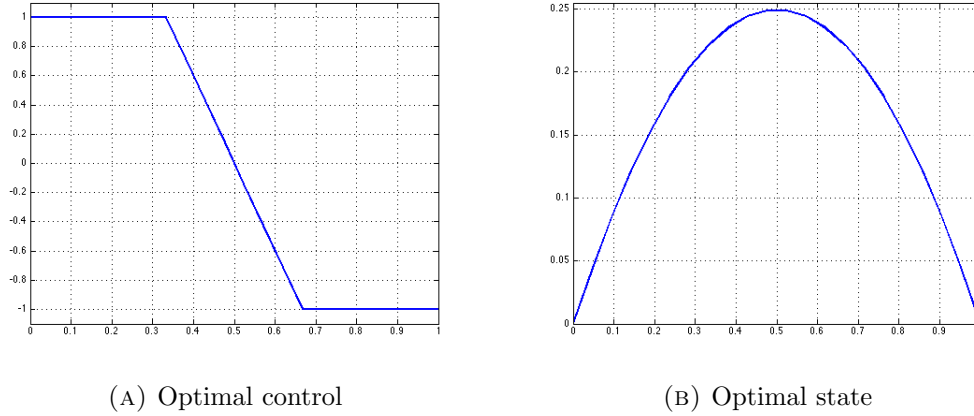
(A) Optimal control   (B) Optimal state

FIGURE 1. Approximation of the optimal pair $(\bar{y}, \bar{u})$, for $\lambda = 0.1$

[3] E. Casas. Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems. *Advances in Computational Mathematics*, 26:137–156, 2007.

[4] E. Casas and M. Mateos. Uniform convergence of the FEM. Applications to state constrained control problems. *J. of Computational and Applied Mathematics*, 21:67–100, 2002.

[5] Eduardo Casas, Mariano Mateos, and Jean-Pierre Raymond. Error estimates for the numerical approximation of a distributed control problem for the steady-state Navier-Stokes equations. *SIAM J. Control Optim.*, 46(3):952–982 (electronic), 2007.

[6] P.G. Ciarlet and L. L. Lions. *Handbook of Numerical Analysis, Vol. II, Part I – Finite Element Methods.* North-Holland, Amsterdam, 1991.

[7] Juan Carlos de los Reyes and Karl Kunisch. A comparison of algorithms for control constrained optimal control of the burgers equation. *Calcolo*, 41(4):203–225, 2004.

[8] Juan Carlos de los Reyes, Christian Meyer, and Boris Vexler. Finite element error analysis for state-constrained optimal control of the Stokes equations. *Control Cybernet.*, 37(2):251–284, 2008.

[9] de los Reyes, J.C., Tröltzsch, F., and Casas, E. Sufficient second-order optimality conditions for semilinear control problems with pointwise state constraints. *submitted*, 2007.

[10] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin, 1998.

[11] C. Meyer and A. Rösch. Superconvergence properties of optimal control problems. *SIAM J. Control and Optimization*, 43:970–985, 2004.

[12] A. Rösch. Error estimates for linear-quadratic control problems with control constraints. *Optim. Methods Softw.*, 21(1):121–134, 2006.

[13] Stefan Volkwein. *Mesh independence of an augmented lagrangean-SQP method in Hilbert spaces and control problems for the Burgers equation.* PhD thesis, Technische Universität Berlin, 1997.

[14] J. Zowe and S. Kurcyusz. Regularity and stability for the mathematical programming problem in Banach spaces. *Appl. Math. Optimization*, 5:49–62, 1979.

E-mail address: pedro.merino@epn.edu.ec

RESEARCH CENTER ON MATHEMATICAL MODELING MODEMAT, EPN-QUITO, ESCUELA POLITÉCNICA NACIONAL LADRÓN DE GUEVARA E11-253 QUITO, ECUADOR