

THE GAUSS-SEIDEL METHOD FOR GENERALIZED NASH EQUILIBRIUM PROBLEMS OF POLYNOMIALS

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ABSTRACT. This paper concerns the generalized Nash equilibrium problem of polynomials (GNEPP). We apply the Gauss-Seidel method and Lasserre type Moment-SOS relaxations to solve GNEPPs. The convergence of the Gauss-Seidel method is known for some special GNEPPs, such as generalized potential games (GPGs). We give a sufficient condition for GPGs and propose a numerical certificate, based on Putinar's Positivstellensatz. Numerical examples for both convex and nonconvex GNEPPs are given for demonstrating the efficiency of the proposed method.

1. Introduction

The generalized Nash equilibrium problem (GNEP) is a kind of games such that the feasible set of each player's strategy depends on other players' strategies. Let N be the number of players. Suppose the i th player's strategy is the variable $x_i \in \mathbb{R}^{n_i}$ (the n_i -dimensional Euclidean space over the real field \mathbb{R}). The vector of all players' strategy is

$$\mathbf{x} := (x_1, \dots, x_N).$$

The total dimension of all players' strategies is $n := n_1 + \dots + n_N$. For convenience, we use x_{-i} to denote the subvector of all players' strategies except the i th one, i.e.,

$$x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N).$$

When the i th player's strategy x_i is considered, we use (x_i, x_{-i}) to represent \mathbf{x} . When (y, x_{-i}) is written, it means that the i th player's strategy is $y \in \mathbb{R}^{n_i}$ while the vector of all other players' strategies is fixed to be x_{-i} . This paper considers GNEPs whose objective and constraining functions are given by polynomials.

Definition 1.1. A generalized Nash equilibrium problem of polynomials (GNEPP) is to find $\mathbf{x} \in \mathbb{R}^n$ such that each x_i is an optimizer of the i th player's optimization problem

$$(1.1) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ \text{s.t.} & g_{i,j}(x_i, x_{-i}) \geq 0 \quad (j = 1, \dots, s_i), \end{cases}$$

where all $f_i(x_i, x_{-i})$ and $g_{i,j}(x_i, x_{-i})$ are polynomial functions in \mathbf{x} . A solution \mathbf{x} satisfying the above is called a generalized Nash equilibrium (GNE).

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Let $g_i = (g_{i,1}, \dots, g_{i,s_i}) : \mathbb{R}^n \rightarrow \mathbb{R}^{s_i}$ be the vector-valued function. The inequality $g_i(x_i, x_{-i}) \geq 0$ is defined componentwisely. Then (1.1) can be rewritten as

$$(1.2) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ \text{s.t.} & g_i(x_i, x_{-i}) \geq 0. \end{cases}$$

For given x_{-i} , the feasible strategy set for the i th player is

$$(1.3) \quad X_i(x_{-i}) := \{x_i \in \mathbb{R}^{n_i} : g_i(x_i, x_{-i}) \geq 0\}.$$

The entire strategy vector \mathbf{x} is said to be a feasible point if each subvector x_i of \mathbf{x} is feasible for (1.2). For instance, the following GNEP with two players

$$(1.4) \quad \begin{array}{l} \min_{x_1 \in \mathbb{R}^1} \quad x_1 \\ \text{s.t.} \quad x_2(x_1 - x_2 - 1) \geq 0, \\ \quad \quad x_1 \geq 0, \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in \mathbb{R}^1} \quad (x_2)^2 - (x_1 - 1)x_2 \\ \text{s.t.} \quad (x_1)^2 + (x_2)^2 \leq 3, \\ \quad \quad x_2 \geq 0 \end{array} \right.$$

is a GNEPP. The dimensions $n_1 = n_2 = 1$ and

$$\begin{aligned} X_1(x_{-1}) &= \{x_1 \in \mathbb{R} : x_2(x_1 - x_2 - 1) \geq 0, x_1 \geq 0\}, \\ X_2(x_{-2}) &= \{x_2 \in \mathbb{R} : (x_1)^2 + (x_2)^2 \leq 3, x_2 \geq 0\}. \end{aligned}$$

For the first player, when $x_2 > 0$, its feasible set is $x_1 \geq x_2 + 1$, and its best strategy is $x_1 = x_2 + 1$. When $x_2 = 0$, the first player's best strategy is $x_1 = 0$. For any fixed x_1 with $x_1^2 \leq 3$, the second player's problem is feasible and its best strategy is $\max((x_1 - 1)/2, 0)$. One can verify that $(0, 0)$ is a GNE for this GNEPP.

Generalized Nash equilibrium problems have broad applications, for instance, in the environmental pollution control [8, 17]. Let N be the number of countries involved in the pollution control and $x_{i,0}$ denote the (gross) emissions from the i th country. Assume that the by-product gross emissions are proportional to the industrial output. The revenue of the i th country depends on $x_{i,0}$. Typically, the revenue is $x_{i,0}(b_i - 1/2x_{i,0})$ with a given parameter b_i . The variable $x_{i,j}$ represents the investment from country i to country j . Let $x_i := (x_{i,0}, \dots, x_{i,N})$. For an investor, the benefit of the investment lies in the *emissions reduction units* $\gamma_{i,j}x_{i,j}$ with given parameters $\gamma_{i,j}$ ($i, j = 1, \dots, N$). The net emission in country i is $x_{i,0} - \sum_{j=1}^N \gamma_{j,i}x_{j,i}$, which is always nonnegative. The accounted-for-emissions for the i th country is $x_{i,0} - \sum_{j=1}^N \gamma_{i,j}x_{i,j}$. It must be kept below or equal a certain prescribed level E_i under the environmental control. The pollution in a country may affect other countries. The pollution damage for the i th country is

$$p_i := x_{i,0} - \sum_{j=1}^N \gamma_{j,i}x_{j,i} + 2 \prod_{k=1}^N (x_{k,0} - \sum_{j=1}^N \gamma_{j,k}x_{j,k}).$$

For given parameters $b_i, \gamma_{i,j}, E_i$, the i th country's optimization problem is

$$(1.5) \quad \begin{cases} \min_{x_i} & -x_{i,0}(b_i - \frac{1}{2}x_{i,0}) + \sum_{j=1}^N x_{i,j} + p_i \\ \text{s.t.} & x_{i,0} \dots x_{i,j} \geq 0, \\ & x_{i,0} - \sum_{j=1}^N \gamma_{i,j}x_{i,j} \leq E_i, \\ & x_{k,0} - \sum_{j=1}^N \gamma_{j,k}x_{j,k} \geq 0 \quad (k = 1, \dots, N). \end{cases}$$

All countries expect to maximize their revenues subtracting investments and pollution damages. Another application of GNEPP is the model for Internet switching (see Example 5.8). More applications for GNEPPs can be found in [1, 9, 10, 47, 62].

1.1 GNEPs and some existing work

The GNEP is an extension of the Nash equilibrium problem (NEP) [38,39]. For NEPs, the feasible set of each player's strategy is independent of other players. The GNEP originated from economics and was studied in [3, 7, 11, 34, 52]. Robinson [49, 50] established the shadow prices for measuring the effectiveness in an optimization-based combat model. Scotti [57] introduced GNEPs into the study of structural design. Recently, GNEPs have been widely used in many different areas outside economics, such as transportation, telecommunications, pollution control. We refer to [2, 8, 45, 59, 63, 64] for related work.

The following is a classical result about existence of solutions for GNEPs [11,17]. We refer to [51] for the notion of outer and inner semicontinuity and quasi-convexity.

Theorem 1.2. *[11, 17] Suppose the GNEP of (1.1) satisfies:*

- (1) *There exist N nonempty, convex and compact sets $K_i \subseteq \mathbb{R}^{n_i}$ such that for every $(x_i, x_{-i}) \in \mathbb{R}^n$ with $x_i \in K_i$ and for every i , the set $X_i(x_{-i})$ is nonempty, closed and convex, $X_i(x_{-i}) \subseteq K_i$, and $X_i(\cdot)$, as a point-to-set map, is both outer and inner semicontinuous.*
- (2) *For every given x_{-i} , the function $f_i(\cdot, x_{-i})$ is quasi-convex on $X_i(x_{-i})$.*

Then, a generalized Nash equilibrium exists.

There are no special existence results for solutions of GNEPPs, to the best of the authors' knowledge. There exists some work for solving GNEPs. Under some convexity assumptions, the GNEP is equivalent to a quasi-variational inequality problem (QVIP) [5, 15, 24, 37, 46]. The Karush-Kuhn-Tucker (KKT) optimality conditions for each player's optimization problem can be used together with the semismooth Newton-type method [13, 14, 16]. A GNEP can be transformed to a NEP with the usage of penalty functions [18, 19, 23, 29]. Gap functions are frequently used for solving GNEPs [60]. A relaxation method for jointly convex GNEPs, based on inexact line search and Nikaido-Isoda functions, is given in [61]. A study on GNEPs with linear coupling constraints and mixed-integer variables is in [56]. Facchinei et al. [20] proposed the Gauss-Seidel method for solving GNEPs. Its main idea is to solve each player's optimization problem alternatively. We also refer to [53, 54] for studies on the Gauss-Seidel method for solving GNEPs with discrete and mixed integer variables. Convergence of the Gauss-Seidel method can be shown for some special GNEPs, such as generalized potential games (GPGs). We refer to [20, 35, 55] for studies on potential games and GPGs. Most of the existing methods assume that each individual player's optimization problem is convex. For more work about GNEPs, we refer to the surveys [17, 22].

1.2 Contributions

In this paper, we use the Gauss-Seidel method introduced in [20] for solving GNEPPs. This method requires to get global minimizers for the occurring optimization problems in each loop. Although the Gauss-Seidel method is not theoretically guaranteed to converge for all GNEPPs, it converges for many problems in the computational practice. For some special GNEPs, the convergence can be guaranteed for the Gauss-Seidel method. Generalized potential games (GPGs) are such GNEPs [20]. We would like to remark that the Gauss-Seidel method can be applied to a GNEPP even if it is not a GPG, while the convergence is not guaranteed. In practice, the method works well for many GNEPPs that are not GPGs.

In [20], it was shown that if a GNEP is in some special forms, then it is a GPG (see section 4). In section 4, we give the first numerical method for verifying GPGs. Our major results are:

- We use the Lasserre type Moment-SOS relaxations [32] to find global minimizers of the occurring polynomial optimization problems in each loop of the Gauss-Seidel method. These relaxations can solve the polynomial optimization problems globally, even if they are nonconvex.
- As demonstrated in section 5, the Gauss-Seidel method works well in practice. Moment-SOS relaxations can be used to verify if a computed solution is a GNE or not. There are no other numerical methods for solving GNEPPs efficiently, especially for nonconvex ones, to the best of the authors' knowledge.
- We give a sufficient condition for checking if a given GNEPP is a GPG or not. Based on it, a numerical certificate is given for checking GPGs. This is the first numerical method that can do this, to the best of the authors' knowledge.

The paper is organized as follows. Some preliminaries about polynomial optimization are given in Section 2. Section 3 gives the Gauss-Seidel method for solving GNEPPs and studies its properties, and the algorithm of solving the occurring polynomial optimization problems globally in the Gauss-Seidel method. Section 4 focuses on generalized potential games. Numerical experiments are given in Section 5.

2. Preliminaries

The symbol \mathbb{N} stands for the set of nonnegative integers, and \mathbb{R} for the real field. The norm $\|\cdot\|$ is the standard Euclidean norm of a vector. For a real number t , $\lceil t \rceil$ (resp., $\lfloor t \rfloor$) denotes the smallest integer not smaller than t (resp., the biggest integer not bigger than t).

The variable $x_i \in \mathbb{R}^{n_i}$ is the strategy of the i th player, and $x_{i,j}$ denotes the j th component of x_i , for $j = 1, \dots, n_i$. In the Gauss-Seidel method, we use $x_i^{(k)}$ to denote the value of x_i in the k th loop. Similarly, $x_{i,j}^{(k)}$ denotes $x_{i,j}$ in the k th iteration. In each loop of the Gauss-Seidel method, we need to solve a polynomial optimization problem about the i th player's strategy vector $x_i := (x_{i,1}, \dots, x_{i,n_i})$. Let $\mathbb{R}[x_i]$ denote the ring of real polynomials in x_i , and for a degree d , $\mathbb{R}[x_i]_d$ denotes the space of all polynomials in x_i whose degrees are at most d . For $x_i := (x_{i,1}, \dots, x_{i,n_i})$ and $\alpha := (\alpha_1, \dots, \alpha_{n_i}) \in \mathbb{N}^{n_i}$, denote

$$x_i^\alpha := x_{i,1}^{\alpha_1} \cdots x_{i,n_i}^{\alpha_{n_i}}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_{n_i}.$$

For an integer $d > 0$, denote the set

$$\mathbb{N}_d^{n_i} := \{\alpha \in \mathbb{N}^{n_i} : |\alpha| \leq d\}.$$

We use $[x_i]_d$ to denote the vector of all monomials in x_i and whose degree is at most d , ordered in the graded alphabetical ordering. For example, if $x_i = (x_{i,1}, x_{i,2})$, then

$$[x_i]_3 = (1, x_{i,1}, x_{i,2}, x_{i,1}^2, x_{i,1}x_{i,2}, x_{i,2}^2, x_{i,1}^3, x_{i,1}^2x_{i,2}, x_{i,1}x_{i,2}^2, x_{i,2}^3).$$

A polynomial $\sigma \in \mathbb{R}[x_i]$ is said to be a sum of squares (SOS) if $\sigma = s_1^2 + s_2^2 + \cdots + s_k^2$ for some polynomials $s_1, \dots, s_k \in \mathbb{R}[x_i]$. The set of all SOS polynomials in x_i is

denoted as $\Sigma[x_i]$. For a degree d , we denote the truncation

$$\Sigma[x_i]_d := \Sigma[x_i] \cap \mathbb{R}[x_i]_d.$$

For a tuple $g = (g_1, \dots, g_t)$ of polynomials in x_i , its quadratic module is the set

$$\text{Qmod}(g) = \Sigma[x_i] + g_1 \cdot \Sigma[x_i] + \dots + g_t \cdot \Sigma[x_i].$$

The truncation of $\text{Qmod}(g)$ with degree $2d$ is the set

$$\text{Qmod}(g)_{2d} = \Sigma[x_i]_{2d} + g_1 \cdot \Sigma[x_i]_{2d - \deg(g_1)} + \dots + g_t \cdot \Sigma[x_i]_{2d - \deg(g_t)}.$$

The tuple g defines the basic closed semi-algebraic set

$$(2.1) \quad \mathcal{S}(g) = \{x_i \in \mathbb{R}^{n_i} : g(x_i) \geq 0\}.$$

The quadratic module $\text{Qmod}(g)$ is said to be *archimedean* if there exists $p \in \text{Qmod}(g)$ such that the set $\mathcal{S}(p)$ is compact. If $\text{Qmod}(g)$ is archimedean, then $\mathcal{S}(g)$ must be compact. Conversely, if $\mathcal{S}(g)$ is compact, say, $\mathcal{S}(g)$ is contained in the ball $\mathcal{S}(R - \|x_i\|^2)$, then $\text{Qmod}(g, R - \|x_i\|^2)$ is archimedean and $\mathcal{S}(g) = \mathcal{S}(g, R - \|x_i\|^2)$. For a polynomial $f \in \mathbb{R}[\mathbf{x}]$, if $f \in \text{Qmod}(g)$, then it is clear that $f \geq 0$ on $\mathcal{S}(g)$. The reverse is not necessarily true. However, when $\text{Qmod}(g)$ is archimedean, if $f > 0$ on $\mathcal{S}(g)$, then $f \in \text{Qmod}(g)$. This conclusion is referred to as Putinar's Positivstellensatz [48]. Interestingly, if $f \geq 0$ on $\mathcal{S}(g)$, we still have $f \in \text{Qmod}(g)$, under some optimality conditions [43].

3. The Gauss-Seidel method for GNEPPs

The Gauss-Seidel method was introduced in [20] for solving GNEPs. The following is the general framework of the Gauss-Seidel method.

Algorithm 3.1. For the GNEP of (1.1), do the following:

Step 1. Choose a feasible starting point $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_N^{(0)})$, a positive regularization parameter $\tau^{(0)}$ and let $k := 0$.

Step 2. If $\mathbf{x}^{(k)}$ satisfies a suitable termination criterion, stop.

Step 3. For $i = 1, \dots, N$, compute a global minimizer $x_i^{(k+1)}$ of the optimization

$$(3.1) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & f_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \dots, x_N^{(k)}) + \tau^{(k)} \|x_i - x_i^{(k)}\|^2 \\ \text{s.t.} & g_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \dots, x_N^{(k)}) \geq 0. \end{cases}$$

Step 4. Choose a new regularization parameter $\tau^{(k+1)} \in [0, \tau^{(k)}]$.

Step 5. Let $\mathbf{x}^{(k+1)} := (x_1^{(k+1)}, \dots, x_N^{(k+1)})$, $k := k + 1$, and go to Step 2.

In practice, Algorithm 3.1 performs well for solving GNEPPs. It can compute equilibria for many problems. This is demonstrated in numerical experiments in Section 5. The GNEPPs are very hard to be solved by other existing methods, to the best of the authors' knowledge. On the other hand, Algorithm 3.1 is not theoretically guaranteed to converge for all GNEPPs. Its convergence can be shown for some special GNEPs, such as GPGs. In the following, we show how to implement Algorithm 3.1 when the defining functions are polynomials. After that, we review some properties of Algorithm 3.1.

3.1 Moment-SOS relaxations for polynomial optimization

We discuss how to implement Algorithm 3.1 when all the objective and constraining functions are given by polynomials. In its Step 3, the sub-optimization (3.1) is a polynomial optimization problem whose variable is $x_i \in \mathbb{R}^{n_i}$. In the following, we give a brief review for using the Lasserre type Moment-SOS hierarchy to solve (3.1). We refer to [31, 32, 42, 43] for related work about polynomial optimization.

For an even degree $2d > 0$, let $\mathbb{R}^{\mathbb{N}_{2d}^{n_i}}$ denote the space of all real vectors that are labeled by $\alpha \in \mathbb{N}_{2d}^{n_i}$. Each $y \in \mathbb{R}^{\mathbb{N}_{2d}^{n_i}}$ is labeled as

$$y = (y_\alpha)_{\alpha \in \mathbb{N}_{2d}^{n_i}}.$$

Such y is called a *truncated multi-sequence* (tms) of degree $2d$. For a polynomial $f = \sum_{\alpha \in \mathbb{N}_{2d}^{n_i}} f_\alpha x_i^\alpha \in \mathbb{R}[x]_{2d}$, define the operation

$$(3.2) \quad \langle f, y \rangle = \sum_{\alpha \in \mathbb{N}_{2d}^{n_i}} f_\alpha y_\alpha.$$

The operation $\langle f, y \rangle$ is linear in y for fixed f and it is linear in f for fixed y . For a polynomial $q \in \mathbb{R}[x_i]_{2d}$ and the integer $t = d - \lceil \deg(q)/2 \rceil$, the outer product $q \cdot [x_i]_t [x_i]_t^T$ is a symmetric matrix of length $\binom{n_i+t}{t}$. It can be expanded as

$$q \cdot [x_i]_t [x_i]_t^T = \sum_{\alpha \in \mathbb{N}_{2d}^{n_i}} x_i^\alpha Q_\alpha,$$

for some symmetric matrices Q_α . We denote

$$(3.3) \quad L_q^{(d)}[y] := \sum_{\alpha \in \mathbb{N}_{2d}^{n_i}} y_\alpha Q_\alpha.$$

It is called the d th *localizing matrix* of q and generated by y . For given q , $L_q^{(d)}[y]$ is linear in y . Clearly, if $q(u) \geq 0$ and $y = [u]_{2d}$, then

$$L_q^{(d)}[y] = q(u)[u]_t [u]_t^T \succeq 0.$$

For instance, if $n_i = d = 2$ and $q = 1 - x_{i,1} - x_{i,1}x_{i,2}$, then

$$L_q^{(2)}[y] = \begin{bmatrix} y_{00} - y_{10} - y_{11} & y_{10} - y_{20} - y_{21} & y_{01} - y_{11} - y_{12} \\ y_{10} - y_{20} - y_{21} & y_{20} - y_{30} - y_{31} & y_{11} - y_{21} - y_{22} \\ y_{01} - y_{11} - y_{12} & y_{11} - y_{21} - y_{22} & y_{02} - y_{12} - y_{13} \end{bmatrix}.$$

When $q = 1$ is the constant one polynomial, the localizing matrix $L_1^{(d)}[y]$ reduces to a moment matrix, which we denote as

$$M_d[y] := L_1^{(d)}[y].$$

When $\mathbf{q} := (q_1, \dots, q_s)$ is a tuple of polynomials, we then define that

$$L_{\mathbf{q}}^{(d)}[y] := \begin{bmatrix} L_{q_1}^{(d)}[y] & & & \\ & L_{q_2}^{(d)}[y] & & \\ & & \ddots & \\ & & & L_{q_s}^{(d)}[y] \end{bmatrix}.$$

In the following, we discuss how to solve (3.1). For convenience, denote

$$(3.4) \quad \begin{cases} f_i^{(k)} := f_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \dots, x_N^{(k)}) + \tau^{(k)} \|x_i - x_i^{(k)}\|^2, \\ g_i^{(k)} := g_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \dots, x_N^{(k)}). \end{cases}$$

They are polynomials in x_i . One can rewrite (3.1) equivalently as

$$(3.5) \quad \begin{cases} \vartheta_{\min} = \min_{x_i \in \mathbb{R}^{n_i}} f_i^{(k)}(x_i) \\ \text{s.t.} \quad g_i^{(k)}(x_i) \geq 0. \end{cases}$$

Denote the degree

$$d_0 := \max\{\lceil \deg(f_i^{(k)})/2 \rceil, \lceil \deg(g_i^{(k)})/2 \rceil\}.$$

For $d = d_0, d_0 + 1, \dots$, the d th moment relaxation for (3.5) is

$$(3.6) \quad \begin{cases} \vartheta_d := \min_y \langle f_i^{(k)}, y \rangle \\ \text{s.t.} \quad M_d[y] \succeq 0, L_{g_i^{(k)}}^{(d)} \succeq 0, \\ y_0 = 1, y \in \mathbb{R}_{2d}^{n_i}. \end{cases}$$

Its dual optimization problem is the SOS relaxation

$$(3.7) \quad \begin{cases} \max \quad \gamma \\ \text{s.t.} \quad f_i^{(k)} - \gamma \in \text{Qmod}(g_i^{(k)})_{2d}. \end{cases}$$

We refer to Section 2 for the truncated quadratic module $\text{Qmod}(g_i^{(k)})_{2d}$. By solving the relaxations (3.6)-(3.7) for $d = d_0, d_0 + 1, \dots$, we get the Moment-SOS hierarchy for solving (3.5). The following is the algorithm.

Algorithm 3.2. (The Moment-SOS hierarchy for solving (3.5)). Let $f_i^{(k)}, g_i^{(k)}$ be as in (3.4). Start with $d := d_0$.

Step 1. Solve the semidefinite relaxation (3.6). If (3.6) is infeasible, then (3.5) has no feasible points and stop; otherwise, solve it for a minimizer y^* and let $t := d_1$, where $d_1 := \lceil \deg(g_i^{(k)})/2 \rceil$.

Step 2. If y^* satisfies the rank condition

$$(3.8) \quad \text{rank } M_t[y^*] = \text{rank } M_{t-d_1}[y^*],$$

then extract $r := \text{rank } M_t(y^*)$ minimizers for (3.5) and stop.

Step 3. If (3.8) fails to hold and $t < d$, let $t := t + 1$ and then go to Step 2; otherwise, let $d := d + 1$ and go to Step 1.

The rank condition (3.8) is called *flat truncation* in the literature [42]. It is a sufficient (and almost necessary) condition for checking convergence of the Moment-SOS hierarchy. Flat truncation is useful for solving truncated moment problems and linear optimization with moment constraints [21, 25, 44]. Indeed, the Moment-SOS hierarchy has finite convergence if and only if the flat truncation is satisfied for some relaxation order, under some generic conditions [42]. When (3.8) holds, the method in [26] can be used to extract r minimizers for (3.5). The method is implemented in the software `GloptPoly 3` [27]. We refer to [26], [42] and [32, Chapter 6] for more details.

The convergence properties of Algorithm 3.2 are as follows. By solving the hierarchy of relaxations (3.6)-(3.7), we can get a monotonically increasing sequence of lower bounds $\{\vartheta_d\}_{d=d_0}^\infty$ for the minimum value ϑ_{\min} , i.e.,

$$\vartheta_{d_0} \leq \vartheta_{d_0+1} \leq \dots \leq \vartheta_{\min}.$$

When $\text{Qmod}(g_i^{(k)})$ is archimedean, we have $\vartheta_d \rightarrow \vartheta_{\min}$ as $d \rightarrow \infty$, as shown in [31]. If $\vartheta_d = \vartheta_{\min}$ for some d , the relaxation (3.6) is said to be exact (or tight) for solving

(3.1). For such a case, the Moment-SOS hierarchy is said to have finite convergence. The Moment-SOS hierarchy has finite convergence when the archimedean and some optimality conditions hold [43]. Although there exist special polynomials such that the Moment-SOS hierarchy fails to have finite convergence, such special problems belong to a set of measure zero in the space of input polynomials [43].

3.2 Some properties of Algorithm 3.1

Although Algorithm 3.1 converges for many problems, it is possible that it does not converge for some special ones. For instance, it is possible that (3.1) becomes infeasible after some loops even if the starting point $\mathbf{x}^{(0)}$ is feasible. The following is such an example.

Example 3.3. Consider the 2-player GNEP

$$(3.9) \quad \begin{array}{l} \min_{x_1 \in \mathbb{R}^1} \quad -x_1 - x_2 \\ \text{s.t.} \quad 0 \leq x_1 \leq 2 \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in \mathbb{R}^1} \quad x_1 x_2 \\ \text{s.t.} \quad x_1 + (x_2)^2 \leq 1. \end{array} \right.$$

If Algorithm 3.1 begins with $(x_1^{(0)}, x_2^{(0)}) = (0, 1)$ and uses the constant $\tau^{(k)} = 0.05$, then $x_1^{(1)} = 2$ and (3.1) is infeasible for $k = 1$ and $i = 2$.

When a GNEP has a shared constraint, i.e., there exists a set $X \subseteq \mathbb{R}^n$ such that $X_i(x_{-i}) = \{x_i : (x_i, x_{-i}) \in X\}$ for all players, then the suboptimization (3.1) is feasible for all k , provided that the initial point $\mathbf{x}^{(0)}$ is feasible [20]. Beyond the concern of infeasibility, the sequence of $\mathbf{x}^{(k)}$ produced by Algorithm 3.1 might be alternating and does not converge. Let's see the following example.

Example 3.4. Consider the 2-player GNEP

$$(3.10) \quad \begin{array}{l} \min_{x_1 \in \mathbb{R}^1} \quad x_1 \\ \text{s.t.} \quad x_1 \geq x_2 \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in \mathbb{R}^1} \quad x_1 x_2 \\ \text{s.t.} \quad (x_1)^2 + (x_2)^2 = 2. \end{array} \right.$$

If Algorithm 3.1 starts with $(x_1^{(0)}, x_2^{(0)}) = (1, 1)$ and uses the constant $\tau^{(k)} = 0.001$, the sub-optimization (3.1) for the first player is

$$\begin{cases} \min_{x_1 \in \mathbb{R}^1} & x_1 + 0.001(x_1 - 1)^2 \\ \text{s.t.} & x_1 \geq 1. \end{cases}$$

Its minimizer $x_1^{(1)} = 1$. After plugging $(x_1^{(1)}, x_2^{(0)})$ into (3.1), the sub-optimization (3.1) for the second player is

$$\begin{cases} \min_{x_2 \in \mathbb{R}^1} & x_2 + 0.001(x_2 - 1)^2 \\ \text{s.t.} & x_2^2 = 1, \end{cases}$$

whose minimizer $x_2^{(1)} = -1$. After one iteration, Algorithm 3.1 produced the point $\mathbf{x}^{(1)} = (1, -1)$. For the loop of $k = 1$, the sub-optimization problem (3.1) for the first player is

$$\begin{cases} \min_{x_1 \in \mathbb{R}^1} & x_1 + 0.001(x_1 - 1)^2 \\ \text{s.t.} & x_1 \geq -1, \end{cases}$$

whose minimizer $x_1^{(2)} = -1$, and the sub-optimization (3.1) for the second player is

$$\begin{cases} \min_{x_2 \in \mathbb{R}^1} & -x_2 + 0.001(x_2 + 1)^2 \\ \text{s.t.} & x_2^2 = 1, \end{cases}$$

whose minimizer $x_2^{(2)} = 1$. So, $(x_1^{(2)}, x_2^{(2)}) = (-1, 1)$. Continuing this process, one can show that $\mathbf{x}^{(k)}$ is alternating in the pattern

$$(1, 1) \longrightarrow (1, -1) \longrightarrow (-1, -1) \longrightarrow (-1, 1) \longrightarrow (1, 1) \longrightarrow \dots$$

Algorithm 3.1 does not converge for this GNEP.

We would like to remark that even for the case that Algorithm 3.1 converges, the limit of $\mathbf{x}^{(k)}$ is not necessarily a GNE for (1.1). This is shown in the following example.

Example 3.5. Consider the GNEP in (1.4). For the first player, when $x_2 > 0$, its feasible set is $x_1 \geq x_2 + 1$, so the sub-optimization (3.1) in the k th loop is

$$(3.11) \quad \begin{cases} \min_{x_1 \in \mathbb{R}^1} & x_1 + \tau^{(k)}(x_1 - x_1^{(k)})^2 \\ \text{s.t.} & x_1 \geq 1 + x_2^{(k)}. \end{cases}$$

For $0 < \tau^{(k)} < 0.5$, the minimizer of (3.11) is $1 + x_2^{(k)}$. For the second player, the sub-optimization (3.1) in the k th loop is

$$(3.12) \quad \begin{cases} \min_{x_2 \in \mathbb{R}^1} & (x_2)^2 - x_2 x_2^{(k)} + \tau^{(k)}(x_2 - x_2^{(k)})^2 \\ \text{s.t.} & (x_2)^2 \leq 3 - (x_2^{(k)} + 1)^2, x_2 \geq 0. \end{cases}$$

When (3.12) is feasible, its minimizer is

$$\min \left\{ \frac{1 + 2\tau^{(k)}}{2 + 2\tau^{(k)}} x_2^{(k)}, \sqrt{3 - (x_2^{(k)} + 1)^2} \right\}.$$

Therefore, for any constant $0 < \tau^{(k)} < 0.5$ or a decreasing $\tau^{(k)}$ with $\tau^{(0)} < 0.5$, if $0 < x_2^{(0)} \leq \sqrt{3} - 1$ (to make (3.12) feasible), then $\mathbf{x}^{(k)} \rightarrow (1, 0)$ as $k \rightarrow \infty$. However, $(1, 0)$ is not a GNE, because when $x_2 = 0$, $x_1 = 0$ is feasible and it is the minimizer. Indeed, $(0, 0)$ is a GNE. This shows that a limit point produced by Algorithm 3.1 is not necessarily a GNE.

In practice, however, the performance of Algorithm 3.1 is good. Under certain conditions, Algorithm 3.1 converges and the limit is a GNE. This requires some assumptions on the feasible sets of (1.2). Let G be a set-valued map defined on a set U , i.e., $G(x)$ is a subset of a range Y , for all $x \in U$. Its domain, $\text{dom } G$, is the set of $x \in U$ such that $G(x) \neq \emptyset$ [4]. The map G is said to be *inner semicontinuous* at $x \in U$ relative to $\text{dom } G$ if for all $y \in G(x)$ and for all sequences $\{x_\ell\} \subseteq \text{dom } G$ such that $x_\ell \rightarrow x$, there exists a sequence of $y_\ell \in G(x_\ell)$ converging to y . The map G is called inner semicontinuous relative to $\text{dom } G$ if it is inner semicontinuous relative to $\text{dom } G$ at every point in $\text{dom } G$. For instance, if the set $X_i(x_{-i}) = \{x_i : (x_i, x_{-i}) \in C_i\}$ for $C_i \subseteq \mathbb{R}^n$ being a polyhedron or a ball, then the set-valued map $x_{-i} \mapsto X_i(x_{-i})$ is inner semicontinuous relative to its domain at all points x_{-i} [51]. However, for the GNEP in (1.4), the set-valued map $x_2 \rightarrow X_1(x_2)$ is not inner semicontinuous at $(0, 0)$ (see the end of this section). We refer to [6, 51] for the inner semicontinuity of set-valued maps. The following is a useful lemma about inner semicontinuity.

Lemma 3.6. *For two closed sets U and V , let $f : U \times V \rightarrow \mathbb{R}$ and $h : U \times V \rightarrow \mathbb{R}^m$ be two continuous functions. For $y \in V$, define the set-valued map*

$$G(y) = \{x \in U : h(x, y) \geq 0\}.$$

Consider two sequences $\{x^{(k)}\} \subseteq \text{dom} G$ and $\{y^{(k)}\} \subseteq V$ such that $x^{(k)} \rightarrow x^*$ and $y^{(k)} \rightarrow y^*$. Suppose $0 \leq \tau^{(k)} \rightarrow 0$ as $k \rightarrow \infty$. Assume that each $x^{(k)}$ is a minimizer of the optimization problem

$$(3.13) \quad \begin{cases} \min_{x \in U} & f(x, y^{(k)}) + \tau^{(k)} \|x - x^{(k-1)}\|^2, \\ \text{s.t.} & h(x, y^{(k)}) \geq 0. \end{cases}$$

If the set-valued map $G(y)$ is inner semicontinuous relative to $\text{dom} G$, then x^* is also a minimizer of

$$(3.14) \quad \begin{cases} \min_{x \in U} & f(x, y^*) \\ \text{s.t.} & h(x, y^*) \geq 0. \end{cases}$$

Proof. We prove it by a contradiction argument. Suppose otherwise that x^* is not a minimizer of (3.14), then there exists $z^* \in G(y^*)$ such that

$$(3.15) \quad f(z^*, y^*) < f(x^*, y^*).$$

Since the mapping G is inner semicontinuous, there exists a sequence of $z^{(k)}$ such that $z^{(k)} \rightarrow z^*$ and $z^{(k)} \in G(y^{(k)})$. The sequence $\{z^{(k)}\}$ is clearly bounded. Because x_k is a minimizer of

$$\begin{cases} \min & f(x, y^{(k)}) + \tau^{(k)} \|x - x^{(k-1)}\|^2 \\ \text{s.t.} & x \in G(y^{(k)}), \end{cases}$$

we have that

$$(3.16) \quad f(z^{(k)}, y^{(k)}) + \tau^{(k)} \|z^{(k)} - x^{(k-1)}\|^2 \geq f(x^{(k)}, y^{(k)}) + \tau^{(k)} \|x^{(k)} - x^{(k-1)}\|^2.$$

Because $f(x, y)$ is continuous, it holds that

$$f(x^{(k)}, y^{(k)}) \rightarrow f(x^*, y^*), \quad f(z_k, y_k) \rightarrow f(z^*, y^*)$$

as $k \rightarrow \infty$. For all $\varepsilon > 0$, there exists K_1 such that

$$\begin{aligned} f(x^{(k)}, y^{(k)}) - f(x^*, y^*) &> -\frac{\varepsilon}{4}, \\ f(z^{(k)}, y^{(k)}) - f(z^*, y^*) &< \frac{\varepsilon}{4} \end{aligned}$$

for all $k > K_1$. Combining the two inequalities, we can get

$$f(x^{(k)}, y^{(k)}) - f(z^{(k)}, y^{(k)}) + f(z^*, y^*) - f(x^*, y^*) > -\frac{\varepsilon}{2}.$$

Therefore, we have

$$(3.17) \quad \begin{aligned} f(z^*, y^*) - f(x^*, y^*) + \frac{\varepsilon}{2} &> f(z^{(k)}, y^{(k)}) - f(x^{(k)}, y^{(k)}) \\ &\geq \tau^{(k)} \left(\|x^{(k)} - x^{(k-1)}\|^2 - \|z^{(k)} - x^{(k-1)}\|^2 \right). \end{aligned}$$

The last inequality follows from (3.16). Because $\{x^{(k)}\}, \{z^{(k)}\}$ are convergent sequences and $\tau^{(k)} \rightarrow 0$, there must exist K_2 such that

$$\tau^{(k)} (\|x^{(k)} - x^{(k-1)}\|^2 - \|z^{(k)} - x^{(k-1)}\|^2) > -\frac{\varepsilon}{2}$$

whenever $k > K_2$. Let $K := \max\{K_1, K_2\}$, then for all $k > K$

$$f(z^*, y^*) - f(x^*, y^*) + \varepsilon > 0.$$

Since ε can be arbitrarily small, the above implies that

$$f(z^*, y^*) - f(x^*, y^*) \geq 0,$$

which contradicts (3.15). Therefore, x^* is a minimizer of (3.14). \square

Lemma 3.6 immediately implies the following result.

Theorem 3.7. *Let $\mathbf{x}^{(k)}$ be the sequence produced by Algorithm 3.1 for the GNEP of (1.1). Assume that $\mathbf{x}^{(k)} \rightarrow \mathbf{x}^*$ and $\tau^{(k)} \rightarrow 0$. If for each i the set-valued map $G_i : x_{-i} \mapsto X_i(x_{-i})$ is inner semicontinuous relative to its domain $\text{dom} G_i$, then the limit \mathbf{x}^* is a GNE for the GNEP of (1.1).*

Remark. Theorem 3.7 assumes that the sequence of $\mathbf{x}^{(k)}$ produced by Algorithm 3.1 converges. However, the theorem does not give a sufficient condition for this sequence to converge. To ensure convergence, we need to assume the GNEPs are GPGs; see Theorems 4.3 and 4.4. There exists a convergence result [55, Lemma 1] that is similar to Lemma 3.6 and Theorem 3.7.

In the proof of Lemma 3.6, it is required that $\tau^{(k)} \rightarrow 0$, which is also assumed in Theorem 3.7. However, in the implementation of Algorithm 3.1, we do not need $\tau^{(k)} \rightarrow 0$. Sometimes, a constant $\tau^{(k)}$ works very well. We refer to Theorem 4.4 and examples in Section 5.

For Examples 3.3 and 3.4, Algorithm 3.1 does not produce a convergent sequence. For Example 3.5, the set-valued map $G_1 : x_2 \mapsto X_1(x_2)$ for the first player is not inner semicontinuous relative to its domain $\text{dom} G_1$. In fact, at the point $(x_1, x_2) = (0, 0)$, it is clear that $x_1 \in G_1(x_2)$. However, for every sequence $\{x_2^{(k)}\}$ such that $0 < x_2^{(k)} \rightarrow x_2 = 0$, $G_1(x_2^{(k)}) = [x_2^{(k)} + 1, \infty)$. Since each $x_2^{(k)} > 0$, there does not exist a sequence $\{x_1^{(k)}\}$ converging to $x_1 = 0$ and $x_1^{(k)} \in G_1(x_2^{(k)}) = [x_2^{(k)} + 1, \infty)$. Therefore, the inner semicontinuity assumption in Theorem 3.7 fails for Example 3.5.

4. Generalized potential games

The Gauss-Seidel method is frequently used for solving GNEPs. However, its convergence is not guaranteed for all of them. One wonders for what kind of GNEPs the Gauss-Seidel method converges. The generalized potential game (GPG) is such a GNEP. The following is the definition of GPGs in [20].

Definition 4.1. ([20]) The GNEP of (1.1) is a generalized potential game if:

- (i) There exists a closed set $\emptyset \neq X \subseteq \mathbb{R}^n$ such that

$$X_i(x_{-i}) \equiv \{x_i \in D_i : (x_i, x_{-i}) \in X\}$$

for all players, where each $D_i \subseteq \mathbb{R}^{n_i}$ is a closed set such that $(D_1 \times \cdots \times D_N) \cap X \neq \emptyset$.

- (ii) There exist a continuous function $P(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and a forcing function $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (i.e., $\lim_{k \rightarrow \infty} \sigma(t_k) = 0$ implies $\lim_{k \rightarrow \infty} t_k = 0$) such that for all $y_i, x_i \in X_i(x_{-i})$

$$(4.1) \quad \begin{aligned} f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) > 0 &\implies \\ P(y_i, x_{-i}) - P(x_i, x_{-i}) &\geq \sigma(f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})). \end{aligned}$$

The item (i) in Definition 4.1 is from the concept of *shared constraint* [17]. It implies that if $\mathbf{x}^{(0)}$ is feasible, then the sub-optimization problem (3.1) is feasible for all k and i . The item (ii) means that there exists a single “dominant function” P that measures the changes on each player’s objective functions [20].

Some special GNEPs can be directly verified as GPGs. For instance, for the GNEP of (1.1), if each objective f_i can be expressed as

$$(4.2) \quad f_i(\mathbf{x}) = f_0(\mathbf{x}) + \sum_{j=1}^M f_{i,j}(x_j)$$

for some functions f_0 and $f_{i,j}$ and the item (i) holds, then the GNEP of (1.1) is a GPG because P, σ can be chosen as

$$P(\mathbf{x}) = f_0(\mathbf{x}) + \sum_{i=1}^N \sum_{j=1}^M f_{i,j}(x_j), \quad \sigma(t) = t.$$

One can easily check that the above $P(x)$ and $\sigma(t)$ satisfy (4.1) [35].

GPGs are extensions of potential games, which were originally introduced for NEPs [35]. They have broad applications [40]. The following is an example of GPG arising from applications.

Example 4.2. The GNEPP from the environmental pollution control, described in the introduction, is a GPG. The functions P and σ can be chosen as

$$P(\mathbf{x}) = 2 \prod_{i=1}^N (x_{i,0} - \sum_{j=1}^N \gamma_{j,i} x_{j,i}) + \sum_{i=1}^N \left[\sum_{j=0}^N x_{i,j} - \sum_{j=1}^N \gamma_{j,i} x_{j,i} - x_{i,0} (b_i - 1/2x_{i,0}) \right],$$

$$\sigma(t) = t.$$

The numerical results of Algorithm 3.1 are shown in the next section.

The following is the convergence result for Algorithm 3.1 when it is applied to solve GPGs.

Theorem 4.3. (*[20, Theorem 5.2]*) *Consider the GNEP of (1.1) such that all the functions are continuous. Assume that (1.1) is a GPG and each set-valued map $G_i : x_{-i} \mapsto X_i(x_{-i})$ is inner semicontinuous relative to its domain. In Algorithm 3.1, suppose each $x_i^{(k+1)}$ is a minimizer of (3.1) and the parameters $\tau^{(k)}$ are updated as*

$$(4.3) \quad \tau^{(k+1)} := \max \left\{ \min \left[\tau^{(k)}, \max_{i=1, \dots, N} (\|x_i^{(k+1)} - x_i^{(k)}\|) \right], 0.1\tau^{(k)} \right\}.$$

Then every limit point of the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ produced by Algorithm 3.1 is a GNE for (1.1).

The updating scheme (4.3) for $\tau^{(k)}$ is a bit complicated. However, if each player's optimization problem (1.2) is convex, then the parameter $\tau^{(k)}$ can be chosen to be constant.

Theorem 4.4. (*[20, Theorem 4.3]*) *Consider the GNEP of (1.1) such that all the functions are continuous. Assume that (1.1) is a GPG and each set-valued map $G_i : x_{-i} \mapsto X_i(x_{-i})$ is inner semicontinuous relative to its domain. Suppose the objectives $f_i(\cdot, x_{-i})$ and the feasible sets $X_i(x_{-i})$ are all convex. In Algorithm 3.1, suppose each $x_i^{(k+1)}$ is a minimizer of (3.1) and the parameter $\tau^{(k)} = \tau > 0$ is a constant. Then every limit point of the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ produced by Algorithm 3.1 is a GNE for (1.1).*

Beyond GPGs, the Gauss-Seidel method has convergence for GNEPs with discrete strategy sets [53] or mixed-integer variables [54]. In general, when (1.1) is not a GPG, the convergence of Algorithm 3.1 is not known very much. We have seen examples in Section 3 such that Algorithm 3.1 fails to converge. On the other hand, the performance of Algorithm 3.1 is actually very good in our computational experiments (see Section 5). In the following, we discuss how to certify that a GNEP is a GPG.

4.1 A certificate for GPGs

Generally, it is hard to check whether a GNEP is a GPG or not. The main challenge is to verify the item (ii) in Definition 4.1. In this subsection, we give a certificate for (4.1) to hold. For the i th player, denote the set

$$(4.4) \quad K_i = \left\{ (x_i, y_i, x_{-i}) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \times \mathbb{R}^{n-n_i} \mid \begin{array}{l} x_i, y_i \in X_i(x_{-i}) \\ f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}) \geq 0 \end{array} \right\}.$$

For convenience, denote the differences of functions

$$(4.5) \quad \begin{cases} \Delta P_i & := P(y_i, x_{-i}) - P(x_i, x_{-i}), \\ \Delta f_i & := f_i(y_i, x_{-i}) - f_i(x_i, x_{-i}). \end{cases}$$

The following lemma is straightforward for verification.

Lemma 4.5. *For the GNEP of (1.1), if the item (i) in Definition 4.1 holds, and there exist polynomials $P \in \mathbb{R}[\mathbf{x}]$, $p_{i,0}, p_{i,1} \in \mathbb{R}[x_i, y_i, x_{-i}]$ ($i = 1, \dots, N$) such that $p_{i,0} \geq 0$, $p_{i,1} \geq 0$ on K_i and*

$$(4.6) \quad \Delta P_i = (p_{i,0} + 1)\Delta f_i + p_{i,1}$$

for all i , then (1.1) is a GPG.

In the equation (4.6), we can replace the constant 1 by any positive number $\epsilon > 0$, up to scaling coefficients. For numerical reasons, we prefer the constant 1. Lemma 4.5 gives a certificate for GPGs. The following are examples of GPGs certified by (4.6).

Example 4.6. Consider the 2-player GNEP with the sets

$$X = \{(x_1, x_2) : 1 \leq x_1, x_2 \leq 10, x_1 \geq x_2\},$$

$$X_1(x_{-1}) = \{x_1 : (x_1, x_2) \in X\}, \quad X_2(x_{-2}) = \{x_2 : (x_1, x_2) \in X\}.$$

The two players' optimization problems are respectively

$$(4.7) \quad \min_{x_1 \in X_1} x_1 + x_2 \quad \Bigg| \quad \min_{x_2 \in X_2} -x_1 x_2.$$

Let $P(x_1, x_2) = (x_1)^3 - x_1 x_2 + x_1$, we have

$$(4.8) \quad \begin{cases} \Delta P_1 & = (y_1 - x_1)[(y_1 - x_1)^2 + 1] + (3y_1 x_1 - x_2)(y_1 - x_1), \\ \Delta P_2 & = -x_1(y_2 - x_2), \\ \Delta f_1 & = y_1 - x_1, \\ \Delta f_2 & = -x_1(y_2 - x_2). \end{cases}$$

The equation (4.6) is satisfied for

$$p_{1,0} = (y_1 - x_1)^2, \quad p_{1,1} = (3y_1 x_1 - x_2)(y_1 - x_1), \quad p_{2,0} = p_{2,1} = 0.$$

It is clear that $p_{1,0}, p_{2,0}, p_{2,1}$ are nonnegative. By the definition of K_1 , $\Delta f_1 \geq 0$, and $3y_1 x_1 - x_2 \geq 3y_1 - x_2 \geq 0$, so $p_{1,1} \geq 0$ on K_1 .

Example 4.7. Consider the 2-player GNEP with the sets

$$X = \{(x_1, x_2) : (x_1)^3 + (x_2)^3 \leq 2, x_1 \geq 6x_2\},$$

$$X_1(x_{-1}) = \{x_1 : (x_1, x_2) \in X\}, \quad X_2(x_{-2}) = \{x_2 : (x_1, x_2) \in X\}.$$

The two players' optimization problems are respectively

$$(4.9) \quad \begin{array}{l} \min_{x_1 \in X_1} \quad (x_1)^2 x_2 + (x_2)^2 x_1 - 4(x_1)^4 \\ \text{s.t.} \quad x_1 \geq 0, \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in X_2} \quad x_1 x_2 - 3(x_2)^2 \\ \text{s.t.} \quad x_2 \geq 0.125. \end{array} \right.$$

For $P(x_1, x_2) = (x_1)^2 x_2 + (x_2)^2 x_1 - 4(x_1)^4$,

$$(4.10) \quad \begin{cases} \Delta P_1 &= \Delta f_1 = (y_1)^2 x_2 + (y_2)^2 x_1 - 4(y_1)^4 \\ &\quad - (x_1)^2 x_2 - (x_2)^2 x_1 + 4(x_1)^4, \\ \Delta P_2 &= (x_1)^2 (y_2 - x_2) + x_1 ((y_2)^2 - (x_2)^2), \\ \Delta f_2 &= x_1 (y_2 - x_2) - 3(y_2)^2 + 3(x_2)^2. \end{cases}$$

The equation (4.6) holds with

$$p_{1,0} = 0, p_{1,1} = 0, p_{2,0} = x_1, p_{2,1} = (y_2 - x_2)[4x_1(y_2 + x_2) + 3(y_2 + x_2) - x_1].$$

Clearly, $p_{1,0}, p_{1,1}, p_{2,0} \geq 0$. Note that

$$\Delta f_2 = (y_2 - x_2)(x_1 - 3(y_2 + x_2)) \geq 0$$

on K_2 . Then, either $x_1 - 3(y_2 + x_2) > 0$ hence $y_2 - x_2 \geq 0$, or $x_1 - 3(y_2 + x_2) = 0$, which forces $y_2 - x_2 = 0$. This is because $x_1 \geq 6y_2$, $x_1 \geq 6x_2$ and $y_2, x_2 > 0$, if $x_1 - 3(y_2 + x_2) = 0$, then the only possible case is $x_1 = 6y_2 = 6x_2$. Thus from

$$4x_1(y_2 + x_2) + 3(y_2 + x_2) - x_1 > x_1(4y_2 + 4x_2 - 1) \geq 0,$$

we know $p_{2,1} \geq 0$ on K_2 .

Example 4.8. Consider the 2-player GNEP with the sets

$$X = \left\{ (x_1, x_2) \left| \begin{array}{l} x_1 = (x_{1,1}, x_{1,2}) \in \mathbb{R}^2, x_2 \in \mathbb{R}, \\ x_{1,1}, x_{1,2}, x_2 \geq 0.5, \\ x_2 - 0.3 \leq x_{1,1} + x_{1,2} \leq x_2 + 0.3 \end{array} \right. \right\},$$

and $X_1(x_{-1}) = \{x_1 : (x_1, x_2) \in X\}$, $X_2(x_{-2}) = \{x_2 : (x_1, x_2) \in X\}$. The optimization problems are respectively

$$(4.11) \quad \begin{array}{l} \min_{x_1 \in X_1} \quad x_{1,1} x_2 + x_{1,2} x_2 \\ \text{s.t.} \quad \|x_1\| = 2 \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in X_2} \quad x_{1,1} \cdot x_{1,2} \cdot x_2 \end{array} \right.$$

For $P(x_1, x_2) = (x_{1,1} + x_{1,2} + 1)^3 x_2$,

$$(4.12) \quad \begin{cases} \Delta P_1 &= x_2((y_{1,1} + y_{1,2} + 1)^3 - (x_{1,1} + x_{1,2} + 1)^3), \\ \Delta P_2 &= (y_2 - x_2)(x_{1,1} + x_{1,2} + 1)^3, \\ \Delta f_1 &= x_2(y_{1,1} + y_{1,2} - x_{1,1} - x_{1,2}), \\ \Delta f_2 &= x_{1,1} x_{1,2} (y_2 - x_2). \end{cases}$$

The equation (4.6) holds with

$$\begin{aligned} p_{1,0} &= (y_{1,1} + y_{1,2} + 1)^2 + (x_{1,1} + x_{1,2} + 1)^2 + (y_{1,1} + y_{1,2})(x_{1,1} + x_{1,2}) \\ &\quad + y_{1,1} + y_{1,2} + x_{1,1} + x_{1,2}, \\ p_{1,1} &= 0, \\ p_{2,0} &= 3x_{1,1} + 3x_{1,2} + 5, \\ p_{2,1} &= (y_2 - x_2)[(x_{1,1})^3 + (x_{1,2})^3 + 3(x_{1,1})^2 + \\ &\quad 3(x_{1,2})^2 + 3x_{1,1} + 3x_{1,2} + 1]. \end{aligned}$$

The equality $\Delta P_1 = (1 + p_{1,0})\Delta f_1$ follows from the identity

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

with $a = y_{1,1} + y_{1,2} + 1$ and $b = x_{1,1} + x_{1,2} + 1$. Clearly, $p_{1,0}, p_{1,1} \geq 0$ on K_1 , and $p_{2,0} \geq 0$ on K_2 . Since $x_{1,1}x_{1,2} > 0$, $\Delta f_2 \geq 0$ implies $y_2 - x_2 \geq 0$, so $p_{2,1} \geq 0$ on K_2 .

4.2 Putinar Positivstellensatz for the certificate

Lemma 4.5 gives a convenient certificate for checking GPGs. One needs to find polynomials $p_{i,0}, p_{i,1}$ and P satisfying (4.6) and $p_{i,0}, p_{i,1} \geq 0$ on K_i . For a polynomial tuple h , we have seen that if $p \in \text{Qmod}(h)$, then $p \geq 0$ on the semialgebraic set $\mathcal{S}(h)$. This motivates us to use Putinar's Positivstellensatz for verifying that.

For the set K_i as in (4.4), let $h_i := (h_{i,t})_{t=1}^{m_i}$ be a tuple of polynomials in $\mathbb{R}[x_i, y_i, x_{-i}]$ such that

$$K_i = \{(x_i, y_i, x_{-i}) : h_i(x_i, y_i, x_{-i}) \geq 0\}.$$

Moreover, let $h_{i,0} = 1$ for all i . When the item (i) in Definition 4.1 holds, the GNEP of (1.1) is a GPG if there exist $P \in \mathbb{R}[\mathbf{x}]$ and $q_{i,0}, q_{i,1} \in \text{Qmod}(h_i)$ such that

$$(4.13) \quad \Delta P_i = (q_{i,0} + 1)\Delta f_i + q_{i,1}$$

for all players. For an even degree $2d$, we parameterize $P, q_{i,0}, q_{i,1}$ as

$$\begin{aligned} P(\mathbf{x}) &= \mathbf{p}^T[\mathbf{x}]_{2d}, \quad q_{i,0} = \sum_{t=0}^{m_i} ([\mathbf{x}, y_i]_{d-d_{it}})^T \cdot Q_{i,0}^t \cdot ([\mathbf{x}, y_i]_{d-d_{it}}) \cdot h_{i,t}, \\ q_{i,1} &= \sum_{t=0}^{m_i} ([\mathbf{x}, y_i]_{d-d_{it}})^T \cdot Q_{i,1}^t \cdot ([\mathbf{x}, y_i]_{d-d_{it}}) \cdot h_{i,t}. \end{aligned}$$

In the above, the degree $d_{it} = \lceil \deg(h_{i,t})/2 \rceil$. One can show that $q_{i,0}, q_{i,1} \in \text{Qmod}(h_i)$ if and only if there exist psd matrices $Q_{i,0}^t, Q_{i,1}^t$ in the above parametrization, for some d [32, Chapter 2]. For notational convenience, denote

$$(4.14) \quad Q := (Q_{i,0}^t, Q_{i,1}^t)_{i=1, \dots, N, t=1, \dots, m_i}.$$

Therefore, the certificate (4.6) in Lemma 4.5 can be checked by solving the semi-definite program

$$(4.15) \quad \begin{cases} \min_{\mathbf{p}, Q} & \sum_{i,t} \text{trace}(Q_{i,0}^t + Q_{i,1}^t) \\ \text{s.t.} & \Delta P_i \equiv (q_{i,0} + 1)\Delta f_i + q_{i,1} \ (\forall i), \\ & P \in \mathbb{R}[\mathbf{x}]_{2d}, \\ & Q_{i,0}^t \succeq 0, Q_{i,1}^t \succeq 0 \ (\forall i, t). \end{cases}$$

The certificate given by solving (4.15) does not require to have priori polynomials $P, q_{i,0}$ and $q_{i,1}$. Instead, the coefficients of these polynomials are variables in (4.15) that are awaiting to be solved numerically.

in a Windows 10 operating system. In the computation, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ is regarded to converge if for some k it holds that

$$(5.1) \quad \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|_{\infty} \leq 10^{-8} \text{ for all } i, j \in \{k-10, \dots, k\}.$$

The point $\mathbf{x}^{(k)}$ is regarded as a GNE with the accuracy parameter $\varepsilon > 0$ if

$$(5.2) \quad |f_i(\mathbf{x}^{(k)}) - f_i^*| \leq \varepsilon$$

for all players, where f_i^* is the minimum value of (1.2) with $x_{-i} = x_{-i}^{(k)}$. Our computational results show that Algorithm 3.1 performs very well for solving GNEPPs, even if for nonconvex ones. First, we see some examples of the GPGs from Section 4.

Example 5.1. Consider the environmental pollution problem in the introduction and Example 4.2. We have seen that it is a GPG. Assume the number of players is $N = 2$ and the parameters $b_1 = b_2 = 2, E_1 = E_2 = 1, \gamma_{1,1} = 0.7, \gamma_{1,2} = 0.9, \gamma_{2,1} = \gamma_{2,2} = 0.8$. We run Algorithm 3.1 with $x_{1,0}^{(0)} = x_{1,1}^{(0)} = \dots = x_{2,2}^{(0)} = 0.5$, and $\tau^{(0)} = 0.1, \tau^{(k+1)}$ updated as in (4.3). After 21 iterations, we get

$$x_{1,0}^{(21)} = 0.9999, x_{1,1}^{(21)} = 0, x_{1,2}^{(21)} = 0, x_{2,0}^{(21)} = 0.7500, x_{2,1}^{(21)} = 0, x_{2,2}^{(21)} = 0.9375.$$

Its accuracy parameter $\varepsilon = 1.7856 \cdot 10^{-8}$. It costs about 7 seconds.

Example 5.2. i) Consider the GNEP in Example 4.6. It is a GPG. All the individual optimization problems are convex. We run Algorithm 3.1 with the initial point $(x_1^{(0)}, x_2^{(0)}) = (3, 2)$ and fixed $\tau^{(k)} = 0.02$, and get a GNE $(2.0000, 2.0000)$ with $\varepsilon = 6.1541 \cdot 10^{-8}$. It runs 12 iterations and costs 2.6289 seconds.

ii) Consider the GNEP in Example 4.7. It is a GPG. We run Algorithm 3.1 with the initial point $(x_1^{(0)}, x_2^{(0)}) = (1, 0.125)$ and fixed $\tau^{(k)} = 0.02$. It returns the GNE $(1.2595, 0.1250)$ with $\varepsilon = 2.2891 \cdot 10^{-9}$. It runs 12 iterations and costs around 2 seconds.

iii) Consider the GNEP in Example 4.8. It is a GPG. All the individual optimization problems are convex. We run Algorithm 3.1 with the initial point $(x_{1,1}^{(0)}, x_{1,2}^{(0)}, x_2^{(0)}) = (1, 1, 2)$ and fixed $\tau^{(k)} = 0.02$. It returns the GNE $(1.3229, 0.5000, 1.5229)$ with $\varepsilon = 1.3631 \cdot 10^{-7}$. It runs 12 costs around 3 seconds.

iv) Consider the GNEP in Example 4.9. It is numerically verified to be a GPG. We run Algorithm 3.1 with the initial point $(x_1^{(0)}, x_2^{(0)}) = (0.2, 0.3)$ and fixed $\tau^{(k)} = 0.02$. For $k = 12$, we get $\mathbf{x}^{(12)} = (0.9539, 0.3)$. The iteration difference is $2.1792 \cdot 10^{-8}$ and the GNE accuracy $\varepsilon = 5.4170 \cdot 10^{-9}$. It costs about 1.6 seconds.

Example 5.3. Consider the 2-player GNEP such that the individual optimization problems are respectively

$$\begin{array}{l|l} \min_{x_1 \in \mathbb{R}^2} & \min_{x_2 \in \mathbb{R}^2} \\ x_{1,1}(x_{1,2} + 2x_{2,1} + 2x_{2,2}) & (x_{1,1})^2 + (x_{1,2})^2 \\ \quad + x_{1,2}(x_{2,1} + x_{2,2}) + 2x_{2,1}x_{2,2} & - (x_{2,1})^2 - (x_{2,2})^2 \\ \text{s.t.} & \text{s.t.} \\ \sum_{i=1}^2 \sum_{j=1}^2 x_{i,j} = 1, & \sum_{i=1}^2 \sum_{j=1}^2 x_{i,j} = 1, \\ x_{1,1} \geq 0, x_{1,2} \geq 0, & x_{2,1} \geq 0, x_{2,2} \geq 0. \end{array}$$

By solving the semidefinite program (4.15), we can numerically check that this GNEP is a GPG. Run Algorithm 3.1 with the initial points $x_1^{(0)} = (0.2, 0.3), x_2^{(0)} = (0.2, 0.3)$ and fixed $\tau^{(k)} = 0.02$. After 19 loops, we get that

$$\mathbf{x}^{(19)} = (0, 0.5, 0, 0.5).$$

as a GNE with accuracy parameter $\varepsilon = 5.1857 \cdot 10^{-7}$. It costs about 5.36 seconds.

Example 5.4. Consider the 2-player GNEP whose optimization problems are

$$\begin{array}{l|l} \min_{x_1 \in \mathbb{R}^2} & -2(x_{1,2})^2 + x_{2,1}x_{1,2} + x_{1,1}x_{2,1} \\ \text{s.t.} & x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} = 1 \\ & x_{1,1}, x_{1,2} \geq 0.1 \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in \mathbb{R}^2} & (x_{2,1})^2 - 2x_{1,2}x_{2,2} \\ & -2x_{1,1}x_{2,2} + (x_{2,2})^2 \\ \text{s.t.} & x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} = 1 \\ & x_{2,1}, x_{2,2} \geq 0.1 \end{array} \right.$$

By solving the semidefinite program (4.15), one can numerically check that this GNEP is a GPG. We run Algorithm 3.1 with

$$\mathbf{x}^{(0)} = (0.25, 0.25, 0.25, 0.25), \quad \tau^{(0)} = 0.1,$$

and $\tau^{(k+1)}$ updated as (4.3). For $k = 12$, we get

$$\mathbf{x}^{(12)} = (0.1000, 0.4000, 0.1000, 0.4000),$$

which is a GNE. The accuracy $\varepsilon = 1.14611 \cdot 10^{-8}$. It costs around 2.7 seconds.

Example 5.5. Consider the GNEP whose optimization problems are

$$\begin{array}{l|l} \min_{x_1 \in \mathbb{R}^2} & (x_{1,1})^2 + (x_{1,2})^2 + x_{1,1} + x_{1,2} \\ \text{s.t.} & \|x_1\|^2 + \|x_2\|^2 \leq 1, \\ & x_{1,1} \geq 0, x_{1,2} \leq 0.5, \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in \mathbb{R}^2} & (x_{2,2})^2 - x_{2,1}x_{2,2} \\ \text{s.t.} & \|x_1\|^2 + \|x_2\|^2 \leq 1, \\ & x_{2,1} \leq 0, 0.3 \leq x_{2,2} \leq 0.8. \end{array} \right.$$

This is a GPG [20]. We run Algorithm 3.1 with

$$x_1^{(0)} = (0.5, 0.5), \quad x_2^{(0)} = (-0.6, 0.6), \quad \tau^{(0)} = 0.1,$$

and $\tau^{(k+1)}$ updated as (4.3). For $k = 16$, we get $\mathbf{x}^{(16)} = (0, -0.5, 0, 0.3)$ as a GNE with accuracy parameter $\varepsilon = 4.1908 \cdot 10^{-10}$. It costs around 4.11 seconds.

Example 5.6. Consider the 3-player GNEP whose optimization problems are

$$\begin{array}{l|l|l} \min_{x_1 \in \mathbb{R}^1} & (x_1 - x_2)^2 & \min_{x_2 \in \mathbb{R}^1} & (x_2 - x_3)^2 & \min_{x_3 \in \mathbb{R}^1} & (x_3 - x_1)^2 \\ \text{s.t.} & \sum_{i=1}^3 (x_i)^2 \leq 10 & \text{s.t.} & x_2 \leq 3 & \text{s.t.} & \sum_{i=1}^3 x_i \leq 6. \end{array}$$

Any feasible point \mathbf{x} with $x_1 = x_2 = x_3$ is a GNE, with optimal value 0 for all players. If we run Algorithm 3.1 with $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0, 1, 2)$ and $\tau^{(k)} = 0$ (which is actually not allowed since we require $\tau > 0$, but we still show the result of $\tau = 0$ in order to show the necessity of a positive τ), then we get an alternating sequence

$$\begin{aligned} (0, 1, 2) &\longrightarrow (1, 1, 2) \longrightarrow (1, 2, 2) \longrightarrow (1, 2, 1) \longrightarrow (2, 2, 1) \\ &\longrightarrow (2, 1, 1) \longrightarrow (2, 1, 2) \longrightarrow (1, 1, 2) \longrightarrow \dots \end{aligned}$$

If we run Algorithm 3.1 with the same initial point $\mathbf{x}^{(0)} = (0, 1, 2)$ but different regularization parameter $\tau^{(k)}$, the computational results are reported in Table 1. We run it for five different $\tau^{(k)}$. Two of them are fixed values 0.1, 0.05, and the other one is $\tau_0 = 0.5$, $\tau^{(k+1)}$ updated as (4.3). In the table, ‘‘Iteration Difference’’ is the value of

$$\max_{291 \leq i < j \leq 300} \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|_\infty$$

since none of these five sequence satisfies (5.1) at the 300 iteration. And ‘‘error’’ means the number that $\mathbf{x}^{(300)}$ can be verified up to as a GNE.

TABLE 1. Computational Results for Example 5.6

$\tau^{(k+1)}$	x_1	x_2	x_3	Iteration Difference	ε
0.1000	1.4289	1.4289	1.4289	$1.9139 \cdot 10^{-5}$	10^{-7}
0.0500	1.4494	1.4494	1.4494	$5.2015 \cdot 10^{-5}$	10^{-7}
(4.3)	1.4116	1.4106	1.4116	0.0020	10^{-6}

Example 5.7. Consider the following 2-player GNEP

$$\begin{array}{l} \min_{x_1 \in \mathbb{R}^2} \quad (x_{1,1})^3 + x_{1,2}x_{2,1} + x_{1,1}x_{1,2} + x_{2,2} \\ \text{s.t.} \quad (x_{1,1})^2 + (x_{1,2})^2 \leq 1 \end{array} \quad \left| \quad \begin{array}{l} \min_{x_2 \in \mathbb{R}^2} \quad -(x_{2,1})^4 + x_{1,1}(x_{2,2})^2 \\ \text{s.t.} \quad x_{1,1} \leq (x_{2,1})^2 + (x_{2,2})^2 \leq 1. \end{array} \right.$$

It can be observed that both objective functions are nonconvex. Further the feasible set of the second player is not convex neither. We run Algorithm 3.1 with $\mathbf{x}^{(0)} = (0.5, 0.5, 0.6, 0.6)$ and fixed $\tau^{(k)} = 0.02$. For $k = 16$, we get

$$\mathbf{x}^{(16)} = (-0.9342, -0.3568, 1.0000, -3.7839 \cdot 10^{-5}),$$

which is a GNE. The accuracy $\varepsilon = 6.1075 \cdot 10^{-8}$. It costs around 3.68 seconds.

Example 5.8. ([16, 30]) Consider the example of a model for Internet switching [16, 30]. Assume there are N users, and the maximum capacity of the buffer is B . The x^i denotes the amount of i th user's "packets" in the buffer. It is clear $x^i \geq 0$ for any i . We also suggest the buffer is managed with "drop-tail" policy, which means if the buffer is full, further packets will be lost and resent. Let $\frac{x_i}{x_1 + \dots + x_N}$ be the *transmission rate* of user i , and $\frac{x_1 + \dots + x_N}{B}$ represent the *congestion level* of the buffer, and $1 - \frac{x_1 + \dots + x_N}{B}$ measure the decrease in the utility of the i th user as the congestion level increases. The i th user's optimization problem is

$$\begin{cases} \min_{x_i} & f_i(\mathbf{x}) = -\frac{x_i}{x_1 + \dots + x_N} \left(1 - \frac{x_1 + \dots + x_N}{B}\right) \\ \text{s.t.} & x_i \geq 0, x_1 + \dots + x_N \leq B. \end{cases}$$

It can be transformed into a polynomial optimization problem by introducing a new variable y_i for each player. The GNEP is then equivalent to that

$$\begin{cases} \min_{x_i, y_i} & -x_i y_i \left(1 - \frac{\sum x_i}{B}\right) \\ \text{s.t.} & x_i \geq 0, x_1 + \dots + x_N \leq B \\ & (x_1 + \dots + x_N) y_i = 1. \end{cases}$$

Here, we consider the case that $B = 1$ and $N = 10$, and run Algorithm 3.1 with the initial point

$$(0.4, \underbrace{0.01, 0.01, \dots, 0.01}_{9 \text{ times}}, \underbrace{1/0.49, 1/0.49, \dots, 1/0.49}_{10 \text{ times}}),$$

and $\tau^{(0)} = 0.1$. The parameters $\tau^{(k)}$ are updated as in (4.3). After 47 iterations, Algorithm 3.1 returned the point (here we only show the result of x_1, \dots, x_{10})

$$(0.09, 0.09, \dots, 0.09).$$

with accuracy parameter $\varepsilon = 1.6344 \cdot 10^{-8}$. It costs around 61.93 seconds.

Example 5.9. ([19, A. 1]) Consider a variation of the GNEP in the last example that we change the constraints of the first player to $0.3 \leq x_i \leq 0.5$. This GNEP

can also be transformed into a GNEPP by introducing a new variable y_i for each player and it is then equivalent to that

$$\begin{array}{l|l} \begin{array}{l} \text{player } i = 1 \\ \min_{x_1, y_1 \in \mathbb{R}} -x_1 y_1 (1 - \frac{\sum x_1}{B}) \\ \text{s.t. } 0.3 \leq x_1 \leq 0.5 \\ (x_1 + \dots + x_N) y_1 = 1 \end{array} & \begin{array}{l} \text{player } i > 1 \\ \min_{x_i, y_i \in \mathbb{R}} -x_i y_i (1 - \frac{\sum x_i}{B}) \\ \text{s.t. } x_1 + \dots + x_N \leq B, x_i \geq 0.001 \\ (x_1 + \dots + x_N) y_i = 1. \end{array} \end{array}$$

Here, we consider the case that $B = 1$ and $N = 10$, the same as in [19]. We run Algorithm 3.1 with the initial point

$$(0.3, \underbrace{0.01, 0.01, \dots, 0.01}_{9 \text{ times}}, \underbrace{1/0.39, 1/0.39, \dots, 1/0.39}_{10 \text{ times}})$$

and $\tau^{(0)} = 0.1$. The parameters $\tau^{(k)}$ are updated as in (4.3). After 47 iterations, Algorithm 3.1 returned the point (here we only show the result of x_1, \dots, x_{10})

$$(0.3, 0.06943, 0.06943, \dots, 0.06943)$$

with accuracy parameter $\varepsilon = 1.1261 \cdot 10^{-8}$. It costs around 60.69 seconds.

Example 5.10. Consider the GNEP which is the same as in Example 5.9 except we change the objective function to

$$f_i(\mathbf{x}) = \frac{x_i}{x_1 + \dots + x_N} (1 - \frac{x_1 + \dots + x_N}{B}).$$

We still consider the case that $B = 1$ and $N = 10$, and the same technique to transform each player's subproblem into polynomial optimization problems. Start from the initial point

$$(0.3, \underbrace{0.01, 0.01, \dots, 0.01}_{9 \text{ times}}, \underbrace{1/0.39, 1/0.39, \dots, 1/0.39}_{10 \text{ times}})$$

with $\tau^{(0)} = 0.1$ and $\tau^{(k)}$ updated as in (4.3). After 44 iterations, Algorithm 3.1 returns the GNE (here we only show the result of x_1, \dots, x_{10})

$$(0.5000, 0.4920, 0.0010, \dots, 0.0010)$$

with the accuracy parameter $\varepsilon = 3.7773 \cdot 10^{-7}$. It took about 59 seconds.

Example 5.11. (Random GNEPPs with joint simplex/ball constraints) We randomly generate objective polynomials for each player with the joint simplex/ball constraint

$$\sum_{i=1}^N \sum_{j=1}^{n_i} x_{i,j} = 1, x_{i,j} \geq 0, \quad \text{or} \quad \|x_1\|^2 + \dots + \|x_N\|^2 \leq 1.$$

We generate 100 random instances and count the number of problems that was solved successfully by Algorithm 3.1. The accuracy parameter is set to be $\varepsilon = 10^{-6}$ for checking $\mathbf{x}^{(k)}$ as a GNE, i.e., we regard $(\mathbf{x}^{(k)})$ as a GNE if (5.2) was satisfied with $\varepsilon = 10^{-6}$. For each instance, we run Algorithm 3.1 for at most 200 loops with $\tau^{(0)} = 0.1$, $\tau^{(k+1)}$ updated as in (4.3). If it does not return a GNE with required accuracy, we regard that it fails to solve the GNEP. The performance of Algorithm 3.1 is reported in Table 2. The number N is the number of players, n_i is the dimension of the i th player's strategy vector, and d is the degree of objective polynomials. The time is measured in seconds.

TABLE 2. Computational Results for Example 5.11

N	(n_1, \dots, n_N)	d	Joint Simplex		Joint Ball	
			Succ. Rate	Ave. Time	Succ. Rate	Ave. Time
3	(2,2,2)	3	100%	9.97	94 %	16.71
3	(2,2,2)	4	92%	46.10	83 %	37.88
3	(3,3,3)	2	95%	11.21	97 %	9.93
3	(3,3,3)	3	92%	36.21	96 %	38.44
3	(3,3,3)	4	84%	98.76	88 %	88.98
4	(3,3,3,3)	2	94%	19.50	96 %	19.10
2	(4,3)	3	97%	13.53	92 %	17.55
2	(4,3)	4	92%	52.54	94 %	55.65
3	(3,2,4)	2	96%	9.43	97 %	9.09
3	(3,2,4)	3	92%	44.53	98 %	26.06
4	(3,2,4,2)	2	93%	19.52	95 %	22.73
4	(3,2,4,2)	3	94%	70.76	96 %	89.46

5.1 Test problems in [19]

We apply Algorithm 3.1 to solve the GNEPs in [19] that are GNEPPs or that can be transformed into GNEPPs. We normalize the objective functions such that the greatest absolute values of the coefficients are equal to one. For example, the problem A.17 in [19] is normalized as follows:

$$\begin{array}{l|l}
 \min_{x_1 \in \mathbb{R}^2} & \frac{1}{38}((x_{1,1})^2 + x_{1,1}x_{1,2} + (x_{1,2})^2) \\
 & + (x_{1,1} + x_{1,2})x_{2,1} - \frac{25x_{1,1}}{38} - x_{1,2} \\
 \text{s.t.} & x_{1,1} + 2x_{1,2} - x_{2,1} \leq 14, \\
 & 3x_{1,1} + 2x_{1,2} + x_{2,1} \leq 30, \\
 & x_1 \geq 0, \\
 \hline
 \min_{x_2 \in \mathbb{R}^1} & \frac{1}{25}(x_{1,1} + x_{1,2})x_{2,1} \\
 & + \frac{1}{25}(x_{2,1})^2 - x_{2,1} \\
 \text{s.t.} & x_{1,1} + 2x_{1,2} - x_{2,1} \leq 14, \\
 & 3x_{1,1} + 2x_{1,2} + x_{2,1} \leq 30, \\
 & x_1 \geq 0.
 \end{array}$$

For the test problem A.2 and A.14, we use the same technique as shown in Example 5.9 to transform these non-polynomial GNEPs into GNEPPs. For the test problem A.10a, we run Algorithm 3.1 with the same initial point as in [19] and yield an alternative sequence that is not convergent. Moreover, we also run Algorithm 3.1 with randomly generated feasible initial points for 100 times and no convergent sequence can be obtained. All the parameters are settled the same as in [19] for each problem. The computational results are shown in Table 3, where e denotes the vector of all ones. All the problems, except problem A.10a, were solved successfully by Algorithm 3.1.

6. Conclusions

This paper discusses how to use the Gauss-Seidel method for solving the generalized Nash equilibrium problems of polynomials. The polynomial optimization in each loop of the Gauss-Seidel method is solved by the Lasserre type Moment-SOS relaxations. The convergence results are presented for general GNEPs and for the special case of GPGs. In particular, we give a certificate for checking GPGs. Numerical experiments show that the Gauss-Seidel method is efficient for solving many GNEPPs, even if the players' optimization problems are nonconvex.

TABLE 3. Computational Results for test problems in [19]

problem	initial point	τ_0	$\tau^{(k+1)}$	iterations	time	ε
A.2	$0.05e$	0.1	(4.3)	27	37.61	$0.73 \cdot 10^{-7}$
A.3	$0.1e$	0.1	(4.3)	46	13.13	$0.10 \cdot 10^{-5}$
A.4	$0.1e$	0.1	(4.3)	12	10.96	$0.32 \cdot 10^{-6}$
A.5	$0.1e$	0.1	(4.3)	25	10.21	$0.16 \cdot 10^{-7}$
A.6	e	0.1	(4.3)	38	11.38	$0.41 \cdot 10^{-6}$
A.7	e	0.1	(4.3)	17	10.74	$0.21 \cdot 10^{-7}$
A.8	$0.5e$	0.1	(4.3)	54	14.84	$0.52 \cdot 10^{-6}$
A.10a	see [19]	0.1	(4.3)	200	not convergent	
A.11	$0.5e$	0.1	(4.3)	37	4.86	$0.11 \cdot 10^{-6}$
A.12	e	0.1	(4.3)	65	7.22	$0.17 \cdot 10^{-7}$
A.13	e	0.1	(4.3)	12	2.01	$0.71 \cdot 10^{-8}$
A.14	$0.1e$	0.1	(4.3)	42	50.50	$0.56 \cdot 10^{-8}$
A.15	e	0.0001	(4.3)	200	45.51	$0.13 \cdot 10^{-5}$
A.17	e	0.001	(4.3)	200	25.82	$0.19 \cdot 10^{-7}$
A.18	e	0.5	(4.3)	200	59.11	$0.29 \cdot 10^{-5}$

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