ON TIGHT PROJECTIVE DESIGNS

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ABSTRACT. It is shown that among all tight designs in $\mathbb{FP}^n \neq \mathbb{RP}^1$, where \mathbb{F} is \mathbb{R} or \mathbb{C} , or \mathbb{H} (quaternions), only 5-designs in \mathbb{CP}^1 [14] have irrational angle set. This is the only case of equal ranks of the first and the last irreducible idempotent in the corresponding Bose-Mesner algebra.

Keywords: projective design, angle set, Bose-Mesner algebra.

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1. INTRODUCTION

A well known theorem of Bannai and Hoggar [3] states that there are no tight t-designs in $\mathbb{FP}^n \neq \mathbb{RP}^1$ if $t \geq 6$. Moreover, a theorem of Hoggar [10] states the same for $t \geq 4$ if $\mathbb{F} \neq \mathbb{R}$. Surprisingly, a tight 5-design in \mathbb{CP}^1 has been constructed in [14], so Hoggar's theorem has to be corrected. The results of [3] and [10] are essentially based on Theorem 2.6(c) [9] that states that the angle set of every tight t-design in $\mathbb{FP}^n \neq \mathbb{RP}^1$ is rational. But it is not rational for the 5-design constructed in [14].

In the present paper we investigate this contradiction and prove that the only cases where the angle set is not rational are

(1)
$$\mathbb{F} = \mathbb{C}, n = 1, t = 5$$
 and

(2) $\mathbb{F} = \mathbb{R}, n = 1, t \neq 1, 2, 3, 5.$

A fortiori, there are no complications in [3] where $t \ge 6$ by assumption.

Our principal observation is that if t = 2s - 1, $s \ge 2$ then the last irreducible idempotent L_s in the corresponding Bose-Mesner algebra is not E_s from the proof of Theorem 2.6(c) [9] (actually, from [18]). Nevertheless, $\operatorname{rk} L_s \neq \operatorname{rk} E_1$, except for our case (1). This "critical inequality" implies the rationality of the angle set, similarly to the argument in [9]. This material is concentrated in Section 4 of the present paper, while Sections 2 and 3 contain all the necessary background and preliminary analysis.

2. Projective t-designs

For the reader's convenience we basically use the same notation as in [8] and other related papers. Let us recall this notation. In particular, let

$$\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}; \quad m = \frac{1}{2}(\mathbb{F} : \mathbb{R}) = \begin{cases} 1/2 & \mathbb{F} = \mathbb{R} \\ 1 & \mathbb{F} = \mathbb{C} \\ 2 & \mathbb{F} = \mathbb{H} \end{cases}$$

The number 2N is nothing but the real (topological) dimension of the \mathbb{F} -linear space \mathbb{F}^{n+1} . The latter consists of all $(n+1) \times 1$ matrices (columns) over \mathbb{F} with the standard addition and multiplication by scalars $\tau \in \mathbb{F}$ from the right (for

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definiteness). As usual, the inner product of $a, b \in \mathbb{F}^{n+1}$ is a^*b where a^* is the row conjugate transpose to a. Accordingly, the set

$$S^{2N-1} = \{a : a^*a = 1\}$$

is the unit sphere in \mathbb{F}^{n+1} . A quotient set of the sphere with respect to the equivalence relation $a_1 \sim a_2 \iff a_1 = a_2\lambda$, $\lambda \in \mathbb{F}$, $|\lambda| = 1$, is the projective space \mathbb{FP}^n . The "inner product" $(\hat{a}, \hat{b}) = |a^*b|^2$ in \mathbb{FP}^n is well-defined through the natural mapping $a \mapsto \hat{a}$ from S^{2N-1} onto \mathbb{FP}^n . Obviously, $(\hat{b}, \hat{a}) = (\hat{a}, \hat{b})$ and $0 \leq (\hat{a}, \hat{b}) \leq 1$ with the equality $(\hat{a}, \hat{b}) = 1$ if and only if $\hat{a} = \hat{b}$. For every nonempty $X \subset \mathbb{FP}^n$ its angle set is

$$A(X) = \{(x, y) : x, y \in X, x \neq y\}$$

The related combinatorial parameters are

$$s = |A(X)|, \quad e = |A(X) \setminus \{0\}|, \quad \epsilon = s - e = |A(X) \cap \{0\}|.$$

Let $P_i^{(\alpha,\beta)}(\tau)$ be the Jacobi polynomials [20] such that

deg
$$P_i^{(\alpha,\beta)} = i$$
, $P_i^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_i}{i!}$ (2.1)

where

$$(\alpha + 1)_i = \prod_{l=1}^{i} (\alpha + l), \quad (\alpha + 1)_0 = 1.$$

In particular, $P_0^{(\alpha,\beta)}(\tau)\equiv 1.$ In what follows we fix

$$\alpha = N - m - 1, \quad \beta = m - 1$$
 (2.2)

and set

$$P_i(\xi) = P_i^{(\alpha,\beta)}(2\xi - 1),$$
 (2.3)

for short. A finite nonempty subset $X \subset \mathbb{FP}^n$ is called a *t*-design if

$$\sum_{x \in X} P_i((x, y)) = 0, \quad y \in X, \quad 1 \le i \le t.$$
(2.4)

Let X be a t-design and let

$$R_e^{\epsilon}(\xi) = \frac{(N)_s}{(m)_s} P_e^{(\alpha+1,\beta+\epsilon)} (2\xi-1).$$

In particular,

$$R_e^{\epsilon}(1) = \frac{(N)_s (N - m + 1)_e}{(m)_s e!}$$
(2.5)

The following theorems are fundamental, see [1], [2], [11]. (Cf. [6] for the spherical designs.)

Theorem A. The inequalities

$$t \le s + e, \quad |X| \ge R_e^{\epsilon}(1)$$

hold, and the equalities

$$t = s + e, \quad |X| = R_e^{\epsilon}(1)$$

are equivalent.

In the latter case the t-design X is called *tight*. Note that t = s + e is equivalent to e = [t/2], $\epsilon = \operatorname{res}_2(t)$.

Theorem B. If X is a tight t-design then A(X) coincides with the set of roots of the polynomial $\xi^{\epsilon} R_{\epsilon}^{e}(\xi)$.

Recall that these roots are simple and lie on (0, 1).

Theorem C. Let X be a subset of \mathbb{FP}^n such that $|X| = R_e^{\epsilon}(1)$ and A(X) coincides with the set of roots of $\xi^{\epsilon} R_e^{\epsilon}(\xi)$, then X is a tight $(2e + \epsilon)$ -design.

The projective t-designs can be characterized as the averaging sets in the sense of [19] for suitable spaces of functions on \mathbb{FP}^n . Usually, these spaces are described in terms of harmonic analysis but we prefer a more elementary approach [15], [16].

We say that a mapping $\phi : \mathbb{FP}^n \to \mathbb{C}$ is a *polynomial function* if it is of the form

$$\phi(\hat{a}) = \psi(a), \quad a \in S^{2N-1},$$

where ψ is a polynomial on \mathbb{F}^{n+1} in real coordinates. This ψ must be invariant with respect to the rotations of \mathbb{F} , i.e. $\psi(a\lambda) = \psi(a)$ for all $\lambda \in \mathbb{F}$, $|\lambda| = 1$. It is not unique but becomes unique if it is required to be homogeneous (which is always possible) of minimal degree. The latter is said to be the *degree* of ϕ . The number $\deg \phi$ is an even integer since $\psi(-a) = \psi(a)$.

Example 2.1. For every $t \in \mathbb{N}$ and every $y \in \mathbb{FP}^n$ the function $\phi_{2t;y}(x) = (x, y)^t$, $x \in \mathbb{FP}^n$, is a polynomial function of degree 2t.

Given $d \in 2\mathbb{N}$, we denote by $\operatorname{Pol}_{\mathbb{F}}(d)$ the space of all polynomial functions of degrees $\leq d$. It has been proven in [16] that the family $\{\phi_{d;y} : y \in \mathbb{FP}^n\}$ spans the whole space $\operatorname{Pol}_{\mathbb{F}}(d)$. We apply this result to prove the following

Proposition 2.2. A finite nonempty set $X \subset \mathbb{FP}^n$ is a tight t-design if and only if

$$\frac{1}{|X|} \sum_{x \in X} \phi(x) = \int_{S^{2N-1}} \tilde{\phi}(a) \, d\sigma(a), \quad \phi \in \operatorname{Pol}_{\mathbb{F}}(2t), \tag{2.6}$$

where $\tilde{\phi}$ is induced by the natural mapping $S^{2N-1} \to \mathbb{FP}^n$ and σ is the normalized Lebesgue measure.

Proof. The identity (2.6) is equivalent to

$$\frac{1}{|X|} \sum_{x \in X} F((x,y)) = \int_{S^{2N-1}} F(|a^*b|^2) \, d\sigma(a), \tag{2.7}$$

where $y = \hat{b}, b \in S^{2N-1}$, F runs over the space Π_t of all univariate polynomials of degrees $\leq t$. By a known integration formula (see [8], Theorem 2.11) one can rewrite (2.7) in the form

$$\frac{1}{|X|} \sum_{x \in X} F((x,y)) = \int_{-1}^{1} F\left(\frac{1+\tau}{2}\right) \Omega_{\alpha,\beta}(\tau) \, d\tau, \quad \psi \in \Pi_t, \tag{2.8}$$

where $\Omega_{\alpha,\beta}(\tau)$ is the normalized Jacobi weight, i.e.

$$\Omega_{\alpha,\beta}(\tau) = c_{\alpha,\beta}(1-\tau)^{\alpha}(1+\tau)^{\beta}, \quad -1 < \tau < 1,$$
(2.9)

with

$$c_{\alpha,\beta} = \left(\int_{-1}^{1} (1-\tau)^{\alpha} (1+\tau)^{\beta} d\tau\right)^{-1} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}.$$
 (2.10)

In turn, (2.8) is equivalent to its restriction to $F = P_i(\xi)$, $1 \le i \le t$, since these polynomials constitute a basis in Π_t . It remains to note that

$$\int_{-1}^{1} P_i\left(\frac{1+\tau}{2}\right) \Omega_{\alpha,\beta}(\tau) \, d\tau = \int_{-1}^{1} P_i^{(\alpha,\beta)}(\tau) \Omega_{\alpha,\beta}(\tau) \, d\tau = 0$$

by (2.3) and the $\Omega_{\alpha,\beta}$ -orthogonality of the system $\left\{P_i^{(\alpha,\beta)}: i \ge 0\right\}$.

Corollary 2.3. Let $X \subset \mathbb{FP}^n$ be a t-design. Then

$$\frac{1}{|X|} \sum_{x \in X} P((u,x))Q((x,v)) = \int_{S^{2N-1}} P(|a^*c|^2)Q(|c^*b|^2) \, d\sigma(c) \tag{2.11}$$

for $u = \hat{a}$, $v = \hat{b}$ and all univariate polynomials P, Q such that $\deg P + \deg Q \leq t$.

Proof. The mapping $x \mapsto P((u, x))Q((x, v)), x \in \mathbb{FP}^n$, is a polynomial function of degree $\leq 2t$.

Corollary 2.4. Let X, P, Q be fixed under the conditions of Corollary 2.3. Then the value

$$\sum_{x \in X} P((u, x))Q((x, v))$$

depends only on the inner product (u, v) of $u, v \in \mathbb{FP}^n$.

Proof. Let $(u_1, v_1) = (u, v)$, i.e. $|a_1^*b_1|^2 = |a^*b|^2$ where $\hat{a}_1 = u_1$, $\hat{b}_1 = v_1$. Without loss of generality one can assume that $a_1^*b_1 = a^*b$. Then there exists a $(n+1)\times(n+1)$ matrix T over \mathbb{F} such that $T^*T = \text{id}$ and $a_1 = Ta$, $b_1 = Tb$. This substitution in (2.11) is equivalent to the change of variable $c \mapsto T^*c$. The latter does not affect the integral since the measure σ is orthogonally invariant.

3. Bose-Mesner Algebra

Let X be a finite nonempty subset of \mathbb{FP}^n and let

$$A'(X) = A(X) \cup \{1\} = \{(x, y) : x, y \in X\},\$$

so that |A'(X)| = s + 1. The $X \times X$ matrices of the form

$$M_F = [F((x,y))]_{x,y \in X}$$
(3.1)

where F runs over all functions $A'(X) \to \mathbb{C}$, constitute a complex linear space $\mathcal{D}(X)$. Its natural basis consists of the matrices

$$\Delta_{\zeta} = \left[\delta_{\zeta,(x,y)} \right]_{x,y \in X}, \quad \zeta \in A'(X), \tag{3.2}$$

thus, dim $\mathcal{D}(X) = s + 1$. The Lagrange interpolation formula allows us to let F in (3.1) run over the polynomial space Π_s , so that we have the isomorphism $F \mapsto M_F$ between Π_s and $\mathcal{D}(X)$. In particular, if F|A(X) = 0 and F(1) = 1 then $M_F = I$, the unit matrix.

According to Corollary 2.4 for $P, Q \in \Pi_s$, the matrix product $M_P M_Q$ belongs to $\mathcal{D}(X)$ if deg $P + \deg Q \leq t$. However, this condition is not fulfilled if t = 2s - 1 and deg $P = \deg Q = s$. Moreover, Corollary 2.4 cannot be extended to this situation if X is tight. Indeed, suppose to the contrary that

$$\sum_{x \in X} (u, x)^s (x, v)^s = \Phi((u, v)) \quad (u, v \in \mathbb{FP}^n)$$

with a function $\Phi: [0,1] \to \mathbb{R}_+$. Setting v = u we obtain

$$\sum_{x \in X} (u, x)^{2s} = \Phi(1), \quad u \in \mathbb{FP}^n.$$

In other words,

$$\sum_{c \in \tilde{X}} |a^*c|^{4s} = \Phi(1), \quad a \in S^{2N-1},$$

where $\tilde{X} \subset S^{2N-1}$ is a complete system of representatives of points $x \in X$, $|\tilde{X}| = |X|$. By integration over a we obtain

$$\Phi(1) = \left(\int_{S^{2N-1}} |a^*c|^{4s} \, d\sigma(a)\right) \cdot |X|$$

since the integral does not depend on c. As a result,

$$\frac{1}{|X|} \sum_{x \in X} \phi_{4s;u}(x) = \int_{S^{2N-1}} \tilde{\phi}_{4s;u}(a) \, d\sigma(a), \tag{3.3}$$

and by linearity, (3.3) extends to the whole space $\operatorname{Pol}_{\mathbb{F}}(4s)$. Thus, X is a 2s-design which is a contradiction since 2s = t + 1.

Nevertheless, under the constraint $u, v \in X$, one can extend Corollary 2.4 to t = 2s - 1 and P, Q such that $\max(\deg P, \deg Q) = s$. This follows from the construction of a basis in $\mathcal{D}(X)$ using the Jacobi polynomials (cf. [6], Remark 7.6).

Lemma 3.1. Let X be a t-design in \mathbb{FP}^n and let $s = \lfloor \frac{t+1}{2} \rfloor$. Then s+1 matrices $M_i = M_{P_i}, 0 \le i \le s$, constitute a basis \mathcal{M} of $\mathcal{D}(X)$ such that

$$M_i M_k = |X| M_i \delta_{ik} \rho_{\mu(i,k)} \tag{3.4}$$

where $\mu(i, k) = \min(i, k)$ and all $\rho_j > 0$.

Proof. The matrices M_i are linearly independent because of the linear independence of the polynomials P_i . Since $|\mathcal{M}| = s + 1$, this is a basis of $\mathcal{D}(X)$. Now note that

$$\int_{S^{2N-1}} P_i(|a^*c|^2) P_k(|c^*b|^2) \, d\sigma(c) = 0, \quad i \neq k,$$

by the addition formula for polynomial functions [15] (cf. [7], [13], [17]). The same formula with i = k yields

$$\int_{S^{2N-1}} P_i(|a^*c|^2) P_i(|c^*b|^2) \, d\sigma(c) = \chi_i P_i(|a^*b|^2) \tag{3.5}$$

where $\chi_i > 0$. Assuming $\mu(i,k) \leq s-1$ (a fortiori, $i+k \leq 2s-1 \leq t$) and using Corollary 2.3 we get (3.4) with

$$\rho_j = \chi_j, \quad 0 \le j \le s - 1. \tag{3.6}$$

In particular, $M_i M_s = M_s M_i = 0$ for $0 \le i \le s - 1$. It remains to consider the case i = k = s.

If t is even the t = 2s and Corollary 2.3 is applicable to i = k = s, so $M_s^2 = |X|M_s\rho_s$ with

$$\rho_s = \chi_s. \tag{3.7}$$

Let t be odd, so t = 2s - 1. Then we decompose the unity matrix I for the basis \mathcal{M} ,

$$I = \sum_{i=0}^{s} \lambda_i M_i, \qquad (3.8)$$

and get $M_s = \lambda_s M_s^2$ multiplying (3.8) by M_s . This yields

$$\lambda_s = \frac{\operatorname{tr} M_s}{\operatorname{tr} M_s^2} = \frac{|X| P_s(1)}{\sum_{x,y} P_s^2((x,y))} > 0,$$
(3.9)

and then $M_s^2 = |X| M_s \rho_s$ with

$$\rho_s = (\lambda_s |X|)^{-1}. \tag{3.10}$$

Remark 3.2. The formulas (3.6) and (3.7) are joined in

$$o_i = \chi_i, \quad 0 \le i \le [t/2],$$
 (3.11)

while (3.10) appears only for t = 2s - 1 in addition to (3.11).

Remark 3.3. The multiplication table (3.4) shows that under conditions of Lemma 3.1 $\mathcal{D}(X)$ is a commutative matrix algebra, the Bose-Mesner algebra of X [4], [5], [6].

In what follows the conditions of Lemma 3.1 are assumed to be fulfilled. By setting

$$L_{i} = \frac{M_{i}}{\rho_{i}|X|} = \frac{1}{\rho_{i}|X|} \left[P_{i}((x,y))\right]_{x,y\in X}$$
(3.12)

the basis \mathcal{M} turns into $\mathcal{L} = \{L_i\}_0^s$ consisting of idempotents $(L_i^2 = L_i)$ which are pairwise orthogonal $(L_i L_k = 0 \text{ for } i \neq k)$. It is important to calculate their ranks. We have

$$\operatorname{rk} L_i = \operatorname{tr} L_i = \rho_i^{-1} P_i(1), \quad 0 \le i \le s,$$

hence,

$$\operatorname{rk} L_{i} = \chi_{i}^{-1} P_{i}(1) = \frac{P_{i}^{2}(1)}{\int_{S^{2N-1}} P_{i}^{2}(|a^{*}c|^{2}) \, d\sigma(c)}, \quad 0 \le i \le [t/2]$$

by (3.11) and (3.5) for a = b. Finally,

$$\operatorname{rk} L_{i} = \frac{\left(P_{i}^{(\alpha,\beta)}(1)\right)^{2}}{\int_{-1}^{1} \left(P_{i}^{(\alpha,\beta)}(\tau)\right)^{2} \Omega_{(\alpha,\beta)}(\tau) \, d\tau}, \quad 0 \le i \le [t/2].$$
(3.13)

In particular, $\operatorname{rk} L_0 = 1$. In addition to (3.13) we have to find $\operatorname{rk} L_s$ in the case t = 2s - 1. Formula (3.10) is not effective to this end since λ_s is unknown. Indeed, in (3.9) we cannot proceed to the formally corresponding integral in the denominator. Instead of this, we return to the decomposition of unity and express $\operatorname{rk} L_s$ through $\operatorname{rk} L_i, 0 \leq i \leq s - 1$. We have

$$I = \sum_{i=0}^{s} L_i,$$

whence,

$$\operatorname{rk} L_s = \operatorname{tr} L_s = |X| - \sum_{i=0}^{s-1} \operatorname{tr} L_i = |X| - \sum_{i=0}^{s-1} \operatorname{rk} L_i.$$

By substitution from (3.13) the last sum can be written as $c_{\alpha,\beta}^{-1}K_{s-1}^{(\alpha,\beta)}(1,1)$, where $K_{s-1}^{(\alpha,\beta)}(\cdot,\cdot)$ is the reproducing kernel of the Jacobi polynomials with respect to the weight $(1 - \tau)^{\alpha}(1 + \tau)^{\beta}$, see [20], Section 4.5. According to (2.10) and formula (4.5.8) from [20] we obtain

$$\operatorname{rk} L_s = |X| - \frac{\Gamma(s + \alpha + \beta + 1)\Gamma(s + \alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)\Gamma(\alpha + 2)\Gamma(s)}$$

With our α, β defined by (2.2)

$$\operatorname{rk} L_{s} = |X| - \frac{\Gamma(N+s-1)\Gamma(N-m+s)\Gamma(m)}{\Gamma(N)\Gamma(N-m+1)\Gamma(m+s-1)\Gamma(s)} = |X| - \frac{(N)_{s-1}(N-m+1)_{s-1}}{(m)_{s-1}(s-1)!}$$
(3.14)

Lemma 3.4. Let X be a tight t-design in \mathbb{FP}^n with t = 2s - 1. Then

$$\operatorname{rk} L_{s} = \frac{(N)_{s-1}(N-m)_{s}}{(m)_{s}(s-1)!}$$
(3.15)

Proof. In this case e = s - 1, $\epsilon = 1$, so (2.5) yields

$$|X| = R_{s-1}^{1}(1) = \frac{(N)_{s}(N-m+1)_{s-1}}{(m)_{s}(s-1)!}$$

The ranks of the other L_i (including L_s if t = 2s) can be explicitly calculated by (3.13), (2.10) and (2.1) combined with (4.33) of [20]. This results in

Lemma 3.5. Let X be a t-design in \mathbb{FP}^n with $s = \left\lfloor \frac{t+1}{2} \right\rfloor$. Then

$$\operatorname{rk} L_{i} = \frac{(N)_{i-1}(N-m)_{i}(N+2i-1)}{(m)_{i}i!}, \quad 0 \le i \le [t/2].$$
(3.16)

Remark 3.6. Formula (3.16) yields the true value $\operatorname{rk} L_0 = 1$ by setting $(\gamma - 1)(\gamma)_{-1} = 1$ for all γ .

Corollary 3.7. The inequality

$$\operatorname{k} L_i > \operatorname{rk} L_{i-1}, \quad 1 \le i \le [t/2],$$
(3.17)

holds, except for $X \subset \mathbb{FP}^1$. In the latter case

$$\operatorname{k} L_i = 2, \quad 1 \le i \le [t/2].$$
 (3.18)

Now note that our idempotents L_i coincide with the matrices E_i from [9] for $0 \le i \le [t/2]$ but $L_s \ne E_s$ if t = 2s - 1, $X \not\subset \mathbb{FP}^1$. Indeed, according to (2.5) from [9],

$$E_i((x,y)) = \frac{1}{|X|} [Q_i((x,y))]_{x,y \in X}, \quad 0 \le i \le s,$$
(3.19)

where $Q_i(\xi)$ is proportional to $P_i(\xi)$ and

$$Q_i(1) = \frac{(N)_{i-1}(N-m)_i(N+2i-1)}{(m)_i i!}, \quad i \ge 0.$$
(3.20)

Hence, E_i are proportional to L_i for all $i, 0 \leq i \leq s$. Moreover, if $0 \leq i \leq [t/2]$ then tr $E_i = Q_i(1) = \text{tr } L_i$ by (3.20) and (3.16). Hence, $E_i = L_i$ for $0 \leq i \leq [t/2]$. However, if t = 2s - 1 (so s = [t/2] + 1) and $X \not\subset \mathbb{FP}^1$ then tr $E_s = Q_s(1) > \text{tr } L_s$, see (3.15). In this case tr $E_s > \text{rk } E_s$, so E_s is not an idempotent. This is an obstacle to the full proof of Theorem 2.6 [9] of the rationality of A(X). To overcome this difficulty, it suffices to change E_s for L_s (when t = 2s - 1, $s \geq 2$) but then the "critical inequality" rk $L_s \neq \text{rk } L_1$ is needed. However, the latter is not always true. We clarify this intricate situation in the next section.

4. The critical inequality and rationality theorem

We prove the following

Theorem 4.1. With t = 2s - 1, $s \ge 2$, the inequality

$$\operatorname{rk} L_s \neq \operatorname{rk} L_1 \tag{4.1}$$

holds for every tight t-design $X \subset \mathbb{FP}^n$, except for a tight 5-design in \mathbb{CP}^1 .

Proof. From (3.15) it follows that

$$\operatorname{rk} L_{s} \geq \frac{(N)_{1}(N-m)_{2}}{(m)_{2} \cdot 1!} = \frac{N(N-m)(N-m+1)}{m(m+1)}$$

since the right side of (3.15) increases with s. On the other hand, (3.16) yields

$$\operatorname{rk} L_{1} = \frac{(N)_{0}(N-m)_{1}(N+1)}{(m)_{1} \cdot 1!} = \frac{(N-m)(N+1)}{m}$$
(4.2)

Hence,

$$\operatorname{rk} L_s - \operatorname{rk} L_1 \ge \frac{(N-m)((N-m)^2 - (m^2 + m + 1))}{m(m+1)}$$

Since N - m = mn we obtain $\operatorname{rk} L_s > \operatorname{rk} L_1$ for $n^2 > 1 + m^{-1} + m^{-2}$, i.e. for $n \ge 3$ if $\mathbb{F} = \mathbb{R}$ and for $n \ge 2$ if $\mathbb{F} = \mathbb{C}$ or \mathbb{H} . It remains to consider two cases.

(1) $\mathbb{F} = \mathbb{R}$, n = 2. Then m = 1/2, N = 3/2, so $\operatorname{rk} L_1 = 5$ by (4.2), while $\operatorname{rk} L_s = 2s$ by (3.15).

(2) \mathbb{F} is arbitrary, n = 1. Then N = 2m, hence

$$\operatorname{rk} L_{s} - \operatorname{rk} L_{1} = \frac{(2m)_{s-1}}{(s-1)!} - (2m+1) = \begin{cases} -1 & \mathbb{F} = \mathbb{R} \\ s-3 & \mathbb{F} = \mathbb{C} \\ \frac{1}{6}s(s+1)(s+2) - 5 & \mathbb{F} = \mathbb{H} \end{cases}$$

We see that $\operatorname{rk} L_s \neq \operatorname{rk} L_1$ with the only exception $\mathbb{F} = \mathbb{C}$, s = 3, so t = 5. \Box

A tight 5-design in \mathbb{CP}^1 should contain 12 points since its parameters are m = 1, $N = 2, s = 3, e = 2, \epsilon = 1$. Accordingly, (2.5) becomes

$$R_2^1(1) = \frac{(2)_3(2)_2}{(1)_3 \cdot 2!} = 12.$$

Such a design has been constructed in [14] as the projective image of an orbit of the binary icosahedral group that is a subgroup of SU(2). Its representatives on the unit sphere $S^3 \subset \mathbb{C}^2$ are

$$a_{1} = \begin{pmatrix} 1\\ 0 \end{pmatrix}, a_{2} = \begin{pmatrix} 0\\ 1 \end{pmatrix}, a_{k} = \begin{cases} \mu \begin{pmatrix} \lambda \eta^{k-3}\\ 1 \end{pmatrix} & 3 \le k \le 7, \\ \mu \begin{pmatrix} \eta^{k-3}\\ -\lambda \end{pmatrix} & 8 \le k \le 12, \end{cases}$$

where

$$\eta = \exp \frac{2\pi i}{5}, \quad \lambda = 2\cos \frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}, \quad \mu = \frac{1}{2\sin \frac{4\pi}{5}} = \sqrt{\frac{5+\sqrt{5}}{10}}.$$

We omit an elementary calculation of the inner products $(x_j, x_k) = |a_j^* a_k|^2$, only noting that

$$\lambda^2 + \lambda - 1 = 0, \quad (2 - \lambda)\mu^2 = 1, \quad |\eta^r - 1|^2 = 0, 2 - \lambda, 3 + \lambda,$$

the latter for $r \equiv 0, \pm 1, \pm 2 \mod 5$, accordingly. A calculation yields,

$$A(X) = \left\{ 0, \frac{5 - \sqrt{5}}{10}, \frac{5 + \sqrt{5}}{10} \right\},$$
(4.3)

so $s = 3, e = 2, \epsilon = 1$. The polynomial

$$\xi R_2^1(\xi) = 4\xi P_2^{(1,1)}(2\xi - 1) = 6\xi(5\xi^2 - 5\xi + 1)$$

annihilates A(X). By Theorem C of Section 2 our X is indeed a tight 5-design.

The angle set (4.3) is not rational. This occurs because of the equality $\operatorname{rk} L_3 = \operatorname{rk} L_1$. In fact, $\operatorname{rk} L_0 = 1$, $\operatorname{rk} L_1 = 3$, $\operatorname{rk} L_2 = 5$, $\operatorname{rk} L_3 = 3$, by (3.15) and (3.16).

In the following corrected form of Theorem 2.6 [9] the case $X \subset \mathbb{RP}^1$ is also included for completeness. In this case, (3.18) makes it possible for A(X) to be not rational, though $\operatorname{rk} L_s \neq \operatorname{rk} L_1$ under the conditions of Theorem 4.1.

Theorem 4.2. Let X be a tight t-design in \mathbb{FP}^n . Then the angle set A(X) is rational, except for two cases: 1) $X \subset \mathbb{CP}^1$, t = 5; 2) $X \subset \mathbb{RP}^1$, $t \neq 1, 2, 3, 5$.

Proof. For $X \not\subset \mathbb{RP}^1$ the proof is the same as in [9] but with L_s instead of E_s when $t = 2s - 1, s \ge 2$, and using our Theorem 4.1 in this case.

Now let $X \subset \mathbb{RP}^1$. Then it is the projective image of a regular (2t + 2)-gon as easily follows from [12]. Therefore,

$$A(X) \setminus \{0\} = \left\{\cos^2 \frac{k\pi}{t+1}\right\}_{1}^{e} = \left\{\frac{1}{2}\left(1 + \cos\frac{2k\pi}{t+1}\right)\right\}_{1}^{e}$$

where e = [t/2]. Since $\cos m\theta$ is a polynomial of $\cos \theta$ with integer coefficients, the set A(X) is rational if and only if the number $\rho = \cos \frac{2\pi}{t+1}$ is rational. Obviously, the latter is true if t = 1, 2, 3, 5. Conversely, let $\rho \in \mathbb{Q}$. Then the complex number $w = \exp(2\pi i/(t+1))$ satisfies the equation $w^2 - 2\rho w + 1 = 0$. On the other hand, this is a primitive root of 1 of degree t + 1. It is known that irreducible (over \mathbb{Q}) equation for w is of degree $\varphi(t+1)$ where φ is the Euler function. Hence, $\varphi(t+1) \leq 2$. If $\varphi(t+1) = 1$ then t = 1. If $\varphi(t+1) = 2$ then $t \in \{2, 3, 5\}$ as easily follows from the classical formula

$$\varphi\left(\prod_{i=1}^r q_i^{\nu_i}\right) = \prod_{i=1}^r q_i^{\nu_i - 1}(q_i - 1)$$

where q_1, \ldots, q_r are prime divisors of t + 1.

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