# ON TIGHT PROJECTIVE DESIGNS 

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#### Abstract

It is shown that among all tight designs in $\mathbb{F P}{ }^{n} \neq \mathbb{R P}^{1}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, or $\mathbb{H}$ (quaternions), only 5 -designs in $\mathbb{C P}^{1}$ [14] have irrational angle set. This is the only case of equal ranks of the first and the last irreducible idempotent in the corresponding Bose-Mesner algebra.

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## 1. Introduction

A well known theorem of Bannai and Hoggar [3] states that there are no tight $t$-designs in $\mathbb{F P}^{n} \neq \mathbb{R P}^{1}$ if $t \geq 6$. Moreover, a theorem of Hoggar [10] states the same for $t \geq 4$ if $\mathbb{F} \neq \mathbb{R}$. Surprisingly, a tight 5 -design in $\mathbb{C P}^{1}$ has been constructed in [14], so Hoggar's theorem has to be corrected. The results of [3] and [10] are essentially based on Theorem 2.6(c) [9] that states that the angle set of every tight $t$-design in $\mathbb{F P}^{n} \neq \mathbb{R} \mathbb{P}^{1}$ is rational. But it is not rational for the 5 -design constructed in 14 .

In the present paper we investigate this contradiction and prove that the only cases where the angle set is not rational are
(1) $\mathbb{F}=\mathbb{C}, n=1, t=5$ and
(2) $\mathbb{F}=\mathbb{R}, n=1, t \neq 1,2,3,5$.

A fortiori, there are no complications in [3] where $t \geq 6$ by assumption.
Our principal observation is that if $t=2 s-1, s \geq 2$ then the last irreducible idempotent $L_{s}$ in the corresponding Bose-Mesner algebra is not $E_{s}$ from the proof of Theorem 2.6(c) [9] (actually, from [18]). Nevertheless, rk $L_{s} \neq \operatorname{rk} E_{1}$, except for our case (1). This "critical inequality" implies the rationality of the angle set, similarly to the argument in [9]. This material is concentrated in Section 4 of the present paper, while Sections 2 and 3 contain all the necessary background and preliminary analysis.

## 2. Projective $t$-designs

For the reader's convenience we basically use the same notation as in 8 and other related papers. Let us recall this notation. In particular, let

$$
\mathbb{F} \in\{\mathbb{R}, \mathbb{C}, \mathbb{H}\} ; \quad m=\frac{1}{2}(\mathbb{F}: \mathbb{R})= \begin{cases}1 / 2 & \mathbb{F}=\mathbb{R} \\ 1 & \mathbb{F}=\mathbb{C} ; \quad N=m(n+1) . \\ 2 & \mathbb{F}=\mathbb{H}\end{cases}
$$

The number $2 N$ is nothing but the real (topological) dimension of the $\mathbb{F}$-linear space $\mathbb{F}^{n+1}$. The latter consists of all $(n+1) \times 1$ matrices (columns) over $\mathbb{F}$ with the standard addition and multiplication by scalars $\tau \in \mathbb{F}$ from the right (for
definiteness). As usual, the inner product of $a, b \in \mathbb{F}^{n+1}$ is $a^{*} b$ where $a^{*}$ is the row conjugate transpose to $a$. Accordingly, the set

$$
S^{2 N-1}=\left\{a: a^{*} a=1\right\}
$$

is the unit sphere in $\mathbb{F}^{n+1}$. A quotient set of the sphere with respect to the equivalence relation $a_{1} \sim a_{2} \Longleftrightarrow a_{1}=a_{2} \lambda, \lambda \in \mathbb{F},|\lambda|=1$, is the projective space $\mathbb{F P}^{n}$. The "inner product" $(\hat{a}, \hat{b})=\left|a^{*} b\right|^{2}$ in $\mathbb{F P}$ " is well-defined through the natural mapping $a \mapsto \hat{a}$ from $S^{2 N-1}$ onto $\mathbb{F P}^{n}$. Obviously, $(\hat{b}, \hat{a})=(\hat{a}, \hat{b})$ and $0 \leq(\hat{a}, \hat{b}) \leq 1$ with the equality $(\hat{a}, \hat{b})=1$ if and only if $\hat{a}=\hat{b}$. For every nonempty $X \subset \mathbb{F P} \mathbb{P}^{n}$ its angle set is

$$
A(X)=\{(x, y): x, y \in X, x \neq y\}
$$

The related combinatorial parameters are

$$
s=|A(X)|, \quad e=|A(X) \backslash\{0\}|, \quad \epsilon=s-e=|A(X) \cap\{0\}| .
$$

Let $P_{i}^{(\alpha, \beta)}(\tau)$ be the Jacobi polynomials [20] such that

$$
\begin{equation*}
\operatorname{deg} P_{i}^{(\alpha, \beta)}=i, \quad P_{i}^{(\alpha, \beta)}(1)=\frac{(\alpha+1)_{i}}{i!} \tag{2.1}
\end{equation*}
$$

where

$$
(\alpha+1)_{i}=\prod_{l=1}^{i}(\alpha+l), \quad(\alpha+1)_{0}=1
$$

In particular, $P_{0}^{(\alpha, \beta)}(\tau) \equiv 1$. In what follows we fix

$$
\begin{equation*}
\alpha=N-m-1, \quad \beta=m-1 \tag{2.2}
\end{equation*}
$$

and set

$$
\begin{equation*}
P_{i}(\xi)=P_{i}^{(\alpha, \beta)}(2 \xi-1) \tag{2.3}
\end{equation*}
$$

for short. A finite nonempty subset $X \subset \mathbb{F} \mathbb{P}^{n}$ is called a $t$-design if

$$
\begin{equation*}
\sum_{x \in X} P_{i}((x, y))=0, \quad y \in X, \quad 1 \leq i \leq t \tag{2.4}
\end{equation*}
$$

Let $X$ be a $t$-design and let

$$
R_{e}^{\epsilon}(\xi)=\frac{(N)_{s}}{(m)_{s}} P_{e}^{(\alpha+1, \beta+\epsilon)}(2 \xi-1)
$$

In particular,

$$
\begin{equation*}
R_{e}^{\epsilon}(1)=\frac{(N)_{s}(N-m+1)_{e}}{(m)_{s} e!} \tag{2.5}
\end{equation*}
$$

The following theorems are fundamental, see [1], [2], [11]. (Cf. [6] for the spherical designs.)

Theorem A. The inequalities

$$
t \leq s+e, \quad|X| \geq R_{e}^{\epsilon}(1)
$$

hold, and the equalities

$$
t=s+e, \quad|X|=R_{e}^{\epsilon}(1)
$$

are equivalent.
In the latter case the $t$-design $X$ is called tight. Note that $t=s+e$ is equivalent to $e=[t / 2], \epsilon=\operatorname{res}_{2}(t)$.

Theorem B. If $X$ is a tight $t$-design then $A(X)$ coincides with the set of roots of the polynomial $\xi^{\epsilon} R_{e}^{\epsilon}(\xi)$.

Recall that these roots are simple and lie on $(0,1)$.
Theorem C. Let $X$ be a subset of $\mathbb{F P}^{n}$ such that $|X|=R_{e}^{\epsilon}(1)$ and $A(X)$ coincides with the set of roots of $\xi^{\epsilon} R_{e}^{\epsilon}(\xi)$, then $X$ is a tight $(2 e+\epsilon)$-design.

The projective $t$-designs can be characterized as the averaging sets in the sense of [19] for suitable spaces of functions on $\mathbb{F P}^{n}$. Usually, these spaces are described in terms of harmonic analysis but we prefer a more elementary approach [15], [16].

We say that a mapping $\phi: \mathbb{F} \mathbb{P}^{n} \rightarrow \mathbb{C}$ is a polynomial function if it is of the form

$$
\phi(\hat{a})=\psi(a), \quad a \in S^{2 N-1}
$$

where $\psi$ is a polynomial on $\mathbb{F}^{n+1}$ in real coordinates. This $\psi$ must be invariant with respect to the rotations of $\mathbb{F}$, i.e. $\psi(a \lambda)=\psi(a)$ for all $\lambda \in \mathbb{F},|\lambda|=1$. It is not unique but becomes unique if it is required to be homogeneous (which is always possible) of minimal degree. The latter is said to be the degree of $\phi$. The number $\operatorname{deg} \phi$ is an even integer since $\psi(-a)=\psi(a)$.

Example 2.1. For every $t \in \mathbb{N}$ and every $y \in \mathbb{F P}^{n}$ the function $\phi_{2 t ; y}(x)=(x, y)^{t}$, $x \in \mathbb{F P}^{n}$, is a polynomial function of degree $2 t$.

Given $d \in 2 \mathbb{N}$, we denote by $\operatorname{Pol}_{\mathbb{F}}(d)$ the space of all polynomial functions of degrees $\leq d$. It has been proven in [16] that the family $\left\{\phi_{d ; y}: y \in \mathbb{F} \mathbb{P}^{n}\right\}$ spans the whole space $\operatorname{Pol}_{\mathbb{F}}(d)$. We apply this result to prove the following

Proposition 2.2. A finite nonempty set $X \subset \mathbb{F P}^{n}$ is a tight $t$-design if and only if

$$
\begin{equation*}
\frac{1}{|X|} \sum_{x \in X} \phi(x)=\int_{S^{2 N-1}} \tilde{\phi}(a) d \sigma(a), \quad \phi \in \operatorname{Pol}_{\mathbb{F}}(2 t) \tag{2.6}
\end{equation*}
$$

where $\tilde{\phi}$ is induced by the natural mapping $S^{2 N-1} \rightarrow \mathbb{F P}{ }^{n}$ and $\sigma$ is the normalized Lebesgue measure.

Proof. The identity (2.6) is equivalent to

$$
\begin{equation*}
\frac{1}{|X|} \sum_{x \in X} F((x, y))=\int_{S^{2 N-1}} F\left(\left|a^{*} b\right|^{2}\right) d \sigma(a) \tag{2.7}
\end{equation*}
$$

where $y=\hat{b}, b \in S^{2 N-1}, F$ runs over the space $\Pi_{t}$ of all univariate polynomials of degrees $\leq t$. By a known integration formula (see [8], Theorem 2.11) one can rewrite (2.7) in the form

$$
\begin{equation*}
\frac{1}{|X|} \sum_{x \in X} F((x, y))=\int_{-1}^{1} F\left(\frac{1+\tau}{2}\right) \Omega_{\alpha, \beta}(\tau) d \tau, \quad \psi \in \Pi_{t} \tag{2.8}
\end{equation*}
$$

where $\Omega_{\alpha, \beta}(\tau)$ is the normalized Jacobi weight, i.e.

$$
\begin{equation*}
\Omega_{\alpha, \beta}(\tau)=c_{\alpha, \beta}(1-\tau)^{\alpha}(1+\tau)^{\beta}, \quad-1<\tau<1 \tag{2.9}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{\alpha, \beta}=\left(\int_{-1}^{1}(1-\tau)^{\alpha}(1+\tau)^{\beta} d \tau\right)^{-1}=\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1)} \tag{2.10}
\end{equation*}
$$

In turn, (2.8) is equivalent to its restriction to $F=P_{i}(\xi), 1 \leq i \leq t$, since these polynomials constitute a basis in $\Pi_{t}$. It remains to note that

$$
\int_{-1}^{1} P_{i}\left(\frac{1+\tau}{2}\right) \Omega_{\alpha, \beta}(\tau) d \tau=\int_{-1}^{1} P_{i}^{(\alpha, \beta)}(\tau) \Omega_{\alpha, \beta}(\tau) d \tau=0
$$


Corollary 2.3. Let $X \subset \mathbb{F P}^{n}$ be at-design. Then

$$
\begin{equation*}
\frac{1}{|X|} \sum_{x \in X} P((u, x)) Q((x, v))=\int_{S^{2 N-1}} P\left(\left|a^{*} c\right|^{2}\right) Q\left(\left|c^{*} b\right|^{2}\right) d \sigma(c) \tag{2.11}
\end{equation*}
$$

for $u=\hat{a}, v=\hat{b}$ and all univariate polynomials $P, Q$ such that $\operatorname{deg} P+\operatorname{deg} Q \leq t$.
Proof. The mapping $x \mapsto P((u, x)) Q((x, v)), x \in \mathbb{F P}^{n}$, is a polynomial function of degree $\leq 2 t$.

Corollary 2.4. Let $X, P, Q$ be fixed under the conditions of Corollary 2.3. Then the value

$$
\sum_{x \in X} P((u, x)) Q((x, v))
$$

depends only on the inner product $(u, v)$ of $u, v \in \mathbb{F P} \mathbb{P}^{n}$.
Proof. Let $\left(u_{1}, v_{1}\right)=(u, v)$, i.e. $\left|a_{1}^{*} b_{1}\right|^{2}=\left|a^{*} b\right|^{2}$ where $\hat{a}_{1}=u_{1}, \hat{b}_{1}=v_{1}$. Without loss of generality one can assume that $a_{1}^{*} b_{1}=a^{*} b$. Then there exists a $(n+1) \times(n+1)$ matrix $T$ over $\mathbb{F}$ such that $T^{*} T=$ id and $a_{1}=T a, b_{1}=T b$. This substitution in (2.11) is equivalent to the change of variable $c \mapsto T^{*} c$. The latter does not affect the integral since the measure $\sigma$ is orthogonally invariant.

## 3. Bose-Mesner algebra

Let $X$ be a finite nonempty subset of $\mathbb{F P}^{n}$ and let

$$
A^{\prime}(X)=A(X) \cup\{1\}=\{(x, y): x, y \in X\}
$$

so that $\left|A^{\prime}(X)\right|=s+1$. The $X \times X$ matrices of the form

$$
\begin{equation*}
M_{F}=[F((x, y))]_{x, y \in X} \tag{3.1}
\end{equation*}
$$

where $F$ runs over all functions $A^{\prime}(X) \rightarrow \mathbb{C}$, constitute a complex linear space $\mathcal{D}(X)$. Its natural basis consists of the matrices

$$
\begin{equation*}
\Delta_{\zeta}=\left[\delta_{\zeta,(x, y)}\right]_{x, y \in X}, \quad \zeta \in A^{\prime}(X) \tag{3.2}
\end{equation*}
$$

thus, $\operatorname{dim} \mathcal{D}(X)=s+1$. The Lagrange interpolation formula allows us to let $F$ in (3.1) run over the polynomial space $\Pi_{s}$, so that we have the isomorphism $F \mapsto M_{F}$ between $\Pi_{s}$ and $\mathcal{D}(X)$. In particular, if $F \mid A(X)=0$ and $F(1)=1$ then $M_{F}=I$, the unit matrix.

According to Corollary 2.4 for $P, Q \in \Pi_{s}$, the matrix product $M_{P} M_{Q}$ belongs to $\mathcal{D}(X)$ if $\operatorname{deg} P+\operatorname{deg} Q \leq t$. However, this condition is not fulfilled if $t=2 s-1$ and $\operatorname{deg} P=\operatorname{deg} Q=s$. Moreover, Corollary 2.4 cannot be extended to this situation if $X$ is tight. Indeed, suppose to the contrary that

$$
\sum_{x \in X}(u, x)^{s}(x, v)^{s}=\Phi((u, v)) \quad\left(u, v \in \mathbb{F P}^{n}\right)
$$

with a function $\Phi:[0,1] \rightarrow \mathbb{R}_{+}$. Setting $v=u$ we obtain

$$
\sum_{x \in X}(u, x)^{2 s}=\Phi(1), \quad u \in \mathbb{F P}^{n}
$$

In other words,

$$
\sum_{c \in \tilde{X}}\left|a^{*} c\right|^{4 s}=\Phi(1), \quad a \in S^{2 N-1}
$$

where $\tilde{X} \subset S^{2 N-1}$ is a complete system of representatives of points $x \in X,|\tilde{X}|=$ $|X|$. By integration over $a$ we obtain

$$
\Phi(1)=\left(\int_{S^{2 N-1}}\left|a^{*} c\right|^{4 s} d \sigma(a)\right) \cdot|X|
$$

since the integral does not depend on $c$. As a result,

$$
\begin{equation*}
\frac{1}{|X|} \sum_{x \in X} \phi_{4 s ; u}(x)=\int_{S^{2 N-1}} \tilde{\phi}_{4 s ; u}(a) d \sigma(a) \tag{3.3}
\end{equation*}
$$

and by linearity, (3.3) extends to the whole space $\operatorname{Pol}_{\mathbb{F}}(4 s)$. Thus, $X$ is a $2 s$-design which is a contradiction since $2 s=t+1$.

Nevertheless, under the constraint $u, v \in X$, one can extend Corollary 2.4 to $t=2 s-1$ and $P, Q$ such that $\max (\operatorname{deg} P, \operatorname{deg} Q)=s$. This follows from the construction of a basis in $\mathcal{D}(X)$ using the Jacobi polynomials (cf. 6], Remark 7.6).

Lemma 3.1. Let $X$ be a $t$-design in $\mathbb{F P}^{n}$ and let $s=\left[\frac{t+1}{2}\right]$. Then $s+1$ matrices $M_{i}=M_{P_{i}}, 0 \leq i \leq s$, constitute a basis $\mathcal{M}$ of $\mathcal{D}(X)$ such that

$$
\begin{equation*}
M_{i} M_{k}=|X| M_{i} \delta_{i k} \rho_{\mu(i, k)} \tag{3.4}
\end{equation*}
$$

where $\mu(i, k)=\min (i, k)$ and all $\rho_{j}>0$.
Proof. The matrices $M_{i}$ are linearly independent because of the linear independence of the polynomials $P_{i}$. Since $|\mathcal{M}|=s+1$, this is a basis of $\mathcal{D}(X)$. Now note that

$$
\int_{S^{2 N-1}} P_{i}\left(\left|a^{*} c\right|^{2}\right) P_{k}\left(\left|c^{*} b\right|^{2}\right) d \sigma(c)=0, \quad i \neq k
$$

by the addition formula for polynomial functions [15] (cf. [7, [13], 17]). The same formula with $i=k$ yields

$$
\begin{equation*}
\int_{S^{2 N-1}} P_{i}\left(\left|a^{*} c\right|^{2}\right) P_{i}\left(\left|c^{*} b\right|^{2}\right) d \sigma(c)=\chi_{i} P_{i}\left(\left|a^{*} b\right|^{2}\right) \tag{3.5}
\end{equation*}
$$

where $\chi_{i}>0$. Assuming $\mu(i, k) \leq s-1$ (a fortiori, $i+k \leq 2 s-1 \leq t$ ) and using Corollary 2.3 we get (3.4) with

$$
\begin{equation*}
\rho_{j}=\chi_{j}, \quad 0 \leq j \leq s-1 \tag{3.6}
\end{equation*}
$$

In particular, $M_{i} M_{s}=M_{s} M_{i}=0$ for $0 \leq i \leq s-1$. It remains to consider the case $i=k=s$.

If $t$ is even the $t=2 s$ and Corollary 2.3 is applicable to $i=k=s$, so $M_{s}^{2}=$ $|X| M_{s} \rho_{s}$ with

$$
\begin{equation*}
\rho_{s}=\chi_{s} \tag{3.7}
\end{equation*}
$$

Let $t$ be odd, so $t=2 s-1$. Then we decompose the unity matrix $I$ for the basis $\mathcal{M}$,

$$
\begin{equation*}
I=\sum_{i=0}^{s} \lambda_{i} M_{i} \tag{3.8}
\end{equation*}
$$

and get $M_{s}=\lambda_{s} M_{s}^{2}$ multiplying (3.8) by $M_{s}$. This yields

$$
\begin{equation*}
\lambda_{s}=\frac{\operatorname{tr} M_{s}}{\operatorname{tr} M_{s}^{2}}=\frac{|X| P_{s}(1)}{\sum_{x, y} P_{s}^{2}((x, y))}>0 \tag{3.9}
\end{equation*}
$$

and then $M_{s}^{2}=|X| M_{s} \rho_{s}$ with

$$
\begin{equation*}
\rho_{s}=\left(\lambda_{s}|X|\right)^{-1} \tag{3.10}
\end{equation*}
$$

Remark 3.2. The formulas (3.6) and (3.7) are joined in

$$
\begin{equation*}
\rho_{i}=\chi_{i}, \quad 0 \leq i \leq[t / 2] \tag{3.11}
\end{equation*}
$$

while (3.10) appears only for $t=2 s-1$ in addition to (3.11).
Remark 3.3. The multiplication table (3.4) shows that under conditions of Lemma 3.1 $\mathcal{D}(X)$ is a commutative matrix algebra, the Bose-Mesner algebra of $X$ [4, [5, [6].

In what follows the conditions of Lemma 3.1 are assumed to be fulfilled. By setting

$$
\begin{equation*}
L_{i}=\frac{M_{i}}{\rho_{i}|X|}=\frac{1}{\rho_{i}|X|}\left[P_{i}((x, y))\right]_{x, y \in X} \tag{3.12}
\end{equation*}
$$

the basis $\mathcal{M}$ turns into $\mathcal{L}=\left\{L_{i}\right\}_{0}^{s}$ consisting of idempotents ( $L_{i}^{2}=L_{i}$ ) which are pairwise orthogonal ( $L_{i} L_{k}=0$ for $i \neq k$ ). It is important to calculate their ranks.

We have

$$
\operatorname{rk} L_{i}=\operatorname{tr} L_{i}=\rho_{i}^{-1} P_{i}(1), \quad 0 \leq i \leq s
$$

hence,

$$
\operatorname{rk} L_{i}=\chi_{i}^{-1} P_{i}(1)=\frac{P_{i}^{2}(1)}{\int_{S^{2 N-1}} P_{i}^{2}\left(\left|a^{*} c\right|^{2}\right) d \sigma(c)}, \quad 0 \leq i \leq[t / 2]
$$

by (3.11) and (3.5) for $a=b$. Finally,

$$
\begin{equation*}
\operatorname{rk} L_{i}=\frac{\left(P_{i}^{(\alpha, \beta)}(1)\right)^{2}}{\int_{-1}^{1}\left(P_{i}^{(\alpha, \beta)}(\tau)\right)^{2} \Omega_{(\alpha, \beta)}(\tau) d \tau}, \quad 0 \leq i \leq[t / 2] \tag{3.13}
\end{equation*}
$$

In particular, rk $L_{0}=1$. In addition to (3.13) we have to find rk $L_{s}$ in the case $t=2 s-1$. Formula (3.10) is not effective to this end since $\lambda_{s}$ is unknown. Indeed, in (3.9) we cannot proceed to the formally corresponding integral in the denominator. Instead of this, we return to the decomposition of unity and express rk $L_{s}$ through $\operatorname{rk} L_{i}, 0 \leq i \leq s-1$. We have

$$
I=\sum_{i=0}^{s} L_{i},
$$

whence,

$$
\operatorname{rk} L_{s}=\operatorname{tr} L_{s}=|X|-\sum_{i=0}^{s-1} \operatorname{tr} L_{i}=|X|-\sum_{i=0}^{s-1} \operatorname{rk} L_{i} .
$$

By substitution from (3.13) the last sum can be written as $c_{\alpha, \beta}^{-1} K_{s-1}^{(\alpha, \beta)}(1,1)$, where $K_{s-1}^{(\alpha, \beta)}(\cdot, \cdot)$ is the reproducing kernel of the Jacobi polynomials with respect to the
weight $(1-\tau)^{\alpha}(1+\tau)^{\beta}$, see [20], Section 4.5. According to (2.10) and formula (4.5.8) from [20] we obtain

$$
\operatorname{rk} L_{s}=|X|-\frac{\Gamma(s+\alpha+\beta+1) \Gamma(s+\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2) \Gamma(\alpha+2) \Gamma(s)}
$$

With our $\alpha, \beta$ defined by (2.2)

$$
\begin{equation*}
\operatorname{rk} L_{s}=|X|-\frac{\Gamma(N+s-1) \Gamma(N-m+s) \Gamma(m)}{\Gamma(N) \Gamma(N-m+1) \Gamma(m+s-1) \Gamma(s)}=|X|-\frac{(N)_{s-1}(N-m+1)_{s-1}}{(m)_{s-1}(s-1)!} \tag{3.14}
\end{equation*}
$$

Lemma 3.4. Let $X$ be a tight $t$-design in $\mathbb{F P}^{n}$ with $t=2 s-1$. Then

$$
\begin{equation*}
\operatorname{rk} L_{s}=\frac{(N)_{s-1}(N-m)_{s}}{(m)_{s}(s-1)!} \tag{3.15}
\end{equation*}
$$

Proof. In this case $e=s-1, \epsilon=1$, so (2.5) yields

$$
|X|=R_{s-1}^{1}(1)=\frac{(N)_{s}(N-m+1)_{s-1}}{(m)_{s}(s-1)!}
$$

The ranks of the other $L_{i}$ (including $L_{s}$ if $t=2 s$ ) can be explicitly calculated by (3.13), (2.10) and (2.1) combined with (4.33) of [20]. This results in
Lemma 3.5. Let $X$ be a $t$-design in $\mathbb{F P}^{n}$ with $s=\left[\frac{t+1}{2}\right]$. Then

$$
\begin{equation*}
\operatorname{rk} L_{i}=\frac{(N)_{i-1}(N-m)_{i}(N+2 i-1)}{(m)_{i} i!}, \quad 0 \leq i \leq[t / 2] \tag{3.16}
\end{equation*}
$$

Remark 3.6. Formula (3.16) yields the true value rk $L_{0}=1$ by setting $(\gamma-1)(\gamma)_{-1}=$ 1 for all $\gamma$.
Corollary 3.7. The inequality

$$
\begin{equation*}
\operatorname{rk} L_{i}>\operatorname{rk} L_{i-1}, \quad 1 \leq i \leq[t / 2] \tag{3.17}
\end{equation*}
$$

holds, except for $X \subset \mathbb{F P}^{1}$. In the latter case

$$
\begin{equation*}
\operatorname{rk} L_{i}=2, \quad 1 \leq i \leq[t / 2] \tag{3.18}
\end{equation*}
$$

Now note that our idempotents $L_{i}$ coincide with the matrices $E_{i}$ from 9 for $0 \leq i \leq[t / 2]$ but $L_{s} \neq E_{s}$ if $t=2 s-1, X \not \subset \mathbb{F P}^{1}$. Indeed, according to (2.5) from 9],

$$
\begin{equation*}
E_{i}((x, y))=\frac{1}{|X|}\left[Q_{i}((x, y))\right]_{x, y \in X}, \quad 0 \leq i \leq s \tag{3.19}
\end{equation*}
$$

where $Q_{i}(\xi)$ is proportional to $P_{i}(\xi)$ and

$$
\begin{equation*}
Q_{i}(1)=\frac{(N)_{i-1}(N-m)_{i}(N+2 i-1)}{(m)_{i} i!}, \quad i \geq 0 \tag{3.20}
\end{equation*}
$$

Hence, $E_{i}$ are proportional to $L_{i}$ for all $i, 0 \leq i \leq s$. Moreover, if $0 \leq i \leq[t / 2]$ then $\operatorname{tr} E_{i}=Q_{i}(1)=\operatorname{tr} L_{i}$ by (3.20) and (3.16). Hence, $E_{i}=L_{i}$ for $0 \leq i \leq[t / 2]$. However, if $t=2 s-1$ (so $s=[t / 2]+1$ ) and $X \not \subset \mathbb{F P}^{1}$ then $\operatorname{tr} E_{s}=Q_{s}(1)>\operatorname{tr} L_{s}$, see (3.15). In this case $\operatorname{tr} E_{s}>\operatorname{rk} E_{s}$, so $E_{s}$ is not an idempotent. This is an obstacle to the full proof of Theorem 2.6 [9] of the rationality of $A(X)$. To overcome this difficulty, it suffices to change $E_{s}$ for $L_{s}$ (when $t=2 s-1, s \geq 2$ ) but then the "critical inequality" $\mathrm{rk} L_{s} \neq \operatorname{rk} L_{1}$ is needed. However, the latter is not always true. We clarify this intricate situation in the next section.

## 4. The critical inequality and Rationality theorem

We prove the following
Theorem 4.1. With $t=2 s-1, s \geq 2$, the inequality

$$
\begin{equation*}
\operatorname{rk} L_{s} \neq \operatorname{rk} L_{1} \tag{4.1}
\end{equation*}
$$

holds for every tight $t$-design $X \subset \mathbb{F P}^{n}$, except for a tight 5-design in $\mathbb{C P}^{1}$.
Proof. From (3.15) it follows that

$$
\operatorname{rk} L_{s} \geq \frac{(N)_{1}(N-m)_{2}}{(m)_{2} \cdot 1!}=\frac{N(N-m)(N-m+1)}{m(m+1)}
$$

since the right side of (3.15) increases with $s$. On the other hand, (3.16) yields

$$
\begin{equation*}
\operatorname{rk} L_{1}=\frac{(N)_{0}(N-m)_{1}(N+1)}{(m)_{1} \cdot 1!}=\frac{(N-m)(N+1)}{m} \tag{4.2}
\end{equation*}
$$

Hence,

$$
\operatorname{rk} L_{s}-\operatorname{rk} L_{1} \geq \frac{(N-m)\left((N-m)^{2}-\left(m^{2}+m+1\right)\right)}{m(m+1)}
$$

Since $N-m=m n$ we obtain $\operatorname{rk} L_{s}>\operatorname{rk} L_{1}$ for $n^{2}>1+m^{-1}+m^{-2}$, i.e. for $n \geq 3$ if $\mathbb{F}=\mathbb{R}$ and for $n \geq 2$ if $\mathbb{F}=\mathbb{C}$ or $\mathbb{H}$. It remains to consider two cases.
(1) $\mathbb{F}=\mathbb{R}, n=2$. Then $m=1 / 2, N=3 / 2$, so $\operatorname{rk} L_{1}=5$ by (4.2), while rk $L_{s}=2 s$ by (3.15).
(2) $\mathbb{F}$ is arbitrary, $n=1$. Then $N=2 m$, hence

$$
\operatorname{rk} L_{s}-\operatorname{rk} L_{1}=\frac{(2 m)_{s-1}}{(s-1)!}-(2 m+1)= \begin{cases}-1 & \mathbb{F}=\mathbb{R} \\ s-3 & \mathbb{F}=\mathbb{C} \\ \frac{1}{6} s(s+1)(s+2)-5 & \mathbb{F}=\mathbb{H}\end{cases}
$$

We see that $\operatorname{rk} L_{s} \neq \operatorname{rk} L_{1}$ with the only exception $\mathbb{F}=\mathbb{C}, s=3$, so $t=5$.
A tight 5 -design in $\mathbb{C P}^{1}$ should contain 12 points since its parameters are $m=1$, $N=2, s=3, e=2, \epsilon=1$. Accordingly, (2.5) becomes

$$
R_{2}^{1}(1)=\frac{(2)_{3}(2)_{2}}{(1)_{3} \cdot 2!}=12 .
$$

Such a design has been constructed in [14] as the projective image of an orbit of the binary icosahedral group that is a subgroup of $S U(2)$. Its representatives on the unit sphere $S^{3} \subset \mathbb{C}^{2}$ are

$$
a_{1}=\binom{1}{0}, a_{2}=\binom{0}{1}, a_{k}= \begin{cases}\mu\binom{\lambda \eta^{k-3}}{1} & 3 \leq k \leq 7 \\ \mu\binom{\eta^{k-3}}{-\lambda} & 8 \leq k \leq 12\end{cases}
$$

where

$$
\eta=\exp \frac{2 \pi i}{5}, \quad \lambda=2 \cos \frac{2 \pi}{5}=\frac{\sqrt{5}-1}{2}, \quad \mu=\frac{1}{2 \sin \frac{4 \pi}{5}}=\sqrt{\frac{5+\sqrt{5}}{10}}
$$

We omit an elementary calculation of the inner products $\left(x_{j}, x_{k}\right)=\left|a_{j}^{*} a_{k}\right|^{2}$, only noting that

$$
\lambda^{2}+\lambda-1=0, \quad(2-\lambda) \mu^{2}=1, \quad\left|\eta^{r}-1\right|^{2}=0,2-\lambda, 3+\lambda
$$

the latter for $r \equiv 0, \pm 1, \pm 2 \bmod 5$, accordingly. A calculation yields,

$$
\begin{equation*}
A(X)=\left\{0, \frac{5-\sqrt{5}}{10}, \frac{5+\sqrt{5}}{10}\right\} \tag{4.3}
\end{equation*}
$$

so $s=3, e=2, \epsilon=1$. The polynomial

$$
\xi R_{2}^{1}(\xi)=4 \xi P_{2}^{(1,1)}(2 \xi-1)=6 \xi\left(5 \xi^{2}-5 \xi+1\right)
$$

annihilates $A(X)$. By Theorem C of Section 2 our $X$ is indeed a tight 5-design.
The angle set (4.3) is not rational. This occurs because of the equality rk $L_{3}=$ rk $L_{1}$. In fact, rk $L_{0}=1, \operatorname{rk} L_{1}=3$, $\operatorname{rk} L_{2}=5$, rk $L_{3}=3$, by (3.15) and (3.16).

In the following corrected form of Theorem $2.6\left[9\right.$ the case $X \subset \mathbb{R} \mathbb{P}^{1}$ is also included for completeness. In this case, (3.18) makes it possible for $A(X)$ to be not rational, though $\operatorname{rk} L_{s} \neq \operatorname{rk} L_{1}$ under the conditions of Theorem4.1.

Theorem 4.2. Let $X$ be a tight t-design in $\mathbb{F P}^{n}$. Then the angle set $A(X)$ is rational, except for two cases: 1) $X \subset \mathbb{C P}^{1}, t=5$; 2) $X \subset \mathbb{R P}^{1}, t \neq 1,2,3,5$.

Proof. For $X \not \subset \mathbb{R P}^{1}$ the proof is the same as in 9 but with $L_{s}$ instead of $E_{s}$ when $t=2 s-1, s \geq 2$, and using our Theorem4.1 in this case.

Now let $X \subset \mathbb{R P}^{1}$. Then it is the projective image of a regular $(2 t+2)$-gon as easily follows from [12]. Therefore,

$$
A(X) \backslash\{0\}=\left\{\cos ^{2} \frac{k \pi}{t+1}\right\}_{1}^{e}=\left\{\frac{1}{2}\left(1+\cos \frac{2 k \pi}{t+1}\right)\right\}_{1}^{e}
$$

where $e=[t / 2]$. Since $\cos m \theta$ is a polynomial of $\cos \theta$ with integer coefficients, the set $A(X)$ is rational if and only if the number $\rho=\cos \frac{2 \pi}{t+1}$ is rational. Obviously, the latter is true if $t=1,2,3,5$. Conversely, let $\rho \in \mathbb{Q}$. Then the complex number $w=\exp (2 \pi i /(t+1))$ satisfies the equation $w^{2}-2 \rho w+1=0$. On the other hand, this is a primitive root of 1 of degree $t+1$. It is known that irreducible (over $\mathbb{Q})$ equation for $w$ is of degree $\varphi(t+1)$ where $\varphi$ is the Euler function. Hence, $\varphi(t+1) \leq 2$. If $\varphi(t+1)=1$ then $t=1$. If $\varphi(t+1)=2$ then $t \in\{2,3,5\}$ as easily follows from the classical formula

$$
\varphi\left(\prod_{i=1}^{r} q_{i}^{\nu_{i}}\right)=\prod_{i=1}^{r} q_{i}^{\nu_{i}-1}\left(q_{i}-1\right)
$$

where $q_{1}, \ldots, q_{r}$ are prime divisors of $t+1$.

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