

# Johnson Type Bounds on Constant Dimension Codes \*

Shu-Tao Xia<sup>†</sup> and Fang-Wei Fu<sup>‡</sup>

## Abstract

Very recently, an operator channel was defined by Koetter and Kschischang when they studied random network coding. They also introduced constant dimension codes and demonstrated that these codes can be employed to correct errors and/or erasures over the operator channel. Constant dimension codes are equivalent to the so-called linear authentication codes introduced by Wang, Xing and Safavi-Naini when constructing distributed authentication systems in 2003. In this paper, we study constant dimension codes. It is shown that Steiner structures are optimal constant dimension codes achieving the Wang-Xing-Safavi-Naini bound. Furthermore, we show that constant dimension codes achieve the Wang-Xing-Safavi-Naini bound if and only if they are certain Steiner structures. Then, we derive two Johnson type upper bounds, say I and II, on constant dimension codes. The Johnson type bound II slightly improves on the Wang-Xing-Safavi-Naini bound. Finally, we point out that a family of known Steiner structures is actually a family of optimal constant dimension codes achieving both the Johnson type bounds I and II.

**keywords:** Constant dimension codes, linear authentication codes, binary constant weight codes, Johnson bounds, Steiner structures, random network coding.

---

\*This research is supported in part by the NSFC-GDSF Joint Fund under Grant No. U0675001, and the open research fund of National Mobile Communications Research Laboratory, Southeast University.

<sup>†</sup>S.-T. Xia is with the Graduate School at Shenzhen of Tsinghua University, Shenzhen, Guangdong 518055, P. R. China. He is also with the National Mobile Communications Research Laboratory, Southeast University, P.R. China. E-mail: xiast@sz.tsinghua.edu.cn

<sup>‡</sup>F.-W. Fu is with the Chern Institute of Mathematics, and The Key Laboratory of Pure Mathematics and Combinatorics, Nankai University, Tianjin 300071, P.R. China. Email: fwfu@nankai.edu.cn

# 1 Introduction

Throughout this paper,  $\mathbb{F}_q$  denotes the finite field with  $q$  elements, where  $q$  is a prime power. Let  $W$  be an  $n$ -dimensional vector space over  $\mathbb{F}_q$  and let  $\mathcal{P}(W)$  denote the set of all subspaces of  $W$ . For any  $A, B \in \mathcal{P}(W)$ , denote

$$A + B = \{a + b : a \in A, b \in B\},$$

that is the smallest subspace containing both  $A$  and  $B$ . It is known [3] that the *dimension distance* between  $A$  and  $B$  defined by

$$d(A, B) = \dim(A + B) - \dim(A \cap B) \quad (1)$$

$$= \dim(A) + \dim(B) - 2 \dim(A \cap B) \quad (2)$$

is a metric for the space  $\mathcal{P}(W)$ . A  $q$ -ary  $(n, M, D)$  or  $(n, M, D)_q$  code  $\mathcal{C}$  is simply a subset of  $\mathcal{P}(W)$  with size  $M$  and *minimum dimension distance*  $D$  which is defined by

$$D = D(\mathcal{C}) = \min_{X \neq Y \in \mathcal{C}} d(X, Y). \quad (3)$$

For any positive integer  $l \leq n$ , let  $\mathcal{P}(W, l)$  denote the set of all  $l$ -dimensional subspaces of  $W$ . For integers  $0 \leq m \leq n$  and  $q \geq 2$ , let

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{i=0}^{m-1} \frac{q^{n-i} - 1}{q^{m-i} - 1}$$

denote the  $q$ -binomial coefficient or *Gaussian binomial coefficient* [7, pp.443-444]. It is well known that  $|\mathcal{P}(W, l)| = \begin{bmatrix} n \\ l \end{bmatrix}_q$ . A  $q$ -ary  $(n, M, 2\delta, l)$  or  $(n, M, 2\delta, l)_q$  constant dimension code is simply a subset of  $\mathcal{P}(W, l)$  with size  $M$  and minimum dimension distance  $2\delta$ . Note that by (2) the dimension distance of any two codewords of a constant dimension code must be an even number and  $1 \leq \delta \leq l$ . An  $(n, M, \geq 2\delta, l)_q$  constant dimension code is a subset of  $\mathcal{P}(W, l)$  with size  $M$  and minimum dimension distance at least  $2\delta$ . For fixed numbers  $n, l, \delta, q$ , denote  $A_q[n, 2\delta, l]$  the maximum number  $M$  of codewords in an  $(n, M, \geq 2\delta, l)_q$  constant dimension code. An  $(n, M, \geq 2\delta, l)_q$  constant dimension code is said to be *optimal* if  $M = A_q[n, 2\delta, l]$ . One of the main research problems on constant dimension codes is to determine  $A_q[n, 2\delta, l]$  and find corresponding optimal constant dimension codes.

Denote  $X^\perp$  the orthogonal complement of  $X \in \mathcal{P}(W)$ . For any two  $l$ -dimensional subspaces  $X, Y \in \mathcal{P}(W, l)$ , since  $X^\perp \cap Y^\perp = (X + Y)^\perp$ , we have

$$\begin{aligned} d(X^\perp, Y^\perp) &= \dim(X^\perp) + \dim(Y^\perp) - 2\dim(X^\perp \cap Y^\perp) \\ &= n - \dim(X) + n - \dim(Y) - 2(n - \dim(X + Y)) \\ &= d(X, Y). \end{aligned} \tag{4}$$

Let  $\mathcal{C} \subseteq \mathcal{P}(W, l)$  be an  $(n, M, 2\delta, l)_q$  constant dimension code. Then by (4) we know that  $\bar{\mathcal{C}} \triangleq \{X^\perp : X \in \mathcal{C}\}$  is an  $(n, M, 2\delta, n - l)_q$  constant dimension code. This implies that

$$A_q[n, 2\delta, l] = A_q[n, 2\delta, n - l]. \tag{5}$$

Hence, we only need to determine  $A_q[n, 2\delta, l]$  for  $l \leq n/2$ .

When studying random network coding [1, 2], Koetter and Kschischang [3] defined a so-called *operator channel* and found that an  $(n, M, \geq 2\delta, l)_q$  constant dimension code  $\mathcal{C}$  could be employed to correct errors and/or erasures over the operator channel, i.e., the errors and/or erasures could be corrected by a minimum dimension distance decoder if the sum of errors and erasures is less than  $\delta$ . Some bounds on  $A_q[n, 2\delta, l]$ , e.g., the Hamming type upper bound, the Gilbert type lower bound, and the Singleton type upper bound, were derived in [3]. It is known that the Hamming type bound is not very good [3] and there exist no non-trivial perfect codes meeting the Hamming type bound [5, 6]. The Singleton type bound developed in [3] is the following:

**Proposition 1** [3, Th.3] (Singleton type bound)

$$A_q[n, 2\delta, l] \leq \begin{bmatrix} n - \delta + 1 \\ l - \delta + 1 \end{bmatrix}_q.$$

Moreover, Koetter and Kschischang [3] designed a class of Reed-Solomon like constant dimension codes and afforded decoding procedures. They showed that these codes were nearly Singleton-type-bound-achieving.

In 2003, Wang, Xing and Safavi-Naini [4] introduced the so-called *linear authentication codes* when constructing distributed authentication systems. They [4, Th.4.1] showed that an  $(n, M, \geq 2\delta, l)_q$  constant dimension code is exactly an  $[n, M, t = n - l, d = \delta]$  linear authentication code over  $\mathbb{F}_q$ . Furthermore, they established an upper bound [4, Th.5.2] on linear authentication codes, which is equivalent to the following bound on constant dimension codes:

**Proposition 2** [4, Th.5.2] (Wang-Xing-Safavi-Naini Bound)

$$A_q[n, 2\delta, l] \leq \frac{\begin{bmatrix} n \\ l-\delta+1 \end{bmatrix}_q}{\begin{bmatrix} l \\ l-\delta+1 \end{bmatrix}_q}.$$

Moreover, Wang, Xing and Safavi-Naini [4] presented some constructions of linear authentication codes (or corresponding constant dimension codes) that are asymptotically close to this bound.

In this paper, we show that Steiner structures are optimal constant dimension codes achieving the Wang-Xing-Safavi-Naini bound in Proposition 2. Furthermore, it is shown that constant dimension codes achieve the Wang-Xing-Safavi-Naini bound if and only if they are certain Steiner structures. Two Johnson type upper bounds, say I and II, on constant dimension codes are derived. The Johnson type bound II slightly improves on the Wang-Xing-Safavi-Naini bound. It is observed that the Wang-Xing-Safavi-Naini bound is always better than the Singleton type bound for nontrivial constant dimension codes. Finally, we point out that a family of known Steiner structures is actually a family of optimal constant dimension codes achieving both the Johnson type bounds I and II.

## 2 Steiner Structures

In this section we first introduce the combinatorial objectives Steiner structures. Then we show that constant dimension codes achieve the Wang-Xing-Safavi-Naini bound if and only if they are certain Steiner structures. This means that Steiner structures are optimal constant dimension codes. Finally we describe the only known family of nontrivial Steiner structures in combinatorics.

Recall that  $W$  is the  $n$ -dimensional vector space over the finite field  $\mathbb{F}_q$  and  $\mathcal{P}(W, l)$  denote the set of all  $l$ -dimensional subspaces of  $W$ . The following definition and proposition on Steiner structures are from [5].

**Definition 1** [5] *A subset  $\mathcal{F} \subseteq \mathcal{P}(W, l)$  is called a Steiner structure  $S[t, l, n]_q$  if each  $t$ -dimensional subspace of  $W$  is contained in exactly one  $l$ -dimensional subspace from  $\mathcal{F}$ . The  $l$ -dimensional subspaces in  $\mathcal{F}$  are called blocks of the Steiner structure  $S[t, l, n]_q$ .*

**Proposition 3** [5] *The total number of blocks in an  $S[t, l, n]_q$  is  $\begin{bmatrix} n \\ t \end{bmatrix}_q / \begin{bmatrix} l \\ t \end{bmatrix}_q$ .*

Below we show that Steiner structures are constant dimension codes.

**Proposition 4** *A Steiner structure  $S[t, l, n]_q$  is an  $(n, M, 2\delta, l)_q$  constant dimension code with  $M = \binom{n}{t}_q / \binom{l}{t}_q$  and  $\delta = l - t + 1$ .*

*Proof:* By Definition 1 and Proposition 3, we only need to show that  $\delta = l - t + 1$ . For any two different blocks  $X, Y \in S[t, l, n]_q$ , since every  $t$ -dimensional subspace is contained in exactly one block of  $S[t, l, n]_q$ , we have  $\dim(X \cap Y) \leq t - 1$ . Thus, by (2),  $d(X, Y) = 2l - 2 \dim(X \cap Y) \geq 2(l - t + 1)$ , which implies that  $\delta \geq l - t + 1$ . On the other hand, let  $V$  be a fixed  $(t - 1)$ -dimensional subspace of  $W$ , choose two  $t$ -dimensional subspaces  $U_1$  and  $U_2$  of  $W$  such that  $V = U_1 \cap U_2$ . Let  $X_1$  and  $X_2$  be the unique blocks in  $S[t, l, n]_q$  such that  $U_1 \subseteq X_1$  and  $U_2 \subseteq X_2$ , respectively. Then,  $V \subseteq X_1 \cap X_2$ , which implies that  $\dim(X_1 \cap X_2) \geq \dim(V) = t - 1$ . Hence, by (2),  $d(X_1, X_2) \leq 2(l - t + 1)$ . Thus,  $\delta \leq l - t + 1$  since  $2\delta$  is the minimum dimension distance of  $S[t, l, n]_q$ . Combining these assertions,  $\delta = l - t + 1$ . This completes the proof.  $\square$

Next we give the necessary and sufficient condition for constant dimension codes to achieve the Wang-Xing-Safavi-Naini bound in Proposition 2.

**Theorem 1** *An  $(n, M, \geq 2\delta, l)_q$  constant dimension code  $\mathcal{C}$  achieves the Wang-Xing-Safavi-Naini bound, i.e.,  $M = \frac{\binom{n}{l-\delta+1}_q}{\binom{l}{l-\delta+1}_q}$ , if and only if  $\mathcal{C}$  is a Steiner structure  $S[l - \delta + 1, l, n]_q$ .*

*Proof:* Since an  $(n, M, 2\delta, l)_q$  constant dimension code is an  $(n, M, \geq 2\delta, l)_q$  constant dimension code, we know from Propositions 2 and 4 that a Steiner structure  $S[l - \delta + 1, l, n]_q$  is an  $(n, M = \frac{\binom{n}{l-\delta+1}_q}{\binom{l}{l-\delta+1}_q}, \geq 2\delta, l)_q$  constant dimension code achieving the Wang-Xing-Safavi-Naini bound.

On the other hand, suppose there exists an  $(n, M = \frac{\binom{n}{l-\delta+1}_q}{\binom{l}{l-\delta+1}_q}, \geq 2\delta, l)_q$  constant dimension code  $\mathcal{C}$  achieving the Wang-Xing-Safavi-Naini bound. Since the dimension distance between any two different codewords of  $\mathcal{C}$  is not small than  $2\delta$ , it follows from (2) that each  $(l - \delta + 1)$ -dimensional subspace could not be contained in two different codewords. Moreover, since each codeword of  $\mathcal{C}$  contains  $\binom{l}{l-\delta+1}_q$  distinct  $(l - \delta + 1)$ -dimensional subspaces, all codewords of  $\mathcal{C}$  contains totally  $M \binom{l}{l-\delta+1}_q = \binom{n}{l-\delta+1}_q$  pairwise different  $(l - \delta + 1)$ -dimensional subspaces. Note that there are totally  $\binom{n}{l-\delta+1}_q$  distinct  $(l - \delta + 1)$ -dimensional subspaces

of  $W$ . Hence, each  $(l - \delta + 1)$ -dimensional subspace is contained in exactly one codeword of  $\mathcal{C}$ . Therefore, regarding the codewords of  $\mathcal{C}$  as blocks,  $\mathcal{C}$  forms a Steiner structure  $S[l - \delta + 1, l, n]_q$  by its definition.  $\square$

Theorem 1 shows that Steiner structures are optimal constant dimension codes. The following corollary follows from Theorem 1 immediately.

**Corollary 1**

$$A_q[n, 2\delta, l] = \frac{\begin{bmatrix} n \\ l-\delta+1 \end{bmatrix}_q}{\begin{bmatrix} l \\ l-\delta+1 \end{bmatrix}_q}$$

if and only if a Steiner structure  $S[l - \delta + 1, l, n]_q$  exists.

It is known [5, 6] that trivial Steiner structures  $S[t, n, n]_q$  and  $S[t, t, n]_q$  exist for all  $t \leq n$ . For nontrivial Steiner structures, by our knowledge, it is only known [5, 6] that  $S[1, l, n]_q$  exists where  $l \mid n$ , and the blocks of  $S[1, l, n]_q$  form a partition of  $W$  (excluding the zero vector). For completeness, we review the construction [5, 6] of such an  $S[1, l, n]_q$  where  $n = kl$  as follows. Let  $e = (q^{kl} - 1)/(q^l - 1)$  and let  $\alpha$  be a primitive element of  $\mathbb{F}_{q^n}$ . Define

$$\langle \alpha^e \rangle = \{1, \alpha^e, \alpha^{2e}, \dots, \alpha^{(q^l-2)e}\}.$$

The cosets

$$C_i = \alpha^i \langle \alpha^e \rangle = \{\alpha^i, \alpha^{i+e}, \alpha^{i+2e}, \dots, \alpha^{i+(q^l-2)e}\}, \quad i = 0, 1, \dots, e - 1$$

are called *cyclotomic classes* of order  $e$ . Let  $E_i = C_i \cup \{0\}$ . Note that  $\mathbb{F}_{q^n}$  is an  $n$ -dimensional vector space over  $\mathbb{F}_q$ . One can verify that  $E_0 = \mathbb{F}_{q^l}$  and the  $E_i$ 's, when viewed as subsets of  $\mathbb{F}_q^n$ , are  $l$ -dimensional subspaces of  $W = \mathbb{F}_q^n$ . Regarding all  $E_i$ 's as blocks, an  $S[1, l, kl]_q$  is obtained.

From Corollary 1, we have the following result.

**Corollary 2** *For any positive integers  $k$  and  $l$ , we have*

$$A_q[kl, 2l, l] = \frac{q^{kl} - 1}{q^l - 1}. \tag{6}$$

### 3 Johnson Type Bound I

In this section, we first review some basic definitions and the Johnson bound I for binary constant weight codes in coding theory. It is shown that a corresponding

binary constant weight code can be obtained from a given constant dimension code. Then, using the Johnson bound I for this corresponding binary constant weight code, we obtain the Johnson type bound I for constant dimension codes. It is observed that this bound is tight in some cases.

Let  $\mathbb{F}_2^n$  be the  $n$ -dimensional vector space over the binary field  $\mathbb{F}_2$ . For any two vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{F}_2^n$ , the Hamming distance  $d_H(\mathbf{a}, \mathbf{b})$  is the number of coordinates in which they differ, the Hamming weight  $w_H(\mathbf{a})$  is the number of nonzero coordinates in  $\mathbf{a}$ . It is known that

$$d_H(\mathbf{a}, \mathbf{b}) = w_H(\mathbf{a}) + w_H(\mathbf{b}) - 2w_H(\mathbf{a} * \mathbf{b}) \quad (7)$$

where

$$\mathbf{a} * \mathbf{b} = (a_1b_1, a_2b_2, \dots, a_nb_n).$$

A binary code  $C$  of length  $n$  is a nonempty subset of  $\mathbb{F}_2^n$ . The minimum distance of  $C$  is the minimum Hamming distance between any two distinct codewords in  $C$ . A binary constant weight code is a binary code such that every codeword has a fixed Hamming weight. Denote  $A(n, 2\delta, w)$  the maximum number of codewords in a binary constant weight code with length  $n$ , weight  $w$  and minimum distance at least  $2\delta$ . We state the Johnson bound I for binary constant weight codes in the following proposition.

**Proposition 5** (Johnson bound I) [8] *If  $w^2 > n(w - \delta)$ , then*

$$A(n, 2\delta, w) \leq \left\lfloor \frac{n\delta}{w^2 - n(w - \delta)} \right\rfloor,$$

where  $\lfloor \cdot \rfloor$  denotes the floor function.

Below we show that a corresponding binary constant weight code can be obtained from a given constant dimension code.

Recall that  $W$  is the  $n$ -dimensional vector space over the finite field  $\mathbb{F}_q$  and  $\mathcal{P}(W, l)$  denote the set of all  $l$ -dimensional subspaces of  $W$ . Let  $\mathbf{0}$  denote the all-zero vector in  $W$  and  $W^* = W \setminus \{\mathbf{0}\}$ . Denote  $N = q^n - 1$ . Suppose all the vectors in  $W^*$  are ordered from 1 to  $N$ . Define the incidence vector of a subset  $X \subseteq W$  by

$$\mathbf{v}_X = (v_1, v_2, \dots, v_N) \in \mathbb{F}_2^N$$

where  $v_i = 1$  if the  $i$ -th vector of  $W^*$  is contained in  $X$ , and  $v_i = 0$  otherwise. For any two  $l$ -dimensional subspaces  $X, Y \in \mathcal{P}(W, l)$ , by (7) it is easy to see that

$$w_H(\mathbf{v}_X) = w_H(\mathbf{v}_Y) = q^l - 1, \quad (8)$$

$$w_H(\mathbf{v}_X * \mathbf{v}_Y) = q^{\dim(X \cap Y)} - 1, \quad (9)$$

$$d_H(\mathbf{v}_X, \mathbf{v}_Y) = 2(q^l - q^{\dim(X \cap Y)}). \quad (10)$$

Let  $\mathcal{C}$  be an  $(n, M, 2\delta, l)_q$  constant dimension code. By (8), the incidence vectors of the codewords in  $\mathcal{C}$  form a binary constant weight code  $\mathbf{C}$ , which is called the *derived binary constant weight code of  $\mathcal{C}$* . From (8), (10) and the definition of constant dimension codes, we have the following result.

**Proposition 6** *Let  $\mathcal{C}$  be an  $(n, M, 2\delta, l)_q$  constant dimension code. Then its derived binary constant weight code  $\mathbf{C}$  has the following parameters: length  $N = q^n - 1$ , size  $M$ , minimum distance  $2(q^l - q^{l-\delta})$ , and weight  $q^l - 1$ .*

Although every constant dimension code corresponds to a binary constant weight code, the reverse proposition may not hold. Given a binary  $(q^n - 1, M, 2(q^l - q^{l-\delta}), q^l - 1)$  constant weight code, since its codewords may not be the incidence vectors of any subspaces, the code may not correspond to any  $(n, M, 2\delta, l)_q$  constant dimension code. From Propositions 5 and 6 we obtain the Johnson type bound I for constant dimension codes.

**Theorem 2** (Johnson type bound I for constant dimension codes)

*If  $(q^l - 1)^2 > (q^n - 1)(q^{l-\delta} - 1)$ , then*

$$A_q[n, 2\delta, l] \leq \left\lfloor \frac{(q^l - q^{l-\delta})(q^n - 1)}{(q^l - 1)^2 - (q^n - 1)(q^{l-\delta} - 1)} \right\rfloor.$$

The Johnson type bound I for constant dimension codes is tight in some cases. By Proposition 4 and Corollary 2, the Steiner structure  $S[1, l, kl]_q$  is a  $(kl, \frac{q^{kl}-1}{q^l-1}, 2l, l)_q$  constant dimension code achieving the Johnson type bound I for constant dimension codes.

**Remark 1** There is another method to obtain a binary constant weight code from a constant dimension code. Let  $\tilde{W}$  be the set of all 1-dimensional subspaces of  $W$ . Denote  $\tilde{N} = \frac{q^n-1}{q-1}$ . We can regard  $\tilde{W}$  as  $PG(n-1, q)$ , the  $(n-1)$ -dimensional projective geometry over  $\mathbb{F}_q$  with  $\tilde{N}$  points [7, Appendix B], where

each  $X \in \mathcal{P}(W, l)$  corresponds to an  $(l-1)$ -flat of  $PG(n-1, q)$ , say  $\tilde{X}$ . Suppose all points in  $PG(n-1, q)$  are ordered from 1 to  $\tilde{N}$ . Define the *punctured incidence vector* of  $X \in \mathcal{P}(W, l)$  as the incidence vector of  $\tilde{X} \in PG(n-1, q)$ . By putting together the punctured incidence vectors of all codewords of an  $(n, M, 2\delta, l)_q$  constant dimension code  $\mathcal{C}$ , we obtain a corresponding binary constant weight code  $\tilde{\mathbf{C}}$  which has length  $\tilde{N} = \frac{q^n-1}{q-1}$ , size  $M$ , minimum distance  $\frac{2(q^l-q^{l-\delta})}{q-1}$ , and weight  $\frac{q^l-1}{q-1}$ . Note that the derived binary constant weight code  $\mathbf{C}$  can be obtained by concatenating  $(q-1)$  times of  $\tilde{\mathbf{C}}$ . Hence, we have

$$A_q[n, 2\delta, l] \leq A\left(\frac{q^n-1}{q-1}, \frac{2(q^l-q^{l-\delta})}{q-1}, \frac{q^l-1}{q-1}\right). \quad (11)$$

However, by employing the Johnson bound I for binary constant weight codes, (11) implies the same results with Theorem 2.

## 4 Johnson Type Bound II

In this section, we derive an upper bound for constant dimension codes. We call this upper bound the Johnson type bound II for constant dimension codes since it is similar to the Johnson bound II for binary constant weight codes [8]. The Johnson type bound II for constant dimension codes slightly improves on the Wang-Xing-Safavi-Naini bound.

Let  $V_1, V_2 \in \mathcal{P}(W)$  and  $V_2 \subseteq V_1$ . Define

$$(V_2|V_1)^\perp = \{\mathbf{a} \in V_1 : \forall \mathbf{b} \in V_2, \mathbf{a}\mathbf{b}^T = 0\},$$

i.e.,  $(V_2|V_1)^\perp$  is the orthogonal complement of  $V_2$  in  $V_1$ . For any  $S \subseteq W$ , denote  $\langle S \rangle$  the minimum subspace containing  $S$ .

### Theorem 3

$$A_q[n, 2\delta, l] \leq \left\lfloor \frac{q^n-1}{q^l-1} A_q[n-1, 2\delta, l-1] \right\rfloor.$$

*Proof:* Suppose  $\mathcal{C}$  is an optimal  $(n, M, \geq 2\delta, l)_q$  constant dimension code with  $M = A_q[n, 2\delta, l]$ . Consider the binary  $M \times (q^n-1)$  matrix, say  $\mathcal{P}$ , whose rows consist of all the codewords of  $\mathbf{C}$ , where  $\mathbf{C}$  is the derived binary constant weight code of  $\mathcal{C}$  in Proposition 6. Denote  $\Delta$  the total number of 1's in the matrix  $\mathcal{P}$ . Since each codeword of  $\mathbf{C}$  has weight  $q^l-1$ ,

$$\Delta = M(q^l-1) = A_q[n, 2\delta, l](q^l-1). \quad (12)$$

On the other hand, we will show that the number of 1's in each column of  $\mathcal{P}$  is not greater than  $A_q[n-1, 2\delta, l-1]$ . Recall that the positions of the incidence vectors are indexed by the non-zero vectors of  $W$ . Without loss of generality, suppose  $\alpha_1 \in W$  is the non-zero vector which indexes the first column of  $\mathcal{P}$ . Let

$$\mathcal{C}_1 = \{X \in \mathcal{C} : \text{the first component of } \mathbf{v}_X \text{ is } 1\}.$$

Hence, the weight of the column indexed by  $\alpha_1$  equals  $|\mathcal{C}_1|$ . Noting that  $\alpha_1 \in X$  for any  $X \in \mathcal{C}_1$ , let  $\mathcal{C}'_1 = \{(\langle \alpha_1 | X \rangle)^\perp : X \in \mathcal{C}_1\}$  and  $W_1 = (\langle \alpha_1 | W \rangle)^\perp$ . Clearly,  $W_1$  is an  $(n-1)$ -dimensional vector space over  $\mathbb{F}_q$  and each element of  $\mathcal{C}'_1$  is an  $(l-1)$ -dimensional subspace of  $W_1$ . Hence,  $\mathcal{C}'_1 \subseteq \mathcal{P}(W_1, l-1)$  is a  $q$ -ary constant dimension code with length  $n-1$ , size  $|\mathcal{C}_1|$ , and dimension  $l-1$ . Moreover, for any two different codewords of  $\mathcal{C}'_1$ , e.g.,  $(\langle \alpha_1 | X \rangle)^\perp$  and  $(\langle \alpha_1 | Y \rangle)^\perp$ , where  $X \neq Y \in \mathcal{C}_1$ ,

$$\begin{aligned} & d((\langle \alpha_1 | X \rangle)^\perp, (\langle \alpha_1 | Y \rangle)^\perp) \\ &= 2(l-1) - 2\dim[(\langle \alpha_1 | X \rangle)^\perp \cap (\langle \alpha_1 | Y \rangle)^\perp] \\ &= 2(l-1) - 2\dim((\langle \alpha_1 | X \cap Y \rangle)^\perp) \\ &= 2l - 2\dim(X \cap Y) \\ &= d(X, Y) \geq 2\delta. \end{aligned}$$

Hence,  $\mathcal{C}'_1$  is an  $(n-1, |\mathcal{C}_1|, \geq 2\delta, l-1)_q$  constant dimension code, which implies that

$$|\mathcal{C}_1| \leq A_q[n-1, 2\delta, l-1].$$

The weight of the column indexed by  $\alpha_1$  is not greater than  $A_q[n-1, 2\delta, l-1]$ . Therefore, by counting the number of 1's for each column of  $\mathcal{P}$ , we have that

$$\Delta \leq (q^n - 1)A_q[n-1, 2\delta, l-1].$$

Combining this with (12) and noting that  $A_q[n, 2\delta, l]$  is an integer, we obtain the required conclusion.  $\square$

Using Theorem 3 recursively, we obtain the Johnson type bound II for constant dimension codes.

**Corollary 3** (Johnson type bound II for constant dimension codes)

$$A_q[n, 2\delta, l] \leq \left\lfloor \frac{q^n - 1}{q^l - 1} \left\lfloor \frac{q^{n-1} - 1}{q^{l-1} - 1} \left\lfloor \dots \left\lfloor \frac{q^{n-l+\delta} - 1}{q^\delta - 1} \right\rfloor \dots \right\rfloor \right\rfloor \right\rfloor.$$

The Johnson type bound II slightly improves on the Wang-Xing-Safavi-Naini bound. Let  $B_S, B_{WXS}, B_J$  denote respectively the Singleton type bound in Proposition 1, the Wang-Xing-Safavi-Naini bound in Proposition 2, and the Johnson type bound II in Corollary 3. For example, letting  $q = 2, n = 6, \delta = 2$  and  $l = 3$ , we have  $B_S = 155, B_{WXS} = 93$  and  $B_J = 90$ . Below we show that the Wang-Xing-Safavi-Naini bound is always better than the Singleton type bound for  $\delta > 1$  and  $n > l$ . Since for  $i = 0, 1, \dots, l - \delta$ ,

$$\frac{(q^{n-l+i+1} - 1)}{(q^{i+1} - 1)} \geq \frac{(q^{n-l+\delta+i} - 1)}{(q^{\delta+i} - 1)} \iff (q^{n-l+i+1} - q^{i+1})(q^{\delta-1} - 1) \geq 0,$$

we have that

$$\begin{aligned} B_S &= \frac{(q^{n-\delta+1} - 1)(q^{n-\delta} - 1) \dots (q^{n-l+1} - 1)}{(q^{l-\delta+1} - 1)(q^{l-\delta} - 1) \dots (q - 1)} \\ &\geq \frac{(q^n - 1)(q^{n-1} - 1) \dots (q^{n-l+\delta} - 1)}{(q^l - 1)(q^{l-1} - 1) \dots (q^\delta - 1)} = B_{WXS} \end{aligned} \quad (13)$$

and the equality holds if and only if  $\delta = 1$  or  $n = l$ . Furthermore, by [3, Lemma 5],  $1 < q^{-l(n-l)} \begin{bmatrix} n \\ l \end{bmatrix}_q < 4$  for  $0 < l < n$ . Using the similar arguments in the proof of [3, Lemma 5], we obtain

$$1 < q^{-m(u-v)} \frac{(q^u - 1)(q^{u-1} - 1) \dots (q^{u-m+1} - 1)}{(q^v - 1)(q^{v-1} - 1) \dots (q^{v-m+1} - 1)} < 4 \quad \text{for } 1 \leq m \leq v < u. \quad (14)$$

Hence, by (13) and (14), it is easy to see that for  $\delta > 1$  and  $0 < l < n$

$$B_{WXS} < B_S < 4q^{(l-\delta+1)(n-l)} < 4B_{WXS}. \quad (15)$$

For example, letting  $n = 100, l/n = 0.4, \delta/n = 0.2$ , it is computed with the *Mathematica* software that

$$\frac{B_S}{B_{WXS}} \approx 3.46, 1.79, 1.45, 1.32, \quad \text{for } q = 2, 3, 4, 5, \text{ respectively.}$$

## 5 Concluding Remarks

In this paper we show that Steiner structures, e.g.,  $S[1, l, kl]_q$ , are optimal constant dimension codes or linear authentication codes, and could be applied in random network coding or distributed authentication systems. Furthermore, it is shown that constant dimension codes achieve the Wang-Xing-Safavi-Naini bound

if and only if they are certain Steiner structures. We derive two Johnson type bounds for constant dimension codes. It would be interesting to construct more constant dimension codes which achieve Johnson type bounds I or II. It is a hard problem to determine  $A_q[n, 2\delta, l]$  in general. However, one can first make efforts to determine  $A_q[n, 4, l]$ ,  $A_q[n, 6, l]$  and  $A_q[n, 2(l - 1), l]$  in the following steps.

## Acknowledgment

The authors would like to thank Professor Cunsheng Ding for reading this paper and giving valuable comments that helped to improve the paper.

## References

- [1] Ho T, Koetter R, Médard M, Karger D, Effros M (2003) The benefits of coding over routing in a randomized setting. In: Proc. IEEE Int. Symp. Inform. Theory, Yokohama, Japan, p. 442
- [2] Ho T, Médard M, Koetter R, Karger D, Effros M, Shi J, Leong B (2006) A random linear network coding approach to multicast. *IEEE Trans. Inform. Theory* 52: 4413–4430
- [3] Koetter R, Kschischang F (2007) Coding for errors and erasures in random network coding. In: Proc. IEEE Int. Symp. Inform. Theory, Nice, France, pp. 791–795. Full version is available online at <http://www.arxiv.org/abs/cs.IT/0703061>
- [4] Wang H, Xing C, Safavi-Naini R (2003) Linear authentication codes: bounds and constructions. *IEEE Trans. Inform. Theory* 49: 866–872
- [5] Schwartz M, Etzion T (2002) Codes and anticodes in the Grassmann graph. *J. of Combinatorial Theory Series A* 97: 27–42
- [6] Etzion T (1998) Perfect byte-correcting codes. *IEEE Trans. Inform. Theory* 44: 3140–3146
- [7] MacWilliams FJ, Sloane NJA (1981) *The Theory of Error-Correcting Codes*. North-Holland, Amsterdam
- [8] Tonchev VD (1998) Codes and designs. In: Pless VC, Huffman WC (eds) *Handbook of Coding Theory*, Chapter 15. North-Holland, Amsterdam