ON SOME OPEN PROBLEMS ON MAXIMAL CURVES

STEFANIA FANALI AND MASSIMO GIULIETTI

Abstract. In this paper we solve three open problems on maximal curves with Frobenius dimension 3. In particular, we prove the existence of a maximal curve with order sequence $(0, 1, 3, q)$.

1. Introduction

Let $\mathbb{F}_{q^2}$ be a finite field with q^2 elements where q is a power of a prime p. An $\mathbb{F}_{q^2}$-rational curve, that is a projective, geometrically absolutely irreducible, non-singular algebraic curve defined over $\mathbb{F}_{q^2}$, is called $\mathbb{F}_{q^2}$-maximal if the number of its $\mathbb{F}_{q^2}$-rational points attains the Hasse-Weil upper bound

$$q^2 + 1 + 2gq$$

where g is the genus of the curve. Maximal curves have interesting properties and have also been investigated for their applications in Coding theory. Surveys on maximal curves are found in [5, 6, 7, 18, 19] and [13, Chapter 10]; see also [3, 4, 8, 15, 17].

For an $\mathbb{F}_{q^2}$-maximal curve $\mathcal{X}$, the *Frobenius linear series* is the complete linear series $\mathcal{D} = |(q+1)P_0|$, where P_0 is any $\mathbb{F}_{q^2}$-rational point of $\mathcal{X}$. The projective dimension r of the Frobenius linear series, called the *Frobenius dimension* of $\mathcal{X}$, is one of the most important birational invariants of maximal curves. No maximal curve with Frobenius dimension 1 exists, whereas the Hermitian curve is the only maximal curve with Frobenius dimension 2. Maximal curves with higher Frobenius dimension have small genus, see Proposition 2.3.

In this paper, we deal with some open problems concerning maximal curves $\mathcal{X}$ with Frobenius dimension 3. For $P \in \mathcal{X}$ denote by $j_i(P)$ the i-th $(\mathcal{D}, P)$-order and by ϵ_i the i-th $\mathcal{D}$-order $(i = 0, \ldots, 3)$. For $i \neq 2$, the values of ϵ_i and $j_i(P)$ are known, see e.g. [13, Prop. 10.6]. More precisely, $\epsilon_0 = 0$, $\epsilon_1 = 1$ and $\epsilon_3 = q$; for an $\mathbb{F}_{q^2}$-rational point $P \in \mathcal{X}$, $j_0(P) = 0$, $j_1(P) = 1$, $j_3(P) = q+1$; for a non-$\mathbb{F}_{q^2}$-rational point P, $j_0(P) = 0$, $j_1(P) = 1$, $j_3(P) = q$.

S. Fanali and M. Giulietti are with the Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli 1, 06123, Perugia, Italy (e-mail: stefania.fanali@dipmat.unipg.it; giuliet@dipmat.unipg.it).

This research was performed within the activity of GNSAGA of the Italian INDAM..

In 1999, Cossidente, Korchmáros and Torres [2] proved that ϵ_2 is either 2 or 3, and that if the latter case holds then $p = 3$. They also showed that for an $\mathbb{F}_{q^2}$-rational point $P \in \mathcal{X}$ only a few possibilities for $j_2(P)$ can occur, namely

$$j_2(P) \in \left\{2, 3, q+1-\left\lfloor \frac{1}{2}(q+1) \right\rfloor, q+1-\left\lfloor \frac{2}{3}(q+1) \right\rfloor \right\}.$$

In [2] it was asked whether the following three cases actually occur for maximal curves with Frobenius dimension 3:

(A) $\epsilon_2 = 3$;
(B) $\epsilon_2 = 2$, $j_2(P) = 3$ for some $\mathbb{F}_{q^2}$-rational point P;
(C) $\epsilon_2 = 2$, $j_2(P) = q+1-\left\lfloor \frac{2}{3}(q+1) \right\rfloor$ for some $\mathbb{F}_{q^2}$-rational point P.

The main result of the paper is the proof that the recently discovered GK-curve [10] defined over $\mathbb{F}_{27^2}$ provides an affirmative answer to question (A), see Theorem 3.5. It is also shown that the curve of equation $Y^{16} = X(X+1)^6$ defined over $\mathbb{F}_{49}$ provides an affirmative answer to both questions (B) and (C), see Theorem 4.1. Finally, in Section 5 we construct an infinite family of maximal curves with $\mathcal{D}$-orders $(0, 1, 2, q)$ having an $\mathbb{F}_{q^2}$-rational point P with $j_2(P) = 3$, see Theorem 5.4.

It should be noted that in [1, Section 4] it is pointed out that due to some results by Homma and Hefez-Kakuta, an interesting geometrical property of a maximal curves $\mathcal{X}$ with Frobenius dimension 3 with $\epsilon_2 = 3$ is that of being a non-reflexive space curve of degree $q+1$ whose tangent surface is also non-reflexive.

The language of function fields will be used throughout the paper. The points of a maximal curve $\mathcal{X}$ will be then identified with the places of the function field $\mathbb{F}_{q^2}(\mathcal{X})$. Places of degree one correspond to $\mathbb{F}_{q^2}$-rational points.

2. Preliminaries

Throughout the paper, p is a prime number, $q = p^n$ is some power of p, $K = \mathbb{F}_{q^2}$ is the finite field with q^2 elements, F is a function field over K such that K is algebraically closed in F, $g(F)$ is the genus of F, $N(F)$ is the number of places of degree 1 of F, $\mathbb{P}(F)$ is the set of all places of F.

For a place P of degree 1, let $H(P)$ be the Weierstrass semigroup at P, that is, the set of non-negative integers i for which there exists $\alpha \in F$ such that the pole divisor $(\alpha)_\infty$ is equal to iP.

For a divisor D of F, let $\mathcal{L}(D)$ be the Riemann-Roch space of D, see e.g. [16, Def. 1.4.4]. The set of effective divisors $|D| = \{\alpha + D \mid \alpha \in \mathcal{L}(D)\}$ is the complete linear series associated to D. The degree n of $|D|$ is the degree of D, whereas the dimension r of $|D|$ is the dimension of the K-linear space $L(D)$ minus 1.

We recall some facts on orders of linear series, for which we refer to [13, Section 7.6]. For a place P of F, an integer j is a $(|D|, P)$-order if there exists a divisor E in $|D|$ with $v_P(E) = j$. There are exactly $r+1$ orders

$$j_0(P) < j_1(P) < \ldots < j_r(P),$$

and $(j_0(P), j_1(P), \ldots, j_r(P))$ is said to be the $(|D|, P)$-order sequence. For all but a finite number of places the $(|D|, P)$-order sequence is the same. Let $(\epsilon_0, \ldots, \epsilon_r)$ be the generic $(|D|, P)$-order sequence, called the $|D|$-order sequence. In general, $j_i(P) \geq \epsilon_i$. The so-called *p-adic criterion* (see e.g. [13, Lemma 7.62]) states that if $\epsilon < p$ is a $|D|-$order, then $0, 1, \ldots, \epsilon - 1$ are also $|D|$-orders.

Let F be a maximal function field, that is, $N(F) = q^2 + 1 + 2gq$. For a place P_0 of degree 1, let $\mathcal{D} = |(q+1)P_0|$ be the Frobenius linear series of F. By the so-called fundamental equation (see e.g. [13, Section 9.8]) the linear series $\mathcal{D}$ does not depend on the choice of P_0. The dimension r of $\mathcal{D}$ is the Frobenius dimension of F. Some facts on the Frobenius linear series of a maximal function field are collected in the following proposition (see [13, Prop. 10.6]).

Proposition 2.1. *Let $\mathcal{D}$ be the Frobenius linear series of a maximal function field F, and let $(\epsilon_0, \ldots, \epsilon_r)$ be the order sequence of $\mathcal{D}$. For a place P of degree 1, let*

$$H(P) = \{0 = m_0(P) < m_1(P) < m_2(P) < \ldots\}.$$

(a) $m_r(P) = q + 1$, $m_{r-1}(P) = q$.
(b) *The $(\mathcal{D}, P)$-orders at a place P of degree 1 are the terms of the sequence*

$$0 < 1 < q + 1 - m_{r-2}(P) < \ldots < q + 1 - m_1(P) < q + 1.$$

(c) $\epsilon_0 = 0$, $\epsilon_1 = 1, \epsilon_r = q$.

The only maximal function field with Frobenius dimension 2 is the Hermitian function field $H = K(x, y)$ with $y^{q+1} = x^q + x$, see e.g. [13, Remark 10.23]. Maximal function fields with Frobenius dimension 3 were investigated in [2]. Corollary 3.5 in [2] states that if $\epsilon_2 = 2$, then for any place P of degree 1

$$j_2(P) \in \left\{2, 3, q + 1 - \left\lfloor \frac{1}{2}(q+1) \right\rfloor, q + 1 - \left\lfloor \frac{2}{3}(q+1) \right\rfloor \right\}.$$

For each value of q there exists a unique maximal function field such that $j_2(P) = q + 1 - \left\lfloor \frac{1}{2}(q+1) \right\rfloor$ holds for some place P (see [2, Remark 3.6]). A number of examples for which $j_2(P) = 2$ occurs are known, see [13, Chapter 10]. So far, no example of a maximal function field with Frobenius dimension 3 having a place P of degree 1 with $j_2(P) \in \left\{3, q + 1 - \left\lfloor \frac{2}{3}(q+1) \right\rfloor \right\}$ appears to have been known in the literature (see [2, Remark 3.9], [1, Section 4]).

A result from [2] that will be useful in the sequel is the following.

Lemma 2.2. [2, Lemma 3.7] *If the Frobenius dimension of a maximal function field is* 3*, then there exists a place P of degree* 1 *with $j_2(P) = \epsilon_2$.*

Maximal function fields with higher Frobenius dimension have smaller genus, as stated in the next result.

Proposition 2.3. [13, Corollary 10.25] *The genus g of a maximal function field with Frobenius dimension r is such that*

$$g \leq \begin{cases} \frac{(2q-(r-1))^2-1}{8(r-1)} & \textit{if } r \textit{ is even,} \\ \\ \frac{(2q-(r-1))^2}{8(r-1)} & \textit{if } r \textit{ is odd.} \end{cases}$$

3. The $\mathcal{D}$-order sequence of the GK function field

Throughout this section, we assume that $q = \bar{q}^3$ with $\bar{q}$ a prime power. Let F be the function field $K(x, y)$, where $y^{\bar{q}+1} = x^{\bar{q}} + x$. Let $u = y\frac{x^{\bar{q}^2-1}-1}{x^{\bar{q}-1}+1}$, and consider the field extension $F(z)/F$ where $z^{\bar{q}^2-\bar{q}+1} = u$. The GK function field is

$$\bar{F} = F(z). \tag{3.1}$$

We first recall some proprieties of $\bar{F}$, for which we refer to [10, Section 2]. The function field $\bar{F}$ is a Kummer extension of F, and in particular $\bar{F}/F$ is Galois of degree $\bar{q}^2 - \bar{q} + 1$. The Galois group Γ of $\bar{F}/F$ consists of all the automorphisms g_u of $\bar{F}$ such that

$$g_u(x) = x, \qquad g_u(y) = y, \qquad g_u(z) = uz,$$

with $u^{\bar{q}^2-\bar{q}+1} = 1$.

The function field $\bar{F}$ is $\mathbb{F}_{q^2}$-maximal. Significantly, for $q > 8$, $\bar{F}$ is the only known function field that is maximal but not a subfield of the Hermitian function field (see [10, Theorem 5]). The genus of $\bar{F}$ is

$$g = \frac{1}{2}(\bar{q}^3 + 1)(\bar{q}^2 - 2) + 1.$$

Also, the only common pole of x, y and z is a place P_0 of degree 1 for which

$$\mathcal{L}((q + 1)P_0) =< 1, x, y, z > .$$

Therefore the Frobenius linear series $\mathcal{D}$ consists of divisors

$$\mathcal{D} = \{div(a_0+a_1x+a_2y+a_3z)+(q+1)P_0 \mid (a_0, a_1, a_2, a_3) \in K^4, (a_0, a_1, a_2, a_3) \neq (0, 0, 0, 0)\}.$$

Let P' be any place of degree 1 of $\bar{F}$. Let P be the place of F lying under P'. Then

$$\begin{cases} e(P' \mid P) = \bar{q}^2 - \bar{q} + 1, & \text{if } P \text{ is either a zero or a pole of } z \\ e(P' \mid P) = 1, & \text{otherwise} \end{cases} . \tag{3.2}$$

We now describe the $(\mathcal{D}, P')$-orders for a place P' of degree 1 of $\bar{F}$.

Proposition 3.1. [10, Section 4] *If P' is such that $e(P' \mid P) = \bar{q}^2 - \bar{q} + 1$, then the Weierstrass semigroup at P is the subgroup generated by $\bar{q}^3 - \bar{q}^2 + \bar{q}$, $\bar{q}^3$, and $\bar{q}^3 + 1$.*

¿From (b) of Proposition 2.1 the following corollary is obtained.

Corollary 3.2. *If P' is such that $e(P' \mid P) = \bar{q}^2 - \bar{q} + 1$, then*

$$(j_0(P'), j_1(P'), j_2(P'), j_3(P')) = (0, 1, \bar{q}^2 - \bar{q} + 1, \bar{q}^3 + 1)$$

Assume now that $e(P' \mid P) = 1$. As this occurs for an infinite number of places P, it is possible to choose P in such a way that there exists $ay + by + c \in F$ such that $v_P(ay + by + c) = \bar{q}$ (see e.g. [13, p. 302]). Then $v_{P'}(ax + by + c) = v_P(ax + by + c) = \bar{q}$ holds. Then by (b) of Proposition 2.1, $\bar{q}^3 - \bar{q} + 1 \in H(P')$. Taking into account that the automorphism group of $\bar{F}$ acts transitively on the set of places of degree 1 with $e(P' \mid P) = 1$ [10, Theorem 7], the following result is obtained.

Proposition 3.3. *If P' is such that $e(P' \mid P) = 1$, then*

$$(j_0(P'), j_1(P'), j_2(P'), j_3(P')) = (0, 1, \bar{q}, \bar{q}^3 + 1)$$

Theorem 3.4. *If q is a cube, then there exists a maximal function field with Frobenius dimension 3 and with $\mathcal{D}$-order sequence $(0, 1, \sqrt[3]{q}, q)$.*

Proof. We prove that the $\mathcal{D}$-order sequence of the GK function field is $(0, 1, \bar{q}, \bar{q}^3)$. By Lemma 2.2, there exists an $\mathbb{F}_{q^2}$-rational place P of $\bar{F}$ such that $j_2(P) = \epsilon_2$. Since the only possibilities for $j_2(P)$ are $\bar{q}$ and $\bar{q}^2 - \bar{q} + 1$, and since $j_2(P) \geq \epsilon_2$ for every $P \in \mathbb{P}(\bar{F})$, the claim follows. □

Therefore, the answer to question (A) in Introduction is obtained.

Theorem 3.5. *There exists a maximal curves over $\mathbb{F}_{27^2}$ with Frobenius dimension 3 and with $\mathcal{D}$-order sequence $(0, 1, 3, 27)$.*

4. On a maximal function field over $\mathbb{F}_{49}$

In [9, Example 6.3] it is shown that for every divisor m of $q^2 - 1$ the function field $F = K(x, y)$ with

$$y^{\frac{q^2-1}{m}} = x(x+1)^{q-1}$$

is a maximal function field with genus $g = \frac{1}{2m}(q + 1 - d)(q - 1)$, where $d = \gcd(m, q + 1)$. In this section we focus on the case $q = 7$ and $m = 3$, whence $d = 1$ and $g = 7$. We are going to prove the following result, which provides an affirmative answer to both questions (B) and (C) in Introduction.

Theorem 4.1. *Let $F = K(z, t)$ be the function field defined over $K = \mathbb{F}_{49}$ by the equation $z^{16} = t(t + 1)^6$. Then the Frobenius dimension of F is 3, the $\mathcal{D}$-order sequence of F is $(0, 1, 2, 7)$, and there exists an $\mathbb{F}_{49}$-rational place P of F such that $j_2(P) = 3 = 8 - \lfloor \frac{2}{3}(8) \rfloor$.*

The function field F is a subfield of the Hermitian function field $H = K(x, y)$ with $y^8 = x^7 + x$. More precisely, $F \cong K(x^6, y^3)$, and H/F is Galois of degree 3 (cf. [9, Example 6.3]). The Galois group of H/F is $\Gamma = \{1, \tau, \tau^2\}$, where $\tau(x) = a^8 x$, $\tau(y) = ay$, with a a primitive cubic root of unity.

Let P_0 (resp. P_∞) be the only zero (resp. pole) of x in H. Let $P_1, \ldots, P_6$ be the zeros of y in H distinct from P_0.

Lemma 4.2. *The only ramification points of H/F are P_0 and P_∞.*

Proof. It is easy to see that for each point P of H distinct from P_0 and P_∞ the stabilizer of P in Γ is trivial. On the other hand, both P_0 and P_∞ are fixed by Γ. □

Let $\bar{P}_0$ and $\bar{P}_\infty$ be the places of F lying under P_0 and P_∞, respectively. Let $\bar{P}_1$ and $\bar{P}_2$ be the two places of F lying under the places P_i of H, $i = 1, \ldots, 6$. Also, let $z = y^3$ and $t = x^6$ in F. Then

$$v_{\bar{P}_0}(z) = \frac{1}{3} v_{P_0}(y^3) = 1, \quad v_{\bar{P}_0}(t+1) = \frac{1}{3} v_{P_0}(x^6 + 1) = 0;$$
$$v_{\bar{P}_\infty}(z) = \frac{1}{3} v_{P_\infty}(y^3) = -7, \quad v_{\bar{P}_\infty}(t+1) = \frac{1}{3} v_{P_\infty}(x^6 + 1) = \frac{6}{3} ord_{P_\infty}(x) = -16;$$
$$\text{for } \ i = 1, 2, \ v_{\bar{P}_i}(z) = v_{P_i}(y^3) = 3, \quad v_{\bar{P}_i}(t+1) = v_{P_i}(x^6 + 1) = 7.$$

To sum up,

$$(z) = 3(\bar{P}_1 + \bar{P}_2) + \bar{P}_0 - 7\bar{P}_\infty, \qquad (t+1) = 8(\bar{P}_1 + \bar{P}_2) - 16\bar{P}_\infty.$$

Proposition 4.3. *Let i, j be non-negative integers such that $3i \geq 8j$. Then $7i - 16j \in H(\bar{P}_\infty)$.*

Proof. Let $\gamma = z^i (t+1)^{-j}$. Then

$$(\gamma) = 3i(\bar{P}_1 + \bar{P}_2) + \bar{P}_0 - 7i\bar{P}_\infty - 8j(\bar{P}_1 + \bar{P}_2) + 16j\bar{P}_\infty,$$

whence

$$(\gamma)_\infty = (7i - 16j)\bar{P}_\infty.$$

□

Corollary 4.4. *The only non-gaps at $\bar{P}_\infty$ that are less than or equal to 8 are $0, 5, 7, 8$.*

Proof. The integers 7 and 8 are non-gaps since F is an $\mathbb{F}_{49}$-maximal function field (see Proposition 2.1). Proposition 4.3 for $i = 3$ and $j = 1$ implies that 5 is a non-gap at $\bar{P}_\infty$. Then it is easy to see that $10, 12, 13$ are non-gaps as well. Therefore, we have 7 non-gaps less than $2g = 14$. Since $g = 7$, this rules out the possibility that there is another positive non-gap less than 7 and distinct from 5. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1 The Frobenius dimension of F is 3 by Corollary 4.4. By the p-adic criterion (see Section 2), the $\mathcal{D}$-order sequence is $(0,1,2,7)$. Again by Corollary 4.4 we have $j_2(\bar{P}_\infty)=3$.

5. An $\mathbb{F}_{q^2}$-maximal function field of genus $\frac{q^2-q+4}{6}$

Througouth this section we assume that that $q \equiv 2 \pmod 3$. We recall some facts about the function field F over K, defined by

$$F = K(x,y) \qquad \text{with } y^{\frac{q+1}{3}} + x^{\frac{q+1}{3}} + 1 = 0.$$

Clearly F is a subfield of the hermitian function field H over K defined by $H = K(z,t)$ with $z^{q+1}+t^{q+1}+1=0$, and therefore F is a maximal function field. Since the equation $Y^{\frac{q+1}{3}} + X^{\frac{q+1}{3}} + 1 = 0$ defines a non-singular plane algebraic curve of degree $\frac{q+1}{3}$, the genus $g(F) = \frac{1}{2}\left(\frac{q+1}{3}-1\right)\left(\frac{q+1}{3}-2\right)$, and therefore $N(F) = q^2+1+q\left(\frac{q+1}{3}-1\right)\left(\frac{q+1}{3}-2\right)$.

It is straightforward to check that the zeros of x are $\frac{q+1}{3}$ distinct places of degree 1. The same holds for y. The pole set of x coincides with the pole set of y, and consists of $\frac{q+1}{3}$ places of degree 1.

For $\alpha,\beta \in \mathbb{F}_{q^2}$ such that $\alpha^{\frac{q+1}{3}} + \beta^{\frac{q+1}{3}} + 1 = 0$, let $P_{\alpha,\beta} \in \mathbb{P}(F)$ denote the common zero of $x-\alpha$ and $y-\beta$. Let $P_{\infty,1},\ldots,P_{\infty,\frac{q+1}{3}}$ be the poles of x (and y). Clearly, $P_{\alpha,\beta}$ is a place of degree 1. Also, for any $\beta \in \mathbb{F}_{q^2}$ with $\beta^{\frac{q+1}{3}}+1=0$, the zero divisor $(y-\beta)_0$ of $y-\beta$ is equal to $\frac{q+1}{3}P_{0,\beta}$.

Henceforth, w is an element in $\mathbb{F}_{q^2}$ such that $w^{\frac{q+1}{3}} = 3$. Let

$$u = wxy \in F.$$

For any place P of F which is a zero of either x or y, $v_P(u)=1$ holds. Moreover, for any common pole P of x and y we have $v_P(u)=-2$. Any other place of F is neither a pole or a zero of u.

Consider the field extension $F(z)/F$ where $z^3=u$. Let

$$\bar{F} = F(z). \tag{5.1}$$

Clearly, u is not a 3-rd power of an element in F. Then $\bar{F}$ is a Kummer extension of F (see [16, Proposition III.7.3]), and in particular $\bar{F}/F$ is Galois of degree 3. The ramification index $e(P' \mid P)$ can be easily computed for any place P' of $\bar{F}$ lying over a place P of F: as $gcd(2,3)=1$, (b) of [16, Proposition III.7.3] gives

$$\begin{cases} e(P' \mid P) = 3, & \text{if } P \text{ is either a zero or a pole of } xy,\\ e(P' \mid P) = 1, & \text{otherwise.} \end{cases} \tag{5.2}$$

By [16, Corollary III.7.4],

$$g(\bar{F}) = 1 + 3(g(F) - 1) + 3\frac{q+1}{3} = \frac{q^2 - q + 4}{6}. \tag{5.3}$$

Now we compute the number N of places of degree 1 of $\bar{F}$. Any place in $\mathbb{P}(\bar{F})$ of degree 1 either lies over some $P_{\infty,i}$, or some $P_{\alpha,\beta}$. By (5.2), any place lying over either $P_{\infty,i}$ or $P_{\alpha,\beta}$ with $\alpha\beta = 0$ is fully ramified. This gives $q+1$ places of degree 1 of $\bar{F}$.

Assume now that $\alpha\beta \neq 0$. Let

$$\varphi_{\alpha,\beta}(T) = T^3 - w\alpha\beta \in \mathbb{F}_{q^2}[T].$$

As $gcd(3,p) = 1$, $\varphi_{\alpha,\beta}(T)$ has 3 distinct roots in the algebraic closure of $\mathbb{F}_{q^2}$. Let λ be any of such roots. Then $\lambda \in \mathbb{F}_{q^2}$ if and only if

$$1 = \lambda^{q^2-1} = (\lambda^{q+1})^{q-1} = \left((w\alpha\beta)^{\frac{q+1}{3}}\right)^{q-1} = \left(3(\alpha\beta)^{\frac{q+1}{3}}\right)^{q-1}, \tag{5.4}$$

that is $3(\alpha\beta)^{\frac{q+1}{3}} \in \mathbb{F}_q$. Taking into account the classical relation

$$(A+B+C) \mid A^3 + B^3 + C^3 - 3ABC,$$

we have that $\alpha^{\frac{q+1}{3}} + \beta^{\frac{q+1}{3}} + 1 = 0$ yields

$$3(\alpha\beta)^{\frac{q+1}{3}} = \alpha^{q+1} + \beta^{q+1} + 1.$$

Then (5.4) follows since $(\alpha^{q+1} + \beta^{q+1} + 1)^q = (\alpha^{q+1} + \beta^{q+1} + 1)$.

By [16, Proposition III.7.3], the minimal polynomial of z over F is $\varphi(T) = T^3 - wxy$. As $wxy \in \mathcal{O}_{P_{\alpha,\beta}}$, Kummer's Theorem [16, Theorem III.3.7] applies, and hence $P_{\alpha,\beta}$ has 3 distinct extensions $P \in \mathbb{P}(\bar{F})$ with $\deg(P) = 1$.

Since F is maximal, the number of pairs (α,β) with $\alpha\beta \neq 0$ and $\alpha^{\frac{q+1}{3}} + \beta^{\frac{q+1}{3}} + 1 = 0$ is

$$q^2 + 1 + q\left(\frac{q+1}{3} - 1\right)\left(\frac{q+1}{3} - 2\right) - (q+1)$$

Therefore, the total number N of places of degree 1 of $\bar{F}$ is

$$N = q + 1 + 3\left(q^2 - q + q\left(\frac{q+1}{3} - 1\right)\left(\frac{q+1}{3} - 2\right)\right)$$

By straightforward computation

$$N = q^2 + 1 + 2q\frac{q^2 - q + 4}{6},$$

whence the following result is obtained.

Theorem 5.1. *$\bar{F}$ is an $\mathbb{F}_{q^2}$-maximal function field.*

Proposition 5.2. *The Frobenius dimension of $\bar{F}$ is equal to* 3.

Proof. The assertion follows from Proposition 2.3. □

Remark 5.3. In [14] $\mathbb{F}_{q^2}$-maximal function fields with Frobenius dimension 3 and genus $\frac{q^2-q+4}{6}$. We are not able to tell whether they are isomorphic to $\bar{F}$ or not.

Fix $\beta \in K$ with $\beta^{\frac{q+1}{3}} = 1$, and let $P = P_{0,\beta} \in \mathbb{P}(F)$. Let $\bar{P}$ be the place of $\bar{F}$ lying over P. The pole divisor of $\frac{x}{y-\beta}$ in F is $\frac{q-2}{3}P$. Whence $\bar{P}$ is the only pole of $\frac{x}{y-\beta}$ in $\bar{F}$, and

$$v_{\bar{P}}\left(\frac{x}{y-\beta}\right) = -(q-2).$$

This means that $j_2(\bar{P}) = 3$.

Taking into account that by the p-adic criterion the third $\mathcal{D}$-order must be equal to 2, the following result is arrived at.

Theorem 5.4. *Let q be odd, $q \equiv 2 \pmod 3$. Then $\bar{F}$ is an $\mathbb{F}_{q^2}$-maximal function field with Frobenius dimension 3 such that*

$$(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) = (0, 1, 2, q),$$

and having an $\mathbb{F}_{q^2}$-rational point P with

$$j_0(P_\infty) = 0, \;\; j_1(P_\infty) = 1, \;\; j_2(P_\infty) = 3, \;\; j_3(P_\infty) = q+1.$$

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