Convergence of the Augmented Decomposition Algorithm

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Abstract We study the convergence of the Augmented Decomposition Algorithm (ADA) proposed in [32] for solving multi-block separable convex minimization problems subject to linear constraints. We show that the global convergence rate of the exact ADA is $o(1/\nu)$ under the assumption that there exists a saddle point. We consider the inexact Augmented Decomposition Algorithm (iADA) and establish global and local convergence results under some mild assumptions, by providing a stability result for the maximal monotone operator \mathcal{T} associated with the perturbation from both primal and dual perspectives. This result implies the local linear convergence of the inexact ADA for many applications such as the *lasso*, total variation reconstruction, exchange problem and many other problems from statistics, machine learning and engineering with ℓ_1 regularization.

Keywords Separable convex minimization \cdot convergence rate \cdot augmented decomposition algorithm \cdot distributed computing

1 Introduction

Consider the following convex optimization problem of minimizing the sum of K separable, potentially nonsmooth convex functions subject to the linear

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constraints

$$\min_{x} \quad f(x) = f_{1}(x_{1}) + \dots + f_{K}(x_{K})
s.t. \quad Ex = E_{1}x_{1} + \dots + E_{K}x_{K} = q,
\quad x_{k} \in X_{k}, \quad k = 1, 2, \dots, K,$$
(1.1)

where every f_k is a closed proper convex function (possibly nonsmooth) and each X_k is a closed convex set in \mathbb{R}^{n_k} . Let $x = (x_1, \ldots, x_K) \in \mathbb{R}^n$ be a partition of the variable x and $X = X_1 \times \cdots \times X_K \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_K} = \mathbb{R}^n$ be the domain of x. For the linear constraint, $E = (E_1, \ldots, E_K) \in \mathbb{R}^{m \times n}$ is a partition of the matrix E consistent with the partition of x and $q \in \mathbb{R}^m$ is a column vector. A linear inequality constraint of the form $Ex \leq q$ can be easily transformed to the equality case by introducing a slack variable $x_{K+1} \geq 0$.

Optimization problems in the form of (1.1) arise in many application areas such as signal processing, statistics and machine learning. [26] summarizes a list of applications arising from many areas when more than two blocks are involved ($K \ge 3$).

Many decomposition algorithms have been proposed to solve the above optimization problem; see [4,5,7,12,27,34,35] and references therein. Among them, the ADMM method is perhaps the most popular approach to solve the decomposition problem due to its suitable parallel implementation and outstanding computational performance. When K = 2, the convergence of the ADMM was well studied in the framework of Douglas-Rachford splitting method [12]. The paper [10] proved the linear convergence of the ADMM when at least one of $f_i(\cdot)$ is strongly convex and E satisfies some additional assumptions. For the $K \geq 3$ case, it was shown in [15] that the global convergence is guaranteed if all objective functions f_k are strongly convex. However, for general convex objective functions, it is acknowledged that the direct extension of the original ADMM may diverge [6]. Therefore, most recent researches have been focused on either analyzing problems with additional assumptions or showing the convergence results for variants of the ADMM; see [21,36].

As an alternative to the ADMM algorithm for multi-block convex optimization problems, a new primal-dual algorithm called the *augmented decomposition algorithm* (ADA) was introduced in [32]. This method is closely related to the decomposition algorithm based on the partial inverses proposed in [34] but is derived from the proximal saddle point algorithm (PSPA) which is associated with a special primal-dual saddle function. It was shown in [32] that the algorithm is guaranteed to converge on the basis of convergence results of the proximal point algorithm (PPA) in [30]. What is more exciting is that the calculation of each iteration in PSPA can be carried out in parallel and its parallel implementation leads to the ADA.

Although the global convergence result for the ADA has been well studied under a general condition, the convergence rate result remained unknown. In the first part of this paper, we focus on the convergence analysis of the ADA applied to problem (1.1). For that, we first provide a detailed proof for its convergence. Then, we show the $O(1/\nu)$ convergence rate in an ergodic sense. Finally, we improve the convergence result from $O(1/\nu)$ to $o(1/\nu)$ in a nonergodic sense. These ideas are inspired by recent works on the ADMM and variants of the proximal method of multiplier [9,18,33].

Then, we consider the inexact ADA (iADA) in the second part. We first establish the global convergence result under certain approximation criteria. Then, under some mild assumptions on the function f_k and the structure of feasible set X_k , we show the local linear convergence of the iADA. This work is invoked by recent convergence rate results for the ADMM algorithm in [10, 21]. However, our proof is different from them in which we show the stability of a maximal monotone operator associated with the saddle function for a variant of (1.1). Denote the Lagrangian function by L for (1.1):

$$L(x,y) = \begin{cases} f(x) + \langle Ex - q, y \rangle, & \forall (x,y) \in X \times \mathbb{R}^m, \\ \infty, & \forall x \notin X. \end{cases}$$
(1.2)

The corresponding maximal monotone operator \mathcal{T}_L [30] is defined by

$$\mathcal{T}_L(x,y) = \{(u,v) | (u,-v) \in \partial L(x,y)\}$$
(1.3)

where $\partial L(x, y)$ denotes the subgradient of the convex-concave function L. The inverse of \mathcal{T}_L is given by

$$\mathcal{T}_{L}^{-1}(u,v) = \{(x,y) | (u,-v) \in \partial L(x,y) \}.$$
(1.4)

A solution to $(0,0) \in \mathcal{T}_L(x,y)$ is a saddle point of L. Classical convergence rate results for PPA [31] rely on the assumption that \mathcal{T}_L^{-1} is Lipschitz continuous at (0,0). This result was extended in [25] for situations in which $\mathcal{T}_L^{-1}(0,0)$ is not a singleton and the following holds:

$$\exists a > 0, \quad \exists \delta > 0: \quad \forall w \in \mathcal{B}((0,0),\delta), \quad \forall z \in \mathcal{T}_L^{-1}w, \quad dist(z,\mathcal{T}_L^{-1}(0,0)) \le a||w||. \quad (1.5)$$

It has been pointed out in many works that understanding the Lipschitzian behavior of \mathcal{T}_L^{-1} at the origin is crucial to the study of the local convergence results for algorithms in the PPA framework; see [8,14,22,23]. For instance, [22] showed the *metric subregularity* defined in [11] of \mathcal{T}_L which is closely related to (1.5) under the so-called second order sufficient condition. However, this result inherently requires the solution uniqueness for problem (1.1). Compared with those assumptions, our assumptions in this part mainly rely on the polyhedral property of the feasible set X and the optimal solution set for (1.1) needs not to be a singleton. Our proof is based on Robinson's celebrated work on the error bound result for polyhedral multifunctions [29] and uses some ideas in the analysis for the satisfaction of a certain error bound condition in [21,24].

Organization The remainder of this paper is organized as follows. Section 2 first summarizes the basic idea of the proximal saddle point algorithm and its implementation, the ADA. Then, we show the convergence result for the ADA and compare it with the ADMM. In Section 3, we introduce the iADA and make some basic assumptions on the problem (1.1) for further discussion.

(2.1)

Section 4 studies the stability results of the maximal monotone operator $\mathcal{T}_{\bar{L}}$. Section 5 establishes the global convergence and local linear convergence rate results of the iADA. Finally, some numerical examples are presented in Section 6 to demonstrate the performance the ADA and iADA.

Notation We use $\langle \cdot, \cdot \rangle$ and $||\cdot||$ to denote the standard inner product and \mathcal{L}_2 -norm in the Euclidean space respectively. For any positive definite matrix $G \in S_{++}^n$ and $x, y \in \mathbb{R}^n$, the inner product $\langle x, y \rangle_G$ is defined by $x^T G y$ and its induced norm is denoted by $|| \cdot ||_G$. For $1 \le q \le \infty$, $|| \cdot ||_q$ represents the \mathcal{L}_q -norm. For any $E \in \mathbb{R}^{m \times n}$, ||E|| denotes the spectral norm, *i.e.*, the largest singular value of E. For any function f, let dom f be the effective domain of the function f and int(dom f) be the interior of dom f. For any point $x \in \mathbb{R}^n$ and a closed convex set $C \subset \mathbb{R}^n$, $dist(x, C) = \min_{y \in C} ||y - x||$.

2 Global convergence of the ADA

In this paper, we make the following standard assumption.

Assumption 2.1 The global minimum of (1.1) is attainable and

$$\operatorname{int}(X) \cap \operatorname{dom} f \cap \{x | Ex = q\} \neq \emptyset.$$

If X is polyhedral, an alternative assumption for (2.1) can be that $X \cap \operatorname{int}(\operatorname{dom} f) \cap \int_{\mathcal{T}} F_{\mathcal{T}}$ J / Ø

$$\cap \operatorname{int}(\operatorname{dom} f) \cap \{x | Ex = q\} \neq \emptyset.$$
(2.2)

Assumption 2.1 guarantees the existence of a saddle point of L. Namely, there exist \bar{x} and \bar{y} such that

$$\bar{x} \in \underset{x \in X}{\operatorname{argmin}} L(x, \bar{y}), \qquad \bar{y} \in \underset{y \in \mathbb{R}^m}{\operatorname{argmax}} L(\bar{x}, y).$$
 (2.3)

The dual function for problem (1.1) is

$$d(y) = \min_{x \in X} L(x, y) = \min_{x \in X} \{ f(x) + \langle y, Ex - q \rangle \}$$
(2.4)

and its associated dual problem is given by

$$\max_{y \in \mathbb{R}^m} d(y). \tag{2.5}$$

Let X^* and Y^* be the optimal solution sets of (1.1) and (2.5) respectively. The set of saddle points for the Lagrangian (1.2) is given by $X^* \times Y^*$.

2.1 Augmented Decomposition Algorithm

Here, we first summarize the basic idea of PSPA and its parallel implementation, the ADA. For that, the original problem (1.1) is equivalently transformed into

$$\min_{x,w} f(x) = f_1(x_1) + \dots + f_K(x_K)$$
s.t. $E_j x_j - w_j = 0, \quad j = 1, \dots, K - 1,$
 $E_K x_K - q - w_K = 0,$
 $w_1 + \dots + w_K = 0,$
 $x_k \in X_k, \quad k = 1, 2, \dots, K.$
(2.6)

If $x = (x_1, \ldots, x_K) \in \mathbb{R}^n$ is an optimal solution of (1.1), then $(x, w) = (x_1, \ldots, x_K, E_1 x_1, \ldots, E_{K-1} x_{K-1}, E_K x_K - q)$ will be an optimal solution to (2.6). Instead of adding a multiplier vector for $w_1 + \cdots + w_K = 0$, [32] introduced W as a subspace of $(\mathbb{R}^m)^K$ which is defined as

$$W = \{ w = (w_1, \dots, w_K) | w_1 + \dots + w_K = 0 \} \subset (\mathbb{R}^m)^K.$$
 (2.7)

The orthogonal complement subspace of W is given by

$$W^{\perp} = \{ w = (w_1, \dots, w_K) | w_1 = \dots = w_K \} \subset (\mathbb{R}^m)^K.$$
 (2.8)

For any $w = (w_1, \ldots, w_K) \in (\mathbb{R}^m)^K$, we use $P_{W^{\perp}}(w)$ to denote the projection of w onto the subspace W^{\perp} . In [32], the author proposed to add increments $u_i \in \mathbb{R}^m, i = 1, \ldots, K$ to the first K linear constraints in (2.6) and in addition, add to $w \in W$ a perturbation $v \in W^{\perp}$. The Lagrangian function associated with this perturbation finally works out in terms of the subspace

$$S = \{(\eta, \zeta) | P_{W^{\perp}}(\eta) = \zeta\} \subseteq (\mathbb{R}^m)^K \times W^{\perp},$$
(2.9)

and the functions

$$L_j(x_j, \eta_j) = \begin{cases} f_j(x_j) + \eta_j \cdot E_j x_j, \text{ if } j = 1, \dots, K-1, \\ f_K(x_K) + \eta_K \cdot (E_K x_K - q), \text{ o.w.} \end{cases}$$
(2.10)

to mean that

$$\bar{L}(w,x,\eta,\zeta) = \begin{cases} \sum_{j=1}^{K} [L_j(x_j,\eta_j) - \eta_j \cdot w_j], \text{ if } (w,x) \in W \times X, (\eta,\zeta) \in S, \\ -\infty, \text{ if } (w,x) \in W \times X, (\eta,\zeta) \notin S, \\ +\infty, \text{ if } (w,x) \notin W \times X. \end{cases}$$
(2.11)

The next lemma shows the relationship between L(x, y) and $\overline{L}(w, x, \eta, \zeta)$.

Lemma 1 If $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ is a saddle point of the Lagrangian function in (2.11), then $\bar{\eta}_1 = \bar{\eta}_2 = \cdot = \bar{\eta}_K$ and $(\bar{x}, \bar{\eta}_1)$ is a saddle point of (1.2). Conversely, let (\bar{x}, \bar{y}) be a saddle point of (1.2), and define $\bar{w} = (E_1 \bar{x}_1, \ldots, E_{K-1} \bar{x}_{K-1}, E_K \bar{x}_K - q) \in (\mathbb{R}^m)^K$, $\bar{\eta} = (\bar{y}, \ldots, \bar{y}) \in (\mathbb{R}^m)^K$ and $\bar{\zeta} = \bar{\eta}$. Then $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ is a saddle point of (2.11).

Proof. The dual problem associated with (2.11) is

$$\max_{(\eta,\zeta)\in S} \{\bar{g}(\eta,\zeta) = \inf_{(w,x)\in W\times X} \bar{L}(w,x,\eta,\zeta)\}$$
(2.12)

with its feasible set given by

$$\{(\eta,\zeta)|\bar{g}(\eta,\zeta)>-\infty\}\subset S.$$

As $w \cdot \eta$ cannot be ∞ , this implies $\eta_1 = \eta_2 = \cdot = \eta_K$. As a consequence, the dual problem reduces to

$$\max_{(\eta,\zeta)\in S} \{\bar{g}(\eta,\zeta) = \inf_{x\in X} f(x) + \langle \eta_1, Ex - q \rangle \}$$
(2.13)

which is equivalent to the dual problem corresponding to (1.2). So we can conclude the first part. The second part is similarly based on the above observation for the dual whose proof is omitted here.

Based on [30], the proximal method of multipliers is derived by adding both primal and dual proximal terms into the Lagrangian (2.11). More explicitly, the proximal saddle point algorithm in [32] can be described as the following:

Generate a sequence of elements $(w^{\nu},x^{\nu})\in W\times X$ and $(\eta^{\nu},\zeta^{\nu})\in S$ by letting

$$\bar{L}^{\nu}(w,x,\eta,\zeta) = \bar{L}(w,x,\eta,\zeta) + \frac{\rho}{2}||w-w^{\nu}||^{2} + \frac{1}{2c}||x-x^{\nu}||^{2} - \frac{1}{2\rho}||\eta-\eta^{\nu}||^{2} - \frac{1}{2\rho}||\zeta-\zeta^{\nu}||^{2}$$
(2.14)

and calculating

$$(w^{\nu+1},x^{\nu+1},\eta^{\nu+1},\zeta^{\nu+1})=$$
 unique saddle point of $\bar{L}^{\nu}(w,x,\eta,\zeta)$

with respect to minimizing over $(w, x) \in W \times X$ and maximizing over $(\eta, \zeta) \in S$. According to [30], the sequence $(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ generated by the above algorithm from any initial $(w^1, x^1) \in W \times X$ and $(\eta^1, \zeta^1) \in S$ is certain to converge to some saddle point $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ of the Lagrangian \bar{L} . With the special structure of the saddle point problem, the calculation of the saddle point in (2.14) can be carried out in the following parallel algorithm ADA. For simplicity, we denote

$$\phi_{k,\rho,c}^{\nu}(x_k) = \begin{cases} f_k(x_k) + \frac{\rho}{4} ||E_k x_k - w_k^{\nu} + \frac{2}{\rho} y_k^{\nu}||_2^2 + \frac{1}{2c} ||x_k - x_k^{\nu}||_2^2, & k = 1, \dots, K-1, \\ f_K(x_K) + \frac{\rho}{4} ||E_K x_K - q - w_K^{\nu} + \frac{2}{\rho} y_K^{\nu}||_2^2 + \frac{1}{2c} ||x_K - x_K^{\nu}||_2^2, & k = K. \end{cases}$$

$$(2.15)$$

Algorithm 1 Augmented decomposition algorithm

1: Given $w^0 \in W, x^0 \in X, y^0 \in (\mathbb{R}^m)^K$ 2: for $\nu = 0, 1, ...$ do 3: $x_k^{\nu+1} = \operatorname{argmin}_{x_k \in X_k} \phi_{k,\rho,c}^{\nu}(x_k), k = 1, ..., K$ 4: $\eta_k^{\nu+1} = \begin{cases} y_k^{\nu} + \frac{\rho}{2} [E_k x_k^{\nu+1} - w_k^{\nu}], \text{ if } k = 1, ..., K - 1 \\ y_K^{\nu} + \frac{\rho}{2} [E_K x_K^{\nu+1} - q - w_K^{\nu}], \text{ if } k = K \end{cases}$ 5: for k = 1, ..., K do 6: $\zeta_k^{\nu+1} = \frac{1}{K} \sum_{j=1}^K \eta_j^{\nu+1}$ 7: 8: $w_k^{\nu+1} = w_k^{\nu} + \frac{1}{\rho} [\eta_k^{\nu+1} - \zeta_k^{\nu+1}]$ 9: 10: $y_k^{\nu+1} = \frac{1}{2} [\eta_k^{\nu+1} + \zeta_k^{\nu+1}]$ 11: end for 12: end for

2.2 Convergence of the ADA

In this subsection, we assume q = 0 for notational simplicity which will not influence the proofs below. Define the matrix

$$G := \begin{pmatrix} \rho I_{mK} & & \\ & \frac{1}{c} I_n & \\ & & \frac{1}{\rho} I_{mK} \\ & & & \frac{1}{\rho} I_{mK} \end{pmatrix}.$$
 (2.16)

Hence $G \succ 0$ and $|| \cdot ||_G$ defines a norm. Let $\hat{u} = (\hat{w}, \hat{x}, \hat{\eta}, \hat{\zeta})$ and $u^{\nu} =$ $(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ where \hat{u} is a saddle point of the Lagrangian function (2.11) and u^{ν} is the current iteration point. The convergence result for ADA was established in [32] on the basis of convergence results for the classic PPA. Here, we import the result and provide an alternative proof for it.

Theorem 2.2 Under Assumption 2.1, for any $\rho > 0$ and c > 0, the sequence $\{(w^{\nu}, x^{\nu}, y^{\nu})\}_{\nu=1}^{\infty}$ generated in $W \times X \times (\mathbb{R}^m)^K$ by the ADA from any starting point converges to some $(\bar{w}, \bar{x}, \bar{y})$ such that (a) (\bar{w}, \bar{x}) solves (2.6), hence \bar{x} solves (1.1),

(b) $\bar{y}_1 = \cdots = \bar{y}_q \in \mathbb{R}^m$, and this common multiplier vector solves (2.5).

Proof. From Assumption 2.1 and Lemma 1, there exists a saddle point $(\hat{w}, \hat{x}, \hat{\eta}, \hat{\zeta}) \in$ $W \times X \times S$ of the Lagrangian function (2.11). For each iteration $\nu + 1$, due to the minimax operation on (2.14), from the primal perspective, we have the following inequality

$$\sum_{k=1}^{K} f_k(x_k) + \sum_{k=1}^{K} \langle \eta_k^{\nu+1}, E_k x_k - w_k \rangle$$

$$\geq \sum_{k=1}^{K} f_k(x_k^{\nu+1}) + \sum_{k=1}^{K} \langle \eta_k^{\nu+1}, E_k x_k^{\nu+1} - w_k^{\nu+1} \rangle + \frac{1}{c} \sum_{k=1}^{K} \langle x_k - x_k^{\nu+1}, x_k^{\nu} - x_k^{\nu+1} \rangle$$

$$+ \rho \sum_{k=1}^{K} \langle w_k - w_k^{\nu+1}, w_k^{\nu} - w_k^{\nu+1} \rangle$$
(2.17)

for any $x \in X$ and $w \in W$. Applying $(w, x) = (\hat{w}, \hat{x})$ to (2.17) and noticing that $E\hat{x}_k = \hat{w}_k, k = 1, \dots, K$, we obtain

$$\min P := \sum_{k=1}^{K} f_k(\hat{x}_k) \ge \sum_{k=1}^{K} f_k(x_k^{\nu+1}) + \sum_{k=1}^{K} \langle \eta_k^{\nu+1}, E_k x_k^{\nu+1} - w_k^{\nu+1} \rangle - \frac{1}{c} \sum_{k=1}^{K} \langle x_k^{\nu+1} - \hat{x}_k, x_k^{\nu} - x_k^{\nu+1} \rangle - \rho \sum_{k=1}^{K} \langle w_k^{\nu+1} - \hat{w}_k, w_k^{\nu} - w_k^{\nu+1} \rangle.$$
(2.18)

Similarly, from the dual perspective and the saddle-point property of $(\hat{w}, \hat{x}, \hat{\eta}, \hat{\zeta})$, the following inequality

$$\min P = \sum_{k=1}^{K} f_k(\hat{x}_k) \leq \sum_{k=1}^{K} f_k(x_k^{\nu+1}) + \sum_{k=1}^{K} \langle \hat{\eta}_k, E_k x_k^{\nu+1} - w_k^{\nu+1} \rangle$$

$$\leq \sum_{k=1}^{K} f_k(x_k^{\nu+1}) + \sum_{k=1}^{K} \langle \eta_k^{\nu+1}, E_k x_k^{\nu+1} - w_k^{\nu+1} \rangle$$

$$+ \frac{1}{\rho} \sum_{k=1}^{K} \langle \eta_k^{\nu+1} - \hat{\eta}_k, \eta_k^{\nu} - \eta_k^{\nu+1} \rangle + \frac{1}{\rho} \sum_{k=1}^{K} \langle \zeta_k^{\nu+1} - \hat{\zeta}_k, \zeta_k^{\nu} - \zeta_k^{\nu+1} \rangle$$

(2.19)

holds. Combining the above two inequalities with the following identity

$$2\langle a-b,c-a\rangle = ||c-b||_2^2 - ||c-a||_2^2 - ||b-a||_2^2,$$
(2.20)

we have

$$\sum_{k=1}^{K} \left(\frac{1}{c} ||x_{k}^{\nu} - \hat{x}_{k}||_{2}^{2} + \rho ||w_{k}^{\nu} - \hat{w}_{k}||_{2}^{2} + \frac{1}{\rho} ||\eta_{k}^{\nu} - \hat{\eta}_{k}||_{2}^{2} + \frac{1}{\rho} ||\zeta_{k}^{\nu} - \hat{\zeta}_{k}||_{2}^{2} \right)$$
$$- \sum_{k=1}^{K} \left(\frac{1}{c} ||x_{k}^{\nu+1} - \hat{x}_{k}||_{2}^{2} + \rho ||w_{k}^{\nu+1} - \hat{w}_{k}||_{2}^{2} + \frac{1}{\rho} ||\eta_{k}^{\nu+1} - \hat{\eta}_{k}||_{2} + \frac{1}{\rho} ||\zeta_{k}^{\nu+1} - \hat{\zeta}_{k}||_{2}^{2} \right)$$
$$\geq \sum_{k=1}^{K} \left(\frac{1}{c} ||x_{k}^{\nu+1} - x_{k}^{\nu}||_{2}^{2} + \rho ||w_{k}^{\nu+1} - w_{k}^{\nu}||_{2}^{2} + \frac{1}{\rho} ||\eta_{k}^{\nu+1} - \eta_{k}^{\nu}||_{2}^{2} + \frac{1}{\rho} ||\zeta_{k}^{\nu+1} - \zeta_{k}^{\nu}||_{2}^{2} \right)$$
(2.21)

which is equivalent with

$$||u^{\nu} - \hat{u}||_{G}^{2} - ||u^{\nu+1} - \hat{u}||_{G}^{2} \ge ||u^{\nu} - u^{\nu+1}||_{G}^{2}.$$
 (2.22)

From this inequality, we can easily conclude that

(i) $\sum_{\nu=0}^{\infty} ||u^{\nu} - u^{\nu+1}||_{G}^{2} < \infty;$ (ii) $\{u^{\nu} = (w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$ lies in a compact region;

(iii) $||u^{\nu} - \hat{u}||_{G}$ is a monotonically non-increasing sequence and thus converges.

From (ii), by passing to a subsequence if necessary, there exists at least one limiting point of $\{(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$, denoted as $\bar{u} = (\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$. It follows from (i) that $x^{\nu} - x^{\nu+1} \to 0$, $w^{\nu} - w^{\nu+1} \to 0$ and $\eta^{\nu} - \eta^{\nu+1} \to 0$. The update rule for w implies that $\bar{\eta}_1 = \cdots = \bar{\eta}_K$ and thus $\bar{y}_1 = \cdots = \bar{y}_K = \bar{\eta}_1$. Since $\eta_k^{\nu+1} = y_k^{\nu} + \frac{\rho}{2} [E_k x_k^{\nu+1} - w_k^{\nu}], E_k \bar{x}_k = \bar{w}_k$ holds and thus $E\bar{x} = 0$ which implies the feasibility of \bar{x} . Due to the optimality condition for each block in iteration $\nu + 1$, we have

$$0 \in \partial f_k(x_k^{\nu+1}) + E_k^T \eta_k^{\nu+1} + \frac{1}{c} (x_k^{\nu+1} - x_k^{\nu}) + N_{X_k}(x_k^{\nu+1}), \quad k = 1, \dots, K$$

By passing to the limit, we obtain

$$0 \in \partial f(\bar{x}) + E^T \bar{\eta}_1 + N_X(\bar{x}).$$

As a result, $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ is a saddle point of the Lagrangian function (2.11). Next, we show the uniqueness of the limit point to complete the proof. Let $\bar{u}^1 = (\bar{w}^1, \bar{x}^1, \bar{\eta}^1, \bar{\zeta}^1)$ and $\bar{u}^2 = (\bar{w}^2, \bar{x}^2, \bar{\eta}^2, \bar{\zeta}^2)$ be any two different limit points of $u^{\nu} = (w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$. By the previous argument, both of them are saddle points of (2.11). From (iii), we know the existence of the following limits

$$\lim_{\nu \to \infty} ||u^{\nu} - \bar{u}^{i}||_{G} = \beta_{i}, \quad i = 1, 2.$$

With the following equality

$$||u^{\nu} - \bar{u}^{1}||_{G}^{2} - ||u^{\nu} - \bar{u}^{2}||_{G}^{2} = -2\langle u^{\nu}, \bar{u}^{1} - \bar{u}^{2}\rangle_{G} + ||\bar{u}^{1}||_{G}^{2} - ||\bar{u}^{2}||_{G}^{2}$$

and by passing to the limit, we have

$$\beta_1^2 - \beta_2^2 = -2\langle \bar{u}^1, \bar{u}^1 - \bar{u}^2 \rangle_G + ||\bar{u}^1||_G^2 - ||\bar{u}^2||_G^2 = -||\bar{u}^1 - \bar{u}^2||_G^2$$

and

$$\beta_1^2 - \beta_2^2 = -2\langle \bar{u}^2, \bar{u}^1 - \bar{u}^2 \rangle_G + ||\bar{u}^1||_G^2 - ||\bar{u}^2||_G^2 = ||\bar{u}^1 - \bar{u}^2||_G^2.$$

Thus we obtain $||\bar{u}^1 - \bar{u}^2||_G = 0$ which implies that the sequence $(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ converges to some saddle point of the Lagrangian function (2.11) and hence (a) and (b) hold.

2.3 Rate of Convergence

In this subsection, we study the global convergence rate for the ADA. We first show the sublinear convergence result of the ADA in an ergodic sense. The proof follows the same idea as that in [33].

Theorem 2.3 Let $\{u^{\nu} = (w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$ in $W \times X \times S$ be the infinite sequence generated by the ADA. For any integer N > 0, define \tilde{x}_N by

$$\tilde{x}_N = \frac{1}{N} \sum_{\nu=1}^N x^{\nu}.$$
(2.23)

Then for any saddle point $\hat{u} = (\hat{w}, \hat{x}, \hat{\eta}, \hat{\zeta}) \in W \times X \times S$ of (2.11),

$$f(\tilde{x}_N) + \langle \hat{\eta}_1, E\tilde{x}_N \rangle - \min P \le \frac{||\hat{u} - u^0||_G^2}{N}.$$
 (2.24)

Proof. For any saddle point $(\hat{w}, \hat{x}, \hat{\eta}, \hat{\zeta}) \in W \times X \times S$ of the Lagrangian function (2.11), it follows from (2.18) and (2.19) that

$$||u^{\nu} - \hat{u}||_{G}^{2} - ||u^{\nu+1} - \hat{u}||_{G}^{2}$$

$$\geq ||u^{\nu} - u^{\nu+1}||_{G}^{2} + \sum_{k=1}^{K} f_{k}(x_{k}^{\nu+1}) + \sum_{k=1}^{K} \langle \hat{\eta}_{k}, E_{k} x_{k}^{\nu+1} \rangle - \min P$$

$$\geq \sum_{k=1}^{K} f_{k}(x_{k}^{\nu+1}) + \sum_{k=1}^{K} \langle \hat{\eta}_{k}, E_{k} x_{k}^{\nu+1} \rangle - \min P.$$
(2.25)

Summing (2.25) for $\nu = 0, 1, \ldots, N - 1$, we obtain

$$||u^{0} - \hat{u}||_{G}^{2}$$

$$\geq \sum_{\nu=0}^{N-1} \{\sum_{k=1}^{K} f_{k}(x_{k}^{\nu+1}) + \sum_{k=1}^{K} \langle \hat{\eta}_{k}, E_{k} x_{k}^{\nu+1} \rangle \} - N \min P \qquad (2.26)$$

$$\geq N[f(\tilde{x}_{N}) + \langle \hat{\eta}_{1}, E\tilde{x}_{N} \rangle - \min P]$$

where the second inequality results from the convexity of $f(\cdot)$ and the fact $\hat{\eta}_1 = \hat{\eta}_2 = \cdots = \hat{\eta}_K$. The assertion (2.24) follows immediately from the above inequality.

Next, we shall prove the $o(1/\nu)$ convergence of the ADA. Motivated by [9, 19], we use the quantity $||u^{\nu} - u^{\nu+1}||_G^2$ as a measure of the convergence rate. In fact, if $||u^{\nu} - u^{\nu+1}||_G^2 = 0$, then $u^{\nu+1}$ is an optimal solution, *i.e.*, $(x^{\nu+1}, \eta_1^{\nu+1}) \in X^* \times Y^*$. More explicitly, $||u^{\nu} - u^{\nu+1}||_G^2 = 0$ implies the following:

$$x^{\nu} = x^{\nu+1}$$
 and $w^{\nu} = w^{\nu+1}$. (2.27)

By the update step for w, we can conclude $\eta_1^{\nu+1} = \cdots = \eta_K^{\nu+1}$. Combining this with $x^{\nu} = x^{\nu+1}$, we obtain

$$0 \in \partial f(x^{\nu+1}) + E^T \eta_1^{\nu+1} + N_X(x^{\nu+1}), \qquad (2.28)$$

or equivalently, $(x^{\nu+1}, \eta_1^{\nu+1}) \in X^* \times Y^*$. Conversely, if the quantity $||u^{\nu} - u^{\nu+1}||_G^2$ is relatively large, $u^{\nu+1}$ should not be close to the optimal solution set. Based on previous analysis, $||u^{\nu} - u^{\nu+1}||_G^2$ is a reasonable measure to quantify the distance between $u^{\nu+1}$ and the optimal solution set.

To show the convergence rate, we first prove the following lemma on the monotonicity property of the iterations:

Lemma 2 Let u^{ν} be defined as in Theorem 2.3. Then

$$||u^{\nu} - u^{\nu+1}||_G^2 \le ||u^{\nu-1} - u^{\nu}||_G^2.$$
(2.29)

Proof. For notational simplicity, for each iteration ν , we introduce

$$\Delta u^{\nu+1} = \begin{pmatrix} \Delta w^{\nu+1} \\ \Delta x^{\nu+1} \\ \Delta \eta^{\nu+1} \\ \Delta \zeta^{\nu+1} \end{pmatrix} = \begin{pmatrix} w^{\nu} - w^{\nu+1} \\ x^{\nu} - x^{\nu+1} \\ \eta^{\nu} - \eta^{\nu+1} \\ \zeta^{\nu} - \zeta^{\nu+1} \end{pmatrix}.$$
 (2.30)

By the optimality of $x_k^{\nu+1}$ in iteration $\nu+1$ and the update rule of $\eta_k^{\nu+1}$, we have

$$\frac{1}{c}(x_k^{\nu} - x_k^{\nu+1}) - E_k^T \eta_k^{\nu+1} \in \partial f_k(x_k^{\nu+1}) + N_{X_k}(x_k^{\nu+1}), \qquad k = 1, \dots, K.$$
(2.31)

Considering the ν -th and ν + 1-th iteration, such optimality yields

$$\underbrace{\frac{1}{c} \langle \Delta x_k^{\nu+1}, \Delta x_k^{\nu} - \Delta x_k^{\nu+1} \rangle}_{(a)} - \langle E_k \Delta x_k^{\nu+1}, \Delta \eta_k^{\nu+1} \rangle \ge 0, \qquad k = 1, \dots, K.$$
(2.32)

For the second term in the above inequality,

$$-\sum_{k=1}^{K} \langle E_{k} \Delta x_{k}^{\nu+1}, \Delta \eta_{k}^{\nu+1} \rangle = \sum_{k=1}^{K} \langle E_{k} x_{k}^{\nu+1} - E_{k} x_{k}^{\nu}, \Delta \eta_{k}^{\nu+1} \rangle$$

$$=\sum_{k=1}^{K} \langle \frac{\eta_{k}^{\nu+1} - y_{k}^{\nu}}{\rho/2} - \frac{\eta_{k}^{\nu} - y_{k}^{\nu-1}}{\rho/2} + w_{k}^{\nu} - w_{k}^{\nu-1}, \Delta \eta_{k}^{\nu+1} \rangle$$

$$=\sum_{k=1}^{K} \langle \frac{\eta_{k}^{\nu+1} - \frac{\eta_{k}^{\nu} + \zeta_{k}^{\nu}}{\rho/2}}{\rho/2} - \frac{\eta_{k}^{\nu} - \frac{\eta_{k}^{\nu-1} + \zeta_{k}^{\nu-1}}{\rho/2}}{\rho/2} + \frac{\eta_{k}^{\nu} - \zeta_{k}^{\nu}}{\rho}, \Delta \eta_{k}^{\nu+1} \rangle$$

$$=\sum_{k=1}^{K} \frac{1}{\rho} \langle \Delta \eta_{k}^{\nu} - \Delta \eta_{k}^{\nu+1}, \Delta \eta_{k}^{\nu+1} \rangle + \sum_{k=1}^{K} \frac{1}{\rho} \langle \Delta \zeta_{k}^{\nu}, \Delta \eta_{k}^{\nu+1} \rangle + \sum_{k=1}^{K} \frac{1}{\rho} \langle \eta_{k}^{\nu} - \zeta_{k}^{\nu}, \Delta \eta_{k}^{\nu+1} \rangle$$

$$=\sum_{k=1}^{K} \frac{1}{\rho} \langle A \eta_{k}^{\nu} - \Delta \eta_{k}^{\nu+1}, \Delta \eta_{k}^{\nu+1} \rangle + \sum_{k=1}^{K} \frac{1}{\rho} \langle \Delta \zeta_{k}^{\nu} - \Delta \zeta_{k}^{\nu+1}, \Delta \zeta_{k}^{\nu+1} \rangle + \sum_{k=1}^{K} \frac{1}{\rho} \langle A \zeta_{k}^{\nu} - \Delta \zeta_{k}^{\nu+1}, \Delta \zeta_{k}^{\nu+1} \rangle + \sum_{(k) \in (k)}^{K} \frac{1}{\rho} \langle A \zeta_{k}^{\nu} - \Delta \zeta_{k}^{\nu+1}, \Delta \zeta_{k}^{\nu+1} \rangle + \sum_{(k) \in (k)}^{K} \frac{1}{\rho} \langle A \zeta_{k}^{\nu} - \Delta \zeta_{k}^{\nu+1}, \Delta \zeta_{k}^{\nu+1} \rangle + \sum_{(k) \in (k)}^{K} \frac{1}{\rho} \langle \eta_{k}^{\nu+1} - \zeta_{k}^{\nu+1}, \eta_{k}^{\nu} - \eta_{k}^{\nu+1} \rangle .$$

For term (d),

$$2\sum_{k=1}^{K} \frac{1}{\rho} \langle \eta_{k}^{\nu+1} - \zeta_{k}^{\nu+1}, \eta_{k}^{\nu} - \eta_{k}^{\nu+1} \rangle$$

$$=\sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu} - \zeta_{k}^{\nu+1}||_{2}^{2} - \sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu} - \eta_{k}^{\nu+1}||_{2}^{2} - \sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu+1} - \zeta_{k}^{\nu+1}||_{2}^{2}$$

$$=\sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu} - \zeta_{k}^{\nu} + \zeta_{k}^{\nu} - \zeta_{k}^{\nu+1}||_{2}^{2} - \sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu} - \eta_{k}^{\nu+1}||_{2}^{2} - \sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu} - \zeta_{k}^{\nu}||_{2}^{2} + \sum_{k=1}^{K} \frac{2}{\rho} \langle \eta_{k}^{\nu} - \zeta_{k}^{\nu}, \zeta_{k}^{\nu} - \zeta_{k}^{\nu+1} \rangle + \frac{1}{\rho} ||\zeta^{\nu} - \zeta^{\nu+1}||_{2}^{2} - \sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu} - \eta_{k}^{\nu+1}||_{2}^{2} - \sum_{k=1}^{K} \frac{1}{\rho} ||\eta_{k}^{\nu+1} - \zeta_{k}^{\nu+1}||_{2}^{2}.$$

$$(2.34)$$

Applying the equality (2.20) to (a), (b) and (c) and combining them with the above transformation for term (d), the inequality (2.32) yields

$$\begin{split} ||\Delta u^{\nu}||_{G}^{2} - ||\Delta u^{\nu+1}||_{G}^{2} &\geq \sum_{k=1}^{K} \frac{1}{c} ||\Delta x_{k}^{\nu} - \Delta x_{k}^{\nu+1}||_{2}^{2} + \sum_{k=1}^{K} \frac{1}{\rho} ||\Delta \eta_{k}^{\nu} - \Delta \eta_{k}^{\nu+1}||_{2}^{2} + \sum_{k=1}^{K} \frac{1}{\rho} ||\Delta \zeta^{\nu} - \Delta \zeta^{\nu+1}||_{2}^{2} + \underbrace{\sum_{k=1}^{K} \frac{1}{\rho} (||\eta_{k}^{\nu} - \eta_{k}^{\nu+1}||_{2}^{2} - ||\zeta_{k}^{\nu} - \zeta_{k}^{\nu+1}||_{2}^{2})}_{\geq 0} \geq 0. \end{split}$$

$$(2.35)$$

The nonnegativity of the last term is a direct result of the definition $\zeta_1^{\nu} = \cdots = \zeta_K^{\nu} = \frac{1}{K} \sum_{j=1}^{K} \eta_j^{\nu}$ and Cauchy–Schwarz inequality. Hence the inequality (2.29) holds.

The following elementary lemma helps to improve the convergence rate from $O(1/\nu)$ to $o(1/\nu)$.

Lemma 3 Suppose a sequence $\{a_{\nu}\}_{\nu=0}^{\infty} \subseteq \mathbb{R}$ satisfies the following: (a) $a_{\nu} \geq 0$; (b) $\sum_{\nu=0}^{\infty} a_{\nu} < \infty$; and (c) a_{ν} is monotonically non-increasing. Then, we have $a_{\nu} = o(1/\nu)$.

Proof. See Lemma 1.1 in [9].

Combining the results from previous two lemmas, we present the $o(1/\nu)$ convergence of the ADA.

Theorem 2.4 Let $\{u^{\nu} = (w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$ in $W \times X \times S$ be the infinite sequence generated by the ADA, then

$$||u^{\nu} - u^{\nu+1}||_G^2 = o(1/\nu) \tag{2.36}$$

holds and thus

$$||x^{\nu} - x^{\nu+1}||_2^2 = o(1/\nu)$$
(2.37)

and

$$\left\|\sum_{k=1}^{K} E_k x_k^{\nu+1}\right\|_2^2 = o(1/\nu).$$
(2.38)

Proof. In the proof of Theorem 2.2, we have shown that

$$\sum_{\nu=0}^{\infty} ||u^{\nu} - u^{\nu+1}||_G^2 < \infty.$$

On the other hand, Lemma 2 proved the non-increasing property of $||u^{\nu} - u^{\nu+1}||_G^2$. Hence, (2.36) follows directly from Lemma 3 and then (2.37) holds.

For the estimate for the constraint in (2.38), we have

$$\begin{split} &||\sum_{k=1}^{K} E_{k} x_{k}^{\nu+1}||_{2}^{2} = ||\sum_{k=1}^{K} (E_{k} x_{k}^{\nu+1} - w_{k}^{\nu})||_{2}^{2} = \frac{4}{\rho^{2}} ||\sum_{k=1}^{K} (\eta_{k}^{\nu+1} - y_{k}^{\nu})||_{2}^{2} \\ &\leq \frac{4K}{\rho^{2}} \sum_{k=1}^{K} ||\eta_{k}^{\nu+1} - \frac{1}{2} (\eta_{k}^{\nu} + \zeta_{k}^{\nu})||_{2}^{2} \\ &= \frac{K}{\rho^{2}} \sum_{k=1}^{K} ||\eta_{k}^{\nu+1} - \eta_{k}^{\nu} + \eta_{k}^{\nu+1} - \zeta_{k}^{\nu+1} + \zeta_{k}^{\nu+1} - \zeta_{k}^{\nu}||_{2}^{2} \\ &\leq \frac{3K}{\rho^{2}} ||\eta^{\nu+1} - \eta^{\nu}||_{2}^{2} + 3K ||w^{\nu+1} - w^{\nu}||_{2}^{2} + \frac{3K}{\rho^{2}} ||\zeta^{\nu+1} - \zeta^{\nu}||_{2}^{2} = o(1/\nu), \end{split}$$

$$(2.39)$$

where the first two equalities result from $w^{\nu} \in W$ and the updating rule for $\eta^{\nu+1}$. This finishes the proof for (2.38).

From Theorem 2.4, a reasonable stopping criterion for the ADA can be either

$$\frac{||x^{\nu} - x^{\nu+1}||}{\max\{1, ||x^{\nu}||\}} \le \epsilon$$
(2.40)

or

$$\frac{||Ex^{\nu+1} - q||}{\max\{1, ||q||\}} \le \epsilon \tag{2.41}$$

for some given tolerance ϵ .

2.4 Relation to the ADMM

The ADA is closely related to the ADMM. Here, we compare the ADA with two variants of ADMM, namely, the Variable Splitting ADMM and the Proximal Jacobian ADMM. For simplicity of notation, we assume q = 0.

Applying the classical two-block ADMM to the transformation in (2.6), [36] proposed the following Variable Splitting ADMM (VSADMM), see Algorithm 2. The convergence result for VSADMM was established on the basis of the

Algorithm 2 Variable Splitting ADMM

 $\begin{array}{ll} \text{1: Given } w^0 \in W, x^0 \in X, y^0 \in (\mathbb{R}^m)^K, \beta > 0 \\ \text{2: for } \nu = 0, 1, \dots \text{ do} \\ \text{3: } & x_k^{\nu+1} = \operatorname{argmin}_{x_k \in X_k} f_k(x_k) + \frac{\beta}{2} ||E_k x_k - w_k^{\nu} + \frac{y_k^{\nu}}{\beta}||_2^2, \quad k = 1, \dots, K, \\ \text{4: } & w^{\nu+1} = \operatorname{argmin}_{w \in W} \frac{\beta}{2} \sum_{k=1}^K ||E_k x_k^{\nu+1} - w_k + \frac{y_k^{\nu}}{\beta}||_2^2, \\ \text{5: } & y_k^{\nu+1} = y_k^{\nu} + \beta [E_k x_k^{\nu+1} - w_k^{\nu+1}], \quad k = 1, \dots, K. \\ \text{6: end for} \end{array}$

classical two-block ADMM. Compared to the ADA, we notice that no proximal

terms exist during the x-update in the VSADMM. Therefore, the full column rank assumption of E_k is necessary for the VSADMM to guarantee the solution uniqueness in each iteration. The w-update step in the VSADMM also differs from that in the ADA as it does not use the information on the previous iteration explicitly.

The Proximal Jacobian ADMM (Prox-JADMM) provided in [9] solves problem (1.1) directly by adding a proximal term in the Jacobian-type ADMM, see Algorithm 3. It is worth noting that the ADA shares the same $o(1/\nu)$ con-

Algorithm 3 Proximal Jacobian ADMM

1: Given $x^0 \in X, \lambda^0 \in \mathbb{R}^m, \beta > 0$ 2: for $\nu = 0, 1, ...$ do 3: for k = 1, ..., K do 4: $x_k^{\nu+1} = \operatorname{argmin}_{x_k \in X_k} f_k(x_k) + \frac{\beta}{2} ||E_k x_k + \sum_{j \neq k} E_j x_j^{\nu} - \frac{\lambda^{\nu}}{\beta} ||_2^2 + \frac{1}{2} ||x_k - x_k^{\nu}||_{P_k}^2,$ 5: end for 6: $\lambda^{\nu+1} = \lambda^{\nu} - \gamma \beta \sum_{k=1}^{K} E_k x_k^{\nu+1},$ 7: end for

vergence rate as the Prox-JADMM. However, the Prox-JADMM requires the constraints E_k , the proximal terms P_k and the damping parameter γ to satisfy certain relationships to guarantee the convergence. Because the convergence results for the ADA are established using a very different approach, we impose no restriction on the proximal terms.

3 The Inexact Augmented Decomposition Algorithm

Here, we first review the general convergence theory of the (inexact-)proximal point algorithm (PPA) developed in [30,31]. Let $\mathcal{T} : \mathcal{X} \rightrightarrows \mathcal{X}$ be a maximally monotone operator. In order to solve the inclusion problem:

$$0 \in \mathcal{T}(z),\tag{3.1}$$

PPA takes the form of

$$z^{k+1} \approx (I + c_k \mathcal{T})^{-1} (z^k), \quad \forall k \ge 0,$$
(3.2)

in the (k+1)-th iteration with a given sequence $c_k \uparrow c_{\infty} \leq \infty$. The convergence result of PPA can be guaranteed as long as the approximation computation satisfies certain criteria; see [30,31]. In addition, the local linear convergence result could be established when \mathcal{T}^{-1} is Lipschitz continuous at the origin. In accordance with the PPA, the inexact version of the ADA comes out naturally as follows in Algorithm 4. The iADA allows the subproblems to be solved inexactly which is very important in many applications as it might be very expensive to solve these subproblems exactly.

Two natural concerns arise for the iADA: (1) the global convergence and (2) the local convergence rate. For that, we make the following assumptions on f for the rest of the paper:

Algorithm 4 Inexact augmented decomposition algorithm

1: Given $w^0 \in W, x^0 \in X, y^0 \in (\mathbb{R}^m)^K$ 2: for $\nu = 0, 1, ...$ do 3: $x_k^{\nu+1} \approx \operatorname{argmin}_{x_k \in X_k} \phi_{k,\rho,c}^{\nu}(x_k), k = 1, ..., K$ 4: $\eta_k^{\nu+1} = \begin{cases} y_k^{\nu} + \frac{\rho}{2} [E_k x_k^{\nu+1} - w_k^{\nu}], \text{ if } k = 1, ..., K - 1 \\ y_K^{\nu} + \frac{\rho}{2} [E_K x_K^{\nu+1} - q - w_K^{\nu}], \text{ if } k = K \end{cases}$ 5: for k = 1, ..., K do 6: $\zeta_k^{\nu+1} = \frac{1}{K} \sum_{j=1}^K \eta_j^{\nu+1}$ 7: 8: $w_k^{\nu+1} = w_k^{\nu} + \frac{1}{\rho} [\eta_k^{\nu+1} - \zeta_k^{\nu+1}]$ 9: 10: $y_k^{\nu+1} = \frac{1}{2} [\eta_k^{\nu+1} + \zeta_k^{\nu+1}]$ 11: end for 12: end for

Assumption 3.1 (a) $f = f_1(x_1) + \cdots + f_K(x_K)$, with each f_k given by

$$f_k(x_k) = g_k(A_k x_k) + h_k(x_k)$$
(3.3)

where g_k and h_k are both closed proper convex functions and A_k 's are some given matrices.

(b) Every g_k is strongly convex and continuously differentiable on $int(dom g_k)$ with a Lipschitz continuous gradient

$$||A_{k}^{T}\nabla g_{k}(A_{k}x_{k}) - A_{k}^{T}\nabla g_{k}(A_{k}x_{k}')|| \leq L_{g}^{k}||A_{k}(x_{k} - x_{k}')||, \qquad \forall x_{k}, x_{k}' \in X_{k}$$
(3.4)

where $L_{q}^{k} \ge 0, k = 1, ..., K$.

(c) The epigraph of each h_k is a polyhedral convex set.

- (d) The feasible sets $X_k, k = 1, ..., K$ are polyhedral convex sets.
- (e) The feasible sets $X_k, k = 1, ..., K$ are compact sets.

Here are several comments on the above assumptions.

- Either g_k or h_k can be absent in f_k . Although g_k is assumed to be strongly convex, we do not impose any condition on A_k . Therefore, f_k is not necessarily strongly convex in general and the optimal solution is not necessarily unique.
- We do not assume any condition for the rank of E_k , k = 1, ..., K which is required to have full column rank in [21]. For the ADMM, this assumption is necessary to ensure that in each iteration, the subproblem for the k-th block is strongly convex. But for the iADA, this assumption is no longer required as there exists a proximal term in each subproblem which makes its optimality attainable and unique.
- The compactness assumption of $X_k, k = 1, ..., K$ will facilitate the proof in Section 4 and is not necessary for the convergence result in Section 5 due to the boundedness of the sequence generated by the iADA.

Based on these assumptions, we can simply write f as

$$f(x) = g(Ax) + h(x) = \sum_{k=1}^{K} g_k(A_k x_k) + \sum_{k=1}^{K} h_k(x_k)$$
(3.5)

where $g(Ax) = \sum_{k=1}^{K} g_k(A_k x_k)$ and $h(x) = \sum_{k=1}^{K} h_k(x_k)$ represent the smooth and nonsmooth parts respectively. In addition, $g(\cdot)$ is strongly convex and $h(\cdot)$ is convex with a polyhedral epigraph. The strong convexity of $g(\cdot)$ implies the following proposition, whose proof is omitted.

Proposition 1 For any x in the solution set X^* , $A_k x_k, k = 1, ..., K$ are constant and hence Ax is constant.

In the next section, we will discuss the stability result of the Lagrangian function under some perturbations which is essential to the local linear convergence result.

4 On the stability results of $\mathcal{T}_{ar{L}}$

In this section, we establish the stability result of the maximal monotone operator $\mathcal{T}_{\bar{L}}$ defined in (4.2) corresponding to the perturbations of both primal and dual solutions under Assumption 3.1. This property serves the key ingredient for the local convergence rate analysis of the iADA.

Recall the definition of $\overline{L}(w, x, \eta, \zeta)$ in (2.11). For each $(w, x, \eta, \zeta) \in W \times X \times S$, $\mathcal{T}_{\overline{L}}(w, x, \eta, \zeta)$ is defined as

 $T_{\bar{L}}(w,x,\eta,\zeta) = \{(v_1,v_2,v_3,v_4) | (v_1,v_2,-v_3,-v_4) \in \partial \bar{L}(w,x,\eta,\zeta)\}, \quad (4.1)$ or equivalently, $T_{\bar{L}}(w,x,\eta,\zeta)$ is the set of $v = (v_1,v_2,v_3,v_4) \in (\mathbb{R}^m)^K \times \mathbb{R}^n \times (\mathbb{R}^m)^K \times (\mathbb{R}^m)^K$ such that

$$\bar{L}(w', x', \eta, \zeta) - \langle w', v_1 \rangle - \langle x', v_2 \rangle + \langle \eta, v_3 \rangle + \langle \zeta, v_4 \rangle
\geq \bar{L}(w, x, \eta, \zeta) - \langle w, v_1 \rangle - \langle x, v_2 \rangle + \langle \eta, v_3 \rangle + \langle \zeta, v_4 \rangle
\geq \bar{L}(w, x, \eta', \zeta') - \langle w, v_1 \rangle - \langle x, v_2 \rangle + \langle \eta', v_3 \rangle + \langle \zeta', v_4 \rangle
\text{for all } (w', x') \in W \times X, (\eta', \zeta') \in S.$$
(4.2)

Any solution to $(0,0,0,0) \in \mathcal{T}_{\bar{L}}(w,x,\eta,\zeta)$ is a saddle point of \bar{L} . Denote $v_1 = (v_{1,1},\ldots,v_{1,K}) \in (\mathbb{R}^m)^K, v_2 = (v_{2,1},\ldots,v_{2,K}) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_K}, v_3 = (v_{3,1},\ldots,v_{3,K}) \in (\mathbb{R}^m)^K$ and $P_{W^{\perp}}(v_4) = (v_4^{\perp},\ldots,v_4^{\perp}) \in (\mathbb{R}^m)^K$. We consider the following perturbed form of problem (2.6):

$$\min_{x,w} f_1(x_1) + \dots + f_K(x_K) - \langle w, v_1 \rangle - \langle x, v_2 \rangle$$
s.t. $E_k x_k - w_k + v_{3,k} + v_4^{\perp} = 0, \quad k = 1, \dots, K - 1,$
 $E_K x_K - q - w_K + v_{3,K} + v_4^{\perp} = 0,$
 $w_1 + \dots + w_K = 0,$
 $x_k \in X_k, \quad k = 1, 2, \dots, K$

$$(4.3)$$

Its corresponding KKT conditions are given by

$$-E_{k}^{T}\eta_{k} + v_{2,k} \in \partial f_{k}(x_{k}) + N_{X_{k}}(x_{k}), \quad k = 1, \dots, K$$

$$-\eta_{k} + \mu = v_{1,k}, \quad k = 1, \dots, K$$

$$E_{k}x_{k} - w_{k} + v_{3,k} + v_{4}^{\perp} = 0, \quad k = 1, \dots, K - 1 \quad (4.4)$$

$$E_{K}x_{K} - q - w_{K} + v_{3,K} + v_{4}^{\perp} = 0,$$

$$w_{1} + \dots + w_{K} = 0.$$

One can easily check that

$$\mathcal{T}_{\bar{L}}^{-1}(v_1, v_2, v_3, v_4) = \text{ set of all } (w, x, \eta, P_{W^{\perp}}(\eta)) \in W \times X \times S$$

such that there exists $\mu \in \mathbb{R}^m$ satisfying that (w, x, η, μ) (4.5)
is a solution of the KKT conditions (4.4).

Based on the above observation, we first study the stability results of the KKT system (4.4) under perturbations considered above. Under Assumption 3.1, every $f_k(x_k)$ is the sum of a smooth function $g_k(A_k x_k)$ and a nonsmooth function $h_k(x_k)$ with a polyhedral epigraph. By introducing a variable $s = (s_1, \ldots, s_K) \in \mathbb{R}^K$, for each k, we can rewrite the polyhedral set $\{(x_k, s_k) : x_k \in X_k, h_k(x_k) \leq s_k\}$ compactly as $C_x^k x_k + C_s^k s_k \geq c_k$ for some matrices $C_x^k \in \mathbb{R}^{j_k \times n_k}, C_s^k \in \mathbb{R}^{j_k \times 1}$ and $c_k \in \mathbb{R}^{j_k \times 1}$, where j_k s are some positive integers with $\sum_{k=1}^K j_k = j$. Then, we can transform (2.6) equivalently into

$$\min_{x,w,s} \sum_{k=1}^{K} g_k(A_k x_k) + s_k$$
s.t. $E_k x_k - w_k = 0, \quad k = 1, \dots, K - 1,$
 $E_K x_K - q - w_K = 0,$
 $w_1 + \dots + w_K = 0,$
 $C_x^k x_k + C_s^k s_k - c_k \ge 0, \quad k = 1, 2, \dots, K.$

$$(4.6)$$

For the perturbed problem (4.3), similarly, we have the following equivalent transformation:

$$\min_{x,w,s} \sum_{k=1}^{K} g_k(A_k x_k) + s_k - \langle w, v_1 \rangle - \langle x, v_2 \rangle
s.t. E_k x_k - w_k + v_{3,k} + v_4^{\perp} = 0, \quad k = 1, \dots, K - 1,
E_K x_K - q - w_K + v_{3,K} + v_4^{\perp} = 0,
w_1 + \dots + w_K = 0,
C_x^k x_k + C_s^k s_k - c_k \ge 0, \quad k = 1, 2, \dots, K.$$
(4.7)

The canonical Lagrangian function for (4.7) is given by

$$L^{v}(w, x, s, \eta, \lambda, \mu) = \sum_{k=1}^{K} g_{k}(A_{k}x_{k}) + s_{k} - \langle w, v_{1} \rangle - \langle x, v_{2} \rangle$$

+
$$\sum_{k=1}^{K-1} \langle E_{k}x_{k} - w_{k} + v_{3,k} + v_{4}^{\perp}, \eta_{k} \rangle + \langle E_{K}x_{K} - q - w_{K} + v_{3,K} + v_{4}^{\perp}, \eta_{K} \rangle$$

-
$$\sum_{k=1}^{K} \langle C_{x}^{k}x_{k} + C_{s}^{k}s_{k} - c_{k}, \lambda_{k} \rangle + \langle w_{1} + \dots + w_{K}, \mu \rangle.$$

(4.8)

We use $Sol(P(v_1, v_2, v_3, v_4))$ to denote the set of saddle points for the Lagrangian function $L^v(w, x, s, \eta, \lambda, \mu)$ defined above corresponding to the perturbed problem (4.7). Let $(v_1, v_2, v_3, v_4) = (0, 0, 0, 0)$, then Sol(P(0, 0, 0, 0)) represents the set of saddle points for the Lagrangian function of problem (4.6). In order to show the stability results for the KKT system (4.4), we define a set-valued mapping \mathcal{M} that assigns the vector $(d, e, f) \in \mathbb{R}^n \times (\mathbb{R}^m)^K \times (\mathbb{R}^m)^K$ to the set of $(w, x, s, \eta, \lambda, \mu) \in (\mathbb{R}^m)^K \times \mathbb{R}^n \times \mathbb{R}^K \times (\mathbb{R}^m)^K \times \mathbb{R}^j \times \mathbb{R}^m$ that satisfy the following equations

$$-E_{k}^{T}\eta_{k} + (C_{x}^{k})^{T}\lambda_{k} = d_{k}, \quad k = 1, ..., K$$

$$-\eta_{k} + \mu = e_{k}, \quad k = 1, ..., K$$

$$w_{k} - E_{k}x_{k} = f_{k}, \quad k = 1, ..., K - 1$$

$$w_{K} - E_{K}x_{K} + q = f_{K}, \qquad (4.9)$$

$$w_{1} + \dots + w_{K} = 0,$$

$$\leq \lambda_{k} \perp C_{x}^{k}x_{k} + C_{s}^{k}s_{k} - c_{k} \geq 0, \quad k = 1, ..., K$$

$$(C_{s}^{k})^{T}\lambda_{k} = 1, \quad k = 1, ..., K.$$

One can easily verify that

0

$$(w, x, s, \eta, \lambda, \mu) \in \mathcal{M}(A^T \nabla g(Ax) - v_2, v_1, v_3 + P_{W^{\perp}}(v_4))$$

if and only if $(w, x, s, \eta, \lambda, \mu) \in Sol(P(v_1, v_2, v_3, v_4)),$

$$(4.10)$$

i.e., a solution of the KKT system of (4.7) is also a saddle point of the Lagrangian function (4.8). By taking $(v_1, v_2, v_3, v_4) = (0, 0, 0, 0)$, we see that $(w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*) \in \mathcal{M}(A^T \nabla g(Ax^*), 0, 0)$ if and only if $(w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*) \in Sol(P(0, 0, 0, 0))$. It is easily seen that \mathcal{M} is a polyhedral multifunction; *i.e.*, the graph of \mathcal{M} is the union of a finitely many polyhedral convex sets. In [29], Robinson established the following proposition that \mathcal{M} enjoys the local upper Lipschitzian continuity property; see also [20].

Proposition 2 There exists a positive scalar θ that depends on A, E, C_x, C_s only, such that for each $(\bar{d}, \bar{e}, \bar{f})$ there is a positive δ' satisfying

$$\begin{split} \mathcal{M}(d,e,f) &\subseteq \mathcal{M}(\bar{d},\bar{e},\bar{f}) + \theta ||(d,e,f) - (\bar{d},\bar{e},\bar{f})||\mathcal{B} \text{ whenever } ||(d,e,f) - (\bar{d},\bar{e},\bar{f})|| \leq \delta' \\ (4.11) \\ \text{where } \mathcal{B} \text{ is the unit Euclidean ball in } (\mathbb{R}^m)^K \times \mathbb{R}^n \times \mathbb{R}^k \times (\mathbb{R}^m)^K \times \mathbb{R}^j \times \mathbb{R}^m. \end{split}$$

Based on this proposition, we claim that

Lemma 4 Suppose Assumptions 2.1 and 3.1 hold. Then there exist positive scalars δ , τ depending on A, E, C_x, C_s only, such that for all $v = (v_1, v_2, v_3, v_4) \in (\mathbb{R}^m)^K \times \mathbb{R}^n \times (\mathbb{R}^m)^K \times (\mathbb{R}^m)^K$ and $||v|| \leq \delta$, any $(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) \in Sol(P(v_1, v_2, v_3, v_4))$, we have

$$dist((w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)), Sol(P(0, 0, 0, 0))) \le \tau ||v||.$$
(4.12)

Proof. By the previous proposition, \mathcal{M} is locally upper Lipschtizian with modulus θ at $(A^T \nabla g(Ax^*), 0, 0)$ for any $x^* \in X^*$. First we show that as $v \to 0$, $A^T \nabla g(Ax(v)) \to A^T \nabla g(Ax^*)$. For that, take a sequence $v^i = (v_1^i, v_2^i, v_3^i, v_4^i) \in (\mathbb{R}^m)^K \times \mathbb{R}^n \times (\mathbb{R}^m)^K \times (\mathbb{R}^m)^K, i = 1, 2, \cdots$, such that $||v^i|| \to 0$. Based on Assumption 3.1(e), the sequence $x(v^i), i = 1, 2, \cdots$ lies in a compact set and so the other sequence $s(v^i)$ and $w(v^i)$ also belong to some compact sets, given the fact $s(v^i) = h(x(v^i))$ and the linear relationship among $x(v^i), v^i$ and $w(v^i)$. By passing to a subsequence if necessary, let $(w^\infty, x^\infty, s^\infty)$ be a cluster point of $\{(w(v^i), x(v^i), s(v^i)\}$. Due to the continuity of $\nabla g(\cdot), (A^T \nabla g(Ax(v^i)) - v_2^i, v_1^i, v_3^i + P_{W^{\perp}}(v_4^i))$ converges to $(A^T \nabla g(Ax^\infty), 0, 0)$ as $i \to \infty$. For all i, $\{(w(v^i), x(v^i), s(v^i), A^T \nabla g(Ax(v^i)) - v_2^i, v_1^i, v_3^i + P_{W^{\perp}}(v_4^i))\}$ lies in the set

$$\{(w, x, s, d, e, f) | (w, x, s, \eta, \lambda, \mu) \in \mathcal{M}(d, e, f) \text{ for some } (\eta, \lambda, \mu) \}$$

which is a closed polyhedral set. By passing to the limit, we can conclude

$$(w^{\infty}, x^{\infty}, s^{\infty}, \eta^{\infty}, \lambda^{\infty}, \mu^{\infty}) \in \mathcal{M}(A^T \nabla g(Ax^{\infty}), 0, 0)$$

for some $(\eta^{\infty}, \lambda^{\infty}, \mu^{\infty}) \in (\mathbb{R}^m)^K \times \mathbb{R}^j \times \mathbb{R}^m$. From Proposition 1, we know $Ax^{\infty} = Ax^*$ for any $x^* \in X^*$ which further implies that $A^T \nabla g(Ax(v)) \to A^T \nabla g(Ax^*)$. Then there exists a positive scalar δ such that for all v satisfying $||v|| \leq \delta$, the following inequality

$$||A^T \nabla g(Ax(v)) - A^T \nabla g(Ax^*)|| + ||v|| \le \delta'$$

holds. Based on Proposition 2, there exists $(w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*) \in \mathcal{M}(A^T \nabla g(Ax^*), 0, 0)$, satisfying

$$\begin{aligned} ||(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) - (w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*)|| \\ &\leq \theta(||A^T \nabla g(Ax(v)) - A^T \nabla g(Ax^*)|| + ||v||). \end{aligned}$$
(4.13)

Since $(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) \in \mathcal{M}(A^T \nabla g(Ax) - v_2, v_1, v_3 + P_{W^{\perp}}(v_4))$, by the definition of \mathcal{M} we have

$$-E_{k}^{T}\eta_{k}(v) + (C_{x}^{k})^{T}\lambda_{k}(v) = A_{k}^{T}\nabla g_{k}(A_{k}x_{k}(v)) - v_{2}, \quad k = 1, \dots, K$$

$$-\eta_{k}(v) + \mu(v) = v_{1,k}, \quad k = 1, \dots, K$$

$$w_{k}(v) - E_{k}x_{k}(v) = v_{3,k} + v_{4}^{\perp}, \quad k = 1, \dots, K - 1$$

$$w_{K}(v) - E_{K}x_{K}(v) + q = v_{3,K} + v_{4}^{\perp},$$

$$w_{1}(v) + \dots + w_{K}(v) = 0,$$

$$0 \leq \lambda_{k}(v) \perp C_{x}^{k}x_{k}(v) + C_{s}^{k}s_{k}(v) - c_{k} \geq 0, \quad k = 1, \dots, K$$

$$(C_{s}^{k})^{T}\lambda_{k}(v) = 1, \quad k = 1, \dots, K.$$

$$(4.14)$$

Similarly, since $(w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*) \in \mathcal{M}(A^T \nabla g(Ax^*), 0, 0)$, it follows that

$$-E_{k}^{T}\eta_{k}^{*} + (C_{x}^{k})^{T}\lambda_{k}^{*} = A_{k}^{T}\nabla g_{k}(A_{k}x_{k}^{*}), \quad k = 1, \dots, K$$

$$-\eta_{k}^{*} + \mu^{*} = 0, \quad k = 1, \dots, K$$

$$w_{k}^{*} - E_{k}x_{k}^{*} = 0, \quad k = 1, \dots, K - 1$$

$$w_{K}^{*} - E_{K}x_{K}^{*} + q = 0, \quad (4.15)$$

$$w_{1}^{*} + \dots + w_{K}^{*} = 0,$$

$$0 \leq \lambda_{k}^{*} \perp C_{x}^{k}x_{k}^{*} + C_{s}^{k}s_{k}^{*} - c_{k} \geq 0, \quad k = 1, \dots, K$$

$$(C_{s}^{k})^{T}\lambda_{k}^{*} = 1, \quad k = 1, \dots, K.$$

Due to the strong convexity of $g_k(\cdot)$ and the Lipschitzian continuity of its derivative $\nabla g_k(\cdot)$ in Assumption 3.1, there exist positive scalars σ_g^k, L_g^k such that for all $x_1^k, x_2^k \in X_k$

$$\langle A_k^T \nabla g_k(A_k x_1^k) - A_k^T \nabla g_k(A_k x_2^k), x_1^k - x_2^k \rangle \ge \sigma_g^k ||A_k x_1^k - A_k x_2^k||^2, \quad (4.16)$$

and

$$||A_k^T \nabla g_k(A_k x_1^k) - A_k^T \nabla g_k(A_k x_2^k)|| \le L_g^k ||A_k x_1^k - A_k x_2^k||.$$
(4.17)

Define $\sigma_g = \min_k \sigma_g^k$ and $L_g = \max_k L_g^k$. Taking $x_1 = x(v), x_2 = x^*$, we obtain

$$\begin{split} \sigma_g \sum_{k=1}^{K} ||A_k(x(v)_k - x_k^*)||^2 \\ &\leq \sum_{k=1}^{K} \langle A_k^T \nabla g_k(A_k x(v)_k) - A_k^T \nabla g_k(A_k x_k^*), x(v)_k - x_k^* \rangle \\ &= \sum_{k=1}^{K} \langle -E_k^T(\eta(v)_k - \eta_k^*) + (C_x^k)^T(\lambda(v)_k - \lambda_k^*) + v_{2,k}, x(v)_k - x_k^* \rangle \\ &= \sum_{k=1}^{K} \langle \lambda(v)_k - \lambda_k^*, C_x^k x(v)_k - C_x^k x_k^* \rangle + \sum_{k=1}^{K} \langle \eta(v)_k - \eta_k^*, -E_k x(v)_k + E_k x_k^* \rangle \\ &+ \sum_{k=1}^{K} \langle v_{2,k}, x(v)_k - x_k^* \rangle \end{split}$$

where the first inequality comes from (4.16) and the equalities come from (4.14) and (4.15). Moreover, we have

$$\begin{split} &\sum_{k=1}^{K} \langle \lambda(v)_{k} - \lambda_{k}^{*}, C_{x}^{k} x(v)_{k} - C_{x}^{k} x_{k}^{*} \rangle \\ &= \sum_{k=1}^{K} \langle \lambda(v)_{k} - \lambda_{k}^{*}, C_{x}^{k} x(v)_{k} - C_{x}^{k} x_{k}^{*} \rangle + \langle \sum_{k=1}^{K} \lambda(v)_{k} - \lambda_{k}^{*}, C_{s}^{k} s(v)_{k} - C_{s}^{k} s_{k}^{*} \rangle \\ &= \sum_{k=1}^{K} \langle \lambda(v)_{k} - \lambda_{k}^{*}, (C_{x}^{k} x(v)_{k} + C_{s}^{k} s(v)_{k} - c_{k}) - (C_{x}^{k} x_{k}^{*} + C_{s}^{k} s_{k}^{*} - c_{k}) \rangle \\ &= -\sum_{k=1}^{K} [\langle \lambda_{k}^{*}, C_{x}^{k} x(v)_{k} + C_{s}^{k} s(v)_{k} - c_{k} \rangle + \langle \lambda(v)_{k}, C_{x}^{k} x_{k}^{*} + C_{s}^{k} s_{k}^{*} - c_{k} \rangle] \leq 0 \end{split}$$

where the first equality follows from the fact that $(C_s^k)^T \lambda(v)_k = (C_s^k)^T \lambda_k^* = 1, k = 1, \ldots, K$ and the last equality and inequality both result from the complementary conditions in (4.14) and (4.15). Consequently, we obtain that

$$\begin{split} \sigma_g \sum_{k=1}^{K} ||A_k(x(v)_k - x_k^*)||^2 \\ &\leq \sum_{k=1}^{K} \langle \eta(v)_k - \eta_k^*, -E_k x(v)_k + E_k x_k^* \rangle + \sum_{k=1}^{K} \langle v_{2,k}, x(v)_k - x_k^* \rangle \\ &= \sum_{k=1}^{K} \langle \mu(v) + v_{1,k} - \mu^*, -w(v)_k + w_k^* + v_{3,k} + v_4^\perp \rangle + \sum_{k=1}^{K} \langle v_{2,k}, x(v)_k - x_k^* \rangle \\ &= \sum_{k=1}^{K} \langle \mu(v) - \mu^*, v_{3,k} + v_4^\perp \rangle + \underbrace{\sum_{k=1}^{K} \langle \mu(v) - \mu^*, -w(v)_k + w_k^* \rangle}_{=0} \\ &+ \sum_{k=1}^{K} \langle v_{1,k}, -w(v)_k + w_k^* + v_{3,k} + v_4^\perp \rangle + \sum_{k=1}^{K} \langle v_{2,k}, x(v)_k - x_k^* \rangle \\ &\leq ||\mu(v) - \mu^*||(||v_3|| + ||v_4||) + ||w(v) - w^*||||v_1|| + ||v_1||(||v_3|| + ||v_4||) + ||(x(v) - x^*)||||v_2|| \\ &\leq ||(w(v), x(v), \mu(v)) - (w^*, x^*, \mu^*)|||v|| + ||v||^2. \end{split}$$

Finally, based on Proposition 2 and the above inequality, we have

$$\begin{split} &||(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) - (w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*)||^2 \\ &\leq \theta^2 (||A^T \nabla g(Ax(v)) - A^T \nabla g(Ax^*)|| + ||v||)^2 \\ &\leq 2\theta^2 (\sum_{k=1}^K ||A_k^T \nabla g_k(A_k x(v)_k) - A_k^T \nabla g_k(A_k x_k^*)||^2 + ||v||^2) \\ &\leq 2\theta^2 (L_g^2 \sum_{k=1}^K ||A_k(x(v)_k - x_k^*)||^2 + ||v||^2) \\ &\leq 2\theta^2 \max\{\frac{L_g^2}{\sigma_g}, 1\} (\sigma_g \sum_{k=1}^K ||A_k(x(v)_k - x_k^*)||^2 + ||v||^2) \\ &\leq 2\theta^2 \max\{\frac{L_g^2}{\sigma_g}, 1\} (||(w(v), x(v), \mu(v)) - (w^*, x^*, \mu^*)||||v|| + 2||v||^2) \\ &\leq 2\theta^2 \max\{\frac{L_g^2}{\sigma_g}, 1\} (||(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) - (w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*)||||v|| + 2||v||^2) \end{split}$$

We see the above inequality is quadratic in $||(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) - (w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*)||/||v||$, so we have

$$||(w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)) - (w^*, x^*, s^*, \eta^*, \lambda^*, \mu^*)|| / ||v|| \leq \tau$$

for some scalar τ depending on θ, L_g, σ_g only. We conclude that

$$dist((w(v), x(v), s(v), \eta(v), \lambda(v), \mu(v)), Sol(P(0, 0, 0, 0))) \le \tau ||v||.$$

In view of the operator $\mathcal{T}_{\bar{L}}$, combining Lemma 4 with the observation in (4.5), we have the following corollary.

Corollary 1 Suppose Assumptions 2.1 and 3.1 hold. Then there exist positive scalars δ , τ depending on A, E, C_x, C_s only, such that for all $v = (v_1, v_2, v_3, v_4) \in (\mathbb{R}^m)^K \times \mathbb{R}^n \times (\mathbb{R}^m)^K \times (\mathbb{R}^m)^K$ and $||v|| \leq \delta$, any $(w(v), x(v), \eta(v), \zeta(v)) \in \mathcal{T}_{L}^{-1}(v)$ satisfies

$$dist((w(v), x(v), \eta(v), \zeta(v)), \mathcal{T}_{\bar{L}}^{-1}(0, 0, 0, 0)) \le 2\tau ||v||.$$
(4.18)

Proof. From Lemma 4 and observation in (4.5) , we know that for any $(w(v), x(v), \eta(v), \zeta(v)) \in \mathcal{T}_{\bar{L}}^{-1}(v)$, there exists a $(w^*, x^*, \eta^*, \zeta^*) \in \mathcal{T}_{\bar{L}}^{-1}(0, 0, 0, 0)$ satisfying that

$$|(w(v), x(v), \eta(v)) - (w^*, x^*, \eta^*)|| \le \tau ||v||.$$

Since $\zeta(v) = P_{W^{\perp}}(\eta(v))$ and $\zeta^* = P_{W^{\perp}}(\eta^*)$, then

$$||(w(v), x(v), \eta(v), \zeta(v)) - (w^*, x^*, \eta^*, \zeta^*)|| \le 2\tau ||v||$$

holds which leads to (4.18).

The compactness assumption of X_k is indeed necessary for Corollary 1. However, if the generated sequence $\{x(v^i)\}$ lies in a compact set for a sequence $\{v^i\}_{i=1}^{\infty}$ converging to the origin, we claim the following result: under Assumptions 2.1 and 3.1(a)-(d), there exist positive scalars δ, τ depending on A, E, C_x, C_s only, when $||v^i|| \leq \delta$ the following

$$dist((w(v^{i}), x(v^{i}), \eta(v^{i}), \zeta(v^{i})), \mathcal{T}_{\bar{L}}^{-1}(0, 0, 0, 0)) \le 2\tau ||v^{i}||$$

$$(4.19)$$

holds. This observation relaxes the compactness assumption for $X_k, k = 1, ..., K$ (Assumption 3.1(e)) when we show the local linear convergence in Theorem 5.2 for the iADA in Section 5.

5 Convergence analysis of the inexact ADA

In this section, we study the convergence results of the inexact ADA for solving the problem (1.1). For that, we first need to adopt the following stopping criterion developed in [30,31] for approximately solving these subproblems

dist
$$(0, \partial \phi_{k,\rho,c}^{\nu}(x_k^{\nu+1})) \le \frac{\epsilon_{\nu}}{cK(\rho||E|| + ||E|| + 1)}, \qquad \sum_{\nu=0}^{\infty} \epsilon_{\nu} < \infty.$$
 (A)

Theorem 5.1 Suppose Assumption 2.1 holds and let $\{(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$ in $W \times X \times S$ be the infinite sequence generated by the ADA with the stopping criterion (A). Then $(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ converges to some saddle point $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ of (2.11) such that

(a) (\bar{w}, \bar{x}) solves (2.6), hence \bar{x} solves (1.1),

(b) $\bar{\eta}_1 = \cdots = \bar{\eta}_K \in \mathbb{R}^m$, and this common multiplier vector solves (2.5).

Proof. In each iteration ν , we denote $(w_0^{\nu+1}, x_0^{\nu+1}, \eta_0^{\nu+1}, \zeta_0^{\nu+1}) = P_{\nu}(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ as the exact saddle point of $\bar{L}^{\nu}(w, x, \eta, \zeta)$ and $(w^{\nu+1}, x^{\nu+1}, \eta^{\nu+1}, \zeta^{\nu+1})$ as the inexact saddle point generated following the stopping criteria (A) respectively. By the update rule, the following estimates hold:

$$\begin{split} ||\eta_0^{\nu+1} - \eta^{\nu+1}|| &\leq \frac{\rho ||E||}{2} ||x_0^{\nu+1} - x^{\nu+1}||, \\ ||\zeta_0^{\nu+1} - \zeta^{\nu+1}|| &\leq \frac{\rho ||E||}{2} ||x_0^{\nu+1} - x^{\nu+1}||, \end{split}$$

and

$$||w_0^{\nu+1} - w^{\nu+1}|| \le ||E||||x_0^{\nu+1} - x^{\nu+1}||.$$

Thus, we can obtain

$$||(w^{\nu+1}, x^{\nu+1}, \eta^{\nu+1}, \zeta^{\nu+1}) - P_{\nu}(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})|| \le (\rho||E|| + ||E|| + 1)||x^{\nu+1} - x_0^{\nu+1}||.$$
(5.1)

Observing that the function $\phi_{k,\rho,c}^{\nu}$ defined in (2.15) is strongly convex with modulus at least $\frac{1}{c}$ and $x_{0,k}^{\nu+1}$ minimize $\phi_{k,\rho,c}^{\nu}(x_k)$, we get

$$||x^{\nu+1} - x_0^{\nu+1}|| \le c \sum_{k=1}^{K} \operatorname{dist}(0, \partial \phi_{k,\rho,c}^{\nu+1}(x_k^{\nu+1})).$$
(5.2)

Combining criterion (A), (5.1) and (5.2), we have

$$||(w^{\nu+1}, x^{\nu+1}, \eta^{\nu+1}, \zeta^{\nu+1}) - P_{\nu}(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})|| \le \epsilon_{\nu}, \text{ with } \sum_{\nu=1}^{\infty} \epsilon_{\nu} < \infty.$$
(5.3)

From Assumption 2.1, there exists a saddle point of the Lagrangian (1.2). Therefore based on the relationship between (1.2) and (2.11) in Lemma 1, there exists at least one saddle point of the Lagrangian function \bar{L} . On the basis of [31], the sequence of elements $(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ generated in this manner from any initial $(w^1, x^1) \in W \times X$ and $(\eta^1, \zeta^1) \in S$ converges to some saddle point $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ of the \bar{L} . Then (\bar{w}, \bar{x}) solves (2.6) and $(\bar{\eta}, \bar{\zeta})$ solves (2.12). By Lemma 1, both (a) and (b) hold.

For the local convergence analysis, we need the following stopping criteria

$$\operatorname{dist}(0, \partial \phi_{k,\rho,c}^{\nu}(x_k^{\nu+1})) \leq \frac{\epsilon_{\nu}'}{cK(\rho||E|| + ||E|| + 1)} \min\{1, ||x_k^{\nu+1} - x_k^{\nu}||\}, \qquad \sum_{\nu=0}^{\infty} \epsilon_{\nu}' < \infty.$$
(B)

The iADA does not impose any condition on the choice of c. We set $c = \rho$ for simplicity of the following analysis. The coefficient $\rho/2$ for the primal proximal term $||w - w^{\nu}||^2$ in (2.14) can be changed to $1/2\rho$ after the rescaling $w' = \rho w$ and such rescaling only applies to the magnitude of w and does not bring any

other changes to the iADA. So this distinction from the standard proximal point method for minimax problems in [30, Section 5] will not influence the following convergence results.

Theorem 5.2 Suppose Assumptions 2.1 and 3.1 hold and let $\{(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$ in $W \times X \times S$ be the infinite sequence generated by the ADA with the stopping criterion (B). Then, $(w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})$ converges to some saddle point $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ of (2.11) and there exists $\{\theta_{\nu}\}$ such that

$$dist((w^{\nu+1}, x^{\nu+1}, \eta^{\nu+1}, \zeta^{\nu+1}), \mathcal{T}_{\bar{L}}^{-1}((0, 0, 0, 0))) \leq \theta_{\nu} dist((w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu}), \mathcal{T}_{\bar{L}}^{-1}((0, 0, 0, 0)))$$

for sufficient large ν and $\lim_{\nu \to \infty} \theta_{\nu} = \frac{2\tau}{\sqrt{(4\tau^2 + \rho^2)}} < 1$ for some τ .

Proof. From Corollary 4.18, we have shown that there exist $\tau, \delta > 0$ such that for all $v = (v_1, v_2, v_3, v_4) \in (\mathbb{R}^m)^K \times \mathbb{R}^n \times (\mathbb{R}^m)^K \times (\mathbb{R}^m)^K$ and $||v|| \leq \delta$, any $(w(v), x(v), \eta(v), \zeta(v)) \in \mathcal{T}_{\bar{L}}^{-1}(v)$ satisfies

$$dist((w(v), x(v), \eta(v), \zeta(v)), \mathcal{T}_{\bar{L}}^{-1}((0, 0, 0, 0))) \le 2\tau ||v||.$$
(5.4)

So this theorem follows from [25, Theorem 2.1].

Remark 1. In Theorem 5.1, we have shown that the sequence $\{u^{\nu} = (w^{\nu}, x^{\nu}, \eta^{\nu}, \zeta^{\nu})\}$ converges to some saddle point $(\bar{w}, \bar{x}, \bar{\eta}, \bar{\zeta})$ of (2.11) and hence $\{u^{\nu}\}$ lies in a compact set. Based on the observation in (4.19) and the proof of [25, Theorem 2.1], the compactness of assumption of X_k (Assumption 3.1(e)) is no longer needed for Theorem 5.2.

Remark 2. When $c \neq \rho$, the local linear convergence still holds while the convergence rate $(\lim_{\nu \to \infty} \theta_{\nu})$ changes.

Next, we provide some well-known examples on which the iADA enjoys the local linear convergence.

Convex regularization. Many problems from empirical risk minimization and variable selection can be written as the following:

$$\min_{x} f(x; (A, b)) + r(x)$$
(5.5)

where $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, $f(\cdot)$ is the loss function which is often strongly convex with Lipschitz continuous gradient and $r(\cdot)$ is a convex regularization term which is possibly nonsmooth (*e.g.*, the ℓ_1 -norm and TV-norm). By adding the constraint x - z = 0, the above problem can be reformulate as

$$\min_{x,z} f(x; (A, b)) + r(z)
s.t. \quad x - z = 0.$$
(5.6)

Exchange problem. Consider a network with K agents exchanging n commodities. Let $x_k \in \mathbb{R}^n$ be the amount of commodities in each agent k and $f_k : \mathbb{R}^n \to \mathbb{R}$ be its corresponding cost function. The exchange problem is given by

$$\min_{\{x_k\}_{k=1}^K} \sum_{k=1}^K f_k(x_k) \qquad \text{s.t. } \sum_{k=1}^K x_k = 0 \tag{5.7}$$

which minimizes the total cost subject to the equilibrium constraint on all K agents. In this special case, $E_k = I$ and q = 0. Optimization problems in this form arise in many areas such as resource allocation [2,38], multi-agent system [39] and image processing [37]. When the cost function f_k in each agent satisfies Assumption 3.1(a)-(c), based on Theorem 5.2, local linear convergence result is valid for the iADA under certain approximation criteria.

6 Numerical Examples

In this section, we demonstrate the linear convergence of both the exact ADA and the inexact ADA by some simple numerical examples. All the computational tasks for numerical experiments are implemented in Matlab 2017b running on a MacBook Pro. Retina, 2.6 GHz Intel Core i7 with 16Gb 2133 MHz LPDDR3 memory.

6.1 The *lasso* problem

We perform some numerical experiments of Algorithm 1 for solving the following *lasso* problem:

$$\min_{x \in \mathbb{R}^d} \frac{1}{2} ||Ax - b||_2^2 + \lambda_1 ||x||_1 \tag{6.1}$$

where $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$ and λ_1 is the regularization parameter. By introducing an auxiliary variable $z \in \mathbb{R}^d$, the above problem is equivalent to

$$\min_{\substack{x,z \in \mathbb{R}^d}} \frac{1}{2} ||Ax - b||_2^2 + \lambda_1 ||z||_1
s.t. \quad x - z = 0.$$
(6.2)

Clearly, (6.2) is a two-block decomposition problem with $f_1(x_1) = \frac{1}{2} ||Ax_1 - b||_2^2$ and $f_2(x_2) = \lambda_1 ||x_2||_1$ by replacing x and z with x_1 and x_2 . Notice that f_1 and f_2 are not necessarily strongly convex. In this case,

$$\phi_{1,\rho,c}^{\nu}(x_1) = \frac{1}{2} ||Ax_1 - b||_2^2 + \frac{\rho}{4} ||x_1 - w_1^{\nu} + \frac{2}{\rho} y_1^{\nu}||_2^2 + \frac{1}{2c} ||x_1 - x_1^{\nu}||_2^2,
\phi_{2,\rho,c}^{\nu}(x_2) = \lambda_1 ||x_2||_1 + \frac{\rho}{4} ||x_2 + w_2^{\nu} - \frac{2}{\rho} y_2^{\nu}||_2^2 + \frac{1}{2c} ||x_2 - x_2^{\nu}||_2^2.$$
(6.3)

For the first block, we can derive that

$$x_1^{\nu+1} = [A^T A + (\frac{\rho}{2} + \frac{1}{c})\mathbf{I}_d]^{-1}(A^T b + \frac{\rho}{2}w_1^{\nu} + \frac{x_1^{\nu}}{c} - y_1^{\nu}).$$
(6.4)

Though it may be time consuming to compute $[A^T A + (\frac{\rho}{2} + \frac{1}{c})\mathbf{I}_d]^{-1}$ when d is large, we only need to compute it at the initialization stage. The special structure of $A^T A + (\frac{\rho}{2} + \frac{1}{c})\mathbf{I}_d$ can be exploited and substantially improve

performance, see [3, Section 4.2]. For the second block, the exact solution to the subproblem in each iteration is given by

$$x_2^{\nu+1} := S(\frac{y_2^{\nu} + x_2^{\nu}/c - \rho w_2^{\nu}/2}{\rho/2 + 1/c}, \frac{\lambda_1}{\rho/2 + 1/c})$$
(6.5)

where the soft thresholding operator S is defined in [3].

We generate the matrix A and 0.05d nonzero entries of the sparse vector $x_0 \in \mathbb{R}^d$ from the standard Gaussian distribution $\mathcal{N}(0, 1)$. We then let the response vector $b \in \mathbb{R}^n$ be given by $b = Ax_0 + \epsilon$ where $\epsilon \sim \mathcal{N}(0, 10^{-3}\mathbf{I}_n)$ and let the regularization parameter λ_1 be $0.1||A^Tb||_{\infty}$. We test the algorithm on two different sets of (n, d): (1000, 4000), (2000, 20000).

In our test, we compare the result of ADA with two other methods for the *lasso* problem: ADMM¹[3] and P-PPA[1]. For the implementation of ADMM, we take a widely-used step-length 1.618 and a fixed penalty parameter 1. For P-PPA, we used the parameters suggested in [1] for solving the *lasso*. For the ADA, we choose the following three pairs of (ρ, c) : (1, 1), (5, 5), (10, 10). In each iteration, we solve both subproblems exactly and the computational time for all three algorithms is nearly the same. For all algorithms, we use the same initial point $(x^0, y^0) = (\mathbf{0}, \mathbf{0})$ and run 300 iterations. For all comparison algorithms, we report the objective value $f(x^{\nu}) = \frac{1}{2} ||Ax_1^{\nu} - b||^2 + \lambda_1 ||x_2^{\nu}||_1$, and the residual norm $||x_1^{\nu} - x_2^{\nu}||$. The convergence results are presented in Figures 1 and 2.

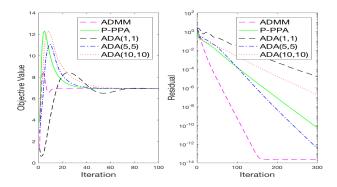


Fig. 1: Convergence results of ADA, ADMM and P-PPA for the *lasso*: (n, d) = (1000, 4000).

From Figure 1, we notice that ADMM performs best in the case (n, d) = (1000, 4000) while ADA achieves comparable performance with P-PPA when $(\rho, c) = (5, 5)$. This suggests that the convergence of ADA becomes slow if the proximal parameter is either too big or too small. When (n, d) = (2000, 20000), P-PPA shows the best convergence and ADA with $(\rho, c) = (10, 10)$ converges a little bit slower. Both ADMM and P-PPA methods use the Gauss-Seidel style update which tends to converge faster in terms of iterations, since it is able

¹ Available at http://web.stanford.edu/boyd/papers/admm/

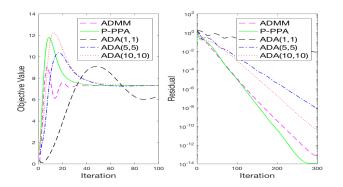


Fig. 2: Convergence results of ADA, ADMM and P-PPA for the *lasso*: (n, d) = (2000, 20000).

to incorporate information from the other coordinates more quickly. However, the Jacobi style update of ADA is more amenable for parallelization.

6.2 The exchange problem

For the exchange problem in (5.7), we consider the quadratic cost function $f_k(x_k) = \frac{1}{2} ||A_k x_k - b_k||^2$ where $A_k \in \mathbb{R}^{p \times n}$ and $b_k \in \mathbb{R}^p$, $k = 1, \ldots, K$. Then, the subproblems in each iteration can be written as

$$x_k^{\nu+1} = \underset{x_k}{\operatorname{argmin}} \frac{1}{2} ||A_k x_k - b_k||^2 + r||x_k - d_k^{\nu}||^2, \quad \forall k = 1, \dots, K,$$
(6.6)

for some $r \in \mathbb{R}_+$ and $d_k^{\nu} \in \mathbb{R}^n$. Notice that the matrices $A_k^T A_k + 2r \mathbf{I}_n, k = 1, \ldots, K$ are positive definite since r > 0. We only have to compute $(A_k^T A_k + 2r \mathbf{I}_n)^{-1}$ for one time before the iterations start. In the experiments, we randomly generate the optimal solution $x_k^*, k = 1, \ldots, K - 1$ by the standard normal distribution and set $x_K^* = -\sum_{k=1}^{K-1} x_k^*$. The matrices $A_k, k = 1, \ldots, K$ are generated from standard Gaussian distribution and we let $b_k = A_k x_k^*$. In this setting, x^* is an optimal solution to (5.7) but not necessarily the unique one, and the optimal value is 0. We set K = 20, n = 1000, p = 800, and none of $f_k(x_k), k = 1, \ldots, K$ is strongly convex. We compare the performance of ADA with VSADMM and Prox-JADMM mentioned in Section 2.4. For the implementation of VSADMM and Prox-JADMM, we use codes provided in [9]. For the proximal parameters of ADA, we set $(\rho, c) = (10, 10)$ in the experiment.

For all of the algorithms, we start from the same initial point $(x^0, y^0) = (\mathbf{0}, \mathbf{0})$ and run 500 iterations. Figure 3 shows the objective function value $\sum_{k=1}^{K} f_k(x_k)$ and the residual $||\sum_{k=1}^{K} x_k||$ of each iteration for the average outcome of 10 random simulations. We can see that ADA shows a better convergence of the objective value compared with VSADMM and is slower than

Prox-JADMM in terms of iterations. However, Prox-JADMM requires extra computational time to update the proximal parameters which is shown in Figure 4. Overall, ADA shows competitive convergence results in this experiment compared with two variants of the classical ADMM method which facilitate parallelization.

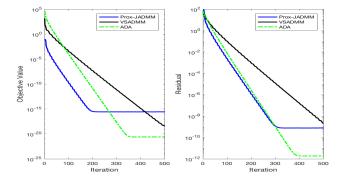


Fig. 3: Exchange Problem: K = 20, n = 1000, p = 800. Convergence results versus iteration.

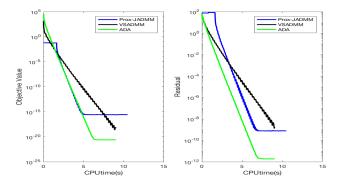


Fig. 4: Exchange Problem: K = 20, n = 1000, p = 800. Convergence results versus time.

6.3 Distributed ℓ_1 -regularized logistic regression

Here, we use iADA to solve the convex regularization problem (5.6) with a modest number of features but a relative large number of training examples.

Many statistical problems belong to this regime, with a large n and a small d dataset. In particular, we consider the following ℓ_1 -regularized logistic regression:

$$\min_{x \in \mathbb{R}^d} F(x) = \sum_{j=1}^n \ell(x; (a_j, b_j)) + \lambda ||x||_1$$
(6.7)

where $(a_j, b_j) \in \mathbb{R}^{d+1}, j = 1, ..., n$ and $\ell(x; (a_j, b_j)) = \log(1 + \exp(-b_j a_j^T x))$. For the purpose of parallel computation, we partition $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ into N blocks

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_N \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b^1 \\ \vdots \\ b^N \end{bmatrix}$$

with $A_i \in \mathbb{R}^{n_i \times d}$ and $b^i \in \mathbb{R}^{n_i}$. Define $\bar{n}_i = \sum_{j=1}^i n_j$ and we notice $\bar{n}_0 = 0$ and $\bar{n}_N = \sum_{j=1}^N n_j = n$. By introducing variables $x_i \in \mathbb{R}^d, i = 1, \ldots, N$, (6.7) can be transformed into the following:

$$\min_{\substack{x_i, z \in \mathbb{R}^d \\ \text{s.t.}}} \sum_{i=1}^N \ell_i(x_i; (A_i, b^i)) + \lambda ||z||_1$$
s.t. $x_i - z = 0, \quad i = 1, \dots, N.$
(6.8)

where $\ell_i(x_i; (A_i, b^i)) = \sum_{j=\bar{n}_{i-1}+1}^{\bar{n}_i} \log(1 + \exp(-b_j a_j^T x_i))$. In our experiment, we use two publicly available datasets: (1) the **w8a** dataset (49749 examples and 300 features) and (2) the **ijcnn1** dataset (49990 examples and 22 feature). The main step of iADA algorithm is given by

$$x_{i}^{\nu+1} \approx \underset{x_{i}}{\operatorname{argmin}} \underbrace{\ell_{i}(x_{i}; (A_{i}, b^{i})) + \frac{\rho}{4} ||x_{i} - w_{x,i}^{\nu} + \frac{2}{\rho} y_{x,i}^{\nu} ||_{2}^{2} + \frac{1}{2c} ||x_{i} - x_{i}^{\nu} ||_{2}^{2}}_{\phi_{i,\rho,c}^{\nu}(x_{i})},$$

$$z^{\nu+1} = \underset{z}{\operatorname{argmin}} \lambda_{1} ||z||_{1} + \frac{\rho}{4} \sum_{i=1}^{N} ||z + w_{z,i}^{\nu} - \frac{2}{\rho} y_{z,i}^{\nu} ||_{2}^{2} + \frac{1}{2c} ||z - z^{\nu} ||_{2}^{2},$$

$$(6.9)$$

where $w_x^{\nu} = (w_{x,1}^{\nu}, \ldots, w_{x,N}^{\nu}) \in \mathbb{R}^{Nd}, y_x^{\nu} = (y_{x,1}^{\nu}, \ldots, y_{x,N}^{\nu}) \in \mathbb{R}^{Nd}$ and $w_z^{\nu}, y_z^{\nu} \in \mathbb{R}^{Nd}$. The x_i update involves an ℓ_2 regularized logistic regression which cannot be solved exactly. Here, we use the L-BFGS algorithm to solve them until the inexact criteria (A) and (B) are satisfied. Such criteria can be checked by identifying the norm of the gradient $||\nabla \phi_{i,\rho,c}^{\nu}(x_i)||$. For the z update, exact solutions can be derived by the soft threshold operator.

For comparison, we consider the inexact ADMM (iADMM) method proposed in [12, 16]. Similar subproblems as (6.9) will arise for x_i and z updates. An analogous inexact criterion as (A) are proposed in [12, 16] to guarantee the convergence of the inexact ADMM and can also be verified by examining the norm of the gradient in the x_i updates.

In the experiment, we set $\epsilon_{\nu} = \frac{1}{\nu^{\gamma}}$ with $\gamma = 1.0, 1.5, 2.0$ to control the inexactness of the x_i updates in both algorithms. We also consider different partitions with N = 20, 50. For the implementation of iADMM, we use a steplength 1.618 and a fixed penalty parameter 10 after tuning. For iADA, we choose the proximal parameters (ρ, c) = (10, 10). Both algorithms terminated when

$$\frac{\sum_{i=1}^{N} ||x_i^{\nu} - z^{\nu}||_2}{N||z^{\nu}||_2} \le 10^{-6} \text{ and } \frac{|F(z^{\nu}) - F(z^*)|}{\max\{1, |F(z^*)|\}} \le 10^{-10}$$

are satisfied. $F(z^*)$ is the optimal solution of (6.8) derived by running iADMM for 2000 iterations.

The computational results are presented in Table 1. The datasets are listed in the first column. The numbers of partitions N and the inexactness parameter γ are given in columns two and three separately. The ∞ symbol in the third column represents the exact x_i updates achieved by setting $\epsilon_{\nu} = 1e - 10$ in all iterations. The average number of iterations (upon round off) for iADA and iADMM are given in the next two columns. The total amount of L-BFGS updates for both methods are presented in columns 6-7 and the average CPU time (in seconds) for these methods are given in the last two columns.

Iteration L-BFGS CPU time Dataset Ν γ iADA iADA iADMM iADMM iADA iADMM 24.001.0274380 70361 83703 30.00 1.51691974108958199 14.1819.15202.044616 65647 15.0220.8716419513349945 54928 15.7017.78150 ∞ w8a 1.017221188460 127538 16.0022.341.514012078594 72673 13.1010.665067909 2.099 88 60368 11.7510.10 10670776276289313.1910.50 ∞ 276 49378 79120 17.02 1.0202 26.801.5114135297414214210.1614.10202.011213431308 4674210.5015.1119073111 73193 23.0222.15186 ∞ ijcnn1 1.0106 228 68001 115891 11.7422.051.5107 11258093 64195 10.97 11.58502.09.27 99 88 57777 50099 10.699583 68652 69291 11.3211.21 ∞

Table 1: Comparison of iADA and iADMM for solving (6.8).

From Table 1, we see that when $\gamma = 1.5$ or 2.0, iADA shows better performance in the case N = 20 while iADMM converges faster when N = 50. For both algorithms, the CPU time is much longer in the case of $\gamma = 1.0$ when the convergence is not guaranteed in theory. Finally, compared with the exact update, it takes more iterations for the inexact version of both algorithms to converge but with shorter CPU time. This phenomenon results from the large number of L-BFGS updates in each iteration of exact ADA and ADMM.

7 Conclusions

In this paper, we study the convergence results of the ADA and its inexact version, the iADA, for solving multi-block separable convex minimization problems subject to linear constraints. First, we prove the global convergence and the $o(1/\nu)$ rate for the exact ADA when there exists a saddle point for the corresponding Lagrangian function. Next, global convergence and local linear convergence for the iADA are established under some mild assumptions and certain approximation criteria.

Before ending this paper, we would like to discuss two possible directions related to the ADA. Firstly, we notice that both the primal PPA [13] and the Augmented Lagrangian Method [17] can be accelerated by utilizing the idea from Nesterov's seminal work [28]. It is natural to ask whether we can accelerate the ADA based on similar techniques since all of them belong to the general PPA framework. Secondly, the applicability of the approximation criteria in (A) and (B) is limited in practice due to the summable requirement and more implementable approximation criteria are needed for practical problems.

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References

- 1. Bai, J., Zhang, H., Li, J.: A parameterized proximal point algorithm for separable convex optimization. Optimization Letters pp. 1–20 (2017)
- Beck, A., Nedic, A., Ozdaglar, A., Teboulle, M.: An O(1/k) Gradient Method for Network Resource Allocation Problems. IEEE Transactions on Control of Network Systems 1(1), 64–73 (2014)
- Boyd, S., Parikh, N., Chu, E., Peleato, B., Eckstein, J.: Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends
 m Machine Learning 3(1), 1–122 (2011)
- Chang, T.H., Nedic, A., Scaglione, A.: Distributed constrained optimization by consensus-based primal-dual perturbation method. IEEE Transactions on Automatic Control 59(6), 1524–1538 (2014)
- Chatzipanagiotis, N., Dentcheva, D., Zavlanos, M.M.: An augmented Lagrangian method for distributed optimization. Mathematical Programming 152(1-2), 405–434 (2015)
- Chen, C., He, B., Ye, Y., Yuan, X.: The direct extension of ADMM for multi-block convex minimization problems is not necessarily convergent. Mathematical Programming 155(1-2), 57–79 (2016)
- Chen, G., Teboulle, M.: A proximal-based decomposition method for convex minimization problems. Mathematical Programming 64(1-3), 81–101 (1994)
- Cui, Y., Sun, D., Toh, K.C.: On the R-superlinear convergence of the KKT residues generated by the augmented Lagrangian method for convex composite conic programming. arXiv preprint arXiv:1706.08800 (2017)

- Deng, W., Lai, M.J., Peng, Z., Yin, W.: Parallel Multi-Block ADMM with o(1/k) Convergence. Journal of scientific computing 71(2), 712–736 (2017)
- Deng, W., Yin, W.: On the global and linear convergence of the generalized alternating direction method of multipliers. Journal of Scientific Computing 66(3), 889–916 (2016)
 Dontchev, A.L.: Implicit functions and solution mappings (2009)
- Eckstein, J., Bertsekas, D.P.: On the Douglas—Rachford splitting method and the proximal point algorithm for maximal monotone operators. Mathematical Programming 55(1), 293–318 (1992)
- Güler, O.: New proximal point algorithms for convex minimization. SIAM Journal on Optimization 2(4), 649–664 (1992)
- 14. Han, D., Sun, D., Zhang, L.: Linear rate convergence of the alternating direction method of multipliers for convex composite quadratic and semi-definite programming. arXiv preprint arXiv:1508.02134 (2015)
- 15. Han, D., Yuan, X.: A note on the alternating direction method of multipliers. Journal of Optimization Theory and Applications **155**(1), 227–238 (2012)
- He, B., Liao, L.Z., Han, D., Yang, H.: A new inexact alternating directions method for monotone variational inequalities. Mathematical Programming 92(1), 103–118 (2002)
- 17. He, B., Yuan, X.: On the acceleration of augmented lagrangian method for linearly constrained optimization. Optimization online ${\bf 3}$ (2010)
- He, B., Yuan, X.: On the O(1/n) Convergence Rate of the Douglas–Rachford Alternating Direction Method. SIAM Journal on Numerical Analysis 50(2), 700–709 (2012)
- 19. He, B., Yuan, X.: On non-ergodic convergence rate of Douglas–Rachford alternating direction method of multipliers. Numerische Mathematik **130**(3), 567–577 (2015)
- Hoffman, A.J.: On approximate solutions of systems of linear inequalities. Selected Papers Of Alan J Hoffman: With Commentary pp. 174–176 (2003)
- Hong, M., Luo, Z.Q.: On the linear convergence of the alternating direction method of multipliers. Mathematical Programming 162(1-2), 165–199 (2017)
- 22. Li, X., Sun, D., Toh, K.C.: A highly efficient semismooth Newton augmented Lagrangian method for solving Lasso problems. arXiv preprint arXiv:1607.05428 (2016)
- Liu, Y.J., Sun, D., Toh, K.C.: An implementable proximal point algorithmic framework for nuclear norm minimization. Mathematical programming 133(1), 399–436 (2012)
- Luo, Z.Q., Tseng, P.: On the convergence rate of dual ascent methods for linearly constrained convex minimization. Mathematics of Operations Research 18(4), 846–867 (1993)
- Luque, F.J.: Asymptotic convergence analysis of the proximal point algorithm. SIAM Journal on Control and Optimization 22(2), 277–293 (1984)
- Ma, S.: Alternating proximal gradient method for convex minimization. Journal of Scientific Computing 68(2), 546–572 (2016)
- Mulvey, J.M., Ruszczyn, A., et al.: A diagonal quadratic approximation method for large scale linear programs. Operations Research Letters 12(4), 205–215 (1992)
- 28. Nesterov, Y.: A method of solving a convex programming problem with convergence rate o (1/k2)
- Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Mathematical Programming at Oberwolfach pp. 206–214 (1981)
- Rockafellar, R.T.: Augmented Lagrangians and applications of the proximal point algorithm in convex programming. Mathematics of operations research 1(2), 97–116 (1976)
- Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM journal on control and optimization 14(5), 877–898 (1976)
- 32. Rockafellar, R.T.: PROBLEM DECOMPOSITION IN BLOCK-SEPARABLE CON-VEX OPTIMIZATION: IDEAS OLD AND NEW. Washington.edu (2017)
- Shefi, R., Teboulle, M.: Rate of convergence analysis of decomposition methods based on the proximal method of multipliers for convex minimization. SIAM Journal on Optimization 24(1), 269–297 (2014)
- Spingarn, J.E.: Applications of the method of partial inverses to convex programming: decomposition. Mathematical Programming 32(2), 199–223 (1985)
- Tseng, P.: Applications of a splitting algorithm to decomposition in convex programming and variational inequalities. SIAM Journal on Control and Optimization 29(1), 119–138 (1991)

- Wang, X., Hong, M., Ma, S., Luo, Z.Q.: Solving multiple-block separable convex minimization problems using two-block alternating direction method of multipliers. arXiv preprint arXiv:1308.5294 (2013)
- Wright, S.J.: Accelerated block-coordinate relaxation for regularized optimization. SIAM Journal on Optimization 22(1), 159–186 (2012)
- Xiao, L., Boyd, S.: Optimal scaling of a gradient method for distributed resource allocation. Journal of optimization theory and applications 129(3), 469–488 (2006)
- You, K., Xie, L.: Network topology and communication data rate for consensusability of discrete-time multi-agent systems. IEEE Transactions on Automatic Control 56(10), 2262-2275 (2011)