# Tikhonov regularization of control-constrained optimal control problems 

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#### Abstract

We consider Tikhonov regularization of control-constrained optimal control problems. We present new a-priori estimates for the regularization error assuming measure and source-measure conditions. In the special case of bang-bang solutions, we introduce another assumption to obtain the same convergence rates. This new condition turns out to be useful in the derivation of error estimates for the discretized problem. The necessity of the just mentioned assumptions to obtain certain convergence rates is analyzed. Finally, a numerical example confirms the analytical findings.


Keywords: Tikhonov regularization, Optimal control, Control constraints, Apriori error estimates, Bang-bang controls.

## 1 Introduction

In this article we study the regularization of the minimization problem

$$
\begin{equation*}
\min _{u \in U_{\mathrm{ad}}} J_{0}(u) \quad \text { with } \quad J_{0}(u):=\frac{1}{2}\|T u-z\|_{H}^{2} \tag{0}
\end{equation*}
$$

for $T: U \rightarrow H$ a given linear and continuous operator between the control space $U:=L^{2}\left(\Omega_{U}\right)$ with scalar product $(\cdot, \cdot)_{U}$ and an arbitrary Hilbert space $H$ where $z \in H$ is a fixed function to be approached. The set $\Omega_{U} \subset \mathbb{R}^{n}$, $n \geq 1$, is a bounded measurable domain and the set of admissible controls $U_{\mathrm{ad}} \subset U$ is given by

$$
\begin{equation*}
U_{\mathrm{ad}}:=\left\{u \in U \mid a(x) \leq u(x) \leq b(x) \text { for almost all } x \in \Omega_{U}\right\} \tag{1}
\end{equation*}
$$

with fixed control bounds $a, b \in L^{\infty}\left(\Omega_{U}\right)$ fulfilling $a \leq b$.
We give two instances of $T$ as solution operators of linear partial differential equations (PDEs):

[^0]Example 1. Let $y$ be the unique weak solution of the Poisson problem

$$
\begin{align*}
-\Delta y=u & \text { in } \Omega \\
y=0 & \text { on } \partial \Omega \tag{2}
\end{align*}
$$

for given $u \in L^{2}(\Omega)$ on some bounded sufficiently regular domain $\Omega \subset \mathbb{R}^{d}$, $d \geq 1$, with boundary $\partial \Omega$.

We set $\Omega_{U}:=\Omega$ and get $y=$ Tu where $T: U=L^{2}(\Omega) \rightarrow H:=L^{2}(\Omega)$ is the weak solution operator associated with problem (2).

Example 2. Consider the heat equation

$$
\begin{array}{rlrl}
\partial_{t} y-\Delta y & =B u & & \text { in } I \times \Omega \\
y & =0 & & \text { in } I \times \partial \Omega  \tag{3}\\
y(0)=0 & & \text { in } \Omega
\end{array}
$$

with a control operator $B: U \rightarrow L^{2}\left(I, H^{-1}(\Omega)\right)$.
We fix a time interval $I:=\left(0, T_{e}\right) \subset \mathbb{R}$ with a given end-time fulfilling $0<T_{e}<\infty$. Furthermore, we assume $\Omega$ to be a domain as in the previous example. Let $T:=S B$ be the control-to-state operator with $S: L^{2}\left(I, H^{-1}(\Omega)\right)$ $\rightarrow H:=L^{2}\left(I, L^{2}(\Omega)\right)$ being the weak solution operator for the heat equation (3). We will discuss it later from (52) onwards.

Let us mention two instances for the control operator B:

1. (Distributed controls) We set $\Omega_{U}:=I \times \Omega$. The control operator $B$ : $U=H \rightarrow L^{2}\left(I, H^{-1}(\Omega)\right)$ is given by $B:=\mathrm{Id}$, i.e., the identity mapping induced by the standard Sobolev imbedding $\iota: L^{2}(\Omega) \hookrightarrow H^{-1}(\Omega)$.
2. (Located controls) Let $\Omega_{U}:=I$ and $g_{1} \in L^{2}(\Omega)$ be a fixed function. The operator $B$ given by

$$
\begin{equation*}
B: L^{2}(I, \mathbb{R}) \rightarrow L^{2}\left(I, H^{-1}(\Omega)\right), \quad u \mapsto\left(t \mapsto u(t) \iota g_{1}\right) \tag{4}
\end{equation*}
$$

with $\iota$ from the previous item maps a control function $u$ depending only on time to a function distributed in space-time.
With little more effort one can consider the case of several fixed functions $g_{1}, \ldots, g_{D}$, replacing $B$ by $u \mapsto\left(t \mapsto \sum_{i=1}^{D} u_{i}(t) \iota g_{i}\right)$ and seeking for control functions $u_{1}, \ldots, u_{D}$. We omit this generalization here to shorten the exposition and refer the interested reader to [Dan16].

To unify the examples just given, it is useful to write $T=S B$ with two continuous linear operators $B: U \rightarrow R$ and $S: R \rightarrow H$ where $R$ is an appropriately chosen function space. This decomposition is always possible for a given $T$ by taking $B=\mathrm{Id}$ and $S=T$ (or vice versa).

Often, the solutions of $\left(\mathbb{\mathbb { P } _ { 0 }}\right)$ possess a special structure: They take values only on the bounds $a$ and $b$ of the admissible set $U_{\text {ad }}$ given in (1) and are therefore called bang-bang solutions.

Theoretical and numerical questions related to this control problem attracted much interest in recent years, see, e.g., [DH12], WW11a, [GY11, (WW11b], WW13], Wac13], Wac14], (Fel03, (Alt+12], AS11], and Sey15. The last four papers are concerned with $T$ being the solution operator of an ordinary differential equation, the first three papers with $T$ being a solution operator of an elliptic PDE as in Example 1, and the remaining references with $T$ being a general linear operator as here. In [Dan16], a brief survey of the content of these and some other related papers is given at the end of the bibliography. For an appropriate discretization of Example 2 we refer to (DHV15], Dan16], and the forthcoming [DH], but see also the numerics section below.

Problem $\left(\mathbb{\mathbb { P } _ { 0 }}\right)$ is in general ill-posed, meaning that a solution does not depend continuously on the datum $z$, see WW11b, p. 1130]. The numerical treatment of a discretized problem version is often challenging or even impossible.

Therefore, we use Tikhonov regularization to overcome these difficulties. The regularized problem is given by

$$
\min _{u \in U_{\mathrm{ad}}} J_{\alpha}(u) \quad \text { with } \quad J_{\alpha}(u):=\frac{1}{2}\|T u-z\|_{H}^{2}+\frac{\alpha}{2}\|u\|_{U}^{2}
$$

where $\alpha>0$ denotes the regularization parameter.
Formally, for $\alpha=0$ problem $\left(\mathbb{\mathbb { P } _ { \alpha }}\right)$ reduces to problem $(\sqrt[\mathbb{P}_{0}]{ })$ also called the limit problem.

Note that for the regularized problems $\left.\mid \mathbb{P}_{\alpha}\right), \alpha>0$, and their discretizations, explicit solution representations are available and can be utilized for numerical implementation; cf. (6), (67) below.

We recall in the next section basic properties of the regularized and the limit problem: Problem $\left(\mathbb{\mathbb { P } _ { \alpha }}\right.$ ) has a solution $\bar{u}_{\alpha}$ for all $\alpha \geq 0$. If $\alpha>0$, the solution is unique. If $\alpha=0$ and the operator $T$ is injective, the solution of the limit problem $\left(\mathbb{\mathbb { P } _ { 0 }}\right.$ ) is unique, too. Note that $T$ is injective in Example 1 and 2.

If $T$ is not injective, the limit problem might have several solutions. By $\hat{u}_{0}$ we denote the solution of the limit problem with minimal $U$ norm, i.e. $\hat{u}_{0}=\operatorname{argmin}\left\{\|u\|_{U} \mid u\right.$ solves $\left.\left.\mathbb{P}_{0}\right\}\right\}$. We close the section by stating a first convergence result, which in particular shows that the regularized solutions $\bar{u}_{\alpha}$ converge to $\hat{u}_{0}$ if $\alpha$ tends to zero.

More convergence results are obtained in the third section if a condition on the smoothness of the limit problem $(\sqrt[\mathbb{P}_{0}]{ })$ is fulfilled. For easy reference, we call this Assumption 7 source-measure condition below. The main result is Theorem 11, where we show convergence rates which improve known ones, see Table 1 for a detailed comparison.

The necessity of the just mentioned smoothness conditions to obtain better convergence rates is a topic which is discussed in the fourth section. We present a new proof of the necessity of the measure condition which motivates another condition, namely (36), in the special case of bang-bang solutions.

This new condition (36) is exploited in the fifth section. We show that the condition implies the same convergence rates as the source-measure condition. The new condition is (almost) necessary to obtain these rates. Finally, it turns out that the new and the old condition coincide if the limit problem is of certain regularity.

The reason to introduce this new condition (36) is that it leads to an improved bound on the decay of smoothness in the weak derivative of the optimal control when $\alpha$ tends to zero. This bound is useful to derive improved convergence rates for the discretization errors of the regularized problem, which we sketch.

The last section is concerned with a numerical example confirming our theoretical findings.

## 2 First results

Lemma 3. The optimal control problem $\left(\mathbb{\mathbb { P } _ { \alpha }}\right)$ admits for fixed $\alpha \geq 0$ at least one solution $\bar{u}_{\alpha} \in U$, which can be characterized by the first order necessary and sufficient optimality condition

$$
\begin{equation*}
\bar{u}_{\alpha} \in U_{\mathrm{ad}}, \quad\left(\alpha \bar{u}_{\alpha}+B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)_{U} \geq 0 \quad \forall u \in U_{\mathrm{ad}} \tag{5}
\end{equation*}
$$

where $B^{*}$ denotes the adjoint operator of $B, \bar{y}_{\alpha}:=T \bar{u}_{\alpha} \in H$ is named optimal state, and the so-called optimal adjoint state $\bar{p}_{\alpha}$ is defined by $\bar{p}_{\alpha}:=$ $S^{*}\left(\bar{y}_{\alpha}-z\right)$.

If $\alpha>0$ or $T$ is injective, the solution $\bar{u}_{\alpha}$ is unique. The quantities $\bar{y}_{\alpha}$ and $\bar{p}_{\alpha}$ are always unique for given $\alpha \geq 0$ even if $\bar{u}_{0}$ is not.

Proof. We have a convex optimization problem with a weakly lower semicontinuous cost functional on the non-empty, bounded, closed, and convex set $U_{\text {ad }}$. Therefore, classic theory as elaborated, e.g., in ET76], guarantees existence and uniqueness. We refer to [Hin+09, Theorem 1.46, p. 66] or Trö05, Satz 2.14] for a proof in our specific setting.

Note that in the case $\alpha=0$, uniqueness of the state $\bar{y}_{0}$ follows from the fact that the cost functional of $\left(\mathbb{\mathbb { P } _ { 0 }}\right)$ with respect to the state, i.e. $y \mapsto$ $\frac{1}{2}\|y-z\|_{H}^{2}$, is strictly convex. Thus by injectivity of $T$, uniqueness of $\bar{u}_{0}$ can be derived since $\bar{y}_{0}=T \bar{u}_{0}$.

As a consequence of the fact that $U_{\mathrm{ad}}$ is a closed and convex set in a Hilbert space we have the following lemma.

Lemma 4. In the case $\alpha>0$ the variational inequality (5) is equivalent to

$$
\begin{equation*}
\bar{u}_{\alpha}=P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} B^{*} \bar{p}_{\alpha}\right) \tag{6}
\end{equation*}
$$

where $P_{U_{\mathrm{ad}}}: U \rightarrow U_{\mathrm{ad}}$ is the orthogonal projection.
Proof. See Hin +09 , Corollary 1.2, p. 70] with $\gamma=\frac{1}{\alpha}$.
We now derive an explicit characterization of optimal controls.
Lemma 5. If $\alpha>0$, then for almost all $x \in \Omega_{U}$ there holds for the optimal control

$$
\bar{u}_{\alpha}(x)= \begin{cases}a(x) & \text { if } B^{*} \bar{p}_{\alpha}(x)+\alpha a(x)>0  \tag{7}\\ -\alpha^{-1} B^{*} \bar{p}_{\alpha}(x) & \text { if } B^{*} \bar{p}_{\alpha}(x)+\alpha \bar{u}_{\alpha}(x)=0, \\ b(x) & \text { if } B^{*} \bar{p}_{\alpha}(x)+\alpha b(x)<0\end{cases}
$$

Suppose $\alpha=0$ is given. Then any optimal control fulfills a.e. in $\Omega_{U}$

$$
\bar{u}_{0}(x) \begin{cases}=a(x) & \text { if } B^{*} \bar{p}_{0}(x)>0  \tag{8}\\ \in[a(x), b(x)] & \text { if } B^{*} \bar{p}_{0}(x)=0 \\ =b(x) & \text { if } B^{*} \bar{p}_{0}(x)<0\end{cases}
$$

Proof. Let us first note that the variational inequality (5) is for $\alpha \geq 0$ equivalent to the following pointwise one:

$$
\begin{equation*}
\forall^{\prime} x \in \Omega_{U} \forall v \in[a(x), b(x)]:\left(\alpha \bar{u}_{\alpha}(x)+B^{*} \bar{p}_{\alpha}(x), v-\bar{u}_{\alpha}(x)\right)_{\mathbb{R}} \geq 0 \tag{9}
\end{equation*}
$$

where " $\forall$ "" denotes "for almost all".
This can be shown via a Lebesgue point argument, see the proof of Trö05, Lemma 2.26]. By cases, one immediately derives (7) and (8) from (9).

As a consequence of 8 we have: If $B^{*} \bar{p}_{0}$ vanishes only on a subset of $\Omega_{U}$ with Lebesgue measure zero, the optimal control $\bar{u}_{0}$ is unique and a bangbang solution: It takes values only on the bounds $a$ and $b$ of the admissible set $U_{\text {ad }}$ given in (1).

If the limit problem $\left(\mathbb{P}_{0}\right)$ admits several solutions, we by $\hat{u}_{0}$ denote the minimal $U$ norm solution, i.e.

$$
\begin{equation*}
\hat{u}_{0}=\operatorname{argmin}\left\{\|u\|_{U} \mid u \text { solves } \mathbb{P}_{0}\right\} \tag{10}
\end{equation*}
$$

Note that this minimization problem has a unique solution since the $U$ norm is strictly convex and the set $\left\{u\right.$ solves $\left(\mathbb{P}_{0}\right\}$ is non-empty, closed and convex in $U$.

The next Theorem establishes convergence $\bar{u}_{\alpha} \rightarrow \hat{u}_{0}$ if $\alpha \rightarrow 0$, which is the reason to highlight the minimal $U$ norm solution among the solutions of ( $\mathbb{P}_{0}$.

Theorem 6. For the solution $\left(\bar{u}_{\alpha}, \bar{y}_{\alpha}\right)$ of $\left(\mathbb{P}_{\alpha}\right)$ with $\alpha>0$ and any solution $\left(\bar{u}_{0}, \bar{y}_{0}\right)$ of $\left(\mathbb{P}_{0}\right)$, there holds

1. The optimal control and the optimal state depend continuously on $\alpha$. More precisely, the inequality

$$
\begin{equation*}
\left\|\bar{y}_{\alpha^{\prime}}-\bar{y}_{\alpha}\right\|_{H}^{2}+\alpha^{\prime}\left\|\bar{u}_{\alpha^{\prime}}-\bar{u}_{\alpha}\right\|_{U}^{2} \leq\left(\alpha-\alpha^{\prime}\right)\left(\bar{u}_{\alpha}, \bar{u}_{\alpha^{\prime}}-\bar{u}_{\alpha}\right)_{U} \tag{11}
\end{equation*}
$$

holds for all $\alpha \geq 0$ and all $\alpha^{\prime} \geq 0$.
2. The regularized solutions converge to the minimal $U$ norm solution $\hat{u}_{0}$, i.e.,

$$
\begin{equation*}
\left\|\bar{u}_{\alpha}-\hat{u}_{0}\right\|_{U} \rightarrow 0 \quad \text { if } \alpha \rightarrow 0 \tag{12}
\end{equation*}
$$

3. The optimal state satisfies the rate of convergence

$$
\begin{equation*}
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}=o(\sqrt{\alpha}) \tag{13}
\end{equation*}
$$

Proof. The Theorem is a collection of classic results from the theory of linear inverse problems with convex constraints (given here by $U_{\text {ad }}$ ) taken from EHN00, Chapter 5.4], see also Neu86.

## 3 Refined convergence rates under additional assumptions

To prove better rates of convergence with respect to $\alpha$, we rely on the following assumption.

Assumption 7 ([WW11b, Assumption 3.1]). Let $\bar{u}_{0}$ be a solution of $\left(\mathbb{P}_{0}\right)$. There exist a set $A \subset \Omega_{U}$, a function $w \in H$ with $T^{*} w \in L^{\infty}\left(\Omega_{U}\right)$ and constants $\kappa>0$ and $C \geq 0$, such that there holds the inclusion

$$
\left\{x \in \Omega_{U} \mid B^{*} \bar{p}_{0}(x)=0\right\} \subset A^{c}
$$

for the complement $A^{c}=\Omega_{U} \backslash A$ of $A$ and in addition

1. (source condition)

$$
\begin{equation*}
\chi_{A^{c}} \bar{u}_{0}=\chi_{A^{c}} P_{U_{\mathrm{ad}}}\left(T^{*} w\right) \tag{14}
\end{equation*}
$$

2. ( $\left(\bar{p}_{0}-\right)$ measure condition)

$$
\begin{equation*}
\forall \epsilon>0: \quad \operatorname{meas}\left(\left\{x \in A\left|0 \leq\left|B^{*} \bar{p}_{0}(x)\right| \leq \epsilon\right\}\right) \leq C \epsilon^{\kappa}\right. \tag{15}
\end{equation*}
$$

with the convention that $\kappa:=\infty$ if the left-hand side of (15) is zero for some $\epsilon>0$.

Source conditions of the form $\bar{u}_{0}=P_{U_{\mathrm{ad}}}\left(T^{*} w\right)$ are well known in the theory of inverse problems with convex constraints, see [Neu86] and [EHN00]. However, since they are usually posed almost everywhere, thus globally, they are unlikely to hold in the optimal control setting with $T$ as in, e.g., Example 1, see WW11a, p. 860].

Similar measure conditions were previously used for control problems with elliptic PDEs, starting with the analysis in [WW11a] and [DH12].

A condition related to the measure condition was also used to establish stability results for bang-bang control problems with autonomous ODEs, see [Fel03, Assumption 2].

In all above-mentioned references, the measure condition (15) is assumed to hold with $A=\Omega_{U}$, thus globally. Together with formula (8) one immediately observes that this implies bang-bang controls.

The combination of both conditions in Assumption 7 was introduced in WW11b and also used in (WW13).

In Theorem 11 we will show that if a solution $\bar{u}_{0}$ of $\left(\mathbb{\mathbb { P } _ { 0 }}\right.$ fulfills Assumption 7. we have convergence $\bar{u}_{\alpha} \rightarrow \bar{u}_{0}$ for $\alpha \rightarrow 0$. From formula (12) in Theorem 6 we conclude $\bar{u}_{0}=\hat{u}_{0}$, which means: If Assumption 7 is valid for a solution of $\left(\mathbb{P}_{0}\right)$, this solution has to be the minimal $U$ norm solution 10$]$.

Key ingredient in our analysis of the regularization error is the following lemma, which has its origin in the proof of WW11b, Theorem 3.5].

Lemma 8. Let Assumption 7,2 be valid for a solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$. Then there holds with some constant $C>0$ independent of $\alpha$ and $u$

$$
\begin{equation*}
C\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa} \leq\left(B^{*} \bar{p}_{0}, u-\bar{u}_{0}\right)_{A} \leq\left(B^{*} \bar{p}_{0}, u-\bar{u}_{0}\right)_{U} \quad \forall u \in U_{\mathrm{ad}} \tag{16}
\end{equation*}
$$

where $(\cdot, \cdot)_{A}$ and $(\cdot, \cdot)_{U}$ denote the scalar products in $L^{2}(A)$ and $U=L^{2}\left(\Omega_{U}\right)$, respectively.

Proof. For $\epsilon>0$ we define $B_{\epsilon}:=\left\{x \in A| | B^{*} \bar{p}_{0} \mid \geq \epsilon\right\}$. Using the (pointwise) optimality condition (9) and Assumption 72, we conclude for some $u \in U_{\text {ad }}$

$$
\begin{aligned}
\int_{\Omega_{U}}\left(B^{*} \bar{p}_{0}, u-\bar{u}_{0}\right)_{\mathbb{R}} & =\int_{\Omega_{U}}\left|B^{*} \bar{p}_{0}\left\|u-\bar{u}_{0}\left|\geq \int_{A}\right| B^{*} \bar{p}_{0}\right\| u-\bar{u}_{0}\right| \\
& \geq \epsilon\left\|u-\bar{u}_{0}\right\|_{L^{1}\left(B_{\epsilon}\right)} \\
& \geq \epsilon\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}-\epsilon\left\|u-\bar{u}_{0}\right\|_{L^{1}\left(A \backslash B_{\epsilon}\right)} \\
& \geq \epsilon\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}-\epsilon\left\|u-\bar{u}_{0}\right\|_{L^{\infty}\left(\Omega_{U}\right)} \operatorname{meas}\left(A \backslash B_{\epsilon}\right) \\
& \geq \epsilon\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}-c \epsilon^{\kappa+1}\left\|u-\bar{u}_{0}\right\|_{L^{\infty}\left(\Omega_{U}\right)}
\end{aligned}
$$

where without loss of generality $c>1$.

Setting $\epsilon:=c^{-2 / \kappa}\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}^{1 / \kappa}\left\|u-\bar{u}_{0}\right\|_{L^{\infty}\left(\Omega_{U}\right)}^{-1 / \kappa}$ yields

$$
\begin{aligned}
\int_{A}\left(B^{*} \bar{p}_{0}, u-\bar{u}_{0}\right)_{\mathbb{R}} & \geq c^{-2 / \kappa}\left(1-\frac{1}{c}\right)\left\|u-\bar{u}_{0}\right\|_{L^{\infty}\left(\Omega_{U}\right)}^{-1 / \kappa}\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa} \\
& \geq c^{-2 / \kappa}\left(1-\frac{1}{c}\right)\|b-a\|_{L^{\infty}\left(\Omega_{U}\right)}^{-1 / \kappa}\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa}
\end{aligned}
$$

by the definition of $U_{\text {ad }}$.
With the previous Lemma, we can now improve the inequality (setting there $\alpha:=0$ ) from general inverse problem theory, since the error in the control in the $L^{1}$ norm now appears on the left-hand side with a factor $\mathrm{C}>0$ independent of $\alpha$. This is in contrast to the error in the $L^{2}$ norm.

Lemma 9. Let Assumption 7.2 hold (with possibly meas $(A)=0$ ) for a solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$. Then there holds for some $C>0$ independent of $\alpha$

$$
\begin{aligned}
&\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}+C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa}+\alpha \| \bar{u}_{\alpha}- \\
& \bar{u}_{0} \|_{U}^{2} \\
& \leq \alpha\left(\bar{u}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U} \quad \forall \alpha>0 .
\end{aligned}
$$

Proof. Adding the necessary condition for $\bar{u}_{\alpha}$ (5) with $u:=\bar{u}_{0}$, i.e.,

$$
0 \leq\left(\alpha \bar{u}_{\alpha}+B^{*} \bar{p}_{\alpha}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U}
$$

to the estimate (16) of Lemma 8 with $u:=\bar{u}_{\alpha}$, we get

$$
\begin{aligned}
C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa} \leq & \left(B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right), \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U}+\alpha\left(\bar{u}_{\alpha}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U} \\
\leq & -\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}+\alpha\left(\bar{u}_{\alpha}-\bar{u}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U} \\
& \quad+\alpha\left(\bar{u}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U} \\
\leq & -\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}-\alpha\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U}^{2}+\alpha\left(\bar{u}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U}
\end{aligned}
$$

The following Lemma is extracted from the proof of WW11b, Lemma 3.2]. It shows how the source condition (Assumption 7.1) is taken into account to reduce the error estimate to the set $A$.

Lemma 10. Let Assumption 7.1 (source condition) be satisfied for a solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$. Then there holds with a constant $C>0$

$$
\left(\bar{u}_{0}, \bar{u}_{0}-u\right)_{U} \leq C\left(\left\|T\left(u-\bar{u}_{0}\right)\right\|_{H}+\left\|u-\bar{u}_{0}\right\|_{L^{1}(A)}\right) \quad \forall u \in U_{\mathrm{ad}}
$$

Proof. The source condition is equivalent to

$$
0 \leq\left(\chi_{A^{c}}\left(\bar{u}_{0}-T^{*} w\right), u-\bar{u}_{0}\right)_{U} \quad \forall u \in U_{\mathrm{ad}}
$$

Using this representation, we can estimate

$$
\begin{aligned}
\left(\bar{u}_{0}, \bar{u}_{0}-u\right)_{U} & \leq\left(\chi_{A^{c}} T^{*} w, \bar{u}_{0}-u\right)_{U}+\left(\chi_{A} \bar{u}_{0}, \bar{u}_{0}-u\right)_{U} \\
& \leq\left(w, T\left(\bar{u}_{0}-u\right)\right)_{H}+\left(-T^{*} w+\bar{u}_{0}, \chi_{A}\left(\bar{u}_{0}-u\right)\right)_{U}
\end{aligned}
$$

Since $T^{*} w \in L^{\infty}\left(\Omega_{U}\right)$, we get the claim.
Using this Lemma, we can now state regularization error estimates. We consider different situations with respect to the fulfillment of parts of Assumption 7 .

Theorem 11. For the regularization error there holds with positive constants $c$ and $C$ indepent of $\alpha>0$ the following, where $\bar{u}_{0}$ is in fact $\hat{u}_{0}$ as noted before Lemma 8.

1. The error in the optimal state fulfills the rate of convergence

$$
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}=o(\sqrt{\alpha}) .
$$

2. Let Assumption 7, 1 be satisfied with $\operatorname{meas}(A)=0$ (source condition holds a.e. on the domain) for a solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$. Then the optimal control converges with the rate

$$
\begin{equation*}
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U} \leq C \sqrt{\alpha} \tag{17}
\end{equation*}
$$

and the optimal state converges with the improved rate

$$
\begin{equation*}
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} \leq C \alpha \tag{18}
\end{equation*}
$$

3. Let Assumption 7.2 be satisfied with $\operatorname{meas}\left(A^{c}\right)=0$ (measure condition holds a.e. on the domain) for a solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$. From (8) we conclude that $\bar{u}_{0}$ is the unique solution of $\left(\mathbb{P}_{0}\right)$. Then the estimates

$$
\begin{align*}
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}\left(\Omega_{U}\right)} & \leq C \alpha^{\kappa}  \tag{19}\\
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U} & \leq C \alpha^{\kappa / 2}  \tag{20}\\
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} & \leq C \alpha^{(\kappa+1) / 2} \tag{21}
\end{align*}
$$

hold true.
If furthermore $\kappa>1$ holds and in addition

$$
\begin{equation*}
T^{*}: \operatorname{range}(T) \rightarrow L^{\infty}\left(\Omega_{U}\right) \quad \text { exists and is continuous } \tag{22}
\end{equation*}
$$

we can improve (21) to

$$
\begin{equation*}
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} \leq C \alpha^{\kappa} \tag{23}
\end{equation*}
$$

4. Let Assumption 7 be satisfied with $\operatorname{meas}(A) \cdot \operatorname{meas}\left(A^{c}\right)>0$ (source and measure condition on parts of the domain) for a solution $\bar{u}_{0}$ of $\mathbb{P}_{0}$ and let in addition $\alpha<1$. Then the estimates

$$
\begin{align*}
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)} & \leq C \alpha^{\min \left(\kappa, \frac{2}{1+1 / \kappa}\right)}  \tag{24}\\
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U} & \leq C \alpha^{\min (\kappa, 1) / 2}  \tag{25}\\
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} & \leq C \alpha^{\min ((\kappa+1) / 2,1)} \tag{26}
\end{align*}
$$

## hold true.

If furthermore $\kappa>1$ and (22) hold, we have the improved estimate

$$
\begin{equation*}
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)} \leq C \alpha^{\kappa} \tag{27}
\end{equation*}
$$

Proof. In this proof, we denote by $C_{1}, \ldots, C_{4}$ positive constants.

1. The estimate is just a repetition of 13 ).
2. Let us recall the estimates of Lemma 9, i.e.,

$$
\begin{equation*}
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}+C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa}+\alpha\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U}^{2} \leq \alpha\left(\bar{u}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U} \tag{28}
\end{equation*}
$$

By Young's inequality we can estimate with a constant $\hat{C}>0$

$$
\begin{equation*}
\hat{C} \alpha\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)} \leq \tilde{C} \alpha^{\kappa+1}+\frac{C}{2}\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa} \tag{29}
\end{equation*}
$$

where $C$ is the same constant as in 28$)$ and $\tilde{C}=\tilde{C}(C, \hat{C}, \kappa)$ is the constant from Young's inequality.

If $A=\Omega_{U}$ up to a set of measure zero, we can combine both estimates taking $\hat{C}:=\left\|\bar{u}_{0}\right\|_{L^{\infty}}$, and move the second summand of 29 to the left. This yields the claim since

$$
\frac{\kappa+1}{1+1 / \kappa}=\kappa
$$

The improved estimate (23) can be obtained with the help of 19 ) as follows

$$
\begin{aligned}
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2} & =\left(T^{*}\left(\bar{y}_{\alpha}-\bar{y}_{0}\right), \bar{u}_{\alpha}-\bar{u}_{0}\right)_{U} \leq C_{1}\left\|T^{*}\left(\bar{y}_{\alpha}-\bar{y}_{0}\right)\right\|_{L^{\infty}}\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}} \\
& \leq C_{2}\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}} \leq C_{3}\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} \alpha^{\kappa}
\end{aligned}
$$

2. +4 . We combine 28 with the estimate of Lemma 10 (with $u:=\bar{u}_{\alpha}$ ), invoke Cauchy's inequality and get

$$
\begin{aligned}
&\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}+C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa}+\alpha\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U}^{2} \\
& \leq \alpha\left(\bar{u}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{U} \leq C_{1} \alpha\left(\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}+\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}\right) \\
& \leq C_{2} \alpha^{2}+\frac{1}{2}\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}+C_{1} \alpha\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}
\end{aligned}
$$

We now move the second addend to the left.
If meas $(A)=0$ (case 2.), we are done. Otherwise (case 4.) we continue estimating, making use of (29), to get

$$
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}^{2}+C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa}+\alpha\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{U}^{2} \leq C_{3} \alpha^{\min (2, \kappa+1)}
$$

from which the claim follows.
To establish formula (27), we integrate (9) over $A$, taking $v:=\bar{u}_{0}(x)$, to end up with

$$
0 \leq\left(\alpha \bar{u}_{\alpha}+B^{*} \bar{p}_{\alpha}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{A} .
$$

By $(\cdot, \cdot)_{A}$ we again denote the scalar product in $L^{2}(A)$.
We add this inequality to the estimate (16) of Lemma 8 with $u:=\bar{u}_{\alpha}$, to get

$$
C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa} \leq\left(B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right), \bar{u}_{0}-\bar{u}_{\alpha}\right)_{A}+\alpha\left(\bar{u}_{\alpha}, \bar{u}_{0}-\bar{u}_{\alpha}\right)_{A} .
$$

Making use of (22) and the convergence rate with $\kappa>1$, we conclude

$$
\left\|B^{*}\left(\bar{p}_{0}-\bar{p}_{\alpha}\right)\right\|_{L^{\infty}\left(\Omega_{U}\right)}=\left\|T^{*}\left(\bar{y}_{0}-\bar{y}_{\alpha}\right)\right\|_{L^{\infty}\left(\Omega_{U}\right)} \leq C_{1}\left\|\bar{y}_{0}-\bar{y}_{\alpha}\right\|_{H} \leq C_{2} \alpha .
$$

Since $\bar{u}_{\alpha} \in L^{\infty}\left(\Omega_{U}\right)$ by (1), combining both estimates gives

$$
\begin{aligned}
C\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}(A)}^{1+1 / \kappa} & \leq C_{3}\left(\left\|B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right)\right\|_{L^{\infty}(A)}+\alpha\right)\left\|\bar{u}_{0}-\bar{u}_{\alpha}\right\|_{L^{1}(A)} \\
& \leq C_{4} \alpha\left\|\bar{u}_{0}-\bar{u}_{\alpha}\right\|_{L^{1}(A)} .
\end{aligned}
$$

Dividing the expression by the norm on the right and taking the $\kappa$ th power, we are done.

Some remarks on the previous theorem are in order.
Let us compare the first with the other cases, where Assumption 7 is taken (partially) into account. In all cases, we get an improved convergence rate for the optimal state.

The second case replicates well known estimates from the theory of inverse problems with convex constraints, see, e.g., Neu86 and EHN00, Theorem 5.19]. However, as indicated in the discussion after Assumption 7 , this situation is unlikely to hold in the context of optimal control problems.

Concerning the "min"-functions in the estimates of case 4, we note that the left argument is chosen if $\kappa<1$, the right one if $\kappa>1$. In the case $\kappa=1$, both expressions coincide. Thus the worse part of Assumption 7 with respect to the rates of cases 2 and 3 dominates the convergence behavior of the regularization errors in the mixed situation of case 4 on the whole domain $\Omega_{U}$. This, however, is not the case locally on $A$, as (27) shows.

As mentioned after Assumption 7, case 3 implies bang-bang controls.

| quantity $\leq C \alpha^{r}$ | here | br | there | assumptions, source |
| :---: | :---: | :---: | :---: | :---: |
| $\left\\|\bar{u}_{\alpha}-\bar{u}_{0}\right\\|_{L^{1}(A)}$ | $r=\kappa$ | $\leftarrow$ | $r=\frac{\kappa}{2-\kappa}$ | $\begin{gathered} \kappa<1 \\ \text { by }(19 / 24) \end{gathered}$ |
| $\left\\|\bar{u}_{\alpha}-\bar{u}_{0}\right\\|_{L^{1}(A)}$ | $r=\kappa$ | $=$ | $r=\kappa$ | $\begin{gathered} \kappa=1 \text { or }(3 . \text { and } \kappa>1) \\ \text { by (19) or (24) } \end{gathered}$ |
| $\left\\|\bar{u}_{\alpha}-\bar{u}_{0}\right\\|_{L^{1}(A)}$ | $r=\kappa$ | $\leftarrow$ | $r=\frac{\kappa+1}{2}$ | 4. and $\kappa>1$ by (27) |
| $\left\\|\bar{u}_{\alpha}-\bar{u}_{0}\right\\|_{U}$ | $r=\frac{\kappa}{2}$ | $\leftarrow$ | $r=\frac{\kappa}{2(2-\kappa)}$ | $\begin{gathered} \quad \kappa<1 \\ \text { by } 20 \text { or } 25) \end{gathered}$ |
| $\left\\|\bar{u}_{\alpha}-\bar{u}_{0}\right\\|_{U}$ | $r=\frac{\kappa}{2}$ | $=$ | $r=\frac{\kappa}{2}$ | $\begin{gathered} \kappa=1 \text { or }(3 . \text { and } \kappa>1) \\ \text { by } 20 \end{gathered}$ |
| $\left\\|\bar{u}_{\alpha}-\bar{u}_{0}\right\\|_{U}$ | $r=\frac{1}{2}$ | $=$ | $r=\frac{1}{2}$ | 4. and $\kappa>1$ <br> by 25 |
| $\left\\|\bar{y}_{\alpha}-\bar{y}_{0}\right\\|_{H}$ | $r=\frac{\kappa+1}{2}$ | $\leftarrow$ | $r=\frac{1}{2-\kappa}$ | $\begin{gathered} \quad \kappa<1 \\ \text { by }(21) \text { or }(26) \end{gathered}$ |
| $\left\\|\bar{y}_{\alpha}-\bar{y}_{0}\right\\|_{H}$ | $r=1$ | $=$ | $r=1$ | $\begin{gathered} \kappa=1 \text { or }(4 . \text { and } \kappa>1) \\ \text { by }(21) \text { or } 26) \end{gathered}$ |
| $\left\\|\bar{y}_{\alpha}-\bar{y}_{0}\right\\|_{H}$ | $r=\kappa$ | $\leftarrow$ | $r=\frac{\kappa+1}{2}$ | 3 . and $\kappa>1$ <br> by 23 |

Table 1: Comparison of convergence rates given in Theorem 11. $3+4$ ("here") with WW11b, Theorem 3.2] ("there"), assuming always (22). The column "br" points to the better rate (i.e. larger $r$ ) unless both coincide (=). We abbreviate by " 3 ." and " 4 ." the assumptions of Theorem 11.3 and 4, respectively.

The condition (22) is fulfilled for Example 1 since $T^{*}: L^{2}(\Omega) \rightarrow H^{2}(\Omega) \cap$ $H^{1}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ by well-known regularity theory and Sobolev imbeddings, see, e.g., Eva98, if $\Omega$ is sufficiently regular.

For Example 2, condition (22) is also valid, see [Dan16, p. 24].
Let us finally compare in Table 1 the cases 3 and 4 with the convergence results of WW11b, Theorem 3.2] to point out which rates stated above are improved. Note for comparison, that (22) is always assumed in WW11b, Theorem 3.2] and $p_{\alpha}$ there is $B^{*} \bar{p}_{\alpha}$ here. If we assume (22), we can estimate $\left\|B^{*}\left(\bar{p}_{0}-\bar{p}_{\alpha}\right)\right\|_{L^{\infty}}=\left\|T^{*}\left(\bar{y}_{0}-\bar{y}_{\alpha}\right)\right\|_{L^{\infty}} \leq C\left\|\bar{y}_{0}-\bar{y}_{\alpha}\right\|_{H}$, and combine this with (21), (23), or (26). Since in WW11b, Theorem 3.2] the convergence rates for $\left\|p_{0}-p_{\alpha}\right\|_{L^{\infty}}$ are obtained in the same way, comparing the state rates gives the same results as comparing $\left\|B^{*}\left(\bar{p}_{0}-\bar{p}_{\alpha}\right)\right\|_{L^{\infty}}$ with $\left\|p_{0}-p_{\alpha}\right\|_{L^{\infty}}$. We therefore omit the latter.

## 4 Necessity of the additional assumptions

Let us now consider the question of necessity of Assumption 7 to obtain convergence rates, thus a converse of Theorem 11 .

We first show that a convergence rate $\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} \leq C \alpha$ implies the source condition (14) to hold at least on $\left\{x \in \Omega_{U} \mid B^{*} \bar{p}_{0}(x)=0\right\}$.

The following Theorem is a for our purposes simplified version of WW13, Theorem 4]. It resembles a necessity result known from inverse problem theory, see, e.g., EHN00, Theorem 5.19] or Neu86. However, in inverse problems, the condition $T \bar{u}_{0}=z$ is typically assumed.

Theorem 12. Let $\hat{u}_{0}$ be the minimal $U$ norm solution of $\mathbb{P}_{0}$ defined in (10). If we assume a convergence rate $\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}=\mathcal{O}(\alpha)$, then there exists a function $w \in H$ such that $\hat{u}_{0}=P_{U_{\mathrm{ad}}}\left(T^{*} w\right)$ holds pointwise a.e. on

$$
\begin{equation*}
K:=\left\{x \in \Omega_{U} \mid B^{*} \bar{p}_{0}(x)=0\right\} . \tag{30}
\end{equation*}
$$

Thus (14) holds on $K$ instead of $A^{c}$.
If even $\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H}=o(\alpha)$, then $\hat{u}_{0}$ vanishes on $K$.
Proof. We integrate the necessary condition (9) over $K$ to obtain

$$
0 \leq\left(\alpha \bar{u}_{\alpha}+T^{*} T\left(\bar{u}_{\alpha}-\hat{u}_{0}\right), u-\bar{u}_{\alpha}\right)_{K} \quad \forall u \in U_{\mathrm{ad}}
$$

Dividing the expression by $\alpha$ and taking the limit we get with the help of (12) the inequality

$$
0 \leq\left(T^{*} \dot{y}_{0}+\hat{u}_{0}, u-\hat{u}_{0}\right)_{K} \quad \forall u \in U_{\mathrm{ad}}
$$

for any weak subsequential limit $\dot{y}_{0}$ of $\frac{1}{\alpha}\left(\bar{y}_{\alpha}-\bar{y}_{0}\right)$, which exists due to the assumption of the Theorem.

Taking $w:=-\dot{y_{0}}$, we obtain the equation $\hat{u}_{0}(x)=P_{[a(x), b(x)]}(w(x))$ pointwise on $K$ by varying $u$. Since $P_{U_{\text {ad }}}$ acts pointwise, we get the claim.

The second assertion follows from the equality $\dot{y}_{0}=0$ in case of $\| \bar{y}_{\alpha}-$ $\bar{y}_{0} \|_{H}=o(\alpha)$.

We next show that if 22 and $\kappa>1$ hold true, convergence as in Theorem 11.3 implies the measure condition (15).

Theorem 13. Let us assume

$$
\begin{equation*}
\exists A \subset \Omega_{U}: \quad\left\{x \in \Omega_{U} \mid B^{*} \bar{p}_{0}(x)=0\right\} \subset A^{c} \tag{31}
\end{equation*}
$$

Let us further assume the $\sigma$-condition

$$
\begin{equation*}
\exists \sigma>0 \forall^{\prime} x \in \Omega_{U}: \quad a \leq-\sigma<0<\sigma \leq b \tag{32}
\end{equation*}
$$

where " $\forall$ '" denotes "for almost all".
If $\kappa>1$ and convergence rates $\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{p}(A)}^{p}+\left\|B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right)\right\|_{L^{\infty}(A)} \leq$ $C \alpha^{\kappa}$ are known to hold for a solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right.$ and some real $p \geq 1$, then the measure condition from Assumption 7 is fulfilled.

Proof. Let us introduce the sets

$$
\begin{aligned}
& A_{0}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}<0 \text { and } \alpha a \geq-B^{*} \bar{p}_{\alpha}\right\}, \\
& A_{1}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}<0 \text { and } \alpha a<-B^{*} \bar{p}_{\alpha}<\alpha b\right\}, \\
& A_{2}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}<0<\alpha b \leq-B^{*} \bar{p}_{\alpha}\right\}, \\
& A_{3}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}>0 \text { and } \alpha a<-B^{*} \bar{p}_{\alpha}<\alpha b\right\}, \\
& A_{4}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}>0>\alpha a \geq-B^{*} \bar{p}_{\alpha}\right\}, \quad \text { and } \\
& A_{5}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}>0 \text { and } \alpha b \leq-B^{*} \bar{p}_{\alpha}\right\} .
\end{aligned}
$$

We also need two subsets of $A_{1}$ and $A_{3}$, respectively, namely by (32)

$$
\begin{aligned}
& \tilde{A}_{1}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}<0 \text { and }-\alpha \frac{\sigma}{2} \leq-B^{*} \bar{p}_{\alpha} \leq \alpha \frac{\sigma}{2}\right\} \subset A_{1}, \quad \text { and } \\
& \tilde{A}_{3}:=\left\{x \in A \mid-B^{*} \bar{p}_{0}>0 \text { and }-\alpha \frac{\sigma}{2} \leq-B^{*} \bar{p}_{\alpha} \leq \alpha \frac{\sigma}{2}\right\} \subset A_{3} .
\end{aligned}
$$

From (31) we conclude $A=A_{0} \cup A_{1} \cup A_{2} \cup A_{3} \cup A_{4} \cup A_{5}$, and from Lemma 5 we infer

$$
\begin{align*}
\int_{A}\left|\bar{u}_{0}-\bar{u}_{\alpha}\right|^{p} & =\int_{A_{1}}\left|a+\alpha^{-1} B^{*} \bar{p}_{\alpha}\right|^{p}+\int_{A_{3}}\left|b+\alpha^{-1} B^{*} \bar{p}_{\alpha}\right|^{p}+\int_{A_{2} \cup A_{4}}|a-b|^{p} \\
& \geq \int_{A_{1}}\left|a+\alpha^{-1} B^{*} \bar{p}_{\alpha}\right|^{p}+\int_{A_{3}}\left|b+\alpha^{-1} B^{*} \bar{p}_{\alpha}\right|^{p} \\
& \geq \int_{\tilde{A}_{1}}\left|a+\alpha^{-1} B^{*} \bar{p}_{\alpha}\right|^{p}+\int_{\tilde{A}_{3}}\left|b+\alpha^{-1} B^{*} \bar{p}_{\alpha}\right|^{p} \\
& \geq\left(\frac{\sigma}{2}\right)^{p} \operatorname{meas}\left(\left\{x \in A| | B^{*} \bar{p}_{\alpha} \left\lvert\, \leq \frac{\sigma}{2} \alpha\right.\right\}\right) . \tag{33}
\end{align*}
$$

Note for the last step that $\tilde{A}_{1} \cup \tilde{A}_{3}=\left\{x \in A| | B^{*} \bar{p}_{\alpha} \left\lvert\, \leq \frac{\sigma}{2} \alpha\right.\right\}$ due to (31).
From $\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{p}(A)}^{p} \leq C \alpha^{\kappa}$ and (33) we conclude

$$
\operatorname{meas}\left(\left\{x \in A\left|\left|B^{*} \bar{p}_{\alpha}\right| \leq C_{1} \alpha\right\}\right) \leq C_{2} \alpha^{\kappa} .\right.
$$

Since $\kappa>1$ and $\left\|B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right)\right\|_{L^{\infty}(A)} \leq C \alpha^{\kappa}$, we get for some arbitrarily chosen $x \in A$ with $\left|B^{*} \bar{p}_{0}(x)\right| \leq \alpha C_{1} / 2$ the estimate

$$
\left|B^{*} \bar{p}_{\alpha}(x)\right| \leq\left|B^{*} \bar{p}_{0}(x)\right|+\left|B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right)(x)\right| \leq \frac{C_{1}}{2}\left(\alpha+\alpha^{\kappa-\epsilon}\right) \leq C_{1} \alpha
$$

for some sufficiently small $\epsilon=\epsilon\left(C_{1}, \kappa\right)>0$. Consequently, we have

$$
\operatorname{meas}\left(\left\{x \in A\left|\left|B^{*} \bar{p}_{0}\right| \leq \frac{C_{1}}{2} \alpha\right\}\right) \leq C_{2} \alpha^{\kappa} .\right.
$$

Concerning the previous Theorem, let us mention the related result [WW13, Theorem 8]. It has the same implication, but assumes 20) and (21), which imply the prerequisites of Theorem 13 in case of 22 .

For the case $\kappa \leq 1$, it is an open question whether the previous Theorem (and likewise WW13, Theorem 8]) is valid.

Note that the $\sigma$-condition (32) is a strengthening of the condition " $a \leq$ $0 \leq b$ almost everywhere". For $\left(\mathbb{P}_{0}\right)$, the problem we finally want to solve, this weaker assumption can always be met by a simple transformation of the variables.

## 5 Bang-bang solutions

In this section, we introduce at first a second measure condition and show that it implies the same convergence results as in Theorem 11.3, thus might replace the $\bar{p}_{0}$-measure condition 15 from Assumption 7 .

We analyze necessity of the condition to obtain convergence rates and show that for bang-bang solutions fulfilling

$$
\begin{equation*}
\operatorname{meas}\left(\left\{x \in \Omega_{U} \mid B^{*} \bar{p}_{0}(x)=0\right\}\right)=0 \tag{34}
\end{equation*}
$$

both measure conditions coincide.
Note that (34) by (8) implies uniqueness of the solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$.
Definition 14 ( $\bar{p}_{\alpha}$-measure condition). If for the set

$$
\begin{equation*}
I_{\alpha}:=\left\{x \in \Omega_{U} \mid \alpha a<-B^{*} \bar{p}_{\alpha}<\alpha b\right\} \tag{35}
\end{equation*}
$$

the condition

$$
\begin{equation*}
\exists \bar{\alpha}>0 \forall 0<\alpha<\bar{\alpha}: \quad \operatorname{meas}\left(I_{\alpha}\right) \leq C \alpha^{\kappa} \tag{36}
\end{equation*}
$$

holds true (with the convention that $\kappa:=\infty$ if the measure in (36) is zero for all $0<\alpha<\bar{\alpha}$ ), we say that the $\bar{p}_{\alpha}$-measure condition is fulfilled.

The equality in the estimate $(33)$ from the proof of Theorem 13 shows that if the $\bar{p}_{\alpha}$-measure condition holds and we assume the additional condition meas $\left(A_{2} \cup A_{4}\right) \leq C \alpha^{\kappa}$ (with $A_{i}$ as in that proof), we get the convergence rate $\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{p}\left(\Omega_{U}\right)}^{p} \leq C \alpha^{\kappa}$ for each $1 \leq p<\infty$ given (34).

Interestingly, these additional conditions are not needed to obtain convergence in the control, as we will now show.

Theorem 15. If the $\bar{p}_{\alpha}$-measure condition (36) and the $\sigma$-condition (32) are fulfilled, the convergence rates

$$
\begin{equation*}
\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{1}\left(\Omega_{U}\right)} \leq C \alpha^{\kappa} \quad \text { and } \quad\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} \leq C \alpha^{(\kappa+1) / 2} \tag{37}
\end{equation*}
$$

hold true for any solution $\bar{u}_{0}$ of $\left(\mathbb{P}_{0}\right)$.
If in addition $\kappa>1$ and 22 is fulfilled, we have the improved estimate

$$
\begin{equation*}
\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{H} \leq C \alpha^{\kappa} \tag{38}
\end{equation*}
$$

Proof. Let $u \in U_{\text {ad }}$ be arbitrarily chosen. For the active set $I_{\alpha}^{c}$ of $\bar{p}_{\alpha}$, which is the complement of the inactive set $I_{\alpha}$ defined in (35), we have by Lemma 5 , making use of the $\sigma$-condition (32), the estimate

$$
\begin{equation*}
\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)_{I_{\alpha}^{c}}=\int_{I_{\alpha}^{c}}\left|B^{*} \bar{p}_{\alpha}\left\|u-\bar{u}_{\alpha} \mid \geq \sigma \alpha\right\| u-\bar{u}_{\alpha} \|_{L^{1}\left(I_{\alpha}^{c}\right)}\right. \tag{39}
\end{equation*}
$$

Invoking the $\bar{p}_{\alpha}$-measure condition (36), we get on the inactive set itself the estimate

$$
\begin{equation*}
\left|\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)_{I_{\alpha}}\right| \leq C \alpha\left\|u-\bar{u}_{\alpha}\right\|_{L^{1}\left(I_{\alpha}\right)} \leq C C_{a b} \alpha^{\kappa+1} \tag{40}
\end{equation*}
$$

with $C_{a b}=\max \left(\|a\|_{\infty},\|b\|_{\infty}\right)$. Consequently, with $L^{1}:=L^{1}\left(\Omega_{U}\right)$ we get

$$
\begin{align*}
\sigma \alpha\left\|u-\bar{u}_{\alpha}\right\|_{L^{1}}-C \alpha^{\kappa+1} & \stackrel{\sqrt[36]{ }}{\leq} \sigma \alpha\left\|u-\bar{u}_{\alpha}\right\|_{L^{1}}-\sigma \alpha\left\|u-\bar{u}_{\alpha}\right\|_{L^{1}\left(I_{\alpha}\right)} \\
& =\sigma \alpha\left\|u-\bar{u}_{\alpha}\right\|_{L^{1}\left(I_{\alpha}^{c}\right)} \\
& \stackrel{39}{\leq}\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)_{I_{\alpha}^{c}}  \tag{41}\\
& =\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)-\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)_{I_{\alpha}} \\
& \leq\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)+C \alpha^{\kappa+1}
\end{align*}
$$

Rearranging terms, we conclude

$$
\begin{equation*}
\sigma \alpha\left\|u-\bar{u}_{\alpha}\right\|_{L^{1}} \leq\left(B^{*} \bar{p}_{\alpha}, u-\bar{u}_{\alpha}\right)+C \alpha^{\kappa+1} \tag{42}
\end{equation*}
$$

Taking $u:=\bar{u}_{0}$ in the previous equation and adding the necessary condition (5) for $\bar{u}_{0}$ for the special case $u:=\bar{u}_{\alpha}$, i.e.,

$$
\begin{equation*}
\left(-B^{*} \bar{p}_{0}, \bar{u}_{0}-\bar{u}_{\alpha}\right) \geq 0 \tag{43}
\end{equation*}
$$

we get the estimate

$$
\begin{align*}
\sigma \alpha\left\|\bar{u}_{0}-\bar{u}_{\alpha}\right\|_{L^{1}} & \leq\left(B^{*}\left(\bar{p}_{\alpha}-\bar{p}_{0}\right), \bar{u}_{0}-\bar{u}_{\alpha}\right)+C \alpha^{\kappa+1} \\
& =-\left\|\bar{y}_{\alpha}-\bar{y}_{0}\right\|_{I}^{2}+C \alpha^{\kappa+1} \tag{44}
\end{align*}
$$

from which the claim follows.
The improved estimate can be established as in the proof of Theorem 11.

The $\bar{p}_{\alpha}$-measure condition (36) is slightly stronger than what actually is necessary in order to obtain the above convergence rates in the control.

Corollary 16. Let $\bar{u}_{0}$ be a solution of $\left(\mathbb{P}_{0}\right.$ and let us assume that the $\sigma$ condition (32) is valid.

If the convergence rate $\left\|\bar{u}_{\alpha}-\bar{u}_{0}\right\|_{L^{p}\left(\Omega_{U}\right)}^{p} \leq C \alpha^{\kappa}$ is known to hold for some real $p \geq 1$ and some real $\kappa>0$, then the measure condition

$$
\begin{equation*}
\operatorname{meas}\left(\left\{x \in \Omega_{U} \mid \alpha(a+\epsilon) \leq-B^{*} \bar{p}_{\alpha}(x) \leq \alpha(b-\epsilon)\right\} \leq \frac{C}{\epsilon^{p}} \alpha^{\kappa}\right. \tag{45}
\end{equation*}
$$

is fulfilled for each $0<\epsilon<\sigma$.
Proof. This follows from the proof of Theorem 13
If the limit problem is of certain regularity, the $\bar{p}_{\alpha}$-measure condition is not stronger than the $\bar{p}_{0}$-measure condition, and, as we show afterwards, both conditions coincide.

Lemma 17. Let Assumption 7 hold with meas $\left(A^{c}\right)=0$ ( $\bar{p}_{0}$-measure condition is valid a.e. on $\Omega_{U}$ ). Let furthermore $\kappa \geq 1$ and (22) be valid. Then the $\bar{p}_{\alpha}$-measure condition (36) is fulfilled.

Proof. Since the set $I_{\alpha}$ from (35) fulfills $I_{\alpha} \subset\left\{x \in \Omega_{U}| | B^{*} \bar{p}_{\alpha}(x) \mid \leq C \alpha\right\}$ with $C=\max \left(\|a\|_{\infty},\|b\|_{\infty}\right)$, we conclude with (22) and Theorem 11 that if $x \in I_{\alpha}$ and $\kappa \geq 1$, we have

$$
\left|B^{*} \bar{p}_{0}(x)\right| \leq\left|B^{*} \bar{p}_{\alpha}(x)\right|+\left|B^{*}\left(\bar{p}_{0}-\bar{p}_{\alpha}\right)(x)\right| \leq C \alpha .
$$

With the $\bar{p}_{0}$-measure condition (15) we obtain the estimate

$$
\operatorname{meas}\left(I_{\alpha}\right) \leq \operatorname{meas}\left(\left\{x \in \Omega_{U}| | B^{*} \bar{p}_{0}(x) \mid \leq C \alpha\right\}\right) \leq C \alpha^{\kappa},
$$

which concludes the proof.
Corollary 18. Let a bang-bang solution be given which fulfills (34). In the case of $\kappa>1$, (22), and the $\sigma$-condition (32), both measure conditions are equivalent.

Proof. One direction of the claim, namely " $\bar{p}_{0}$-m.c. $\Rightarrow \bar{p}_{\alpha}$-m.c.", has already been shown in Lemma 17

For the other direction, we know from Theorem 15 that the convergence rates (37) and (38) hold, which by (22) and Theorem 13 imply the $\bar{p}_{0}$-measure condition.

Let us now consider the situation that the optimal adjoint state fulfills the regularity

$$
\begin{equation*}
\exists C>0:\left\|\partial_{x} B^{*} \bar{p}_{\alpha}\right\|_{L^{\infty}\left(\Omega_{U}\right)} \leq C \tag{46}
\end{equation*}
$$

with a constant $C>0$ independent of $\alpha$ and $\partial_{x}$ denoting the weak differential operator. This bound is valid, e.g., for Example 22 (located controls), see [Dan16, p. 30] for a proof.

Furthermore, we assume the Sobolev regularity $a, b \in W^{1, \infty}\left(\Omega_{U}\right)$ for the control bounds.

Since the orthogonal projection possesses for $f \in W^{1, \infty}\left(\Omega_{U}\right)$ the property

$$
\left\|\partial_{x} P_{U_{\mathrm{ad}}}(f)\right\|_{L^{\infty}\left(\Omega_{U}\right)} \leq\left\|\partial_{x} f\right\|_{L^{\infty}\left(\Omega_{U}\right)}+\left\|\partial_{x} a\right\|_{L^{\infty}\left(\Omega_{U}\right)}+\left\|\partial_{x} b\right\|_{L^{\infty}\left(\Omega_{U}\right)}
$$

see, e.g. Zie89, Corollary 2.1.8], we obtain with the projection formula (6) and the constant

$$
\begin{equation*}
C_{a b}:=\left\|\partial_{x} a\right\|_{L^{\infty}\left(\Omega_{U}\right)}+\left\|\partial_{x} b\right\|_{L^{\infty}\left(\Omega_{U}\right)} \tag{47}
\end{equation*}
$$

a bound on the derivative of the optimal control, namely

$$
\begin{equation*}
\left\|\partial_{x} \bar{u}_{\alpha}\right\|_{L^{\infty}\left(\Omega_{U}\right)} \leq \frac{1}{\alpha}\left\|\partial_{x} B^{*} \bar{p}_{\alpha}\right\|_{L^{\infty}\left(\Omega_{U}\right)}+C_{a b} \stackrel{\sqrt[46]{\leq}}{\leq} C \frac{1}{\alpha} \tag{48}
\end{equation*}
$$

if $\alpha>0$ is sufficiently small.
If the $\bar{p}_{\alpha}$-measure condition (36) is valid, this decay of smoothness in dependence of $\alpha$ can be relaxed in weaker norms, as the following Lemma shows.

Lemma 19 (Smoothness decay in the derivative). Let the $\bar{p}_{\alpha}$-measure condition (36) be fulfilled and the regularity condition (46) be valid as well as a, $b \in W^{1, \infty}\left(\Omega_{U}\right)$. Then there holds with the constant $C_{a b}$ defined in (47) for sufficiently small $\alpha>0$ and each $p$ with $1 \leq p<\infty$ the inequality

$$
\begin{equation*}
\left\|\partial_{x} \bar{u}_{\alpha}\right\|_{L^{p}\left(\Omega_{U}\right)} \leq C \max \left(C_{a b}, \alpha^{\kappa / p-1}\right) \tag{49}
\end{equation*}
$$

with a constant $C>0$ independent of $\alpha$.
Note that $C_{a b}=0$ in the case of constant control bounds $a$ and $b$.
Proof. We invoke (36) and (48) to get the estimate

$$
\begin{aligned}
\left\|\partial_{x} \bar{u}_{\alpha}\right\|_{L^{p}\left(\Omega_{U}\right)}^{p} & \leq \operatorname{meas}\left(I_{\alpha}\right)\left\|\partial_{x} \bar{u}_{\alpha}\right\|_{L^{\infty}\left(\Omega_{U}\right)}^{p}+\operatorname{meas}\left(\Omega_{U}\right) C_{a b}^{p} \\
& \leq C \max \left(\alpha^{\kappa-p}, C_{a b}^{p}\right)
\end{aligned}
$$

with the set $I_{\alpha}$ from (35).
Let us now briefly sketch an application of the previous lemma in the numerical analysis of a suitable finite element discretization of Example 2, 2 (located controls), which has been analyzed in detail recently in [Dan16], see also [DH], founded on a novel discretization scheme proposed in [DHV15].

Discretizing the regularized problem $\left(\mathbb{P}_{\alpha}\right)$ in space and time, one ends up with a problem $\left(\mathbb{P}_{k h}\right)$ depending on the regularization parameter $\alpha>0$ and the grid sizes $k$ and $h$ for the time and space grid, respectively, related to the finite element discretization. This discretization is used later in the numerics section, where it is described in more detail.

One can show that this problem has again a unique solution $\bar{u}_{\alpha, k h}$ and that the error fulfills $\left(\bar{u}_{0}\right.$ the unique solution of $\left(\mathbb{P}_{0}\right)$ )

$$
\begin{align*}
&\left\|\bar{u}_{0}-\bar{u}_{\alpha, k h}\right\|_{U}^{2}+\left\|\bar{u}_{0}-\bar{u}_{\alpha, k h}\right\|_{L^{1}\left(\Omega_{U}\right)} \\
& \quad \leq C\left(\alpha+h^{2}+k^{2} \max \left(1, C_{a b}, \alpha^{\kappa / 2-1}\right)\right)^{\kappa} \tag{50}
\end{align*}
$$

with $C>0$ independent of $\alpha, k$, and $h$, see [Dan16, Theorem 77].
Here, Lemma 19 is used in the proof to get the factor $\alpha^{\kappa / 2-1}$. This factor is obviously better (if $\alpha \rightarrow 0$ ) than $\alpha^{-1}$ which one would get using estimate (48) only.

Using Lemma 19 one can also show error estimates for other quantities, e.g., an adjoint state error.

## 6 A numerical example

To validate numerically the new convergence rates for the regularization error given in Theorem 113 , we construct in the next subsection a known (unique) solution $\bar{u}_{0}$ together with its problem data.

The problem data (but not the solution $\bar{u}_{0}$ ) is used to numerically solve the regularized problem $\left(\mathbb{P}_{\alpha}\right)$. We describe in the next but one subsection how this approximation is computed.

In the last subsection, we present and discuss the numerical results.

### 6.1 A limit problem with given unique solution

To build a concrete instance of the limit problem $\mathbb{P}_{0}$, we consider the situation of Example 22 (heat equation with located controls).

We modify the heat equation (3) in that we take into account a fixed initial value $y(0)=y_{0}$, which can be interpreted as a modification of $z$ in $\left(\mathbb{P}_{0}\right)$, more precisely $z=y_{d}-S\left(0, y_{0}\right)$, where $S(f, g)$ denotes the solution of the heat equation

$$
\begin{align*}
& \partial_{t} y-\Delta y=f \text { in } I \times \Omega \\
& y=0  \tag{51}\\
& \text { in } I \times \partial \Omega \\
& y(0)=g \\
& \text { in } \Omega
\end{align*}
$$

We consider a given exact solution of the limit problem $\mathbb{P}_{0}$ which we denote by $(\bar{u}, \bar{y}, \bar{p})$, thus omitting the index for $\alpha=0$. Please take care of the fact that $\bar{y}=S\left(B \bar{u}, y_{0}\right)$ in what follows, which is not $T \bar{u}$. In the numerical procedure described below, we of course only make use of the problem data. The solution triple $(\bar{u}, \bar{y}, \bar{p})$ is only used to evaluate the error norms.

To understand the construction of the test example, let us elaborate in more detail the weak formulations of the solution operator $S$ and its adjoint for Example 2, 2. It also motivates the discretization schemes stated below.

With the space

$$
W(I):=\left\{v \in L^{2}\left(I, H_{0}^{1}(\Omega)\right) \mid v_{t} \in L^{2}\left(I, H^{-1}(\Omega)\right)\right\},
$$

the operator

$$
\begin{equation*}
S: L^{2}\left(I, H^{-1}(\Omega)\right) \times L^{2}(\Omega) \rightarrow W(I), \quad(f, g) \mapsto y:=S(f, g), \tag{52}
\end{equation*}
$$

denotes the weak solution operator associated with the heat equation (51), which is defined as follows. For $(f, g) \in L^{2}\left(I, H^{-1}(\Omega)\right) \times L^{2}(\Omega)$ the function $y \in W(I)$ with $\langle\cdot, \cdot\rangle:=\langle\cdot, \cdot\rangle_{H^{-1}(\Omega) H_{0}^{1}(\Omega)}$ satisfies the two equations

$$
\begin{align*}
y(0) & =g  \tag{53a}\\
\int_{0}^{T}\left\langle\partial_{t} y(t), v(t)\right\rangle+a(y(t), v(t)) d t & =\int_{0}^{T}\langle f(t), v(t)\rangle d t  \tag{53b}\\
& \forall v \in L^{2}\left(I, H_{0}^{1}(\Omega)\right) .
\end{align*}
$$

Note that by the embedding $W(I) \hookrightarrow C\left([0, T], L^{2}(\Omega)\right)$, see, e.g., Eva98, Theorem 5.9.3], the first relation is meaningful.
In the preceding equation, the bilinear form $a: H^{1}(\Omega) \times H^{1}(\Omega) \rightarrow \mathbb{R}$ is given by

$$
a(f, g):=\int_{\Omega} \nabla f(x) \nabla g(x) d x .
$$

The equations (53) yield an operator $S$ in the sense of (52):
Lemma 20 (Properties of the solution operator $S$ ).

1. For every $(f, g) \in L^{2}\left(I, H^{-1}(\Omega)\right) \times L^{2}(\Omega)$ a unique state $y \in W(I)$ satisfying (53) exists. Thus the operator $S$ from (52) exists. Furthermore the state fulfills

$$
\begin{equation*}
\|y\|_{W(I)} \leq C\left(\|f\|_{L^{2}\left(I, H^{-1}(\Omega)\right)}+\|g\|_{L^{2}(\Omega)}\right) . \tag{54}
\end{equation*}
$$

2. Consider the bilinear form $A: W(I) \times W(I) \rightarrow \mathbb{R}$ given by

$$
\begin{equation*}
A(y, v):=\int_{0}^{T}-\left\langle v_{t}, y\right\rangle+a(y, v) d t+\langle y(T), v(T)\rangle \tag{55}
\end{equation*}
$$

with $\langle\cdot, \cdot\rangle:=\langle\cdot, \cdot\rangle_{H^{-1}(\Omega) H_{0}^{1}(\Omega)}$. Then for $y \in W(I)$, equation (53) is equivalent to

$$
\begin{equation*}
A(y, v)=\int_{0}^{T}\langle f, v\rangle d t+(g, v(0))_{L^{2}(\Omega)} \quad \forall v \in W(I) . \tag{56}
\end{equation*}
$$

Furthermore, $y$ is the only function in $W(I)$ fulfiling equation (56).

Proof. This can be derived using standard results, see Dan16, Lemma 1].
One can show with standard results, see, e.g., Dan16, Lemma 2], that the optimal adjoint state $\bar{p}_{\alpha} \in W(I)$ is the unique weak solution defined and uniquely determined by the adjoint equation

$$
\begin{equation*}
A(v, \bar{p})=\int_{0}^{T}\left\langle\bar{y}-y_{d}, v\right\rangle_{H^{-1}(\Omega) H_{0}^{1}(\Omega)} d t \quad \forall v \in W(I) . \tag{57}
\end{equation*}
$$

This equation corresponds to the backward heat equation

$$
\begin{align*}
-\partial_{t} \bar{p}-\Delta \bar{p} & =\bar{y}-y_{d} & & \text { in } I \times \Omega, \\
\bar{p} & =0 & & \text { on } I \times \partial \Omega,  \tag{58}\\
\bar{p}(T) & =0 & & \text { on } \Omega .
\end{align*}
$$

Let us now construct the test example.
We make use of the fact that instead of the linear control operator $B$, given by (4), we can also use an affine linear control operator

$$
\begin{equation*}
\tilde{B}: U \rightarrow L^{2}\left(I, H^{-1}(\Omega)\right), \quad u \mapsto g_{0}+B u \tag{59}
\end{equation*}
$$

where $g_{0}$ is a fixed function of certain regularity since $g_{0}$ can be interpreted as a modification of $z$.

With a space-time domain $\Omega \times I:=(0,1)^{2} \times(0,0.5)$, thus $T_{e}=0.5$, we choose the optimal control to be the lower bound of the admissible set, i.e., $\bar{u}:=a_{1}:=-0.2$. For the upper bound we set $b_{1}:=0.2$.

With the function

$$
g_{1}\left(x_{1}, x_{2}\right):=\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right)
$$

the optimal adjoint state is chosen as

$$
\bar{p}\left(t, x_{1}, x_{2}\right):=\left(T_{e}-t\right)^{1 / \kappa} g_{1}\left(x_{1}, x_{2}\right)
$$

for some fixed $\kappa>0$ specified below.
With the constant $a:=2$, we take for the optimal state

$$
\bar{y}\left(t, x_{1}, x_{2}\right):=\cos \left(\frac{t}{T_{e}} 2 \pi a\right) \cdot g_{1}\left(x_{1}, x_{2}\right)
$$

from which we derive by (58)

$$
-\partial_{t} \bar{p}-\Delta \bar{p}=\frac{1}{\kappa}\left(T_{e}-t\right)^{1 / \kappa-1} g_{1}-\left(T_{e}-t\right)^{1 / \kappa} \Delta g_{1}=\bar{y}-y_{d}
$$

which gives $y_{d}$.
We also get the initial value of the optimal state $\bar{y}$ :

$$
y_{0}\left(x_{1}, x_{2}\right)=\bar{y}\left(0, x_{1}, x_{2}\right)=g_{1}\left(x_{1}, x_{2}\right) .
$$

Finally we obtain

$$
\begin{aligned}
g_{0} & =\partial_{t} \bar{y}-\Delta \bar{y}-B \bar{u} \\
& =g_{1} 2 \pi\left(-\frac{a}{T_{e}} \sin \left(\frac{t}{T_{e}} 2 \pi a\right)+\pi \cos \left(\frac{t}{T_{e}} 2 \pi a\right)\right)-g_{1} \cdot \bar{u}
\end{aligned}
$$

This example fulfills the measure condition (15) of Assumption 7 with $\operatorname{meas}\left(A^{c}\right)=0$ and exponent $\kappa$ from the definition of $\bar{p}$.

### 6.2 Discretization of the regularized problem

We now describe the discretized regularized optimal control problem $\left(\mathbb{P}_{k h}\right)$ which is solved as an approximation for $\left(\mathbb{\mathbb { P } _ { \alpha }}\right)$.

Consider a partition $0=t_{0}<t_{1}<\cdots<t_{M}=T_{e}$ of the time interval $\bar{I}=\left[0, T_{e}\right]$. With $I_{m}=\left[t_{m-1}, t_{m}\right)$ we have $\left[0, T_{e}\right)=\bigcup_{m=1}^{M} I_{m}$. Furthermore, let $t_{m}^{*}=\frac{t_{m-1}+t_{m}}{2}$ for $m=1, \ldots, M$ denote the interval midpoints. By $0=: t_{0}^{*}<t_{1}^{*}<\cdots<t_{M}^{*}<t_{M+1}^{*}:=T_{e}$ we get a second partition of $\bar{I}$, the so-called dual partition, namely $\left[0, T_{e}\right)=\bigcup_{m=1}^{M+1} I_{m}^{*}$, with $I_{m}^{*}=\left[t_{m-1}^{*}, t_{m}^{*}\right)$. The grid width of the first mentioned (primal) partition is given by the parameters $k_{m}=t_{m}-t_{m-1}$ and $k=\max _{1 \leq m \leq M} k_{m}$. Here and in what follows we assume $k<1$. We also denote by $k$ (in a slight abuse of notation) the grid itself.

On these partitions of the time interval, we define the Ansatz and test spaces of the Petrov-Galerkin schemes. These schemes will replace the continuous-in-time weak formulations of the state equation and the adjoint equation, i.e., (56) and (57), respectively. To this end, we define at first for an arbitrary Banach space $X$ the semidiscrete function spaces

$$
\begin{align*}
P_{k}(X) & :=\left\{v \in C([0, T], X)|v|_{I_{m}} \in \mathcal{P}_{1}\left(I_{m}, X\right)\right\} \hookrightarrow H^{1}(I, X)  \tag{60a}\\
P_{k}^{*}(X) & :=\left\{v \in C([0, T], X)|v|_{I_{m}^{*}} \in \mathcal{P}_{1}\left(I_{m}^{*}, X\right)\right\} \hookrightarrow H^{1}(I, X) \tag{60b}
\end{align*}
$$

and

$$
\begin{equation*}
Y_{k}(X):=\left\{v:[0, T] \rightarrow X^{*}|v|_{I_{m}} \in \mathcal{P}_{0}\left(I_{m}, X\right)\right\} \tag{60c}
\end{equation*}
$$

Here, $\mathcal{P}_{i}(J, X), J \subset \bar{I}, i \in\{0,1\}$, is the set of polynomial functions in time with degree of at most $i$ on the interval $J$ with values in $X$.

Note that we can extend the bilinear form $A$ of (55) in its first argument to $W(I) \cup Y_{k}\left(H_{0}^{1}(\Omega)\right)$, thus consider the operator

$$
\begin{equation*}
A: W(I) \cup Y_{k}\left(H_{0}^{1}(\Omega)\right) \times W(I) \rightarrow \mathbb{R}, \quad A \text { given by } 55 \tag{61}
\end{equation*}
$$

Using continuous piecewise linear functions in space, we can formulate fully discretized variants of the state and adjoint equation.

We consider a regular triangulation $\mathcal{T}_{h}$ of $\Omega$ with mesh size $h:=\max _{T \in \mathcal{T}_{h}}$ $\operatorname{diam}(T)$, see, e.g., BS08, Definition (4.4.13)], and $N=N(h)$ triangles. We
assume that $h<1$. We also denote by $h$ (in a slight abuse of notation) the grid itself.

With the space

$$
\begin{equation*}
X_{h}:=\left\{\phi_{h} \in C^{0}(\bar{\Omega})\left|\phi_{h}\right|_{T} \in \mathcal{P}_{1}(T, \mathbb{R}) \quad \forall T \in \mathcal{T}_{h}\right\} \tag{62}
\end{equation*}
$$

we define $X_{h 0}:=X_{h} \cap H_{0}^{1}(\Omega)$ to discretize $H_{0}^{1}(\Omega)$.
We fix fully discrete ansatz and test spaces, derived from their semidiscrete counterparts from 60), namely

$$
\begin{equation*}
P_{k h}:=P_{k}\left(X_{h 0}\right), \quad P_{k h}^{*}:=P_{k h}^{*}\left(X_{h 0}\right), \quad \text { and } Y_{k h}:=Y_{k}\left(X_{h 0}\right) . \tag{63}
\end{equation*}
$$

With these spaces, we introduce fully discrete state and adjoint equations as follows.

Definition 21 (Fully discrete adjoint equation). For $h \in L^{2}\left(I, H^{-1}(\Omega)\right)$ find $p_{k h} \in P_{k h}$ such that

$$
\begin{equation*}
A\left(\tilde{y}, p_{k h}\right)=\int_{0}^{T}\langle h(t), \tilde{y}(t)\rangle_{H^{-1}(\Omega) H_{0}^{1}(\Omega)} d t \quad \forall \tilde{y} \in Y_{k h} \tag{64}
\end{equation*}
$$

Definition 22 (Fully discrete state equation). For $(f, g) \in L^{2}\left(I, H^{-1}(\Omega)\right) \times$ $L^{2}(\Omega)$ find $y_{k h} \in Y_{k h}$, such that

$$
\begin{equation*}
A\left(y_{k h}, v_{k h}\right)=\int_{0}^{T}\left\langle f(t), v_{k h}(t)\right\rangle_{H^{-1}(\Omega) H_{0}^{1}(\Omega)} d t+\left(g, v_{k h}(0)\right) \quad \forall v_{k h} \in P_{k h} \tag{65}
\end{equation*}
$$

Existence and uniqueness of these two schemes follow as in the semidiscrete case discussed in DHV15] or Dan16, section 2.1.2]. For error estimates of the two schemes, we refer again to [Dan16] or [DH].

We are now able to introduce the discretized optimal control problem which reads

$$
\begin{align*}
& \min _{y_{k h} \in Y_{k h}, u \in U_{\mathrm{ad}}} J\left(y_{k h}, u\right)=\min \frac{1}{2}\left\|y_{k h}-y_{d}\right\|_{I}^{2}+\frac{\alpha}{2}\|u\|_{U}^{2},  \tag{kh}\\
& \text { s.t. } y_{k h}=S_{k h}\left(B u, y_{0}\right)
\end{align*}
$$

where $\alpha, B, y_{0}, y_{d}$, and $U_{\mathrm{ad}}$ are chosen as for $\left(\mathbb{\mathbb { P } _ { \alpha }}\right)$ and $S_{k h}$ is the solution operator associated to the fully discrete state equation (65). Recall that the space $Y_{k h}$ was introduced in (63).

For every $\alpha>0$, this problem admits a unique solution triple ( $\bar{u}_{k h}, \bar{y}_{k h}$, $\left.\bar{p}_{k h}\right)$ where $\bar{y}_{k h}=S_{k h}\left(B \bar{u}_{k h}, y_{0}\right)$ and $\bar{p}_{k h}$ denotes the discrete adjoint state which is the solution of the fully discrete adjoint equation (64) with righthand side $h:=\bar{y}_{k h}-y_{d}$. The first order necessary and sufficient optimality condition for problem $\left(\widetilde{\mathbb{P}_{k h}}\right)$ is given by

$$
\begin{equation*}
\bar{u}_{k h} \in U_{\mathrm{ad}}, \quad\left(\alpha \bar{u}_{k h}+B^{*} \bar{p}_{k h}, u-\bar{u}_{k h}\right)_{U} \geq 0 \quad \forall u \in U_{\mathrm{ad}}, \tag{66}
\end{equation*}
$$

which can be rewritten as

$$
\begin{equation*}
\bar{u}_{k h}=P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha} B^{*} \bar{p}_{k h}\right) . \tag{67}
\end{equation*}
$$

The above mentioned facts can be proven in the same way as for the continuous problem $\left(\mathbb{P}_{\alpha}\right)$.

Note that the control space $U$ is not discretized in the formulation $\sqrt[\mathbb{P}_{k h}]{ }$. In the numerical treatment, the relation (67) is instead exploited to get a discrete control. This approach is called Variational Discretization and was introduced in Hin05], see also Hin+09, Chapter 3.2.5] for further details.

### 6.3 Numerical results

We solve numerically the regularized discretized problem $\left(\mathbb{P}_{k h}\right)$ as an approximation of the limit problem $\left(\mathbb{P}_{0}\right)$ in the situation of Example 2.2 with data given in the last but one subsection. Recall that we denote by $\bar{u}_{k h}$ the former, by $\bar{u}$ the latter.

We investigate the behavior of the error $\left\|\bar{u}_{k h}-\bar{u}\right\|$ if $\alpha \rightarrow 0$ for fixed small discretization parameters $k$ and $h$ and different values of the parameter $\kappa$ from the measure condition (15).

To solve $\left(\mathbb{P}_{k h}\right)$, a fixed-point iteration on the equation (67) is performed:
Each fixed-point iteration is initialized with the starting value $u_{k h}^{(0)}:=a_{1}$ which is the lower bound of the admissible set. As a stopping criterion for the fixed-point iteration, we require for the discrete adjoint states belonging to the current and the last iterate that

$$
\left\|B^{*}\left(p_{k h}^{(i)}-p_{k h}^{(i-1)}\right)\right\|_{L^{\infty}(\Omega \times I)}<t_{0}
$$

where $t_{0}:=10^{-5}$ is a prescribed threshold.
We end up with what we denote by $u_{k h}$ in the tables below: An approximation of $\bar{u}_{k h}$.

The idea is that if $\alpha$ is not to small in comparison to $k$ and $h$, we expect to have $\left\|u_{k h}-\bar{u}\right\| \approx\left\|\bar{u}_{\alpha}-\bar{u}\right\|$, which means that the influence of the discretization is negligible in relation to the influence of the regularization.

Here, we report only on the errors in the optimal control since we observed no or only poor convergence in the error of the optimal state and adjoint state, respectively. This might be due to the fact that the influence of the space- and time-discretization error is much larger than that of the regularization error. This phenomenon was also observed for elliptic problems, compare WW11a.

We consider a fixed fine space-time mesh with $\mathrm{Nh}=\left(2^{5}+1\right)^{2}$ nodes in space and $\mathrm{Nk}=\left(2^{11}+1\right)$ nodes in time. The regularization parameter $\alpha=2^{-\ell}$ is considered for $\ell=1,2,3,4,5,6$.

The problem is solved for different values of $\kappa$, namely $\kappa=0.3,0.5,1$, and 2 .

Let us remark that the convergence of the fixed-point iteration for our example does not depend on the starting value.

As one can see from the experimental order of convergence (EOC) in the Tables 2, 3, 4, and 5, the new convergence rates of Theorem [1],3, more precisely $\sqrt{19)}$ and 20 , can be observed numerically. It seems that they cannot be improved any further. In Figure 1, the convergence of the computed optimal control to the limit control is depicted if $\alpha \rightarrow 0$.

|  | $\left\\|\bar{u}-u_{k h}\right\\|$ | $\left\\|\bar{u}-u_{k h}\right\\|$ | EOC | EOC |
| :---: | :---: | :---: | :---: | :---: |
| $\ell$ | $L^{1}(I, \mathbb{R})$ | $L^{2}(I, \mathbb{R})$ | $L^{1}$ | $L^{2}$ |
| 1 | 0.09417668 | 0.13354708 | $/$ | $/$ |
| 2 | 0.08837777 | 0.12648809 | 0.09 | 0.08 |
| 3 | 0.07681662 | 0.11533688 | 0.20 | 0.13 |
| 4 | 0.06212895 | 0.10353644 | 0.31 | 0.16 |
| 5 | 0.05008158 | 0.09264117 | 0.31 | 0.16 |
| 6 | 0.04011694 | 0.08237596 | 0.32 | 0.17 |

Table 2: Errors and EOC in the control ( $\kappa=0.3, \alpha \rightarrow 0, h, k$ fixed).

|  | $\left\\|\bar{u}-u_{k h}\right\\|$ | $\left\\|\bar{u}-u_{k h}\right\\|$ | EOC | EOC |
| :---: | :---: | :---: | :---: | :---: |
| $\ell$ | $L^{1}(I, \mathbb{R})$ | $L^{2}(I, \mathbb{R})$ | $L^{1}$ | $L^{2}$ |
| 1 | 0.07912861 | 0.11494852 | $/$ | $/$ |
| 2 | 0.05957289 | 0.09753159 | 0.41 | 0.24 |
| 3 | 0.04204449 | 0.08187630 | 0.50 | 0.25 |
| 4 | 0.02963509 | 0.06865675 | 0.50 | 0.25 |
| 5 | 0.02084162 | 0.05749818 | 0.51 | 0.26 |
| 6 | 0.01463170 | 0.04811089 | 0.51 | 0.26 |

Table 3: Errors and EOC in the control ( $\kappa=0.5, \alpha \rightarrow 0, h, k$ fixed $)$.

## References

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|  | $\left\\|\bar{u}-u_{k h}\right\\|$ | $\left\\|\bar{u}-u_{k h}\right\\|$ | EOC | EOC |
| :---: | :---: | :---: | :---: | :---: |
| $\ell$ | $L^{1}(I, \mathbb{R})$ | $L^{2}(I, \mathbb{R})$ | $L^{1}$ | $L^{2}$ |
| 1 | 0.04006495 | 0.07304858 | $/$ | $/$ |
| 2 | 0.02000722 | 0.05160925 | 1.00 | 0.50 |
| 3 | 0.00998774 | 0.03646496 | 1.00 | 0.50 |
| 4 | 0.00498724 | 0.02576440 | 1.00 | 0.50 |
| 5 | 0.00249053 | 0.01820019 | 1.00 | 0.50 |
| 6 | 0.00123906 | 0.01282180 | 1.01 | 0.51 |

Table 4: Errors and EOC in the control ( $\kappa=1, \alpha \rightarrow 0, h, k$ fixed $)$.


Figure 1: Optimal control $\bar{u}$ (solid) and computed counterpart $u_{k h}$ (dashed) over time after level $\ell(\kappa=1, \alpha \rightarrow 0, h, k$ fixed $)$.
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|  | $\left\\|\bar{u}-u_{k h}\right\\|$ | $\left\\|\bar{u}-u_{k h}\right\\|$ | EOC | EOC |
| :---: | :---: | :---: | :---: | :---: |
| $\ell$ | $L^{1}(I, \mathbb{R})$ | $L^{2}(I, \mathbb{R})$ | $L^{1}$ | $L^{2}$ |
| 1 | 0.01081546 | 0.03305084 | $/$ | $/$ |
| 2 | 0.00279478 | 0.01690248 | 1.95 | 0.97 |
| 3 | 0.00074507 | 0.00878066 | 1.91 | 0.94 |
| 4 | 0.00020543 | 0.00463711 | 1.86 | 0.92 |
| 5 | 0.00005823 | 0.00246523 | 1.82 | 0.91 |
| 6 | 0.00001564 | 0.00125068 | 1.90 | 0.98 |

Table 5: Errors and EOC in the control ( $\kappa=2, \alpha \rightarrow 0, h, k$ fixed $)$.
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