# Self-dual Permutation Codes of Finite Groups in Semisimple Case * 

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#### Abstract

The existence and construction of self-dual codes in a permutation module of a finite group for the semisimple case are described from two aspects, one is from the point of view of the composition factors which are self-dual modules, the other one is from the point of view of the Galois group of the coefficient field.


Key words. Finite group, permutation code, self-dual module, selfdual code.

## 1 Introduction

Let $F$ be a finite field of order $q$ which is a power of a prime integer; and let $X$ be a finite set. By $F X$ we denote the $F$-vector space with the basis $X$ and with the usual scalar product as its standard inner product. Any subspace $C$ of $F X$ is just the usual linear code over $F$. In coding-theoretic notation, with respect to the standard inner product, the orthogonal subspace $C^{\perp}$ of a linear code $C$ is called the dual code of $C$; and $C$ is called a self-orthogonal code if $C \subseteq C^{\perp}$; and $C$ is called a self-dual code if $C=C^{\perp}$.

If $X$ is a group, then $F X$ is an algebra with multiplication induced by the multiplication of the group $X$, which is called the group algebra of the group $X$ over $F$; and any left ideal $C$ of $F X$ is said to be a group code. It is an interesting question to find conditions such that a group algebra has a self-dual group codes. More generally, this question can be extended to the group algebras over finite rings.

In 9, finite abelian groups are considered and some results on the nonexistence of self-dual group codes are shown. For the direct product of a finite 2 -group and a finite $2^{\prime}$-group, reference [4] showed when the self-dual group codes do not exist. Using the representation theory of finite groups, for group algebras over finite Galois rings reference [11] gave a complete answer for this

[^0]question. In particular, it is an easy conclusion that there is no self-dual code for finite groups of odd order.

Thus it is reasonable to consider the self-dual extended group codes for finite groups of odd order. And [7] obtained some interesting conditions for the existence of such self-dual codes in characteristic 2: one is from the point of view of self-dual modules, another one is an elementary number-theoretical condition; and 77 also showed some constructions of such codes.

Extending group codes, 3 discussed the so-called permutation codes of finite groups. If $G$ is a finite group and $X$ is a finite $G$-set, then $F X$ is called a permutation $F G$-module, which has the standard inner product with respect to the basis $X$; any $F G$-submodule $C$ of $F X$ is said to be an $F G$-permutation code. If $X$ is a transitive $G$-set, the permutation cades of $F X$ is called transitive permutation codes. View the base set of the group $G$ as a left regular $G$-set, then the group codes are just the permutation codes of $F G$. Some important codes are permutation codes in natural ways, but may not be group codes; e.g. the so-called multiple-cyclic codes; see [3] for details. Moreover, the research of permutation codes is of interests from the point of view of automorphism groups of linear codes, for: any permutation automorphism of a linear code is just a permutation of the standard basis of the linear code. In [3] some conditions are obtained for the non-existence of the self-dual transitive permutation codes of finite groups. And it is also an easy conclusion that there is no self-dual transitive permutation code for finite groups of odd order.

In this paper we discuss the existence and construction of self-dual permutation codes for the semisimple case. The outline is as follows.

Throughout the paper, $F$ denotes a finite field of order $q$, and $G$ denotes a finite group of order coprime to $q$, and any $F G$-module is finite-dimensional.

In $\S 2$, we first make observations on the related module-theoretical aspects, and then turn to the permutation codes. Since $F G$ is a semisimple algebra (Maschke's theorem), any $F G$-module $V$ is decomposed into a direct sum of irreducible $F G$-modules with the collection of the irreducible summands is unique determined up to isomorphism; any irreducible $F G$-module $W$ which appears in the direct sum is called a composition factor of $V$, and the number of the direct summands which are isomorphic to $W$ is called the multiplicity of $W$ in $V$. The dual space $V^{*}:=\operatorname{Hom}_{F}(V, F)$ consisting of all the linear form of $V$ is an $F G$-module with $G$-action: $(g \varphi)(v)=\varphi\left(g^{-1} v\right), \forall g \in G, \varphi \in \operatorname{Hom}_{F}(V, F)$, $v \in V$. We call $V$ a self-dual $F G$-module if $V \cong V^{*}$. So, "self-dual module" and "self-dual code" are different concepts. After the module-theoretical results which we need are obtained, we turn to coding-theoretical notation, and show that, for even $q$ and odd $|G|$, an $F G$-permutation module $F X$ has self-dual permutation codes if and only if any self-dual composition factor of the $F G$-module $F X$ has even multiplicity. For odd $q$, only a sufficient condition is obtained.

In $\S 3$, we discuss transitive permutation codes, i.e. codes of an permutation module $F X$ with a transitive $G$-set $X$. We first reduce the existence of the so-called self-dual extended transitive permutation codes to the existence of such transitive permutation codes $C$ of $F X$ that $C^{\perp}=C \oplus F$. And we show that, for a transitive $G$-set $X$ with length $n=|X|$, if the integer $q$ as an element of the multiplicative group $\mathbb{Z}_{n}^{\times}$has odd order, then there is a permutation code $C$ of $F X$ such that $C^{\perp}=C \oplus F$. It is easy to see that this elementary numbertheoretical condition is similar to that in [7. However, the situation of transitive
permutation codes is more delicate than that of group codes, so that we take a way different from [7] to treat our cases; and we obtained no necessary and sufficient conditions, though some more results are shown in $\S 3$ which seem interesting.

## 2 Self-dual modules and self-dual codes

We adopt the usual notation about linear forms, bilinear forms etc. from the usual linear algebra. A bilinear form $f(-,-)$ on an $F G$-module $V$ is said to be $G$-invariant if

$$
f(g(u), g(v))=f(u, v), \quad \forall u, v \in V
$$

Let $V$ be an $F G$-module with a $G$-invariant non-degenerate bilinear form $\langle-,-\rangle$. Let $U, W$ be submodules of $V$. Denote

$$
\begin{aligned}
& \operatorname{Ann}_{W}^{l}(U)=\{w \in W \mid\langle w, u\rangle=0, \forall u \in U\} \\
& \operatorname{Ann}_{W}^{r}(U)=\{w \in W \mid\langle u, w\rangle=0, \forall u \in U\}
\end{aligned}
$$

in particular, denote $U^{\perp}=\operatorname{Ann}_{V}^{r}(U)$ and ${ }^{\perp} U=\operatorname{Ann}_{V}^{l}(U)$. From the $G$ invariancy of $\langle-,-\rangle$, it is easy to see that $\mathrm{Ann}_{W}^{l}(U)$ and $\mathrm{Ann}_{W}^{r}(U)$ are $F G$ submodules. Note that $\operatorname{Ann}_{W}^{l}(U)=\operatorname{Ann}_{W}^{r}(U)$ and ${ }^{\perp} U=U^{\perp}$ once $\langle-,-\rangle$ is symmetric. For any $v_{0} \in V$ we have the linear form $\left\langle-, v_{0}\right\rangle: V \rightarrow F, v \mapsto\left\langle v, v_{0}\right\rangle ;$ and restricting it to $U$, we have the linear form $\left.\left\langle-, v_{0}\right\rangle\right|_{U}$ on $U$ and it is easy to check that

$$
\begin{equation*}
V \longrightarrow U^{*},\left.\quad v_{0} \longmapsto\left\langle-, v_{0}\right\rangle\right|_{U} \tag{1}
\end{equation*}
$$

is a surjective $F G$-homomorphism with kernel $U^{\perp}$; thus we have an exact sequence of $F G$-homomorphisms:

$$
\begin{equation*}
0 \longrightarrow U^{\perp} \longrightarrow V \longrightarrow U^{*} \longrightarrow 0 \tag{2}
\end{equation*}
$$

in particular, $\operatorname{dim} V=\operatorname{dim} U+\operatorname{dim} U^{\perp}$ because $\operatorname{dim} U=\operatorname{dim} U^{*}$. Restricting the bilinear form $\langle-,-\rangle$ to the $F G$-submodule $U$, we get a $G$-invariant symmetric bilinear form on $U$. If the restricted bilinear form on $U$ is non-degenerate (equivalently, $\operatorname{Ann}_{U}^{r}(U)=U \cap U^{\perp}=0$ ), we say that $U$ is a non-degenerate submodule. On the other hand, if the restricted bilinear form on $U$ is zero (equivalently, $U \subseteq U^{\perp}$ ), we say, in module-theoretical notation, that $U$ is an isotropic submodule.

Recall that any $F G$-module $V$ is written into a direct sum of irreducible modules, and the irreducible direct summands are partitioned by isomorphism, hence $V=V_{1} \oplus \cdots \oplus V_{h}$, with every $V_{i}$ consisting of the irreducible direct summands which are isomorphic to one and the same irreducible module $W_{i}$, but $V_{i}$ and $V_{j}$ for $i \neq j$ have no composition factors in common; thus $V_{i} \cong m_{i} W_{i}$ with $m_{i}$ being the multiplicity of $W_{i}$ in $V$, and $V_{i}$ is called the homogeneous component of $V$ associated with the irreducible module $W_{i}$, and $V=V_{1} \oplus \cdots \oplus V_{h}$ is called the canonical decomposition (or homogeneous decomposition) of $V$, see [10, §2.6]; the canonical decomposition of $V$ is unique, so that for any submodule $U$ of $V$ we have

$$
\begin{equation*}
U=\left(U \cap V_{1}\right) \oplus \cdots \oplus\left(U \cap V_{h}\right) \tag{3}
\end{equation*}
$$

Lemma 1. Let $V$ be an $F G$-module with a $G$-invariant non-degenerate bilinear form; and $U$ be an $F G$-submodule.
(1) If $U$ is non-degenerate then $U$ is an self-dual $F G$-module.
(2) If $U$ is irreducible, then $U$ is either non-degenerate or isotropic.
(3) If $U$ is a homogeneous component associated with an irreducible module $W$, then $W$ is self-dual if and only if $U$ is non-degenerate. $W$ is not self-dual if and only if $U$ is isotropic.

Proof. (1). The non-degeneracy of $U$ implies $U \cap U^{\perp}=0$; thus from that $\operatorname{dim} V=\operatorname{dim} U+\operatorname{dim} U^{\perp}$ we get $V=U^{\perp} \oplus U$, and it follows from the exact sequence (2) that $U \cong V / U^{\perp} \cong U^{*}$.
(2). Because $U \cap U^{\perp}$ is an $F G$-submodule of $U$, the irreducibility of $U$ implies that either $U \cap U^{\perp}=0$ or $U \cap U^{\perp}=U$.
(3). From the exact sequence (2) and the semi-simplicity, we have that $V=U^{\perp} \oplus U^{\prime}$ with $U^{\prime} \cong U^{*}$. Since $F G$ is an Frobenius algebra, it is known (e.g. see [12]) that the dual modules of all the composition factors of $U$ are just all the composition factors of $U^{*}$. Thus $U^{\prime}$ is a homogeneous component too. Thus the conclusions follows from the uniqueness of the homogeneous decomposition.

Remark. It is well-known that "there is a $G$-invariant non-degenerate bilinear form on a $F G$-module $V$ if and only if $V$ is a self-dual $F G$-module". The necessity is a special case of Lemma 1(1); and the sufficiency follows that, with an $F G$-isomorphism $\alpha: V \rightarrow V^{*}$, the composition map

$$
\begin{array}{ccccc}
V \times V & \longrightarrow & V^{*} \times V & \longrightarrow & F, \\
\left(v, v^{\prime}\right) & \longmapsto & \left(\alpha(v), v^{\prime}\right) & \longmapsto & \alpha(v)\left(v^{\prime}\right) .
\end{array}
$$

is a $G$-invariant non-degenerate bilinear form on $V$. For more details, please see [6, Ch.VII, §8].

Lemma 2. Let $V$ be an $F G$-module with a $G$-invariant non-degenerate symmetric bilinear form; let $U$ be an isotropic $F G$-submodule of $V$. Then the following are equivalent:
(i) $U^{\perp}=U$;
(ii) $\operatorname{dim} U=\operatorname{dim} V / 2$;
(iii) the collection of the composition factors of $U$ and the dual modules of the composition factors of $U$ is the collection of the composition factors of $V$.

Proof. (i) $\Leftrightarrow$ (ii) is obvious since $\operatorname{dim} V=\operatorname{dim} U^{\perp}+\operatorname{dim} U$.
(i) $\Leftrightarrow$ (iii). Similar to the proof for Lemma $1(3), V=U^{\perp} \oplus U^{\prime}$ with $U^{\prime} \cong U^{*}$; but now $U \subseteq U^{\perp}$ by hypothesis, so the equivalence is obvious.

Recall from the usual linear algebra that, for an $F G$-module $V$, any bilinear form $f$ on $V$ corresponds to exactly one linear form $\bar{f}$ on the tensor product space $V \otimes_{F} V: \bar{f}\left(v \otimes v^{\prime}\right)=f\left(v, v^{\prime}\right)$; in other words, the dual space $\left(V \otimes_{F} V\right)^{*}$ is identified with the space of all the bilinear forms on $V$. As usually, $V \otimes_{F} V$ is an $F G$-module by diagonal action of $G$, hence $\left(V \otimes_{F} V\right)^{*}$ is also an $F G$-module by diagonal action of $G$; and the space of all the $G$-invariant bilinear forms is identified with the subspace of all the $G$-fixed points of $\left(V \otimes_{F} V\right)^{*}$, denoted by $\left(\left(V \otimes_{F} V\right)^{*}\right)^{G}$.

On the other hand, $G$ acts on the space $\operatorname{Hom}_{F}(V, V)$ of all the linear transformations of $V$ in the following way:

$$
(g \alpha)(v)=g\left(\alpha\left(g^{-1} v\right), \quad \forall g \in G, \alpha \in \operatorname{Hom}(V, V), v \in V ;\right.
$$

and the subspace $\operatorname{Hom}_{F G}(V, V)$ of all the $F G$-endomorphisms of $V$ is just the set of all the $G$-fixed points of $\operatorname{Hom}_{F}(V, V)$.

Lemma 3. Let $V$ be an $F G$-module with a $G$-invariant non-degenerate symmetric bilinear form $\langle-,-\rangle$. For any linear transformation $\alpha \in \operatorname{Hom}_{F}(V, V)$ define

$$
\varphi_{\alpha}(u, v)=\langle\alpha(u), v\rangle, \quad \forall u, v \in V
$$

Then $\varphi_{\alpha}$ is a bilinear form on $V$, and

$$
\varphi: \quad \operatorname{Hom}_{F}(V, V) \longrightarrow\left(V \otimes_{F} V\right)^{*}, \quad \alpha \longmapsto \varphi_{\alpha}
$$

is an FG-isomorphism, and:
(1) $\varphi_{\alpha}$ is $G$-invariant if and only if $\alpha$ is an $F G$-endomorphism;
(2) $\varphi_{\alpha}$ is non-degenerate if and only if $\alpha$ is a non-degenerate transformation;
(3) $\varphi_{\alpha}$ is a symmetric if and only if $\alpha$ is a symmetric transformation.

Proof. It is easy to check that $\varphi_{\alpha}$ is a bilinear form on $V$, and that $\varphi$ is a linear map; and that $\varphi$ is injective because $\langle-,-\rangle$ is non-degenerate, hence $\varphi$ is bijective since $\operatorname{dim} \operatorname{Hom}_{F}(V, V)=\operatorname{dim}\left(V \otimes_{F} V\right)^{*}$. Next, for any $g \in G$, any $\alpha \in \operatorname{Hom}_{F}(V, V)$, and any $u, v \in V$, we have

$$
\begin{aligned}
\varphi_{g \alpha}(u \otimes v) & =\langle(g \alpha)(u), v\rangle=\left\langle g \alpha\left(g^{-1} u\right), v\right\rangle=\left\langle\alpha\left(g^{-1} u\right), g^{-1} v\right\rangle \\
& =\varphi_{\alpha}\left(g^{-1} u \otimes g^{-1} v\right)=\varphi_{\alpha}\left(g^{-1}(u \otimes v)\right)=\left(g \varphi_{\alpha}\right)(u \otimes v) .
\end{aligned}
$$

So $\varphi$ is an $F G$-isomorphism. Hence we have the following isomorphism

$$
\begin{equation*}
\operatorname{Hom}_{F G}(V, V) \stackrel{\cong}{\leftrightarrows}\left(\left(V \otimes_{F} V\right)^{*}\right)^{G}, \quad \alpha \longmapsto \varphi_{\alpha} ; \tag{4}
\end{equation*}
$$

that is, (1) holds. The (2) and (3) can be verified straightforwardly.
Let $V$ and $V^{\prime}$ be $F G$-modules equipped with $G$-invariant bilinear forms $f$ and $f^{\prime}$ respectively. We say that an $F G$-homomorphism $\alpha: V \rightarrow V^{\prime}$ is compatible with the bilinear forms $f$ and $f^{\prime}$ if $f^{\prime}(\alpha(u), \alpha(v))=f(u, v)$ for all $u, v \in V$.

If $f$ is a non-degenerate bilinear form on $V$, then any $F G$-homomorphism $\alpha$ : $V \rightarrow V^{\prime}$ which is compatible with $f$ and $f^{\prime}$ must be injective; for: $\alpha(u)=0 \mathrm{im}-$ plies that for any $v \in V$ we have that $f(u, v)=f^{\prime}(\alpha(u), \alpha(v))=f^{\prime}(0, \alpha(v))=$ 0 , hence $u=0$ by the non-degeneracy of the form $f$.

Lemma 4. Assume that $q$ is even, and $V$ is a self-dual irreducible $F G$-module. If both $f$ and $f^{\prime}$ are $G$-invariant non-degenerate symmetric bilinear forms on $V$, then there is an $F G$-automorphism $\beta: V \rightarrow V$ which is compatible with $f$ and $f^{\prime}$.

Proof. Apply the isomorphism (4) to the $F G$-module $V$ with the $G$ invariant non-degenerate symmetric bilinear form $f$. Since $V$ is irreducible, by the Schur's lemma, $\tilde{F}:=\operatorname{Hom}_{F G}(V, V)$ is a finite dimensional division $F$ algebra, hence $\tilde{F}$ is a field extension of $F$ as it is finite. By the commutativity
of $\tilde{F}$, it is easy to check that the sum and the product of any two symmetric transformations in $\tilde{F}$ are still symmetric transformations, so all the symmetric transformations in $\tilde{F}$ form a subfield $\hat{F}$ of $\tilde{F}$.

By Lemma 3, for the $G$-invariant non-degenerate symmetric bilinear form $f^{\prime}$, there is an $\alpha \in \hat{F}-\{0\}$ such that

$$
f^{\prime}(u, v)=\varphi_{\alpha}(u, v)=f(\alpha(u), v), \quad \forall u, v \in V
$$

Since $\hat{F}$ is a finite field of characteristic 2 , the map $\hat{F} \rightarrow \hat{F}, \lambda \mapsto \lambda^{2}$, is an automorphism of $\hat{F}$. So there is a $\beta \in \hat{F}$ such that $\beta^{2}=\alpha^{-1}$. Then $\beta: V \rightarrow V$ is an $F G$-automorphism of $V$ and a symmetric transformation with respect to the bilinear form $f$; and, noting that $\alpha \beta=\beta \alpha$, for any $u, v \in V$ we have

$$
\left.f^{\prime}(\beta(u), \beta(v))=f(\alpha(\beta(u)), \beta(v))=f((\beta \alpha \beta)(u)), v\right)=f(u, v)
$$

That is, $\beta$ is compatible with the bilinear form $f$ and $f^{\prime}$.
Theorem 1. Let $F$ be a finite field of characteristic 2 and $G$ be a finite group of odd order. Let $V$ be an $F G$-module with a $G$-invariant non-degenerate symmetric bilinear form. Then the following are equivalent:
(i) every self-dual composition factor of $V$ has even multiplicity;
(ii) there is an $F G$-submodule $U$ of $V$ such that $U^{\perp}=U$.

Proof. We denote $\langle-,-\rangle$ for the $G$-invariant non-degenerate symmetric bilinear form on $V$.
(ii) $\Rightarrow$ (i). This is an easy consequence of Lemma 2 (i) $\Rightarrow$ (iii).
(i) $\Rightarrow$ (ii). Let $W$ be an irreducible $F G$-submodule of $V$.

Case 1: $W \subseteq W^{\perp}$. By the exact sequence (2), we have a submodule $W^{\prime}$ of $V$ such that $V=W^{\perp} \oplus W^{\prime}$ and the homomorphism (1) induces an isomorphism

$$
W^{\prime} \xrightarrow{\cong} W^{*},\left.\quad w^{\prime} \longmapsto\left\langle w^{\prime},-\right\rangle\right|_{W} .
$$

Therefore, the matrix of the symmetric bilinear form $\left.\langle-,-\rangle\right|_{W^{\prime} \oplus W}$ restricted to $W^{\prime} \oplus W$ is as follows

$$
\left(\begin{array}{cc}
0 & A \\
A^{T} & *
\end{array}\right)
$$

where $A$ is the matrix of the bilinear form $W^{\prime} \times W \rightarrow F,\left(w^{\prime}, w\right) \mapsto\left\langle w^{\prime}, w\right\rangle$ and $A^{T}$ denotes the transpose of $A$; so $A$ is invertible, and hence $W^{\prime} \oplus W$ is a non-degenerate submodule of $V$. Then

$$
V=\left(W^{\prime} \oplus W\right) \oplus\left(W^{\prime} \oplus W\right)^{\perp}
$$

and $\left(W^{\prime} \oplus W\right)^{\perp}$ is also non-degenerate submodule.
If $W$ is not a self-dual module, then $W^{\prime} \cong W^{*}$ is not self-dual, and hence $\left(W^{\prime} \oplus W\right)^{\perp}$ also satisfies the condition (i). Otherwise, $W$ is a self-dual module, and $W^{\prime} \cong W^{*} \cong W$ is a self-dual module too, hence $\left(W^{\prime} \oplus W\right)^{\perp}$ still satisfies the condition (i). In a word, by induction, there is a submodule $S$ of $\left(W^{\prime} \oplus W\right)^{\perp}$ such that $\operatorname{Ann}_{\left(W^{\prime} \oplus W\right)^{\perp}}(S)=S$. Take $U=W \oplus S$; then it is easy to check that $U^{\perp}=U$ and (ii) holds.

Case 2: $W \nsubseteq W^{\perp}$. Then $W$ is non-degenerate, i.e. $V=W \oplus W^{\perp}$, and $W$ is a self-dual module, see Lemma 1(2). By the condition (i), there is a direct decomposition $W^{\perp}=\tilde{W} \oplus U$ such that $\tilde{W} \cong W$, and $V=W \oplus \tilde{W} \oplus U$.

If $\tilde{W} \subseteq \tilde{W}^{\perp}$, then it is reduced to Case 1 and the (ii) holds by induction. So we assume that $\tilde{W} \nsubseteq \tilde{W}^{\perp}$, and hence $\tilde{W}$ is also non-degenerate. Since $W \perp \tilde{W}$, the submodule $W \oplus \tilde{W}$ is non-degenerate too.

Let $f$ and $\tilde{f}$ denote the restrictions of $\langle-,-\rangle$ on $W$ and on $\tilde{W}$ respectively; so $f$ and $\tilde{f}$ are $G$-invariant non-degenerate symmetric bilinear forms on $W$ and $\tilde{W}$ respectively. Let $\alpha: W \rightarrow \tilde{W}$ be an $F G$-isomorphism. Then $\alpha$ induces a $G$-invariant non-degenerate symmetric bilinear form $f^{\prime}$ on $W$ as follows:

$$
f^{\prime}(u, w):=\tilde{f}(\alpha(u), \alpha(w)), \quad \forall u, w \in W
$$

By Lemma 4, there is an $F G$-automorphism $\beta: W \rightarrow W$ which is compatible with $f$ and $f^{\prime}$, i.e.

$$
f^{\prime}(\beta(u), \beta(w))=f(u, w), \quad \forall u, w \in W
$$

Let $\gamma=\alpha \beta$. Then $\gamma: W \rightarrow \tilde{W}$ is an $F G$-isomorphism, and for any $u, w \in W$ we have

$$
\tilde{f}(\gamma(u), \gamma(w))=\tilde{f}(\alpha(\beta(u)), \alpha(\beta(w)))=f^{\prime}(\beta(u), \beta(w))=f(u, w)
$$

that is, $\gamma$ is an $F G$-isomorphism compatible with the bilinear forms $f$ and $\tilde{f}$. Let

$$
W^{\prime}=\{w+\gamma(w) \mid w \in W\} \subseteq W \oplus \tilde{W}
$$

It is a routine to check that $W^{\prime}$ is a submodule and $W^{\prime} \cong W$; but, noting that $W \perp \tilde{W}$ and char $F=2$, for any $u+\gamma(u) \in W^{\prime}$ and $w+\gamma(w) \in W^{\prime}$ with $u, w \in W$ we have

$$
\begin{aligned}
\langle u+\gamma(u), w+\gamma(w)\rangle & =\langle u, w\rangle+\langle\gamma(u), \gamma(w)\rangle \\
& =f(u, w)+\tilde{f}(\gamma(u), \gamma(w)) \\
& =f(u, w)+f(u, w)=0 .
\end{aligned}
$$

So $W^{\prime} \cong W$ is an irreducible $F G$-submodule of $V$ and $W^{\prime} \subseteq W^{\prime \perp}$, and it is reduced to the Case 1 and (ii) holds by induction again.

Remark. In the proof of Theorem 1, Lemma 4 is quoted only in Case 2 where $W$ and $\tilde{W}$ are self-dual composition factors of $V$. Thus, as a consequence of the proof, we have the following conclusion.

Proposition 1. Let $G$ be a finite group of order coprime to the characteristic (not necessary 2) of the finite field $F$, and $V$ be an $F G$-module with a $G$-invariant non-degenerate symmetric bilinear form. If $V$ has no self-dual composition factor, then $V$ has a submodule $U$ such that $U^{\perp}=U$.

Now we turn to permutation codes. Let $X$ be a finite set; by $\operatorname{Sym}(X)$ we denote the group of all the permutations of $X$. If there is a group homomorphism $G \rightarrow \operatorname{Sym}(X)$, then $X$ is called a $G$-set. In that case, any $g \in G$ is mapped to a permutation: $X \rightarrow X, x \mapsto g x$. Hence, $\left(g g^{\prime}\right) x=g\left(g^{\prime} x\right)$ for all $g, g^{\prime} \in G$ and $x \in X$; and $1 x=x$ for all $x \in X$.

Let $F X=\left\{\sum_{x \in X} a_{x} x \mid a_{x} \in F\right\}$ be the vector space over $F$ with basis $X$. Extending the $G$-action on $X$ linearly, $F X$ becomes an $F G$-module, called an $F G$-permutation module with permutation basis $X$, please cf. [1, §12].

We say that $C$ is an $F G$-permutation code of $F X$, denoted by $C \leq F X$, if $C$ is an $F G$-submodule of the $F G$-permutation module $F X$; and a permutation code $C$ is said to be irreducible if $C$ is an irreducible $F G$-submodule of $F X$. Further, if $X$ is a transitive $G$-set, then any $F G$-permutation code $C$ of $F X$ is said to be a transitive permutation code.

Recall that, for a linear code $C$ of length $n$ over $F$, a permutation of the components of a word of length $n$ is said to be a permutation automorphism of $C$ if the permutation keeps every code word of $C$ still a code word. By $\operatorname{PAut}(C)$ we denote the automorphism group of $C$ consisting of all the permutation automorphisms of $C$. It is easy to see that $C$ is an $F G$-permutation code of a $G$-permutation set $X$ of cardinality $n$ if and only if there is a group homomorphism of $G$ to $\operatorname{PAut}(C)$.

There is a so-called scalar product of any two words of $F X$ as follows:

$$
\left\langle\mathbf{w}, \mathbf{w}^{\prime}\right\rangle=\sum_{x \in X} w_{x} w_{x}^{\prime}, \quad \forall \mathbf{w}=\sum_{x \in X} w_{x} x, \mathbf{w}^{\prime}=\sum_{x \in X} w_{x}^{\prime} x \in F X
$$

which is obvious a non-degenerate symmetric bilinear form on $F X$, we call it the standard inner product on $F X$ with respect to the permutation basis $X$. Moreover, the standard inner product is $G$-invariant, since for any $g \in G$ and any words $\mathbf{w}=\sum_{x \in X} w_{x} x$ and $\mathbf{w}^{\prime}=\sum_{x \in X} w_{x}^{\prime} x$ of $F X$, we have

$$
\begin{aligned}
\left\langle g(\mathbf{w}), g\left(\mathbf{w}^{\prime}\right)\right\rangle & =\left\langle g\left(\sum_{x \in X} w_{x} x\right), g\left(\sum_{x \in X} w_{x}^{\prime} x\right)\right\rangle \\
& =\left\langle\sum_{x \in X} w_{x}(g x), \sum_{x \in X} w_{x}^{\prime}(g x)\right\rangle=\sum_{x \in X} w_{x} w_{x}^{\prime} \\
& =\left\langle\mathbf{w}, \mathbf{w}^{\prime}\right\rangle
\end{aligned}
$$

equivalently,

$$
\left\langle g(\mathbf{w}), \mathbf{w}^{\prime}\right\rangle=\left\langle\mathbf{w}, g^{-1}\left(\mathbf{w}^{\prime}\right)\right\rangle, \quad \forall g \in G, \forall \mathbf{w}, \mathbf{w}^{\prime} \in F X
$$

Thus, $F X$ is a self-dual $F G$-module. In fact, we can make the duality more precisely. Just like the formula (11), the standard inner product induces an isomorphism

$$
F X \xrightarrow{\cong}(F X)^{*}, \quad \mathbf{u} \longmapsto \mathbf{u}^{*}:=\langle\mathbf{u},-\rangle
$$

where

$$
\mathbf{u}^{*}: \quad F X \longrightarrow F, \quad \mathbf{w} \longmapsto \mathbf{u}^{*}(\mathbf{w})=\langle\mathbf{u}, \mathbf{w}\rangle ;
$$

and

$$
X^{*}:=\left\{x^{*} \mid x \in X\right\}
$$

is a $G$-set with $G$-action

$$
g\left(x^{*}\right)=\left(g^{-1} x\right)^{*}, \quad \forall g \in G, x^{*} \in X^{*},
$$

such that $(F X)^{*}$ is an $F G$-permutation module of the $G$-set $X^{*}$, and $\mathbf{u} \mapsto \mathbf{u}^{*}$ is a permutation isomorphism.

Let $F X$ be an $F G$-permutation module. For any permutation code $C$ of $F X$, since $C$ is an $F G$-submodule, $C^{\perp}=\{\mathbf{w} \in F X \mid\langle\mathbf{c}, \mathbf{w}\rangle=0, \forall \mathbf{c} \in C\}$ is an $F G$ submodule again, i.e. $C^{\perp}$ is a permutation code again. In coding-theoretical notation, $C^{\perp}$ is said to be the dual permutation code of $C$.

An $F G$-permutation code $C \leq F X$ is said to be self-orthogonal if $C \subseteq C^{\perp}$. And a permutation code $C \leq F X$ is said to be self-dual if $C=C^{\perp}$.

With the coding-theoretical notation introduced above, from Theorem 1 and Proposition 1, we have the following results at once.

Theorem 2. Let $F$ be a finite field of characteristic 2, and $G$ be a finite group of odd order, and $X$ be a finite $G$-set. Then the following are equivalent:
(i) every self-dual composition factor of $F X$ has even multiplicity;
(ii) there is a self-dual $F G$-permutation code $C$ of $F X$.

Proposition 2. Let $G$ be a finite group of order coprime to the characteristic (not necessary 2) of the field $F$, and $X$ be a finite $G$-set. If $F X$ has no self-dual composition factor, then there is a self-dual $F G$-permutation code of $F X$.

## 3 Self-dual extended transitive permutation codes

If a $G$-set $X=\left\{x_{0}\right\}$ contains of only one element, then $X$ is said to be the trivial $G$-set and the permutation module $F X \cong F$ is just the trivial $F G$-module, which is obviously a self-dual module.

An elementary known fact is that, in the semisimple case, for any transitive $G$-set $X$ the trivial $F G$-module $F$ is a composition factor of multiplicity 1 of the $F G$-permutation module $F X$; e.g. see [3, Lemma 1]; we denote the unique trivial submodule of $F X$ by $F$ if there is no confusion, thus $F X=F \oplus F^{\perp}$. By Theorem 1, $F X$ has no self-dual codes.

Let $X$ be a transitive $G$-set. Let $\hat{X}=X \bigcup\left\{x_{0}\right\}$ be the disjoint union of $X$ with a trivial $G$-set $\left\{x_{0}\right\}$, i.e. $x_{0} \notin X$. Then $F \hat{X}=F X \oplus F x_{0}$, and any permutation code $C$ of $F \hat{X}$ is said to be an extended transitive permutation code of $F X$.

Lemma 5. Notation as above, and let $n=|X|$. The following are equivalent:
(i) there is a permutation code $C$ of $F X$ such that $C^{\perp}=C \oplus F$ and, as an element of the field $F,-n$ has a square root in $F$;
(ii) there is a self-dual permutation code $\hat{C}$ of $F \hat{X}$.

Proof. Let $e=\sum_{x \in X} x$; then $F e$ is the unique submodule of $F X$ which is isomorphic to $F$, so $F x_{0} \oplus F e$ is the homogeneous component of $F \hat{X}$ associated with the trivial module $F$. Noting that $F x_{0} \perp F e$ and $\left\langle x_{0}, x_{0}\right\rangle=1$ and $\langle e, e\rangle=$ $n \neq 0$ (because $n\left||G|\right.$ which is coprime to $q=|F|$ ), we see that $F x_{0} \oplus F e$ is a non-degenerate submodule of $F \hat{X}$. Thus

$$
F \hat{X}=\left(F x_{0} \oplus F e\right) \oplus\left(F x_{0} \oplus F e\right)^{\perp}
$$

and

$$
\left(F x_{0} \oplus F e\right)^{\perp}=\left(F x_{0}\right)^{\perp} \cap(F e)^{\perp}=F X \cap(F e)^{\perp}=\operatorname{Ann}_{F X}(F e)
$$

(ii) $\Rightarrow$ (i). By the formula (3) we have

$$
\hat{C}=\left(\hat{C} \cap\left(F x_{0} \oplus F e\right)\right) \oplus\left(\hat{C} \cap \operatorname{Ann}_{F X}(F e)\right) .
$$

From the condition (ii) that $\hat{C}^{\perp}=\hat{C}$, by Lemma 2(ii), we have

$$
\operatorname{dim}\left(\hat{C} \cap\left(F x_{0} \oplus F e\right)\right)=1, \quad \operatorname{dim}\left(\hat{C} \cap \operatorname{Ann}_{F X}(F e)\right)=\frac{n-1}{2}
$$

Set $C=\hat{C} \cap \operatorname{Ann}_{F X}(F e)$; it is easy to check that, $C$ is a permutation code of $F X$ and $C^{\perp}=C \oplus F e$ in $F X$. On the other hand, for $C \cap\left(F x_{0} \oplus F e\right)$ which is a one-dimensional subspace, we assume that $\lambda \in F$ such that

$$
\hat{C} \cap\left(F x_{0} \oplus F e\right)=F \cdot\left(\lambda x_{0}+e\right) ;
$$

then $\left\langle\lambda x_{0}+e, \lambda x_{0}+e\right\rangle=0$; i.e.

$$
0=\left\langle\lambda x_{0}, \lambda x_{0}\right\rangle+\langle e, e\rangle=\lambda^{2}+n
$$

that is, $\lambda^{2}=-n$.
(i) $\Rightarrow$ (ii). In $F X$, since $\operatorname{dim} C+\operatorname{dim} C^{\perp}=n$, from the condition that $C^{\perp}=$ $C \oplus F e$ we have that $\operatorname{dim} C=\frac{n-1}{2}$. Turn to $F \hat{X}$, set $\lambda \in F$ such that $\lambda^{2}=-n$ and $\hat{C}:=F \cdot\left(\lambda x_{0}+e\right) \oplus C$; as shown above, the 1-dimensional submodule $F \cdot\left(\lambda x_{0}+e\right)$ of $F x_{0} \oplus F e$ is isotropic, hence $\hat{C}$ is an isotropic submodule. But $\operatorname{dim} \hat{C}=\frac{n+1}{2}$; and by Lemma 2, $\hat{C}$ is a self-dual permutation code of $F \hat{X}$.

Remark. In the above lemma, the condition " $-n$ has a square root in $F$ " in (i) always satisfies for characteristic 2.

For any positive integer $n$ we denote $\mathbb{Z}_{n}$ the residue ring of the integer ring $\mathbb{Z}$ modulo $n$, and denote $\mathbb{Z}_{n}^{\times}$the multiplicity group consisting of all the invertible elements of $\mathbb{Z}_{n}$. So $q$ is considered as an element of $\mathbb{Z}_{n}^{\times}$, and we can speak of the order of $q$ in the group $\mathbb{Z}_{n}^{\times}$.

Lemma 6. Let $n$ be an odd integer coprime to $q$. The following are equivalent:
(i) The order of $q$ in $\mathbb{Z}_{n}^{\times}$is odd.
(ii) For any prime $p \mid n$ the order of $q$ in $\mathbb{Z}_{p}^{\times}$is odd.

Proof. Let $n=p_{1}^{m_{1}} \cdots p_{k}^{m_{k}}$. By Chinese Remainder Theorem we have the following isomorphism about the multiplicative groups:

$$
\mathbb{Z}_{n}^{\times} \xrightarrow{\cong} \mathbb{Z}_{p_{1}^{m_{1}}}^{\times} \times \cdots \times \mathbb{Z}_{p_{k}^{m_{k}}}^{\times}, \quad a \longmapsto(a, \cdots, a)
$$

The order of $q \in \mathbb{Z}_{n}^{\times}$is odd if and only if the order $q \in \mathbb{Z}_{p_{i}^{m_{i}}}^{\times}$is odd for every $i=1, \cdots, k$. Consider the exact sequence of multiplication groups:

$$
1 \longrightarrow 1+p_{i} \mathbb{Z}_{p_{i}^{m_{i}}} \xrightarrow{\text { incl }} \mathbb{Z}_{p_{i}^{m_{i}}}^{\times} \xrightarrow{\rho} \mathbb{Z}_{p_{i}}^{\times} \longrightarrow 1
$$

where "incl" is the inclusion map and $\rho$ is the natural map:

$$
\mathbb{Z}_{p_{i}^{m_{i}}}^{\times} \longrightarrow \mathbb{Z}_{p_{i}}^{\times}, \quad a \longmapsto a .
$$

Since the order $\left|1+p_{i} \mathbb{Z}_{p_{i}^{m_{i}}}\right|=p_{i}^{m_{i}-1}$ is odd, the order of $q \in \mathbb{Z}_{p_{i}^{m_{i}}}^{\times}$is odd if and only if the order of $q \in \mathbb{Z}_{p_{i}}^{\times}$is odd.

Recall that $F$ is a finite field of order $q$. For any positive integer $n$, in a suitable extension we can take a primitive $n$ 'th root $\xi_{n}$ of unity, and the extension $F\left(\xi_{n}\right)$ is independent of the choice of $\xi_{n}$; and the order of the Galois group $\left|\operatorname{Gal}\left(F\left(\xi_{n}\right) / F\right)\right|=\left|F\left(\xi_{n}\right): F\right|$ is just the order of $q$ in the multiplicative group $\mathbb{Z}_{n}^{\times}$. As a consequence we have the following at once.

Corollary 1. Let $n$ be an odd integer coprime to $q$. The following are equivalent:
(i). The extension degree $\left|F\left(\xi_{n}\right): F\right|$ is odd.
(ii). For any prime $p \mid n$ the extension degree $\left|F\left(\xi_{p}\right): F\right|$ is odd.

Let $H$ be a subgroup of the group $G$, and let $Y$ be a finite $H$-set; then $F Y$ is an $F H$-permutation module. We have the induced $F G$-module

$$
\operatorname{Ind}_{H}^{G}(F Y)=F G \bigotimes_{F H} F Y=\bigoplus_{t \in T} t \otimes F Y
$$

where $T$ is a representative set of the left cosets of $G$ over $H$; and $\operatorname{Ind}_{H}^{G}(F Y)$ is a vector space with basis

$$
X:=\operatorname{Ind}_{H}^{G}(Y)=\bigcup_{t \in T} t \otimes Y=\bigcup_{t \in T}\{t \otimes y \mid y \in Y\}
$$

which is a $G$-set with $G$-action as follows:

$$
g(t \otimes y)=t_{g} \otimes t_{g}^{-1} g t y, \quad \forall g \in G, t \in T, y \in Y
$$

where $t_{g}$ is the representative of the unique left coset $t_{g} H$ such that $g t \in t_{g} H$, or equivalently $t_{g}^{-1} g t \in H$. We say that $\operatorname{Ind}_{H}^{G}(F Y)$ is the induced $F G$-permutation module with the induced $G$-set $\operatorname{Ind}_{H}^{G}(Y)$.

Lemma 7. Notation as above; and let $D$ be an FH-permutation code of the FH-permutation module FY; then

$$
\operatorname{Ind}_{H}^{G}(D)^{\perp}=\operatorname{Ind}_{H}^{G}\left(D^{\perp}\right) .
$$

Proof. It is obvious that the induced module $C:=\operatorname{Ind}_{H}^{G}(D)$ is a submodule of $\operatorname{Ind}_{H}^{G}(F Y)=\bigoplus_{t \in T} t \otimes F Y$, and we have a direct decomposition of $F$-spaces:

$$
\operatorname{Ind}_{H}^{G}(D)=\bigoplus_{t \in T} t \otimes D
$$

where each $t \otimes D$ is an $F$-subspace of $t \otimes F Y$. Each $t \otimes F Y$ is an $F$-space with bases $t \otimes Y$, hence with the standard inner product:

$$
\left\langle\sum_{y \in Y} a_{y}(t \otimes y), \quad \sum_{y \in Y} b_{y}(t \otimes y)\right\rangle=\sum_{y \in Y} a_{y} b_{y}
$$

and

$$
F Y \longrightarrow t \otimes F Y, \quad \sum_{y \in Y} a_{y} y \longmapsto \sum_{y \in Y} a_{y}(t \otimes y),
$$

is an isometric $F$-isomorphism. With respect to the isometries, it is clear that $(t \otimes D)^{\perp}=t \otimes D^{\perp}$; hence

$$
\operatorname{Ind}_{H}^{G}(D)^{\perp}=\bigoplus_{t \in T}(t \otimes D)^{\perp}=\bigoplus_{t \in T} t \otimes D^{\perp}=\operatorname{Ind}_{H}^{G}\left(D^{\perp}\right)
$$

Lemma 8. Let $p$ be an odd prime which is coprime to $q$ such that the order of $q$ in $\mathbb{Z}_{p}^{\times}$is odd. Let $A$ be a finite abelian p-group, and $H$ be a finite group of odd order which acts on the group $A$. Then there is a group code $C \leq F A$ which is stable by the action of $H$ and $C^{\perp}=C \oplus F$, where $F$ denotes the unique trivial module of FA.

Proof. Let $|A|=n$ which is a power of $p$; take a primitive $n$ 'th root $\xi$ of unity, and denote $\tilde{F}=\underset{\tilde{F}}{F}(\xi)$. Then $\tilde{F} A$ is a splitting semisimple commutative algebra. Let $\Gamma=\operatorname{Gal}(\tilde{F} / F)$ denote the Galois group of $\tilde{F}=F(\xi)$ over $F$; by Lemma 6 and its corollary, $|\Gamma|$ is odd.

Let $A^{*}$ denote the set of all the irreducible characters of $A$ over $\tilde{F}$ (i.e. all the homomorphisms $\left.\chi: A \rightarrow \tilde{F}^{\times}\right)$. With the usual multiplication of functions, $A^{*}$ is an abelian group and $A^{*} \cong A$. Note that for any integer $k$,

$$
\chi^{k}(a)=\chi\left(a^{k}\right), \quad \forall \chi \in A^{*}, a \in A
$$

in particular, $\chi^{-1}(a)=\chi\left(a^{-1}\right)$.
Each $\chi \in A^{*}$ corresponds exactly one irreducible module $\tilde{F} e_{\chi}$ of $\tilde{F} A$, where

$$
e_{\chi}=\frac{1}{n} \sum_{a \in A} \chi\left(a^{-1}\right) a
$$

is a primitive idempotent of the algebra $\tilde{F} A$. And we have the direct decomposition of irreducible $\tilde{F} A$-modules:

$$
\tilde{F} A=\bigoplus_{\chi \in A^{*}} \tilde{F} e_{\chi}
$$

For $\chi, \psi \in A^{*}$ and $\lambda, \mu \in \tilde{F}$, the standard inner product

$$
\left\langle\lambda e_{\chi}, \mu e_{\psi}\right\rangle=n \lambda \mu \cdot\left(\chi \mid \psi^{-1}\right)
$$

where ( $\chi \mid \psi^{-1}$ ) denotes the usual inner product of characters:

$$
\left(\chi \mid \psi^{-1}\right)=\frac{1}{n} \sum_{a \in A} \chi(a) \psi^{-1}\left(a^{-1}\right)=\frac{1}{n} \sum_{a \in A} \chi(a) \psi(a)
$$

By the orthogonal relations of characters,

$$
\left\langle\tilde{F} e_{\chi}, \tilde{F} e_{\psi}\right\rangle= \begin{cases}\tilde{F}, & \text { if } \chi=\psi^{-1} \\ 0, & \text { otherwise }\end{cases}
$$

Any submodule $\tilde{C}$ of $\tilde{F} A$ corresponds exactly to a subset $B \subseteq A^{*}$ such that

$$
\tilde{C}=\bigoplus_{\chi \in B} \tilde{F} e_{\chi}
$$

Thus

$$
\tilde{C}^{\perp}=\bigoplus_{\psi \notin B^{-1}} \tilde{F} e_{\psi}
$$

where $B^{-1}:=\left\{\chi^{-1} \mid \chi \in B\right\}$; in particular, $\tilde{C}$ is self-orthogonal code if and only if $B \cap B^{-1}=\emptyset$.

Recall that $\Gamma=\operatorname{Gal}(\tilde{F} / F)$ is a cyclic group generated by the following automorphism

$$
\gamma: F(\xi) \longrightarrow F(\xi), \quad \lambda \longmapsto \lambda^{q}
$$

The group $\Gamma$ acts on $\tilde{F}$ hence acts on the $\operatorname{ring} \tilde{F} A$ in the following way:

$$
\gamma\left(\sum_{a \in A} \lambda_{a} a\right)=\sum_{a \in A} \gamma\left(\lambda_{a}\right) a, \quad \forall \sum_{a \in A} \lambda_{a} a \in \tilde{F} A .
$$

We denote $(\tilde{F} A)^{\Gamma}$ the subring consisting of all the $\Gamma$-fixed elements of $\tilde{F} A$. It is obvious that $(\tilde{F} A)^{\Gamma}=F A$.

And $\Gamma$ acts on the set $\left\{e_{\chi} \mid \chi \in A^{*}\right\}$ of the primitive idempotents of $\tilde{F} A$ :

$$
\gamma\left(e_{\chi}\right)=\gamma\left(\frac{1}{n} \sum_{a \in A} \chi\left(a^{-1}\right) a\right)=\frac{1}{n} \sum_{a \in A} \gamma\left(\chi\left(a^{-1}\right)\right) a=e_{\gamma(\chi)},
$$

where $\gamma(\chi) \in A^{*}$ is the composition homomorphism

$$
A \xrightarrow{\chi} \tilde{F} \xrightarrow{\gamma} \tilde{F}, \quad a \longmapsto \gamma(\chi(a))=(\chi(a))^{q},
$$

i.e. $\gamma(\chi)=\chi^{q}$. In this way, $\Gamma$ acts on the abelian group $A^{*}$.

On the other hand, $H$ acts on the ring $\tilde{F} A$ :

$$
h\left(\sum_{a \in A} \lambda_{a} a\right)=\sum_{a \in A} \lambda_{a} h(a), \quad \forall \sum_{a \in A} \lambda_{a} a \in \tilde{F} A
$$

Similarly, $H$ acts on the set $\left\{e_{\chi} \mid \chi \in A^{*}\right\}$ of the primitive idempotents of $\tilde{F} A$ :
$h\left(e_{\chi}\right)=h\left(\frac{1}{n} \sum_{a \in A} \chi\left(a^{-1}\right) a\right)=\frac{1}{n} \sum_{a \in A} \chi\left(a^{-1}\right) h(a)=\frac{1}{n} \sum_{b \in A} \chi\left(h^{-1}\left(b^{-1}\right)\right) b=e_{h(\chi)}$,
where $h(\chi) \in A^{*}$ is the composition homomorphism

$$
A \xrightarrow{h^{-1}} A \xrightarrow{\chi} \tilde{F}, \quad a \longmapsto \chi\left(h^{-1}(a)\right) .
$$

In this way, $H$ acts on the abelian group $A^{*}$.
In a word, $\Gamma \times H$ acts on the ring $\tilde{F} A$, and the action induces the action of $\Gamma \times H$ on the abelian group $A^{*}$.

Let $C \leq F A$ be an $H$-stable submodule; denote $\tilde{C}=\tilde{F} \otimes_{F} C$. Then $\tilde{C}$ is a both $H$-stable and $\Gamma$-stable submodule of $\tilde{F} A$ such that $\tilde{C}^{\Gamma}=C$. Let $B \subset A^{*}$
be the subset such that $\tilde{C}=\bigoplus_{\chi \in B} \tilde{F} e_{\chi}$. Since $\tilde{C}$ is $H$-stable, we see that $B$ is $H$-stable; and similarly, $B$ is $\Gamma$-stable. So $B$ is a $(\Gamma \times H)$-stable subset of $A^{*}$.

Conversely, if $B$ is a $(\Gamma \times H)$-stable subset of $A^{*}$, then $\tilde{C}=\bigoplus_{\chi \in B} \tilde{F} e_{\chi}$ is a ( $\Gamma \times H$ )-stable submodule of $\tilde{F} A$, and $\tilde{C}^{\Gamma}$ is an $H$-stable submodule of $F A$.

Let $\Omega$ be a non-trivial $(\Gamma \times H)$-orbit of $A^{*}$, i.e. $1 \notin \Omega$. Let $\chi \in \Omega$, then $\chi \neq 1$ hence the order of $\chi$ is a power of $p$, say $p^{\ell}$ (recall that $A^{*} \cong A$ is an abelian $p$-group). We claim that $\chi^{-1} \notin \Omega$. Suppose it is not the cases, then there is $\gamma^{i} \in \Gamma$ and $h \in H$ such that $\gamma^{i} h(\chi)=\chi^{-1}$, and

$$
h(\chi)=\gamma^{-i}\left(\chi^{-1}\right)=\chi^{(-1)\left(-q^{i}\right)}=\chi^{q^{i}}
$$

thus $\langle\gamma\rangle \times\langle h\rangle$ acts on the cyclic group $\langle\chi\rangle$ of order $p^{\ell}$, and $\gamma^{i} h$ acts on $\langle\chi\rangle$ as the nvolution $\chi \mapsto \chi^{-1}$; but the automorphism group $\operatorname{Aut}(\langle\chi\rangle)$ is a cyclic group, hence the product $\gamma^{i} h$ of the two automorphisms $\gamma^{i}$ and $h$ of odd order still has odd order; it contradicts to that the $\chi \mapsto \chi^{-1}$ is an involution.

The involution $\tau: A^{*} \rightarrow A^{*}, \chi \mapsto \chi^{-1}$, commutes with both $\Gamma$ and $H$ clearly. So $\tau$ permutes all the $(\Gamma \times H)$-orbits of $A^{*}$. For any non-trivial orbit $\Omega \neq\{1\}$, since $\tau(\chi) \notin \Omega$ for any $\chi \in \Omega$, the subset $\tau(\Omega)$ is an orbit different from $\Omega$. Thus we can partition all the non-trivial orbits into two collections $B$ and $B^{-1}=\left\{\chi^{-1} \mid \chi \in B\right\}$, and we get the disjoint union

$$
A^{*}=\{1\} \bigcup B \bigcup B^{-1}
$$

Then the code $\tilde{C}=\bigoplus_{\chi \in B} \tilde{F} e_{\chi}$ is $H$-stable and $\tilde{C}^{\perp}=\tilde{C} \oplus \tilde{F}$; hence the code $C=\tilde{C}^{\Gamma}$ of $F A$ is $H$-stable and $C^{\perp}=C \oplus F$.
Theorem 3. Let $G$ be a finite group of odd order, and $X$ be a finite transitive $G$-set and $n=|X|$. Assume that $q=|F|$ is coprime to $n$, and the order of $q$ in the multiplicative group $\mathbb{Z}_{n}^{\times}$is odd. Then there is a permutation code $C \leq F X$ such that $C^{\perp}=C \oplus F$.

Proof. We prove it by induction on the order of $G$. It is trivial for $|G|=1$. Assume $|G|>1$. Let $x_{1} \in X$ and denote $G_{1}$ the stabilizer of $x_{1}$ in $G$. Then $G_{1}$ is a subgroup and $F X=\operatorname{Ind}_{G_{1}}^{G}(F)$. Since $G$ is solvable by Feit-Thompson Odd Theorem, a minimal normal subgroup $A$ of $G$ is an elementary abelian $p$-group, where $p$ is a prime. Since $A$ is normal, the product $A G_{1}$ is a subgroup of $G$. There are three cases.

Case 1: $A G_{1}=G_{1}$. Then $A \subseteq G_{1}$, and hence $A$ is contained in every conjugate of $G_{1}$ as $A$ is normal. Thus $A$ acts trivially on $X$, and $X$ is a $G / A$-set and $F X$ is a permutation module over $G / A$. Since $|G / A|<|G|$, the conclusion holds by induction.

Case 2: $A G_{1}=G$. Since $A \cap G_{1}$ is both normal in $G_{1}$ and in $A$, we have that $A \cap G_{1}$ is normal in $A G_{1}=G$; but $A$ is a minimal normal subgroup of $G$, so $A \cap G_{1}=1$. Then we have a bijection

$$
\beta: A \longrightarrow X, \quad a \longmapsto a\left(x_{1}\right) .
$$

Let $A$ acts on $A$ by left translation, and let $G_{1}$ acts on $A$ by conjugation; hence $G=A G_{1}$ is mapped into the group $\operatorname{Sym}(A)$ of all the permutations of $A$ :

$$
(b h)(a)=b h a h^{-1}, \quad \forall a, b \in A, h \in H
$$

Noting that $G_{1}$ stabilizes $x_{1}$, we have that

$$
\beta((b h)(a))=\left(b h a h^{-1}\right)\left(x_{1}\right)=b h a\left(x_{1}\right)=(b h) \beta(a) .
$$

Thus, mapping $b h \in G$ to the permutation $a \mapsto b h a h^{-1}$ of $A$ is an action of $G$ on $A$, and $\beta$ is an isomorphism of $G$-sets. Then $n=|A|$ hence $p \mid n$, so $p$ is coprime to $q$, and by the assumption of the lemma, the order of $q$ in $\mathbb{Z}_{p}^{\times}$is odd (see Lemma 6). The conclusion is derived from Lemma 8.

Case 3: $G_{1} \supsetneqq A G_{1} \supsetneqq G$. In this case,

$$
F X \cong \operatorname{Ind}_{G_{1}}^{G}(F)=\operatorname{Ind}_{A G_{1}}^{G} \operatorname{Ind}_{G_{1}}^{A G_{1}}(F)
$$

Let $Y=\left\{g x_{1} \mid g \in A G_{1}\right\}$, which is an $A G_{1}$-set and $\operatorname{Ind}_{G_{1}}^{A G_{1}}(F) \cong F Y$. By induction, there is a code $D \leq F Y$ such that $D^{\perp}=D \oplus F e_{Y}$ where $e_{Y}=$ $\sum_{y \in Y} y$. Turn to the permutation module $F X=\operatorname{Ind}_{A G_{1}}^{G}(F Y)$, by Lemma 7, we have
$\operatorname{Ind}_{A G_{1}}^{G}(D)^{\perp}=\operatorname{Ind}_{A G_{1}}^{G}\left(D^{\perp}\right)=\operatorname{Ind}_{A G_{1}}^{G}\left(D \oplus F e_{Y}\right)=\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus \operatorname{Ind}_{A G_{1}}^{G}\left(F e_{Y}\right)$.
Noting that, $F e_{Y}$ is the unique trivial module of $F Y$, and

$$
\operatorname{Ind}_{A G_{1}}^{G}\left(F e_{Y}\right)=\bigoplus_{t \in G / A G_{1}} t \otimes F e_{Y}
$$

by induction again, there is a code $E \leq \operatorname{Ind}_{A G_{1}}^{G}\left(F e_{Y}\right)$ such that

$$
\operatorname{Ann}_{\operatorname{Ind}_{A G_{1}}^{G}\left(F e_{Y}\right)}(E)=E \oplus F e_{X}
$$

where $e_{X}=\sum_{x \in X} x$. So we can write $\operatorname{Ind}_{A G_{1}}^{G}\left(F e_{Y}\right)=E^{\prime} \oplus E \oplus F e_{X}$, and have

$$
\operatorname{Ind}_{A G_{1}}^{G}(D)^{\perp}=\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus \operatorname{Ind}_{A G_{1}}^{G}\left(F e_{Y}\right)=\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E^{\prime} \oplus E \oplus F e_{X}
$$

Let

$$
C=\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E
$$

which is a permutation code of $F X$ and

$$
\begin{aligned}
C^{\perp} & =\operatorname{Ind}_{A G_{1}}^{G}(D)^{\perp} \bigcap E^{\perp}=\operatorname{Ann}_{F X}\left(\operatorname{Ind}_{A G_{1}}^{G}(D)\right) \bigcap \operatorname{Ann}_{F X}(E) \\
& =\left(\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E^{\prime} \oplus E \oplus F e_{X}\right) \bigcap \operatorname{Ann}_{\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E^{\prime} \oplus E \oplus F e_{X}}(E) \\
& =\left(\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E^{\prime} \oplus E \oplus F e_{X}\right) \bigcap\left(\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E \oplus F e_{X}\right) \\
& =\operatorname{Ind}_{A G_{1}}^{G}(D) \oplus E \oplus F e_{X} \\
& =C \oplus F e_{X}
\end{aligned}
$$

As a consequence of Theorem and Lemma 5 (cf. its remark), we get the followings at once.

Corollary 2. Assume that $q=|F|$ is even and $|G|$ is odd and $X$ is a transitive $G$-set and $n=|X|$. If the order of $q$ in the multiplicity group $\mathbb{Z}_{n}^{\times}$is odd, then there is a self-dual extended code of FX.
Corollary 3. Assume that $|G|$ is odd and $X$ is a transitive $G$-set and $n=|X|$. If $q=|F|$ is coprime to $n$ and the order of $q$ in the multiplicity group $\mathbb{Z}_{n}^{\times}$is odd, and $-n$ has square root in $F$, then there is a self-dual extended code of $F X$.

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