# Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array* 

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#### Abstract

Let $\mathbb{K}$ denote a field. Let $d$ denote a nonnegative integer and consider a sequence $p=\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ consisting of scalars taken from $\mathbb{K}$. We call $p$ a parameter array whenever: (PA1) $\theta_{i} \neq \theta_{j}, \theta_{i}^{*} \neq \theta_{j}^{*}$ if $i \neq j,(0 \leq i, j \leq d)$; (PA2) $\varphi_{i} \neq 0, \phi_{i} \neq 0(1 \leq i \leq d) ;(\mathrm{PA} 3) \varphi_{i}=\phi_{1} \sum_{h=0}^{i-1} \frac{\theta_{h}-\theta_{d-h}}{\theta_{0}-\theta_{d}}+\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i-1}-\theta_{d}\right)$ $(1 \leq i \leq d) ;(\mathrm{PA} 4) \phi_{i}=\varphi_{1} \sum_{h=0}^{i-1} \frac{\theta_{h}-\theta_{d-h}}{\theta_{0}-\theta_{d}}+\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{d-i+1}-\theta_{0}\right)(1 \leq i \leq d) ;$ (PA5) $\left(\theta_{i-2}-\theta_{i+1}\right)\left(\theta_{i-1}-\theta_{i}\right)^{-1},\left(\theta_{i-2}^{*}-\theta_{i+1}^{*}\right)\left(\theta_{i-1}^{*}-\theta_{i}^{*}\right)^{-1}$ are equal and independent of $i$ for $2 \leq i \leq d-1$. In [13] we showed the parameter arrays are in bijection with the isomorphism classes of Leonard systems. Using this bijection we obtain the following two characterizations of parameter arrays. Assume $p$ satisfies PA1, PA2. Let $A, B, A^{*}, B^{*}$ denote the matrices in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which have entries $A_{i i}=\theta_{i}, B_{i i}=\theta_{d-i}$, $A_{i i}^{*}=\theta_{i}^{*}, B_{i i}^{*}=\theta_{i}^{*}(0 \leq i \leq d), A_{i, i-1}=1, B_{i, i-1}=1, A_{i-1, i}^{*}=\varphi_{i}, B_{i-1, i}^{*}=\phi_{i}$ $(1 \leq i \leq d)$, and all other entries 0 . We show the following are equivalent: (i) $p$ satisfies PA3-PA5; (ii) there exists an invertible $G \in \operatorname{Mat}_{d+1}(\mathbb{K})$ such that $G^{-1} A G=B$ and $G^{-1} A^{*} G=B^{*}$; (iii) for $0 \leq i \leq d$ the polynomial


$$
\sum_{n=0}^{i} \frac{\left(\lambda-\theta_{0}\right)\left(\lambda-\theta_{1}\right) \cdots\left(\lambda-\theta_{n-1}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{n-1}^{*}\right)}{\varphi_{1} \varphi_{2} \cdots \varphi_{n}}
$$

is a scalar multiple of the polynomial

$$
\sum_{n=0}^{i} \frac{\left(\lambda-\theta_{d}\right)\left(\lambda-\theta_{d-1}\right) \cdots\left(\lambda-\theta_{d-n+1}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{n-1}^{*}\right)}{\phi_{1} \phi_{2} \cdots \phi_{n}}
$$

We display all the parameter arrays in parametric form. For each array we compute the above polynomials. The resulting polynomials form a class consisting of the $q$-Racah, $q$ Hahn, dual $q$-Hahn, $q$-Krawtchouk, dual $q$-Krawtchouk, quantum $q$-Krawtchouk, affine $q$-Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito, and Orphan polynomials. The Bannai/Ito polynomials can be obtained from the $q$-Racah polynomials by letting $q$ tend to -1 . The Orphan polynomials have maximal degree 3 and exist for $\operatorname{char}(\mathbb{K})=2$ only. For each of the polynomials listed above we give the orthogonality, 3 -term recurrence, and difference equation in terms of the parameter array.

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## 1 Introduction

In this paper we continue to develop the theory of Leonard pairs and Leonard systems [6], [13], 14], 15], 16] [17, [18, [19. We briefly summarize our results so far. In [13] we introduced the notion of a Leonard pair and the closely related notion of a Leonard system (see Section 2 below.) We classified the Leonard systems. In the process we introduced the split decomposition for Leonard systems. Moreover we showed that every Leonard pair satisfies two cubic polynomial relations which we call the tridiagonal relations. The tridiagonal relations generalize both the cubic $q$-Serre relations and the Dolan-Grady relations. In 6] we introduced a generalization of a Leonard pair (resp. system) which we call a tridiagonal pair (resp. system.) We extended some of our results on Leonard pairs and systems to tridiagonal pairs and systems. For instance we showed that every tridiagonal system has a split decomposition. Moreover we showed that every tridiagonal pair satisfies an appropriate pair of tridiagonal relations. We did not get a classification of tridiagonal systems and to our knowledge this remains an open problem. In [14] we introduced the tridiagonal algebra. This is an associative algebra on two generators subject to a pair of tridiagonal relations. We showed that every tridiagonal pair induces on the underlying vector space the structure of an irreducible module for a tridiagonal algebra. Given an irreducible finite dimensional module for a tridiagonal algebra, we displayed sufficient conditions for it to be induced from a Leonard pair in this fashion. We also showed each sequence of Askey-Wilson polynomials gives a basis for an appropriate infinite dimensional irreducible tridiagonal algebra module. In [15] we began with an arbitrary Leonard pair, and exhibited 24 bases for the underlying vector space which we found attractive. For each of these bases we computed the matrices which represent the Leonard pair. We found each of these matrices is tridiagonal, diagonal, upper bidiagonal or lower bidiagonal. We computed the transition matrix for sufficiently many ordered pairs of bases in our set of 24 to enable one to readily find the transition matrix for any ordered pair of bases in our set of 24 . In the survey [16] we gave a number of examples of Leonard pairs. We used these examples to illustrate how Leonard pairs arise in representation theory, combinatorics, and the theory of orthogonal polynomials. The paper [17] is another survey. In [18] we introduced the notion of a parameter array. We showed that the classification of Leonard systems mentioned above gives a bijection from the set of isomorphism classes of Leonard systems to the set of parameter arrays. We introduced the $T D-D$ canonical form and the $L B-U B$ canonical form for Leonard systems. For a Leonard system in $T D-D$ canonical form the associated Leonard pair is represented by a tridiagonal and diagonal matrix, subject to a certain normalization. For a Leonard system in $L B-U B$ canonical form the associated Leonard pair is represented by a lower bidiagonal and upper bidiagonal matrix, subject to a certain normalization. We showed every Leonard system is isomorphic to a unique Leonard system which is in $T D-D$ canonical form and a unique Leonard system which is in $L B-U B$ canonical form. We described these canonical forms using the associated parameter array. In [19] we obtained two characterizations of Leonard pairs based on the split decomposition.

We now give an overview of the present paper. We first review our bijection between the set of isomorphism classes of Leonard systems and the set of parameter arrays. We then use this bijection to obtain two characterizations of Leonard systems. The first characterization
involves bidiagonal matrices and is given in Theorem 3.2. The second characterization involves polynomials and is given in Theorem 4.1. We view Theorem 4.1 as a variation on a theorem of D. Leonard [2, p. 260], [9]. In Section 5 we display all the parameter arrays. For each parameter array we display the corresponding polynomials from our second characterization. These corresponding polynomials form a class consisting of the $q$-Racah, $q$-Hahn, dual $q$-Hahn, $q$-Krawtchouk, dual $q$-Krawtchouk, quantum $q$-Krawtchouk, affine $q$ Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito, and Orphan polynomials. The Bannai/Ito polynomials can be obtained from the $q$-Racah polynomials by letting $q$ tend to -1 . The Orphan polynomials have maximal degree 3 and exist for $\operatorname{char}(\mathbb{K})=2$ only. For each of the polynomials listed above we give the orthogonality, 3-term recurrence, and difference equation in terms of the parameter array. We conclude the paper with an open problem.

We now recall the definition of a parameter array. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 [18, Definition 10.1] Let d denote a nonnegative integer. By a parameter array over $\mathbb{K}$ of diameter $d$ we mean a sequence of scalars $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ taken from $\mathbb{K}$ which satisfy the following conditions (PA1)-(PA5).
(PA1) $\theta_{i} \neq \theta_{j}, \quad \theta_{i}^{*} \neq \theta_{j}^{*} \quad$ if $\quad i \neq j, \quad(0 \leq i, j \leq d)$.
$($ PA2 $) \varphi_{i} \neq 0, \quad \phi_{i} \neq 0 \quad(1 \leq i \leq d)$.
(PA3) $\varphi_{i}=\phi_{1} \sum_{h=0}^{i-1} \frac{\theta_{h}-\theta_{d-h}}{\theta_{0}-\theta_{d}}+\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i-1}-\theta_{d}\right) \quad(1 \leq i \leq d)$.
(PA4) $\phi_{i}=\varphi_{1} \sum_{h=0}^{i-1} \frac{\theta_{h}-\theta_{d-h}}{\theta_{0}-\theta_{d}}+\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{d-i+1}-\theta_{0}\right) \quad(1 \leq i \leq d)$.
(PA5) The expressions

$$
\begin{equation*}
\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_{i}}, \quad \frac{\theta_{i-2}^{*}-\theta_{i+1}^{*}}{\theta_{i-1}^{*}-\theta_{i}^{*}} \tag{1}
\end{equation*}
$$

are equal and independent of $i$ for $2 \leq i \leq d-1$.
We now turn our attention to Leonard systems.

## 2 Parameter arrays and Leonard systems

We recall the notion of a Leonard system and discuss how these objects are related to parameter arrays. Our account will be brief; for more detail see [13, [15], [18, [16]. Let $d$ denote a nonnegative integer. Let $\operatorname{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices which have entries in $\mathbb{K}$. We index the rows and columns by $0,1, \ldots, d$. Let $\mathcal{A}$
denote a $\mathbb{K}$-algebra isomorphic to $\operatorname{Mat}_{d+1}(\mathbb{K})$. An element $A \in \mathcal{A}$ is called multiplicity-free whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Let $A$ denote a multiplicity-free element of $\mathcal{A}$. Let $\theta_{0}, \theta_{1}, \ldots, \theta_{d}$ denote an ordering of the eigenvalues of $A$, and for $0 \leq i \leq d$ put

$$
E_{i}=\prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A-\theta_{j} I}{\theta_{i}-\theta_{j}}
$$

where $I$ denotes the identity of $\mathcal{A}$. We observe $A E_{i}=\theta_{i} E_{i}(0 \leq i \leq d)$; (ii) $E_{i} E_{j}=\delta_{i j} E_{i}$ $(0 \leq i, j \leq d)$; (iii) $\sum_{i=0}^{d} E_{i}=I$. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by $A$. Using (i)-(iii) we find $E_{0}, E_{1}, \ldots, E_{d}$ form a basis for the $\mathbb{K}$-vector space $\mathcal{D}$. We call $E_{i}$ the primitive idempotent of $A$ associated with $\theta_{i}$. By a Leonard system in $\mathcal{A}$ we mean a sequence $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$ which satisfies the following (i)-(v).
(i) Each of $A, A^{*}$ is a multiplicity-free element of $\mathcal{A}$.
(ii) $E_{0}, E_{1}, \ldots, E_{d}$ is an ordering of the primitive idempotents of $A$.
(iii) $E_{0}^{*}, E_{1}^{*}, \ldots, E_{d}^{*}$ is an ordering of the primitive idempotents of $A^{*}$.
$\begin{array}{ll}\text { (iv) } E_{i}^{*} A E_{j}^{*}= \begin{cases}0, & \text { if }|i-j|>1 ; \\ \neq 0, & \text { if }|i-j|=1\end{cases} & (0 \leq i, j \leq d) . \\ \text { (v) } E_{i} A^{*} E_{j}= \begin{cases}0, & \text { if }|i-j|>1 ; \\ \neq 0, & \text { if }|i-j|=1\end{cases} & (0 \leq i, j \leq d) .\end{array}$
We call $\mathcal{A}$ the ambient algbebra of $\Phi$ and say $\Phi$ is over $\mathbb{K}$ [13, Definition 1.4].
Let $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$ denote a Leonard system in $\mathcal{A}$. Then each of the following is a Leonard system in $\mathcal{A}$ :

$$
\begin{aligned}
\Phi^{*} & :=\left(A^{*} ; A ;\left\{E_{i}^{*}\right\}_{i=0}^{d} ;\left\{E_{i}\right\}_{i=0}^{d}\right), \\
\Phi^{\downarrow} & :=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{d-i}^{*}\right\}_{i=0}^{d}\right), \\
\Phi^{\Downarrow} & :=\left(A ; A^{*} ;\left\{E_{d-i}^{d}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right) .
\end{aligned}
$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$
\begin{gather*}
*^{2}=\downarrow^{2}=\Downarrow^{2}=1,  \tag{2}\\
\Downarrow *=* \downarrow, \quad \downarrow *=* \Downarrow, \quad \downarrow \downarrow=\Downarrow \downarrow . \tag{3}
\end{gather*}
$$

The group generated by the symbols $*, \downarrow, \Downarrow$ subject to the relations (2), (3) is the dihedral group $D_{4}$. We recall $D_{4}$ is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of $D_{4}$ on the set of all Leonard systems.
Let $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$ denote a Leonard system in $\mathcal{A}$. In order to describe $\Phi$ we define some parameters. For $0 \leq i \leq d$ let $\theta_{i}$ (resp. $\theta_{i}^{*}$ ) denote the eigenvalue of $A$ (resp. $\left.A^{*}\right)$ associated with $E_{i}\left(\right.$ resp. $\left.E_{i}^{*}.\right)$ We call $\theta_{0}, \theta_{1}, \ldots, \theta_{d}\left(\right.$ resp. $\left.\theta_{0}^{*}, \theta_{1}^{*}, \ldots, \theta_{d}^{*}\right)$ the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. Let $V$ denote an irreducible left $\mathcal{A}$-module.

By a decomposition of $V$ we mean a sequence $U_{0}, U_{1}, \ldots, U_{d}$ consisting of 1-dimensional subspaces of $V$ such that

$$
\left.V=U_{0}+U_{1}+\cdots+U_{d} \quad \quad \text { (direct sum }\right)
$$

By [13, Theorem 3.2] there exists a unique decomposition $U_{0}, U_{1}, \ldots, U_{d}$ of $V$ such that both

$$
\begin{align*}
\left(A-\theta_{i} I\right) U_{i} & =U_{i+1} & & (0 \leq i \leq d-1), \quad\left(A-\theta_{d} I\right) U_{d}=0  \tag{4}\\
\left(A^{*}-\theta_{i}^{*} I\right) U_{i} & =U_{i-1} & & (1 \leq i \leq d), \quad\left(A^{*}-\theta_{0}^{*} I\right) U_{0}=0 \tag{5}
\end{align*}
$$

Pick any integer $i(1 \leq i \leq d)$. Then $\left(A^{*}-\theta_{i}^{*} I\right) U_{i}=U_{i-1}$ and $\left(A-\theta_{i-1} I\right) U_{i-1}=U_{i}$. Apparently $U_{i}$ is an eigenspace for $\left(A-\theta_{i-1} I\right)\left(A^{*}-\theta_{i}^{*} I\right)$ and the corresponding eigenvalue is a nonzero scalar in $\mathbb{K}$. We denote this eigenvalue by $\varphi_{i}$. We call $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{d}$ the first split sequence of $\Phi$. We let $\phi_{1}, \phi_{2}, \ldots, \phi_{d}$ denote the first split sequence of $\Phi^{\Downarrow}$ and call this the second split sequence of $\Phi$.

We recall the notion of isomorphism for Leonard systems. Let $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$ denote a Leonard system in $\mathcal{A}$ and let $\sigma: \mathcal{A} \rightarrow \mathcal{A}^{\prime}$ denote an isomorphism of $\mathbb{K}$-algebras. We write $\Phi^{\sigma}=\left(A^{\sigma} ; A^{* \sigma} ;\left\{E_{i}^{\sigma}\right\}_{i=0}^{d} ;\left\{E_{i}^{* \sigma}\right\}_{i=0}^{d}\right)$ and observe $\Phi^{\sigma}$ is a Leonard system in $\mathcal{A}^{\prime}$. Let $\Phi$ and $\Phi^{\prime}$ denote any Leonard systems over $\mathbb{K}$. By an isomorphism of Leonard systems from $\Phi$ to $\Phi^{\prime}$ we mean an isomorphism of $\mathbb{K}$-algbras from the ambient algebra of $\Phi$ to the ambient algebra of $\Phi^{\prime}$ such that $\Phi^{\sigma}=\Phi^{\prime}$. We say $\Phi$ and $\Phi^{\prime}$ are isomorphic whenever there exists an isomorphism of Leonard systems from $\Phi$ to $\Phi^{\prime}$.

Theorem 2.1 [13, Theorem 1.9] Let d denote a nonnegative integer and let $\left(\theta_{i}, \theta_{i}^{*}, i=\right.$ $\left.0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a sequence of scalars taken from $\mathbb{K}$. Then the following (i), (ii) are equivalent.
(i) The sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$.
(ii) There exists a Leonard system $\Phi$ over $\mathbb{K}$ which has eigenvalue sequence $\theta_{0}, \theta_{1}, \ldots, \theta_{d}$, dual eigenvalue sequence $\theta_{0}^{*}, \theta_{1}^{*}, \ldots, \theta_{d}^{*}$, first split sequence $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{d}$ and second split sequence $\phi_{1}, \phi_{2}, \ldots, \phi_{d}$.

Suppose (i), (ii) hold. Then $\Phi$ is unique up to isomorphism of Leonard systems.
Let $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$ denote a Leonard system. By the parameter array of $\Phi$ we mean the sequence ( $\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d$ ) where $\theta_{0}, \theta_{1}, \ldots, \theta_{d}\left(\right.$ resp. $\left.\theta_{0}^{*}, \theta_{1}^{*}, \ldots, \theta_{d}^{*}\right)$ is the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$ and $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{d}$ (resp. $\phi_{1}, \phi_{2}, \ldots, \phi_{d}$ ) is the first split sequence (resp. second split sequence ) of $\Phi$. By Theorem 2.1 the map which sends a given Leonard system to its parameter array induces a bijection from the set of isomorphism classes of Leonard systems over $\mathbb{K}$ to the set of parameter arrays over $\mathbb{K}$.

Earlier we mentioned an action of $D_{4}$ on the set of Leonard systems. The above bijection induces an action of $D_{4}$ on the set of parameter arrays. This action is described as follows.

Lemma 2.2 [13, Theorem 1.11] Let $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$ denote a Leonard system and let $p=\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote the corresponding parameter array.
(i) The parameter array of $\Phi^{*}$ is $p^{*}$ where $p^{*}:=\left(\theta_{i}^{*}, \theta_{i}, i=0 \ldots d ; \varphi_{j}, \phi_{d-j+1}, j=1 \ldots d\right)$.
(ii) The parameter array of $\Phi^{\downarrow}$ is $p^{\downarrow}$ where $p^{\downarrow}:=\left(\theta_{i}, \theta_{d-i}^{*}, i=0 \ldots d ; \phi_{d-j+1}, \varphi_{d-j+1}, j=\right.$ $1 . . . d)$.
(iii) The parameter array of $\Phi^{\Downarrow}$ is $p^{\Downarrow}$ where $p^{\Downarrow}:=\left(\theta_{d-i}, \theta_{i}^{*}, i=0 \ldots d ; \phi_{j}, \varphi_{j}, j=1 \ldots d\right)$.

## 3 Parameter arrays and bidiagonal matrices

In this section we characterize the parameter arrays in terms of bidiagonal matrices. We will refer to the following set-up.

Definition 3.1 Let d denote a nonnegative integer and let ( $\left.\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a sequence of scalars taken from $\mathbb{K}$. We assume this sequence satisfies PA1 and PA2.

Theorem 3.2 With reference to Definition 3.1, the following (i), (ii) are equivalent.
(i) The sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ satisfies PA3-PA5.
(ii) There exists an invertible matrix $G \in \operatorname{Mat}_{d+1}(\mathbb{K})$ such that both

$$
\begin{align*}
& G^{-1}\left(\begin{array}{ccccccc}
\theta_{0} & & & & & \mathbf{0} \\
1 & \theta_{1} & & & & & \\
& 1 & \theta_{2} & & & \\
& & \cdot & \cdot & & \\
& & & & \cdot & \cdot & \\
\mathbf{0} & & & & 1 & \theta_{d}
\end{array}\right) G=\left(\begin{array}{cccccc}
\theta_{d} & & & & & \mathbf{0} \\
1 & \theta_{d-1} & & & & \\
& 1 & \theta_{d-2} & & & \\
& & \cdot & . & & \\
& & & & \cdot & \\
\mathbf{0} & & & & 1 & \theta_{0}
\end{array}\right),  \tag{6}\\
& G^{-1}\left(\begin{array}{cccccc}
\theta_{0}^{*} & \varphi_{1} & & & & \mathbf{0} \\
& \theta_{1}^{*} & \varphi_{2} & & & \\
& & \theta_{2}^{*} & \cdot & & \\
& & & \cdot & . & \\
& & & & \cdot & \varphi_{d} \\
\mathbf{0} & & & & & \theta_{d}^{*}
\end{array}\right) G=\left(\begin{array}{cccccc}
\theta_{0}^{*} & \phi_{1} & & & & \mathbf{0} \\
& \theta_{1}^{*} & \phi_{2} & & & \\
& & \theta_{2}^{*} & \cdot & & \\
& & & \cdot & \cdot & \\
& & & & \cdot & \phi_{d} \\
\mathbf{0} & & & & & \theta_{d}^{*}
\end{array}\right) . \tag{7}
\end{align*}
$$

Proof: $(i) \Rightarrow(i i)$ The sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ satisfies PA1-PA5 and is therefore a parameter array over $\mathbb{K}$. By Theorem 2.1 there exists a Leonard system over $\mathbb{K}$ which has eigenvalue sequence $\theta_{0}, \theta_{1}, \ldots, \theta_{d}$, dual eigenvalue sequence $\theta_{0}^{*}, \theta_{1}^{*}, \ldots, \theta_{d}^{*}$, first split sequence $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{d}$ and second split sequence $\phi_{1}, \phi_{2}, \ldots, \phi_{d}$. We denote this system by $\Phi=\left(A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d} ;\left\{E_{i}^{*}\right\}_{i=0}^{d}\right)$. Let $\mathcal{A}$ denote the ambient algebra of $\Phi$ and let $V$ denote an irreducible left $\mathcal{A}$-module. Let $U_{0}, U_{1}, \ldots, U_{d}$ denote the decomposition of $V$ which satisfies (4), (5). For $0 \leq i \leq d$ let $u_{i}$ denote a nonzero vector in $U_{i}$ and observe $u_{0}, u_{1}, \ldots, u_{d}$ is a basis for $V$. Normalizing this basis we may assume $\left(A-\theta_{i} I\right) u_{i}=u_{i+1}$ for $0 \leq i \leq d-1$ and $\left(A-\theta_{d} I\right) u_{d}=0$. Since $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{d}$ is the first split sequence of $\Phi$ we have $\left(A^{*}-\theta_{i}^{*} I\right) u_{i}=\varphi_{i} u_{i-1}(1 \leq i \leq d),\left(A^{*}-\theta_{0}^{*} I\right) u_{0}=0$. Applying these comments to $\Phi^{\Downarrow}$ we find there exists a basis $v_{0}, v_{1}, \ldots, v_{d}$ for $V$ such that $\left(A-\theta_{d-i}\right) v_{i}=v_{i+1}(0 \leq i \leq d-1)$, $\left(A-\theta_{0}\right) v_{d}=0$ and $\left(A^{*}-\theta_{i}^{*}\right) v_{i}=\phi_{i} v_{i-1}(1 \leq i \leq d),\left(A^{*}-\theta_{0}^{*}\right) v_{0}=0$. Let $G \in \operatorname{Mat}_{d+1}(\mathbb{K})$
denote the transition matrix from the basis $u_{0}, u_{1}, \ldots, u_{d}$ to the basis $v_{0}, v_{1}, \ldots, v_{d}$, so that $v_{j}=\sum_{i=0}^{d} G_{i j} u_{i}$ for $0 \leq j \leq d$. Using this and elementary linear algebra we find $G$ is invertible and satisfies (6), (7).
$(i i) \Rightarrow(i)$ We apply Theorem [2.1] We show condition (ii) holds in that theorem. In order to do this we invoke some results from [19. Consider the following matrices in $\operatorname{Mat}_{d+1}(\mathbb{K})$ :

$$
A=\left(\begin{array}{cccccc}
\theta_{0} & & & & & \mathbf{0} \\
1 & \theta_{1} & & & & \\
& 1 & \theta_{2} & & & \\
& & \cdot & \cdot & & \\
& & & \cdot & \cdot & \\
\mathbf{0} & & & & 1 & \theta_{d}
\end{array}\right), \quad A^{*}=\left(\begin{array}{cccccc}
\theta_{0}^{*} & \varphi_{1} & & & & \mathbf{0} \\
& \theta_{1}^{*} & \varphi_{2} & & & \\
& & \theta_{2}^{*} & \cdot & \\
& & & \cdot & . & \\
& & & & \cdot & \varphi_{d} \\
\mathbf{0} & & & & & \theta_{d}^{*}
\end{array}\right) .
$$

We observe $A$ (resp. $A^{*}$ ) is multiplicity-free, with eigenvalues $\theta_{0}, \theta_{1}, \ldots, \theta_{d}$ (resp. $\theta_{0}^{*}, \theta_{1}^{*}, \ldots$, $\theta_{d}^{*}$.) For $0 \leq i \leq d$ we let $E_{i}$ (resp. $E_{i}^{*}$ ) denote the primitive idempotent for $A$ (resp. $A^{*}$ ) associated with $\theta_{i}$ (resp. $\theta_{i}^{*}$.) By [19, Lemma 6.2, Theorem 6.3] the sequence ( $A ; A^{*} ;\left\{E_{i}\right\}_{i=0}^{d}$; $\left\{E_{i}^{*}\right\}_{i=0}^{d}$ ) is a Leonard system in $\operatorname{Mat}_{d+1}(\mathbb{K})$. Let us call this system $\Phi$. By the construction $\Phi$ has eigenvalue sequence $\theta_{0}, \theta_{1}, \ldots, \theta_{d}$ and dual eigenvalue sequence $\theta_{0}^{*}, \theta_{1}^{*}, \ldots, \theta_{d}^{*}$. From the form of $A$ and $A^{*}$ we find $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{d}$ is the first split sequence for $\Phi$. By the last line of [19, Theorem 6.3] we find $\phi_{1}, \phi_{2}, \ldots, \phi_{d}$ is the second split sequence for $\Phi$. Now Theorem [2.1(ii) holds; applying that theorem we find ( $\left.\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$. In particular $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ satisfies PA3-PA5.

The matrix $G$ from Theorem 3.2(ii) will be discussed further in Section 10.

## 4 Parameter arrays and polynomials

In this section we characterize the parameter arrays in terms of polynomials. We will use the following notation. Let $\lambda$ denote an indeterminate, and let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of all polynomials in $\lambda$ which have coefficients in $\mathbb{K}$. For the rest of this paper all polynomials which we discuss are assumed to lie in $\mathbb{K}[\lambda]$.

We view the following theorem as a variation on a theorem of D. Leonard [2, p. 260], [9].
Theorem 4.1 With reference to Definition 3.1, the following (i), (ii) are equivalent.
(i) The sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ satisfies PA3-PA5.
(ii) For $0 \leq i \leq d$ the polynomial

$$
\begin{equation*}
\sum_{n=0}^{i} \frac{\left(\lambda-\theta_{0}\right)\left(\lambda-\theta_{1}\right) \cdots\left(\lambda-\theta_{n-1}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{n-1}^{*}\right)}{\varphi_{1} \varphi_{2} \cdots \varphi_{n}} \tag{8}
\end{equation*}
$$

is a scalar multiple of the polynomial

$$
\begin{equation*}
\sum_{n=0}^{i} \frac{\left(\lambda-\theta_{d}\right)\left(\lambda-\theta_{d-1}\right) \cdots\left(\lambda-\theta_{d-n+1}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{n-1}^{*}\right)}{\phi_{1} \phi_{2} \cdots \phi_{n}} \tag{9}
\end{equation*}
$$

Proof: Let us abbreviate

$$
A=\left(\begin{array}{cccccc}
\theta_{0} & & & & & \mathbf{0} \\
1 & \theta_{1} & & & & \\
& 1 & \theta_{2} & & & \\
& & \cdot & \cdot & & \\
& & & \cdot & \cdot & \\
\mathbf{0} & & & & 1 & \theta_{d}
\end{array}\right), \quad A^{*}=\left(\begin{array}{cccccc}
\theta_{0}^{*} & \varphi_{1} & & & & \mathbf{0} \\
& \theta_{1}^{*} & \varphi_{2} & & & \\
& & \theta_{2}^{*} & \cdot & & \\
& & & \cdot & \cdot & \\
& & & & \cdot & \varphi_{d} \\
\mathbf{0} & & & & & \theta_{d}^{*}
\end{array}\right)
$$

and

$$
B=\left(\begin{array}{cccccc}
\theta_{d} & & & & & \mathbf{0} \\
1 & \theta_{d-1} & & & & \\
& 1 & \theta_{d-2} & & & \\
& & \cdot & \cdot & & \\
& & & \cdot & \cdot & \\
\mathbf{0} & & & & 1 & \theta_{0}
\end{array}\right), \quad B^{*}=\left(\begin{array}{cccccc}
\theta_{0}^{*} & \phi_{1} & & & & \mathbf{0} \\
& \theta_{1}^{*} & \phi_{2} & & & \\
& & \theta_{2}^{*} & \cdot & & \\
& & & \cdot & \cdot & \\
& & & & \cdot & \phi_{d} \\
\mathbf{0} & & & & & \theta_{d}^{*}
\end{array}\right)
$$

We let $T, T^{*}, T^{\Downarrow}$ denote the matrices in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which have entries $T_{i j}=\prod_{h=0}^{j-1}\left(\theta_{i}-\theta_{h}\right)$, $T_{i j}^{*}=\prod_{h=0}^{j-1}\left(\theta_{i}^{*}-\theta_{h}^{*}\right), T_{i j}^{\Downarrow}=\prod_{h=0}^{j-1}\left(\theta_{d-i}-\theta_{d-h}\right)$ for $0 \leq i, j \leq d$. Each of $T, T^{*}, T^{\Downarrow}$ is lower triangular with diagonal entries nonzero so these matrices are invertible. Let $D$ (resp. $D^{\Downarrow}$ ) denote the diagonal matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has $i i$ th entry $\varphi_{1} \varphi_{2} \cdots \varphi_{i}\left(\right.$ resp. $\left.\phi_{1} \phi_{2} \cdots \phi_{i}\right)$ for $0 \leq i \leq d$. Each of $D, D^{\Downarrow}$ is invertible. We let $Z$ denote the matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has $i j$ th entry 1 if $i+j=d$ and 0 if $i+j \neq d$, for $0 \leq i, j \leq d$. Observe $Z^{2}=I$ so $Z$ is invertible. We let $H$ (resp. $H^{*}$ ) denote the diagonal matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has $i i$ th entry $\theta_{i}$ (resp. $\theta_{i}^{*}$ ) for $0 \leq i \leq d$. One verifies $T A=H T$ so $A=T^{-1} H T$. One verifies $Z T^{\Downarrow} B=H Z T^{\Downarrow}$ so $B=T^{\Downarrow-1} Z H Z T^{\Downarrow}$. One verifies $D A^{*} D^{-1} T^{* t}=T^{* t} H^{*}$ so $A^{*}=D^{-1} T^{* t} H^{*} T^{*-1 t} D$. Similarly $B^{*}=D^{\Downarrow-1} T^{* t} H^{*} T^{*-1 t} D^{\Downarrow}$. For $0 \leq i \leq d$ let $f_{i}$ denote the polynomial in (8). Let $\mathcal{P}$ denote the matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has $i j$ th entry $f_{j}\left(\theta_{i}\right)$ for $0 \leq i, j \leq d$. From the form of (8) we find $\mathcal{P}=T D^{-1} T^{* t}$. For $0 \leq i \leq d$ let $f_{i}^{\Downarrow}$ denote the polynomial in (9). Let $\mathcal{P}^{\Downarrow}$ denote the matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has $i j$ th entry $f_{j}^{\Downarrow}\left(\theta_{i}\right)$ for $0 \leq i, j \leq d$. From the form of (9) we find $\mathcal{P}^{\Downarrow}=Z T^{\Downarrow} D^{\Downarrow-1} T^{* t}$.
$(i) \Rightarrow(i i)$ By Theorem 3.2 there exists an invertible matrix $G \in \operatorname{Mat}_{d+1}(\mathbb{K})$ such that $G^{-1} A G=B$ and $G^{-1} A^{*} G=B^{*}$. Evaluating $G^{-1} A^{*} G=B^{*}$ using $A^{*}=D^{-1} T^{* t} H^{*} T^{*-1 t} D$ and $B^{*}=D^{\Downarrow-1} T^{* t} H^{*} T^{*-1 t} D^{\Downarrow}$ we find $T^{*-1 t} D G D^{\Downarrow-1} T^{* t}$ commutes with $H^{*}$. Since $H^{*}$ is diagonal with diagonal entries mutually distinct we find $T^{*-1 t} D G D^{\Downarrow-1} T^{* t}$ is diagonal. We denote this diagonal matrix by $F$ and observe $G=D^{-1} T^{* t} F T^{*-1 t} D^{\Downarrow}$. In this product each factor is upper triangular (or diagonal) so $G$ is upper triangular. Recall $G^{-1} A G=B$; evaluating this using $A=T^{-1} H T$ and $B=T^{\Downarrow-1} Z H Z T^{\Downarrow}$ we find $T G T^{\Downarrow-1} Z$ commutes with $H$. Since $H$ is diagonal with diagonal entries mutually distinct we find $T G T^{\Downarrow-1} Z$ is diagonal. We denote this diagonal matrix by $Y$ and observe $T G=Y Z T^{\Downarrow}$. In this equation we compute the entries in column 0 . To aid in this calculation we recall $G$ is upper triangular and observe $T_{i 0}=1, T_{i 0}^{\Downarrow}=1$ for $0 \leq i \leq d$. Computing the column 0 entries in $T G=Y Z T^{\Downarrow}$ using these facts we find $Y_{i i}=G_{00}$ for $0 \leq i \leq d$. Apparently $Y=G_{00} I$. We remark $G_{00} \neq 0$ since $Y$ is invertible by the construction. Dividing $G$ by $G_{00}$ we may assume $G_{00}=1$. Now $Y=I$ so $G=T^{-1} Z T^{\Downarrow}$. Recall $T^{*-1 t} D G D^{\Downarrow-1} T^{* t}$ is diagonal. We evaluate this expression using $G=T^{-1} Z T^{\Downarrow}$ and find $T^{*-1 t} D T^{-1} Z T^{\Downarrow} D^{\Downarrow-1} T^{* t}$ is diagonal.

But $T^{*-1 t} D T^{-1} Z T^{\Downarrow} D^{\Downarrow-1} T^{* t}=\mathcal{P}^{-1} \mathcal{P}^{\Downarrow}$ so $\mathcal{P}^{-1} \mathcal{P}^{\Downarrow}$ is diagonal. Taking the inverse we find $\mathcal{P}^{\Downarrow-1} \mathcal{P}$ is diagonal. For $0 \leq i \leq d$ let $\alpha_{i}$ denote the $i i$ entry of this diagonal matrix. From the definition of $\mathcal{P}$ and $\mathcal{P}^{\Downarrow}$ we find $f_{i}\left(\theta_{j}\right)=\alpha_{i} f_{i}^{\Downarrow}\left(\theta_{j}\right)$ for $0 \leq i, j \leq d$. Recall $\theta_{0}, \theta_{1}, \ldots, \theta_{d}$ are mutually distinct, and that each of $f_{i}, f_{i}^{\Downarrow}$ has degree $i$ for $0 \leq i \leq d$. From these comments we find $f_{i}=\alpha_{i} f_{i}^{\Downarrow}$ for $0 \leq i \leq d$.
$(i i) \Rightarrow(i)$ We show Theorem 3.2(ii) holds. To do this we exhibit an invertible matrix $G \in \operatorname{Mat}_{d+1}(\mathbb{K})$ such that $A G=G B$ and $A^{*} G=G B^{*}$. We define $G=T^{-1} Z T^{\Downarrow}$. Observe $G$ is invertible. The equation $A G=G B$ is routinely verified by evaluating $A, B, G$ using $A=T^{-1} H T, B=T^{\Downarrow-1} Z H Z T^{\Downarrow}, G=T^{-1} Z T^{\Downarrow}$. We now show $A^{*} G=G B^{*}$. For $0 \leq i \leq d$ there exists $\alpha_{i} \in \mathbb{K}$ such that $f_{i}=\alpha_{i} f_{i}^{\Downarrow}$. It follows $\mathcal{P}=\mathcal{P}^{\Downarrow} \operatorname{diag}\left(\alpha_{0}, \alpha_{1}, \ldots, \alpha_{d}\right)$. From this and since $H^{*}$ is diagonal we find $\mathcal{P} H^{*} \mathcal{P}^{-1}=\mathcal{P}^{\Downarrow} H^{*} \mathcal{P}^{\Downarrow-1}$. In this equation we multiply both sides on the left by $T^{-1}$ and on the right by $Z T^{\Downarrow}$ to obtain $T^{-1} \mathcal{P} H^{*} \mathcal{P}^{-1} Z T^{\Downarrow}=$ $T^{-1} \mathcal{P}^{\Downarrow} H^{*} \mathcal{P}^{\Downarrow-1} Z T^{\Downarrow}$. In this equation the left side is equal to $A^{*} G$ and the right side is equal to $G B^{*}$ so $A^{*} G=G B^{*}$. We have now shown $G$ satisfies Theorem 3.2(ii). Applying that theorem we find $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ satisifes PA3-PA5.

We finish this section with a comment.
Lemma 4.2 Referring to Theorem 4.1, assume the equivalent conditions (i), (ii) from that theorem hold. Then for $0 \leq i \leq d$ the scalar referred to in condition (ii) is equal to

$$
\frac{\phi_{1} \phi_{2} \cdots \phi_{i}}{\varphi_{1} \varphi_{2} \cdots \varphi_{i}}
$$

Proof: Compare the coefficient of $\lambda^{i}$ in (8), (9).

## 5 The parameter arrays

In this section we display all the parameter arrays over $\mathbb{K}$. We will use the following notatation.

Definition 5.1 Let $p=\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$. By a base for $p$, we mean a nonzero scalar $q$ in the algebraic closure of $\mathbb{K}$ such that $q+q^{-1}+1$ is equal to the common value of (1) for $2 \leq i \leq d-1$. We remark on the uniqueness of the base. Suppose $d \geq 3$. If $q$ is a base for $p$ then so is $q^{-1}$ and $p$ has no other base. Suppose $d<3$. Then any nonzero scalar in the algebraic closure of $\mathbb{K}$ is a base for $p$.

Definition 5.2 Let $p=\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$. For $0 \leq i \leq d$ we let $f_{i}$ denote the following polynomial in $\mathbb{K}[\lambda]$.

$$
\begin{equation*}
f_{i}=\sum_{n=0}^{i} \frac{\left(\lambda-\theta_{0}\right)\left(\lambda-\theta_{1}\right) \cdots\left(\lambda-\theta_{n-1}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{n-1}^{*}\right)}{\varphi_{1} \varphi_{2} \cdots \varphi_{n}} . \tag{10}
\end{equation*}
$$

We call $f_{0}, f_{1}, \ldots, f_{d}$ the polynomials which correspond to $p$.

We now display all the parameter arrays over $\mathbb{K}$. For each displayed array $\left(\theta_{i}, \theta_{i}^{*}, i=\right.$ $\left.0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ we give a base and present $f_{i}\left(\theta_{j}\right)$ for $0 \leq i, j \leq d$, where $f_{0}, f_{1}, \ldots, f_{d}$ are the corresponding polynomials. Our presentation is organized as follows. In each of Example 5.35 .15 below we give a family of parameter arrays over $\mathbb{K}$. In Theorem 5.16 we show every parameter array over $\mathbb{K}$ is contained in at least one of these families.

In each of Example 5.3 5.15 below the following implicit assumptions apply: d denotes a nonnegative integer, the scalars $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ are contained in $\mathbb{K}$, and the scalars $q, h, h^{*} \ldots$ are contained in the algebraic closure of $\mathbb{K}$.

Example 5.3 (q-Racah) Assume

$$
\begin{align*}
\theta_{i} & =\theta_{0}+h\left(1-q^{i}\right)\left(1-s q^{i+1}\right) q^{-i}  \tag{11}\\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right)\left(1-s^{*} q^{i+1}\right) q^{-i} \tag{12}
\end{align*}
$$

for $0 \leq i \leq d$ and

$$
\begin{align*}
\varphi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(1-r_{1} q^{i}\right)\left(1-r_{2} q^{i}\right),  \tag{13}\\
\phi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(r_{1}-s^{*} q^{i}\right)\left(r_{2}-s^{*} q^{i}\right) / s^{*} \tag{14}
\end{align*}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}, q, s, s^{*}, r_{1}, r_{2}$ are nonzero and $r_{1} r_{2}=s s^{*} q^{d+1}$. Assume none of $q^{i}, r_{1} q^{i}, r_{2} q^{i}, s^{*} q^{i} / r_{1}, s^{*} q^{i} / r_{2}$ is equal to 1 for $1 \leq i \leq d$ and that neither of $s q^{i}, s^{*} q^{i}$ is equal to 1 for $2 \leq i \leq 2 d$. Then $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{4} \phi_{3}\left(\begin{array}{cc}
q^{-i}, s^{*} q^{i+1}, q^{-j}, s q^{j+1} & q, q) \\
r_{1} q, r_{2} q, q^{-d} & q)
\end{array}\right.
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are $q$-Racah polynomials.
Example 5.4 (q-Hahn) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+h\left(1-q^{i}\right) q^{-i} \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right)\left(1-s^{*} q^{i+1}\right) q^{-i}
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(1-r q^{i}\right) \\
\phi_{i} & =-h h^{*} q^{1-i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(r-s^{*} q^{i}\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}, q, s^{*}$, r are nonzero. Assume none of $q^{i}, r q^{i}, s^{*} q^{i} / r$ is equal to 1 for $1 \leq i \leq d$ and that $s^{*} q^{i} \neq 1$ for $2 \leq i \leq 2 d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=\right.$ $1 \ldots d)$ is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} \phi_{2}\left(\left.\begin{array}{c}
q^{-i}, s^{*} q^{i+1}, q^{-j} \\
r q, q^{-d}
\end{array} \right\rvert\, q, q\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are $q$-Hahn polynomials.

Example 5.5 (Dual q-Hahn) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+h\left(1-q^{i}\right)\left(1-s q^{i+1}\right) q^{-i} \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right) q^{-i}
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(1-r q^{i}\right) \\
\phi_{i} & =h h^{*} q^{d+2-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(s-r q^{i-d-1}\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}, q, r, s$ are nonzero. Assume none of $q^{i}, r q^{i}, s q^{i} / r$ is equal to 1 for $1 \leq i \leq d$ and that $s q^{i} \neq 1$ for $2 \leq i \leq 2 d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=\right.$ $1 \ldots$. ) is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} \phi_{2}\left(\left.\begin{array}{c}
q^{-i}, q^{-j}, s q^{j+1} \\
r q, q^{-d}
\end{array} \right\rvert\, q, q\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are dual $q$-Hahn polynomials.
Example 5.6 (Quantum q-Krawtchouk) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}-s q\left(1-q^{i}\right), \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right) q^{-i}
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =-r h^{*} q^{1-i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right) \\
\phi_{i} & =h^{*} q^{d+2-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(s-r q^{i-d-1}\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h^{*}, q, r, s$ are nonzero. Assume neither of $q^{i}, s q^{i} / r$ is equal to 1 for $1 \leq i \leq d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{2} \phi_{1}\left(\left.\begin{array}{c}
q^{-i}, q^{-j} \\
q^{-d}
\end{array} \right\rvert\, q, s r^{-1} q^{j+1}\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are quantum $q$-Krawtchouk polynomials.
Example 5.7 (q-Krawtchouk) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+h\left(1-q^{i}\right) q^{-i} \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right)\left(1-s^{*} q^{i+1}\right) q^{-i}
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right) \\
\phi_{i} & =h h^{*} s^{*} q\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}, q, s^{*}$ are nonzero. Assume $q^{i} \neq 1$ for $1 \leq i \leq d$ and that $s^{*} q^{i} \neq 1$ for $2 \leq i \leq 2 d$. Then the sequence ( $\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d$ ) is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} \phi_{2}\left(\left.\begin{array}{c}
q^{-i}, s^{*} q^{i+1}, q^{-j} \\
0, q^{-d}
\end{array} \right\rvert\, q, q\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are $q$-Krawtchouk polynomials.
Example 5.8 (Affine q-Krawtchouk) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+h\left(1-q^{i}\right) q^{-i}, \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right) q^{-i}
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)\left(1-r q^{i}\right) \\
\phi_{i} & =-h h^{*} r q^{1-i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}, q$, r are nonzero. Assume neither of $q^{i}$, rq${ }^{i}$ is equal to 1 for $1 \leq i \leq d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} \phi_{2}\left(\left.\begin{array}{cc}
q^{-i}, & 0, \\
r q, & q^{-j} \\
r q & q^{-d}
\end{array} \right\rvert\, q, q\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are affine $q$-Krawtchouk polynomials.
Example 5.9 (Dual q-Krawtchouk) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+h\left(1-q^{i}\right)\left(1-s q^{i+1}\right) q^{-i} \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(1-q^{i}\right) q^{-i}
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h h^{*} q^{1-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right) \\
\phi_{i} & =h h^{*} s q^{d+2-2 i}\left(1-q^{i}\right)\left(1-q^{i-d-1}\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}, q, s$ are nonzero. Assume $q^{i} \neq 1$ for $1 \leq i \leq d$ and $s q^{i} \neq 1$ for $2 \leq i \leq 2 d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base $q$. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} \phi_{2}\left(\left.\begin{array}{c}
q^{-i}, q^{-j}, s q^{j+1} \\
0, q^{-d}
\end{array} \right\rvert\, q, q\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are dual $q$-Krawtchouk polynomials.

Example 5.10 (Racah) Assume

$$
\begin{align*}
\theta_{i} & =\theta_{0}+h i(i+1+s)  \tag{15}\\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*} i\left(i+1+s^{*}\right) \tag{16}
\end{align*}
$$

for $0 \leq i \leq d$ and

$$
\begin{align*}
\varphi_{i} & =h h^{*} i(i-d-1)\left(i+r_{1}\right)\left(i+r_{2}\right)  \tag{17}\\
\phi_{i} & =h h^{*} i(i-d-1)\left(i+s^{*}-r_{1}\right)\left(i+s^{*}-r_{2}\right) \tag{18}
\end{align*}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}$ are nonzero and that $r_{1}+r_{2}=s+s^{*}+d+1$. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than d. Assume none of $r_{1}, r_{2}, s^{*}-r_{1}, s^{*}-r_{2}$ is equal to $-i$ for $1 \leq i \leq d$ and that neither of $s, s^{*}$ is equal to $-i$ for $2 \leq i \leq 2 d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base 1 . The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{4} F_{3}\left(\begin{array}{cc|c}
-i, & i+1+s^{*},-j, j+1+s & 1 \\
r_{1}+1, & r_{2}+1, \quad-d & 1
\end{array}\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are Racah polynomials.
Example 5.11 (Hahn) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+s i \\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*} i\left(i+1+s^{*}\right)
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h^{*} \operatorname{si}(i-d-1)(i+r) \\
\phi_{i} & =-h^{*} \operatorname{si}(i-d-1)\left(i+s^{*}-r\right)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h^{*}, s$ are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than $d$. Assume neither of $r, s^{*}-r$ is equal to $-i$ for $1 \leq i \leq d$ and that $s^{*} \neq-i$ for $2 \leq i \leq 2 d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base 1. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} F_{2}\left(\begin{array}{cc|c}
-i, i+1+s^{*},-j & 1 \\
r+1, & -d & 1
\end{array}\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are Hahn polynomials.
Example 5.12 (Dual Hahn) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+h i(i+1+s), \\
\theta_{i}^{*} & =\theta_{0}^{*}+s^{*} i
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =h s^{*} i(i-d-1)(i+r) \\
\phi_{i} & =h s^{*} i(i-d-1)(i+r-s-d-1)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $h, s^{*}$ are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than $d$. Assume neither of $r, s-r$ is equal to $-i$ for $1 \leq i \leq d$ and that $s \neq-i$ for $2 \leq i \leq 2 d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base 1. The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{3} F_{2}\left(\begin{array}{c|c}
-i,-j, j+1+s & 1 \\
r+1, & -d
\end{array}\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are dual Hahn polynomials.
Example 5.13 (Krawtchouk) Assume

$$
\begin{aligned}
\theta_{i} & =\theta_{0}+s i \\
\theta_{i}^{*} & =\theta_{0}^{*}+s^{*} i
\end{aligned}
$$

for $0 \leq i \leq d$ and

$$
\begin{aligned}
\varphi_{i} & =r i(i-d-1) \\
\phi_{i} & =\left(r-s s^{*}\right) i(i-d-1)
\end{aligned}
$$

for $1 \leq i \leq d$. Assume $r, s, s^{*}$ are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than $d$. Assume $r \neq s s^{*}$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base 1 . The corresponding polynomials $f_{i}$ satisfy

$$
f_{i}\left(\theta_{j}\right)={ }_{2} F_{1}\left(\begin{array}{c|c}
-i,-j \\
-d & r^{-1} s s^{*}
\end{array}\right)
$$

for $0 \leq i, j \leq d$. These $f_{i}$ are Krawtchouk polynomials.
Example 5.14 (Bannai/Ito) Assume

$$
\begin{align*}
\theta_{i} & =\theta_{0}+h\left(s-1+(1-s+2 i)(-1)^{i}\right)  \tag{19}\\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(s^{*}-1+\left(1-s^{*}+2 i\right)(-1)^{i}\right) \tag{20}
\end{align*}
$$

for $0 \leq i \leq d$ and

$$
\begin{align*}
& \varphi_{i}= \begin{cases}-4 h h^{*} i\left(i+r_{1}\right), & \text { if } i \text { even, d even; } \\
-4 h h^{*}(i-d-1)\left(i+r_{2}\right), & \text { if } i \text { odd, d even; } \\
-4 h h^{*} i(i-d-1), & \text { if } i \text { even, } d \text { odd; } \\
-4 h h^{*}\left(i+r_{1}\right)\left(i+r_{2}\right), & \text { if } i \text { odd, d odd, }\end{cases}  \tag{21}\\
& \phi_{i}= \begin{cases}4 h h^{*} i\left(i-s^{*}-r_{1}\right), & \text { if } i \text { even, d even; } \\
4 h h^{*}(i-d-1)\left(i-s^{*}-r_{2}\right), & \text { if } i \text { odd, d even; } \\
-4 h h^{*} i(i-d-1), & \text { if } i \text { even, d odd; } \\
-4 h h^{*}\left(i-s^{*}-r_{1}\right)\left(i-s^{*}-r_{2}\right), & \text { if } i \text { odd, d odd }\end{cases} \tag{22}
\end{align*}
$$

for $1 \leq i \leq d$. Assume $h, h^{*}$ are nonzero and that $r_{1}+r_{2}=-s-s^{*}+d+1$. Assume the characteristic of $\mathbb{K}$ is either 0 or an odd prime greater than d/2. Assume neither of $r_{1},-s^{*}-r_{1}$ is equal to $-i$ for $1 \leq i \leq d, d-i$ even. Assume neither of $r_{2},-s^{*}-r_{2}$ is equal to $-i$ for $1 \leq i \leq d$, $i$ odd. Assume neither of $s, s^{*}$ is equal to $2 i$ for $1 \leq i \leq d$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$ which has base -1 . We call the corresponding polynomials from Definition 5.2 the Bannai/Ito polynomials [2, $p$. 260].

Example 5.15 (Orphan) For this example assume $\mathbb{K}$ has characteristic 2. For notational convenience we define some scalars $\gamma_{0}, \gamma_{1}, \gamma_{2}, \gamma_{3}$ in $\mathbb{K}$. We define $\gamma_{i}=0$ for $i \in\{0,3\}$ and $\gamma_{i}=1$ for $i \in\{1,2\}$. Assume

$$
\begin{align*}
\theta_{i} & =\theta_{0}+h\left(s i+\gamma_{i}\right)  \tag{23}\\
\theta_{i}^{*} & =\theta_{0}^{*}+h^{*}\left(s^{*} i+\gamma_{i}\right) \tag{24}
\end{align*}
$$

for $0 \leq i \leq 3$. Assume $\varphi_{1}=h h^{*} r, \varphi_{2}=h h^{*}, \varphi_{3}=h h^{*}\left(r+s+s^{*}\right)$ and $\phi_{1}=h h^{*}\left(r+s\left(1+s^{*}\right)\right)$, $\phi_{2}=h h^{*}, \phi_{3}=h h^{*}\left(r+s^{*}(1+s)\right)$. Assume each of $h, h^{*}, s, s^{*}, r$ is nonzero. Assume neither of $s, s^{*}$ is equal to 1 and that $r$ is equal to none of $s+s^{*}, s\left(1+s^{*}\right), s^{*}(1+s)$. Then the sequence $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots 3 ; \varphi_{j}, \phi_{j}, j=1 \ldots 3\right)$ is a parameter array over $\mathbb{K}$ which has diameter 3 and base 1. We call the corresponding polynomials from Definition5.2 the Orphan polynomials.

Theorem 5.16 Every parameter array over $\mathbb{K}$ is listed in at least one of the Examples 5.35 .15

Proof: Let $p:=\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$. We show this array is given in at least one of the Examples 5.3.5.15. We assume $d \geq 1$; otherwise the result is trivial. For notational convenience let $\tilde{K}$ denote the algebraic closure of $\mathbb{K}$. Let $q$ denote a base for $p$ as in Definition 5.1. For $d<3$ we may assume $q \neq 1$ and $q \neq-1$ in view of the remark in Definition 5.1. By PA5 and Definition 5.1 both

$$
\begin{array}{r}
\theta_{i-2}-[3]_{q} \theta_{i-1}+[3]_{q} \theta_{i}-\theta_{i+1}=0, \\
\theta_{i-2}^{*}-[3]_{q} \theta_{i-1}^{*}+[3]_{q} \theta_{i}^{*}-\theta_{i+1}^{*}=0 \tag{26}
\end{array}
$$

for $2 \leq i \leq d-1$, where $[3]_{q}:=q+q^{-1}+1$. We divide the argument into the following four cases. (I) $q \neq 1, q \neq-1$; (II) $q=1$ and $\operatorname{char}(\mathbb{K}) \neq 2$; (III) $q=-1$ and $\operatorname{char}(\mathbb{K}) \neq 2$; (IV) $q=1$ and $\operatorname{char}(\mathbb{K})=2$.

Case I: $q \neq 1, q \neq-1$.
By (25) there exist scalars $\eta, \mu, h$ in $\tilde{\mathbb{K}}$ such that

$$
\begin{equation*}
\theta_{i}=\eta+\mu q^{i}+h q^{-i} \quad(0 \leq i \leq d) \tag{27}
\end{equation*}
$$

By (26)) there exist scalars $\eta^{*}, \mu^{*}, h^{*}$ in $\tilde{\mathbb{K}}$ such that

$$
\begin{equation*}
\theta_{i}^{*}=\eta^{*}+\mu^{*} q^{i}+h^{*} q^{-i} \quad(0 \leq i \leq d) \tag{28}
\end{equation*}
$$

Observe $\mu, h$ are not both 0 ; otherwise $\theta_{1}=\theta_{0}$ by (27). Similarly $\mu^{*}, h^{*}$ are not both 0 . For $1 \leq i \leq d$ we have $q^{i} \neq 1$; otherwise $\theta_{i}=\theta_{0}$ by (27). Setting $i=0$ in (27]), (28) we obtain

$$
\begin{align*}
\theta_{0} & =\eta+\mu+h  \tag{29}\\
\theta_{0}^{*} & =\eta^{*}+\mu^{*}+h^{*} \tag{30}
\end{align*}
$$

We claim there exists $\tau \in \tilde{\mathbb{K}}$ such that both

$$
\begin{align*}
\varphi_{i} & =\left(q^{i}-1\right)\left(q^{d-i+1}-1\right)\left(\tau-\mu \mu^{*} q^{i-1}-h h^{*} q^{-i-d}\right)  \tag{31}\\
\phi_{i} & =\left(q^{i}-1\right)\left(q^{d-i+1}-1\right)\left(\tau-h \mu^{*} q^{i-d-1}-\mu h^{*} q^{-i}\right) \tag{32}
\end{align*}
$$

for $1 \leq i \leq d$. Since $q \neq 1$ and $q^{d} \neq 1$ there exists $\tau \in \tilde{\mathbb{K}}$ such that (31) holds for $i=1$. In the equation of PA4, we eliminate $\varphi_{1}$ using (31) at $i=1$, and evaluate the result using (27), (28), and [13, Lemma 10.2] in order to obtain (32) for $1 \leq i \leq d$. In the equation of PA3, we eliminate $\phi_{1}$ using (32) at $i=1$, and evaluate the result using (27), (28), and [13), Lemma 10.2] in order to obtain (31) for $1 \leq i \leq d$. We have now proved the claim. We now break the argument into subcases. For each subcase our argument is similar. We will discuss the first subcase in detail in order to give the idea; for the remaining subcases we give the essentials only.
Subcase $q$-Racah: $\mu \neq 0, \mu^{*} \neq 0, h \neq 0, h^{*} \neq 0$. We show $p$ is listed in Example 5.3. Define

$$
\begin{equation*}
s:=\mu h^{-1} q^{-1}, \quad s^{*}:=\mu^{*} h^{*-1} q^{-1} \tag{33}
\end{equation*}
$$

Eliminating $\eta$ in (27) using (29) and eliminating $\mu$ in the result using the equation on the left in (33), we obtain (11) for $0 \leq i \leq d$. Similarly we obtain (12) for $0 \leq i \leq d$. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains scalars $r_{1}, r_{2}$ such that both

$$
\begin{equation*}
r_{1} r_{2}=s s^{*} q^{d+1}, \quad \quad r_{1}+r_{2}=\tau h^{-1} h^{*-1} q^{d} \tag{34}
\end{equation*}
$$

Eliminating $\mu, \mu^{*}, \tau$ in (31), (32) using (331) and the equation on the right in (341), and evaluating the result using the equation on the left in (34), we obtain (13), (14) for $1 \leq i \leq d$. By the construction each of $h, h^{*}, q, s, s^{*}$ is nonzero. Each of $r_{1}, r_{2}$ is nonzero by the equation on the left in (34). The remaining inequalities mentioned below (14) follow from PA1, PA2 and (11)-(14). We have now shown $p$ is listed in Example 5.3.
We now give the remaining subcases of Case I. We list the essentials only.
Subcase $q$-Hahn: $\mu=0, \mu^{*} \neq 0, h \neq 0, h^{*} \neq 0, \tau \neq 0$. Definitions:

$$
s^{*}:=\mu^{*} h^{*-1} q^{-1}, \quad r:=\tau h^{-1} h^{*-1} q^{d}
$$

Subcase dual $q$-Hahn: $\mu \neq 0, \mu^{*}=0, h \neq 0, h^{*} \neq 0, \tau \neq 0$. Definitions:

$$
s:=\mu h^{-1} q^{-1}, \quad r:=\tau h^{-1} h^{*-1} q^{d} .
$$

Subcase quantum $q$-Krawtchouk: $\mu \neq 0, \mu^{*}=0, h=0, h^{*} \neq 0, \tau \neq 0$. Definitions:

$$
s:=\mu q^{-1}, \quad r:=\tau h^{*-1} q^{d} .
$$

Subcase $q$-Krawtchouk: $\mu=0, \mu^{*} \neq 0, h \neq 0, h^{*} \neq 0, \tau=0$. Definition:

$$
s^{*}:=\mu^{*} h^{*-1} q^{-1} .
$$

Subcase affine $q$-Krawtchouk: $\mu=0, \mu^{*}=0, h \neq 0, h^{*} \neq 0, \tau \neq 0$. Definition:

$$
r:=\tau h^{-1} h^{*-1} q^{d} .
$$

Subcase dual $q$-Krawtchouk: $\mu \neq 0, \mu^{*}=0, h \neq 0, h^{*} \neq 0, \tau=0$. Definition:

$$
s:=\mu h^{-1} q^{-1}
$$

We have a few more comments concerning Case I. Earlier we mentioned that $\mu, h$ are not both 0 and that $\mu^{*}, h^{*}$ are not both 0 . Suppose one of $\mu, h$ is 0 and one of $\mu^{*}, h^{*}$ is 0 . Then $\tau \neq 0$; otherwise $\varphi_{1}=0$ by (31) or $\phi_{1}=0$ by (32). Suppose $\mu^{*} \neq 0, h^{*}=0$. Replacing $q$ by $q^{-1}$ we obtain $\mu^{*}=0, h^{*} \neq 0$. Suppose $\mu^{*} \neq 0, h^{*} \neq 0, \mu \neq 0, h=0$. Replacing $q$ by $q^{-1}$ we obtain $\mu^{*} \neq 0, h^{*} \neq 0, \mu=0, h \neq 0$. By these comments we find that after replacing $q$ by $q^{-1}$ if necessary, one of the above subcases holds. This completes our argument for Case I.
Case II: $q=1$ and $\operatorname{char}(\mathbb{K}) \neq 2$.
By (25) and since $\operatorname{char}(\mathbb{K}) \neq 2$, there exist scalars $\eta, \mu, h$ in $\tilde{\mathbb{K}}$ such that

$$
\begin{equation*}
\theta_{i}=\eta+(\mu+h) i+h i^{2} \quad(0 \leq i \leq d) \tag{35}
\end{equation*}
$$

Similarly there exist scalars $\eta^{*}, \mu^{*}, h^{*}$ in $\tilde{\mathbb{K}}$ such that

$$
\begin{equation*}
\theta_{i}^{*}=\eta^{*}+\left(\mu^{*}+h^{*}\right) i+h^{*} i^{2} \quad(0 \leq i \leq d) \tag{36}
\end{equation*}
$$

Observe $\mu, h$ are not both 0 ; otherwise $\theta_{1}=\theta_{0}$. Similarly $\mu^{*}, h^{*}$ are not both 0 . For any prime $i$ such that $i \leq d$ we have $\operatorname{char}(\mathbb{K}) \neq i$; otherwise $\theta_{i}=\theta_{0}$ by (35). Therefore $\operatorname{char}(\mathbb{K})$ is 0 or a prime greater than $d$. Setting $i=0$ in (35), (36) we obtain

$$
\begin{equation*}
\theta_{0}=\eta, \quad \quad \theta_{0}^{*}=\eta^{*} \tag{37}
\end{equation*}
$$

We claim there exists $\tau \in \tilde{\mathbb{K}}$ such that both

$$
\begin{align*}
\varphi_{i} & =i(d-i+1)\left(\tau-\left(\mu h^{*}+h \mu^{*}\right) i-h h^{*} i(i+d+1)\right)  \tag{38}\\
\phi_{i} & =i(d-i+1)\left(\tau+\mu \mu^{*}+h \mu^{*}(1+d)+\left(\mu h^{*}-h \mu^{*}\right) i+h h^{*} i(d-i+1)\right) \tag{39}
\end{align*}
$$

for $1 \leq i \leq d$. There exists $\tau \in \tilde{\mathbb{K}}$ such that (38) holds for $i=1$. In the equation of PA4, we eliminate $\varphi_{1}$ using (38) at $i=1$, and evaluate the result using (35), (36), and [13, Lemma 10.2] in order to obtain (39) for $1 \leq i \leq d$. In the equation of PA3, we eliminate $\phi_{1}$ using (39) at $i=1$, and evaluate the result using (35), (36), and [13, Lemma 10.2] in order to obtain (38) for $1 \leq i \leq d$. We have now proved the claim. We now break the argument into subcases.

Subcase Racah: $h \neq 0, h^{*} \neq 0$. We show $p$ is listed in Example 5.10. Define

$$
\begin{equation*}
s:=\mu h^{-1}, \quad s^{*}:=\mu^{*} h^{*-1} . \tag{40}
\end{equation*}
$$

Eliminating $\eta, \mu$ in (35) using (37), (40) we obtain (15) for $0 \leq i \leq d$. Eliminating $\eta^{*}, \mu^{*}$ in (36) using (37), (40) we obtain (16) for $0 \leq i \leq d$. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains scalars $r_{1}, r_{2}$ such that both

$$
\begin{equation*}
r_{1} r_{2}=-\tau h^{-1} h^{*-1}, \quad \quad r_{1}+r_{2}=s+s^{*}+d+1 \tag{41}
\end{equation*}
$$

Eliminating $\mu, \mu^{*}, \tau$ in (38), (39) using (40) and the equation on the left in (41) we obtain (17), (18) for $1 \leq i \leq d$. By the construction each of $h, h^{*}$ is nonzero. The remaining inequalities mentioned below (18) follow from PA1, PA2 and (15)-(18). We have now shown $p$ is listed in Example 5.10.

We now give the remaining subcases of Case II. We list the essentials only.
Subcase Hahn: $h=0, h^{*} \neq 0$. Definitions:

$$
s=\mu, \quad s^{*}:=\mu^{*} h^{*-1}, \quad r:=-\tau \mu^{-1} h^{*-1}
$$

Subcase dual Hahn: $h \neq 0, h^{*}=0$. Definitions:

$$
s:=\mu h^{-1}, \quad s^{*}=\mu^{*}, \quad r:=-\tau h^{-1} \mu^{*-1} .
$$

Subcase Krawtchouk: $h=0, h^{*}=0$. Definitions:

$$
s:=\mu, \quad s^{*}:=\mu^{*}, \quad r:=-\tau .
$$

Case III: $q=-1$ and $\operatorname{char}(\mathbb{K}) \neq 2$.
We show $p$ is listed in Example 5.14. By (25) and since char $(\mathbb{K}) \neq 2$, there exist scalars $\eta, \mu, h$ in $\tilde{K}$ such that

$$
\begin{equation*}
\theta_{i}=\eta+\mu(-1)^{i}+2 h i(-1)^{i} \quad(0 \leq i \leq d) \tag{42}
\end{equation*}
$$

Similarly there exist scalars $\eta^{*}, \mu^{*}, h^{*}$ in $\tilde{\mathbb{K}}$ such that

$$
\begin{equation*}
\theta_{i}^{*}=\eta^{*}+\mu^{*}(-1)^{i}+2 h^{*} i(-1)^{i} \quad(0 \leq i \leq d) \tag{43}
\end{equation*}
$$

Observe $h \neq 0$; otherwise $\theta_{2}=\theta_{0}$ by (42). Similarly $h^{*} \neq 0$. For any prime $i$ such that $i \leq d / 2$ we have $\operatorname{char}(\mathbb{K}) \neq i$; otherwise $\theta_{2 i}=\theta_{0}$ by (42). By this and since char $(\mathbb{K}) \neq 2$ we find $\operatorname{char}(\mathbb{K})$ is either 0 or an odd prime greater than $d / 2$. Setting $i=0$ in (42), (43) we obtain

$$
\begin{equation*}
\theta_{0}=\eta+\mu, \quad \theta_{0}^{*}=\eta^{*}+\mu^{*} \tag{44}
\end{equation*}
$$

We define

$$
\begin{equation*}
s:=1-\mu h^{-1}, \quad s^{*}=1-\mu^{*} h^{*-1} . \tag{45}
\end{equation*}
$$

Eliminating $\eta$ in (42) using (44) and eliminating $\mu$ in the result using (45) we find (19) holds for $0 \leq i \leq d$. Similarly we find (20) holds for $0 \leq i \leq d$. We now define $r_{1}, r_{2}$. First assume $d$ is odd. Since $\tilde{\mathbb{K}}$ is algebraically closed it contains $r_{1}, r_{2}$ such that

$$
\begin{equation*}
r_{1}+r_{2}=-s-s^{*}+d+1 \tag{46}
\end{equation*}
$$

and such that

$$
\begin{equation*}
4 h h^{*}\left(1+r_{1}\right)\left(1+r_{2}\right)=-\varphi_{1} \tag{47}
\end{equation*}
$$

Next assume $d$ is even. Define

$$
\begin{equation*}
r_{2}:=-1+\frac{\varphi_{1}}{4 h h^{*} d} \tag{48}
\end{equation*}
$$

and define $r_{1}$ so that (46) holds. We have now defined $r_{1}, r_{2}$ for either parity of $d$. In the equation of PA4, we eliminate $\varphi_{1}$ using (47) or (48), and evaluate the result using (19), (20), and [13, Lemma 10.2] in order to obtain (22) for $1 \leq i \leq d$. In the equation of PA3, we eliminate $\phi_{1}$ using (22) at $i=1$, and evaluate the result using (19), (20), and [13), Lemma 10.2] in order to obtain (21) for $1 \leq i \leq d$. We mentioned each of $h, h^{*}$ is nonzero. The remaining inequalities mentioned below (22) follow from PA1, PA2 and (19)-(22). We have now shown $p$ is listed in Example 5.14.
Case IV: $q=1$ and $\operatorname{char}(\mathbb{K})=2$.
We show $p$ is listed in Example 5.15. We first show $d=3$. Recall $d \geq 3$ since $q=1$. Suppose $d \geq 4$. By (25) we have $\sum_{j=0}^{3} \theta_{j}=0$ and $\sum_{j=1}^{4} \theta_{j}=0$. Adding these sums we find $\theta_{0}=\theta_{4}$ which contradicts PA1. Therefore $d=3$. We claim there exist nonzero scalars $h, s$ in $\mathbb{K}$ such that (23) holds for $0 \leq i \leq 3$. Define $h=\theta_{0}+\theta_{2}$. Observe $h \neq 0$; otherwise $\theta_{0}=\theta_{2}$. Define $s=\left(\theta_{0}+\theta_{3}\right) h^{-1}$. Observe $s \neq 0$; otherwise $\theta_{0}=\theta_{3}$. Using these values for $h, s$ we find (23) holds for $i=0,2,3$. By this and $\sum_{j=0}^{3} \theta_{j}=0$ we find (23) holds for $i=1$. We have now proved our claim. Similarly there exist nonzero scalars $h^{*}, s^{*}$ in $\mathbb{K}$ such that (24) holds for $0 \leq i \leq 3$. Define $r:=\varphi_{1} h^{-1} h^{*-1}$. Observe $r \neq 0$ and that $\varphi_{1}=h h^{*} r$. In the equation of PA4, we eliminate $\varphi_{1}$ using $\varphi_{1}=h h^{*} r$ and evaluate the result using (23), (24) and (13), Lemma 10.2] in order to obtain $\phi_{1}=h h^{*}\left(r+s\left(1+s^{*}\right)\right), \phi_{2}=h h^{*}, \phi_{3}=h h^{*}\left(r+s^{*}(1+s)\right)$. In the equation of PA3, we eliminate $\phi_{1}$ using $\phi_{1}=h h^{*}\left(r+s\left(1+s^{*}\right)\right)$ and evaluate the result using (23), (24) and [13, Lemma 10.2] in order to obtain $\varphi_{2}=h h^{*}, \varphi_{3}=h h^{*}\left(r+s+s^{*}\right)$. We mentioned each of $h, h^{*}, s, s^{*}, r$ is nonzero. Observe $s \neq 1$; otherwise $\theta_{1}=\theta_{0}$. Similarly $s^{*} \neq 1$. Observe $r \neq s+s^{*}$; otherwise $\varphi_{3}=0$. Observe $r \neq s\left(1+s^{*}\right)$; otherwise $\phi_{1}=0$. Observe $r \neq s^{*}(1+s)$; otherwise $\phi_{3}=0$. We have now shown $p$ is listed in Example 5.15, We are done with Case IV and the proof is complete.

## 6 The orthogonality relation in terms of the parameter array

Some facts about the polynomials in Examples 5.3-5.15 can be expressed in a uniform and attractive manner by writing things in terms of the associated parameter array. We illustrate this by giving the orthogonality relation, the three-term recurrence, and the difference equation in terms of the parameter array. We start with the orthogonality relation. In order to state the result we define some scalars.

Definition 6.1 Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$. For $0 \leq i \leq d$ we let $k_{i}$ equal

$$
\frac{\varphi_{1} \varphi_{2} \cdots \varphi_{i}}{\phi_{1} \phi_{2} \cdots \phi_{i}}
$$

times

$$
\frac{\left(\theta_{0}^{*}-\theta_{1}^{*}\right)\left(\theta_{0}^{*}-\theta_{2}^{*}\right) \cdots\left(\theta_{0}^{*}-\theta_{d}^{*}\right)}{\left(\theta_{i}^{*}-\theta_{0}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{i-1}^{*}\right)\left(\theta_{i}^{*}-\theta_{i+1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{d}^{*}\right)}
$$

For $0 \leq i \leq d$ we let $k_{i}^{*}$ equal

$$
\frac{\varphi_{1} \varphi_{2} \cdots \varphi_{i}}{\phi_{d} \phi_{d-1} \cdots \phi_{d-i+1}}
$$

times

$$
\frac{\left(\theta_{0}-\theta_{1}\right)\left(\theta_{0}-\theta_{2}\right) \cdots\left(\theta_{0}-\theta_{d}\right)}{\left(\theta_{i}-\theta_{0}\right) \cdots\left(\theta_{i}-\theta_{i-1}\right)\left(\theta_{i}-\theta_{i+1}\right) \cdots\left(\theta_{i}-\theta_{d}\right)} .
$$

We observe $k_{0}=1, k_{0}^{*}=1$. We define

$$
\nu=\frac{\left(\theta_{0}-\theta_{1}\right)\left(\theta_{0}-\theta_{2}\right) \cdots\left(\theta_{0}-\theta_{d}\right)\left(\theta_{0}^{*}-\theta_{1}^{*}\right)\left(\theta_{0}^{*}-\theta_{2}^{*}\right) \cdots\left(\theta_{0}^{*}-\theta_{d}^{*}\right)}{\phi_{1} \phi_{2} \cdots \phi_{d}} .
$$

Theorem 6.2 [15, Lines (128), (129)]. Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$ and let $f_{0}, f_{1}, \ldots, f_{d}$ denote the corresponding polynomials from Definition 5.2. Then both

$$
\begin{align*}
\sum_{r=0}^{d} f_{i}\left(\theta_{r}\right) f_{j}\left(\theta_{r}\right) k_{r}^{*}=\delta_{i j} \nu k_{i}^{-1} & (0 \leq i, j \leq d)  \tag{49}\\
\sum_{r=0}^{d} f_{r}\left(\theta_{i}\right) f_{r}\left(\theta_{j}\right) k_{r}=\delta_{i j} \nu k_{i}^{*-1} & (0 \leq i, j \leq d) \tag{50}
\end{align*}
$$

The scalars $k_{i}, k_{i}^{*}, \nu$ are from Definition 6.1.
We have a comment.
Lemma 6.3 With reference to Definition 6.1, both

$$
\begin{equation*}
\nu=\sum_{r=0}^{d} k_{r}, \quad \nu=\sum_{r=0}^{d} k_{r}^{*} . \tag{51}
\end{equation*}
$$

Proof: To get the equation on the left in (51) set $i=0, j=0$ in (50) and observe $f_{r}\left(\theta_{0}\right)=1$ for $0 \leq r \leq d$. To get the equation on the right in (51) set $i=0, j=0$ in (49) and observe $f_{0}=1$.

## 7 The three-term recurrence in terms of the parameter array

In this section we give a three-term recurrence satisfied by the polynomials in Example 5.35.15. We express the result in terms of the associated parameter array. In order to state the result we define some scalars.

Definition 7.1 Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$. We define

$$
\begin{equation*}
b_{i}=\varphi_{i+1} \frac{\left(\theta_{i}^{*}-\theta_{0}^{*}\right)\left(\theta_{i}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{i-1}^{*}\right)}{\left(\theta_{i+1}^{*}-\theta_{0}^{*}\right)\left(\theta_{i+1}^{*}-\theta_{1}^{*}\right) \cdots\left(\theta_{i+1}^{*}-\theta_{i}^{*}\right)} \quad(0 \leq i \leq d-1) \tag{52}
\end{equation*}
$$

and $b_{d}=0$. We define

$$
\begin{equation*}
c_{i}=\phi_{i} \frac{\left(\theta_{i}^{*}-\theta_{d}^{*}\right)\left(\theta_{i}^{*}-\theta_{d-1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{i+1}^{*}\right)}{\left(\theta_{i-1}^{*}-\theta_{d}^{*}\right)\left(\theta_{i-1}^{*}-\theta_{d-1}^{*}\right) \cdots\left(\theta_{i-1}^{*}-\theta_{i}^{*}\right)} \quad(1 \leq i \leq d) \tag{53}
\end{equation*}
$$

and $c_{0}=0$. We define

$$
\begin{equation*}
a_{i}=\theta_{0}-c_{i}-b_{i} \quad(0 \leq i \leq d) \tag{54}
\end{equation*}
$$

Theorem 7.2 Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$ and let $f_{0}, f_{1}, \ldots, f_{d}$ denote the corresponding polynomials from Definition 5.2. For $0 \leq i, j \leq d$ we have

$$
\begin{equation*}
\theta_{j} f_{i}\left(\theta_{j}\right)=c_{i} f_{i-1}\left(\theta_{j}\right)+a_{i} f_{i}\left(\theta_{j}\right)+b_{i} f_{i+1}\left(\theta_{j}\right) \tag{55}
\end{equation*}
$$

where $f_{-1}, f_{d+1}$ are indeterminates and where the $a_{i}, b_{i}, c_{i}$ are from Definition 7.1.
Proof: Let $\mathcal{P}$ denote the matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has $i j$ th entry $f_{j}\left(\theta_{i}\right)$ for $0 \leq i, j \leq d$. Let $K^{*}$ denote the diagonal matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has entries $K_{i i}^{*}=k_{i}^{*}$ for $0 \leq i \leq d$, where the $k_{i}^{*}$ are from Definition 6.1. Let $H$ denote the diagonal matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has entries $H_{i i}=\theta_{i}$ for $0 \leq i \leq d$. Let $C$ denote the following matrix in Mat $\operatorname{Mat}_{d}(\mathbb{K})$.

$$
C=\left(\begin{array}{cccccc}
a_{0} & b_{0} & & & & \mathbf{0} \\
c_{1} & a_{1} & b_{1} & & & \\
& c_{2} & \cdot & \cdot & & \\
& & \cdot & \cdot & \cdot & \\
& & & \cdot & \cdot & b_{d-1} \\
\mathbf{0} & & & & c_{d} & a_{d}
\end{array}\right)
$$

We define $P^{*}=\mathcal{P}^{t} K^{*}$. By [15, Line (118)] we have $C P^{*}=P^{*} H$. By this and since $K^{*}, H$ are diagonal we find $C \mathcal{P}^{t}=\mathcal{P}^{t} H$. In this equation we expand each side using matrix multiplication and routinely obtain (55).

We finish this section with a comment.
Lemma 7.3 With reference to Definition 6.1 and Definition 7.1 .

$$
k_{i}=\frac{b_{0} b_{1} \cdots b_{i-1}}{c_{1} c_{2} \cdots c_{i}} \quad(0 \leq i \leq d)
$$

Proof: Compare the formulae for the $k_{i}, b_{i}, c_{i}$ given in Definition 6.1] and Definition 7.1 .

## 8 The difference equation in terms of the parameter array

In this section we give a difference equation satisfied by the polynomials in Example 5.3 5.15 We express the result in terms of the associated parameter array. In order to state the result we define some scalars.

Definition 8.1 Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$. We define

$$
b_{i}^{*}=\varphi_{i+1} \frac{\left(\theta_{i}-\theta_{0}\right)\left(\theta_{i}-\theta_{1}\right) \cdots\left(\theta_{i}-\theta_{i-1}\right)}{\left(\theta_{i+1}-\theta_{0}\right)\left(\theta_{i+1}-\theta_{1}\right) \cdots\left(\theta_{i+1}-\theta_{i}\right)} \quad(0 \leq i \leq d-1)
$$

and $b_{d}^{*}=0$. We define

$$
c_{i}^{*}=\phi_{d-i+1} \frac{\left(\theta_{i}-\theta_{d}\right)\left(\theta_{i}-\theta_{d-1}\right) \cdots\left(\theta_{i}-\theta_{i+1}\right)}{\left(\theta_{i-1}-\theta_{d}\right)\left(\theta_{i-1}-\theta_{d-1}\right) \cdots\left(\theta_{i-1}-\theta_{i}\right)} \quad(1 \leq i \leq d)
$$

and $c_{0}^{*}=0$. We define

$$
a_{i}^{*}=\theta_{0}^{*}-c_{i}^{*}-b_{i}^{*} \quad(0 \leq i \leq d) .
$$

Theorem 8.2 Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$ and let $f_{0}, f_{1}, \ldots, f_{d}$ denote the corresponding polynomials from Definition 5.2. For $0 \leq i, j \leq d$ we have

$$
\begin{equation*}
\theta_{i}^{*} f_{i}\left(\theta_{j}\right)=c_{j}^{*} f_{i}\left(\theta_{j-1}\right)+a_{j}^{*} f_{i}\left(\theta_{j}\right)+b_{j}^{*} f_{i}\left(\theta_{j+1}\right), \tag{56}
\end{equation*}
$$

where $\theta_{-1}, \theta_{d+1}$ are indeterminates and the $a_{j}^{*}, b_{j}^{*}, c_{j}^{*}$ are from Definition 8.1.
Proof: By Lemma 2.2(i) the sequence ( $\left.\theta_{i}^{*}, \theta_{i}, i=0 \ldots d ; \varphi_{j}, \phi_{d-j+1}, j=1 \ldots d\right)$ is a parameter array over $\mathbb{K}$. Let $f_{0}^{*}, f_{1}^{*}, \ldots, f_{d}^{*}$ denote the corresponding polynomials from Definition 5.2, so that

$$
\begin{equation*}
f_{i}^{*}=\sum_{n=0}^{i} \frac{\left(\lambda-\theta_{0}^{*}\right)\left(\lambda-\theta_{1}^{*}\right) \cdots\left(\lambda-\theta_{n-1}^{*}\right)\left(\theta_{i}-\theta_{0}\right)\left(\theta_{i}-\theta_{1}\right) \cdots\left(\theta_{i}-\theta_{n-1}\right)}{\varphi_{1} \varphi_{2} \cdots \varphi_{n}} \tag{57}
\end{equation*}
$$

for $0 \leq i \leq d$. Applying Theorem 7.2 to ( $\left.\theta_{i}^{*}, \theta_{i}, i=0 \ldots d ; \varphi_{j}, \phi_{d-j+1}, j=1 \ldots d\right)$ and $f_{0}^{*}, f_{1}^{*}, \ldots, f_{d}^{*}$ we find that for $0 \leq i, j \leq d$,

$$
\begin{equation*}
\theta_{j}^{*} f_{i}^{*}\left(\theta_{j}^{*}\right)=c_{i}^{*} f_{i-1}^{*}\left(\theta_{j}^{*}\right)+a_{i}^{*} f_{i}^{*}\left(\theta_{j}^{*}\right)+b_{i}^{*} f_{i+1}^{*}\left(\theta_{j}^{*}\right), \tag{58}
\end{equation*}
$$

where $f_{-1}^{*}, f_{d+1}^{*}$ are indeterminates. Comparing (10) and (57) we find

$$
\begin{equation*}
f_{i}\left(\theta_{j}\right)=f_{j}^{*}\left(\theta_{i}^{*}\right) \quad(0 \leq i, j \leq d) \tag{59}
\end{equation*}
$$

Evaluating (58) using (59) and reindexing the result we obtain (56).
We finish this section with a comment.
Lemma 8.3 With reference to Definition 6.1 and Definition 8.1,

$$
k_{i}^{*}=\frac{b_{0}^{*} b_{1}^{*} \cdots b_{i-1}^{*}}{c_{1}^{*} c_{2}^{*} \cdots c_{i}^{*}} \quad(0 \leq i \leq d)
$$

Proof: Similar to the proof of Lemma 7.3.

## 9 Some useful formulae

In this section we give alternative formulae for the scalars $a_{i}, b_{i}, c_{i}$ from Definition 7.1. To avoid trivialities we assume the diameter $d \geq 1$. We begin with the $a_{i}$.

Theorem 9.1 [13, Lemma 5.1] With reference to Definition 7.1, let us assume $d \geq 1$. Then

$$
\begin{align*}
a_{0} & =\theta_{0}+\frac{\varphi_{1}}{\theta_{0}^{*}-\theta_{1}^{*}}  \tag{60}\\
a_{i} & =\theta_{i}+\frac{\varphi_{i}}{\theta_{i}^{*}-\theta_{i-1}^{*}}+\frac{\varphi_{i+1}}{\theta_{i}^{*}-\theta_{i+1}^{*}} \quad(1 \leq i \leq d-1),  \tag{61}\\
a_{d} & =\theta_{d}+\frac{\varphi_{d}}{\theta_{d}^{*}-\theta_{d-1}^{*}} \tag{62}
\end{align*}
$$

Lemma 9.2 With reference to Definition 7.1, assume $d \geq 1$. Then

$$
\begin{equation*}
c_{i}\left(\theta_{i-1}^{*}-\theta_{i}^{*}\right)-b_{i}\left(\theta_{i}^{*}-\theta_{i+1}^{*}\right)=\left(\theta_{1}-\theta_{0}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)+\varphi_{1} \tag{63}
\end{equation*}
$$

for $0 \leq i \leq d$, where $\theta_{-1}^{*}, \theta_{d+1}^{*}$ denote indeterminates.
Proof: Setting $\lambda=\theta_{1}$ in (10) we find

$$
\begin{equation*}
f_{i}\left(\theta_{1}\right)=1+\frac{\left(\theta_{1}-\theta_{0}\right)\left(\theta_{i}^{*}-\theta_{0}^{*}\right)}{\varphi_{1}} \quad(0 \leq i \leq d) \tag{64}
\end{equation*}
$$

Setting $j=1$ in (55) and evaluating the result using (54), (64) we obtain (631).
Theorem 9.3 With reference to Definition 7.1, assume $d \geq 1$. Then

$$
\begin{align*}
& b_{0}=\frac{\varphi_{1}}{\theta_{1}^{*}-\theta_{0}^{*}}  \tag{65}\\
& b_{i}=\frac{\left(\theta_{0}-a_{i}\right)\left(\theta_{i}^{*}-\theta_{i-1}^{*}\right)+\left(\theta_{0}-\theta_{1}\right)\left(\theta_{0}^{*}-\theta_{i}^{*}\right)+\varphi_{1}}{\theta_{i+1}^{*}-\theta_{i-1}^{*}} \quad(1 \leq i \leq d-1)  \tag{66}\\
& b_{d}=0  \tag{67}\\
& c_{0}=0  \tag{68}\\
& c_{i}=\frac{\left(\theta_{0}-a_{i}\right)\left(\theta_{i}^{*}-\theta_{i+1}^{*}\right)+\left(\theta_{0}-\theta_{1}\right)\left(\theta_{0}^{*}-\theta_{i}^{*}\right)+\varphi_{1}}{\theta_{i-1}^{*}-\theta_{i+1}^{*}} \quad(1 \leq i \leq d-1)  \tag{69}\\
& c_{d}=\frac{\phi_{d}}{\theta_{d-1}^{*}-\theta_{d}^{*}} . \tag{70}
\end{align*}
$$

Proof: Lines (67), (68) are clear. To get (651) set $i=0$ in (52). To get (70) set $i=d$ in (53). To get (66), (69), solve the linear system (54), (631) for $b_{i}, c_{i}$.

Results similar to Theorem 9.1, Lemma 9.2, and Theorem 9.3 hold for the $a_{i}^{*}, b_{i}^{*}, c_{i}^{*}$.

## 10 Remarks

We conclude this paper with a few remarks.
Let $\left(\theta_{i}, \theta_{i}^{*}, i=0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$ and let $f_{0}, f_{1}, \ldots, f_{d}$ denote the corresponding polynomials from Definition 5.2. Applying Theorem 4.1 with $\lambda=\theta_{d}$ and using Lemma 4.2 we find

$$
\begin{equation*}
f_{i}\left(\theta_{d}\right)=\frac{\phi_{1} \phi_{2} \cdots \phi_{i}}{\varphi_{1} \varphi_{2} \cdots \varphi_{i}} \quad(0 \leq i \leq d) \tag{71}
\end{equation*}
$$

Let the scalars $k_{i}$ be as in Definition 6.1. Comparing (71) with the formulae for $k_{i}$ given in Definition 6.1 we find

$$
k_{i} f_{i}\left(\theta_{d}\right)=\frac{\left(\theta_{0}^{*}-\theta_{1}^{*}\right)\left(\theta_{0}^{*}-\theta_{2}^{*}\right) \cdots\left(\theta_{0}^{*}-\theta_{d}^{*}\right)}{\left(\theta_{i}^{*}-\theta_{0}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{i-1}^{*}\right)\left(\theta_{i}^{*}-\theta_{i+1}^{*}\right) \cdots\left(\theta_{i}^{*}-\theta_{d}^{*}\right)}
$$

for $0 \leq i \leq d$.
We describe the matrix $G$ from Theorem [3.2, We use the following notation. Let $\left(\theta_{i}, \theta_{i}^{*}, i=\right.$ $\left.0 \ldots d ; \varphi_{j}, \phi_{j}, j=1 \ldots d\right)$ denote a parameter array over $\mathbb{K}$ and let $q$ denote a base for this array. To keep things simple we assume $q \neq 1, q \neq-1$. For nonegative integers $r, s, t$ such that $r+s+t \leq d$ we define

$$
[r, s, t]_{q}:=\frac{(q ; q)_{r+s}(q ; q)_{r+t}(q ; q)_{s+t}}{(q ; q)_{r}(q ; q)_{s}(q ; q)_{t}(q ; q)_{r+s+t}}
$$

where

$$
(a ; q)_{n}:=(1-a)(1-a q) \cdots\left(1-a q^{n-1}\right) \quad n=0,1,2, \ldots
$$

We comment $[r, s, t]_{q} \in \mathbb{K}$ [15, Definition 13.1]. Let $S$ denote the upper triangular matrix in $\operatorname{Mat}_{d+1}(\mathbb{K})$ which has entries

$$
S_{i j}=\left(\theta_{0}-\theta_{d}\right)\left(\theta_{0}-\theta_{d-1}\right) \cdots\left(\theta_{0}-\theta_{d-j+i+1}\right)[i, j-i, d-j]_{q}
$$

for $0 \leq i \leq j \leq d$. Then for $G \in \operatorname{Mat}_{d+1}(\mathbb{K}), G$ satisfies Theorem 3.2(ii) if and only if there exists a nonzero $\alpha \in \mathbb{K}$ such that $G=\alpha S$ [15, Theorem 15.2]. Similar results hold for $q=1$ and $q=-1$ [15, Lemma 13.2].

## 11 Open problems

Problem 11.1 Generalize Theorem 4.1 so that it applies to polynomial sequences of infinite length. Use this result to characterize the polynomials of the Askey scheme.

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