

Difference Sets with Few Character Values

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Abstract The known families of difference sets can be subdivided into three classes: difference sets with Singer parameters, cyclotomic difference sets, and difference sets with $\gcd(v, n) > 1$. It is remarkable that all the known difference sets with $\gcd(v, n) > 1$ have the so-called character divisibility property. In 1997, Jungnickel and Schmidt posed the problem of constructing difference sets with $\gcd(v, n) > 1$ that do not satisfy this property. In an attempt to attack this problem, we use difference sets with three nontrivial character values as candidates, and get some necessary conditions.

Keywords Association schemes · Character divisible property · Character values · Difference sets

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1 Introduction

Let G be a finite abelian group of order v and exponent m . A subset D of size k in G is called a (v, k, λ) difference set if each nonidentity element of G can be represented

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as $d_1 d_2^{-1}$, $d_1, d_2 \in D$ in exactly λ ways. The order of D is defined to be $n = k - \lambda$. For a subset A of G , we set $A^{(-1)} = \{g^{-1} \mid g \in A\}$; also we use the same A to denote the group ring element $\sum_{g \in A} g \in \mathbb{Z}[G]$. Then, it is not hard to see that a k -subset D of a group G of order v is a (v, k, λ) difference set in G if and only if it satisfies the following equation in the group ring $\mathbb{Z}[G]$:

$$DD^{(-1)} = n + \lambda G.$$

Besides group rings, character theory is another very fruitful tool in the study of difference sets. For a finite abelian group G , we use \widehat{G} to denote its character group, and χ_0 the principal character. The **Fourier inversion formula** below will be used frequently.

Lemma 1 *Let G be an abelian group of order v . If $A = \sum_{g \in G} a_g g \in \mathbb{Z}[G]$, then*

$$a_h = \frac{1}{v} \sum_{\chi \in \widehat{G}} \chi(A) \chi(h^{-1})$$

for all $h \in G$, where $\chi(A) = \sum_{g \in A} a_g \chi(g)$.

One useful consequence of the inversion formula is as follows. Let G be an abelian group of finite order, and let A and B be two elements of $\mathbb{Z}[G]$. Then $A = B$ if and only if $\chi(A) = \chi(B)$ for all characters χ of G . The next result is a standard characterization of difference sets by using their character values.

Proposition 1 *Let G be an abelian group of order v and $\chi \in \widehat{G}$. Let k and λ be positive integers satisfying $k(k-1) = \lambda(v-1)$. Then a k -subset D of G is a (v, k, λ) difference set in G if and only if*

$$\chi(D) \overline{\chi(D)} = \begin{cases} n, & \text{if } \chi \neq \chi_0, \\ k^2, & \text{if } \chi = \chi_0. \end{cases}$$

For more background on difference sets, the interested reader may refer to [4, 17].

The known families of difference sets can be subdivided into three classes: difference sets with Singer parameters, cyclotomic difference sets, and difference sets with $\gcd(v, n) > 1$. There are five known families of difference sets with $\gcd(v, n) > 1$, namely Hadamard difference sets, the McFarland and the Spence family, a series similar to Spence difference sets discovered by Davis and Jedwab [9], and a series generalizing Hadamard difference sets found by Chen [8]. We say a difference set D has **character divisibility property** if $\sqrt{n} \mid \chi(D)$ for each nonprincipal character χ of G . It is remarkable that all the known difference sets with $\gcd(v, n) > 1$ have this property. So it is natural to ask the following problem, which was posed by Jungnickel and Schmidt in their survey paper [12].

Research Problem: Construct difference sets with $\gcd(v, n) > 1$ that do not have the character divisibility property.

This current project makes an attempt to attack this problem. For a difference set D , we define

$$X = X(D) = \{\chi(D) \mid \chi \in \widehat{G}, \chi \neq \chi_0\},$$

which is the set of character values $\chi(D)$, where χ ranges over all the nonprincipal characters of G . We will use difference sets with $|X| = 3$ as candidates and derive some necessary conditions. With the aid of computer, some infinite classes of plausible parameters satisfying all the necessary conditions has been found. Besides, we find several classes of parameters which meet almost all the necessary conditions. We list these parameters as a supportive evidence for the existence of difference sets without character divisibility property.

Related to this paper, there are some similar results on graphs. In [5, 6, 7], Bridges and Mena studied the multiplicative design, which is related to a family of three eigenvalue graphs. Ma [13] considered the subset of a group with few character values under the name of polynomial addition sets. Overall, these works are particular cases of the problem of graphs with few eigenvalues, see [2, Chapter 15].

This paper is organized as follows. In Section 2, we obtain some necessary conditions for the existence of difference sets with exactly three nontrivial character values. Then, according to Lemma 2, we split our discussion into three cases, which are handled separately in Sections 3-5. Another three special cases are considered in Section 6. A brief conclusion will be given in the last section.

2 Necessary Conditions

In the following, we will always assume that $\chi(D)$ takes exactly three nontrivial character values, denoted by a, b and c , when χ ranges over all the nonprincipal characters of G .

Here we fix some notation. Write \mathbb{Z}_m^* for the multiplicative group of units in \mathbb{Z}_m . For each $t \in \mathbb{Z}_m^*$, define $\sigma_t \in \text{Gal}(\mathcal{Q}(\xi_m)/\mathcal{Q})$ by $\sigma_t(\xi_m) = \xi_m^t$, and every element in $\text{Gal}(\mathcal{Q}(\xi_m)/\mathcal{Q})$ is of this form. For each $\chi \in \widehat{G}$, define $\chi^t(x) = \sigma_t(\chi(x))$ for all $x \in G$, which is also a character of G . For a subset U of \widehat{G} , set $U^{(t)} = \{\chi^t \mid \chi \in U\}$. We define

$$U_z = \{\chi \in \widehat{G} \setminus \{\chi_0\} \mid \chi(D) = z\}$$

for each z in $\{a, b, c\}$. Then, U_a, U_b and U_c form a partition of nonprincipal characters of G . For each character χ of G , χ^{-1} is also a character of G , and $\chi^{-1}(D) = \sigma_{-1}(\chi(D))$, therefore we have

$$\{\sigma_{-1}(a), \sigma_{-1}(b), \sigma_{-1}(c)\} = \{a, b, c\}.$$

Then, at least one element of $\{a, b, c\}$ is fixed by σ_{-1} . Without loss of generality, we assume that $c = \sigma_{-1}(c)$, i.e., c is a real number. Then, we see that $b = \sigma_{-1}(a) = \bar{a}$, and $U_a^{(-1)} = U_b$.

Let χ be a character in U_c , that is $\chi(D) = c$. Together with $\chi(D)\overline{\chi(D)} = n$, we obtain $c = \pm\sqrt{n}$. By taking $G \setminus D$ instead of D , which is also a difference set, we may

assume that $c = \sqrt{n}$. Clearly neither of a, \bar{a} is equal to $\pm\sqrt{n}$. Since $\text{Gal}(Q(\xi_m)/Q)$ is abelian, we have

$$\sigma_{-1}(\chi^t(D)) = \sigma_t \sigma_{-1}(\chi(D)) = \chi^t(D),$$

for each $t \in \mathbb{Z}_m^*$. It now follows that $\sigma_t(\sqrt{n}) = \chi^t(D) = \sqrt{n}$ for each $t \in \mathbb{Z}_m^*$. Hence the number \sqrt{n} is an integer and $U_c^{(t)} = U_c$ for each $t \in \mathbb{Z}_m^*$.

Similarly, it is easy to see that the subgroup $T := \{t \in \mathbb{Z}_m^* \mid \sigma_t(a) = a\}$ has index 2 in \mathbb{Z}_m^* and $U_a^{(t)} = U_a, U_b^{(t)} = U_b$ for each $t \in T$. Consequently, for each $\chi \in \widehat{G}$ and $t \in T$, we have $\chi(D) = \chi^t(D) = \chi(D^{(t)})$. By the inversion formula, we infer that D is fixed by T , namely, $D^{(t)} = D$ for each $t \in T$. On the other hand, we see that $Q(a)$ is a quadratic subfield of $Q(\xi_m)$, by the fundamental theorem of Galois theory. Then, $Q(a) = Q(\sqrt{d})$ for some squarefree integer d . Now we recall some well-known results about quadratic and cyclotomic fields, which can be found in any standard textbook on algebraic number theory, e.g. [11].

The ring of algebraic integers of $Q(\sqrt{d})$ is $\mathbb{Z}[1, \beta]$ with

$$\beta = \begin{cases} \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod{4}, \\ (-1 + \sqrt{d})/2, & \text{if } d \equiv 1 \pmod{4}, \end{cases}$$

and the discriminant of $Q(\sqrt{d})$ is

$$\Delta_d = \begin{cases} 4d, & \text{if } d \equiv 2, 3 \pmod{4}, \\ d, & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

The discriminant of $Q(\xi_m)$ is equal to

$$(-1)^{\phi(m)/2} \frac{m^{\phi(m)}}{\prod_{p|m} p^{\phi(m)/(p-1)}},$$

which has the same prime divisor with m ; unless $m \equiv 2 \pmod{4}$ in which case it has the same odd prime divisor with m . Because a prime p ramifies in a field if and only if p divides the discriminant of this field, we see that each prime divisor of Δ_d is a divisor of m . It follows that $d|m$, and $4|m$ when Δ_d is even.

For each prime $p|m$, we denote the Sylow p -subgroup of G by G_p , and write $G = G_p \times W$, where W is a subgroup of order w and w is coprime to p . We also assume that $|G_p| = p^s$ for some integer s . Now we come to our first result, which is about the discriminant of the quadratic field $Q(a)$.

Lemma 2 *The discriminant Δ_d of the quadratic field $Q(a)$ has only one prime divisor. Moreover, we have*

$$\Delta_d = \begin{cases} -p, & \text{if } d = -p, \text{ where } p \text{ is a prime } \equiv 3 \pmod{4}, \\ -8, & \text{if } d = -2, \\ -4, & \text{if } d = -1, \end{cases}$$

and the above three cases are the only possibilities.

Proof Take any prime p dividing Δ_d , and assume that $p^s \parallel v$. For each nonprincipal character χ which is principal on G_p , $\chi(D)$ is in the field $Q(\xi_{\text{ord}(\chi)})$ whose discriminant is coprime to p . It follows that $\chi(D) = \sqrt{n}$. Using the inversion formula, we can check that the homomorphic image of D in $\bar{G} = G/G_p$ is

$$\bar{D} = \sqrt{n} + \frac{k - \sqrt{n}}{w} \bar{G}, \quad (1)$$

where $w = v/p^s$. It follows that

$$w \mid (k - \sqrt{n}),$$

equivalently, $v \mid p^s(k - \sqrt{n})$. If there is another prime divisor q of Δ_d , then $v \mid q^r(k - \sqrt{n})$ with $q^r \parallel v$. Because $\gcd(p^s, q^r) = 1$, $v \mid (k - \sqrt{n})$ which is false. Therefore Δ_d has only one prime divisor. It follows that either $\Delta_d = \pm p$ for an odd prime p or $\Delta_d = \pm 2^r$ for some integer $r \geq 2$. Correspondingly, we have

- (a) $d = p^* = \left(\frac{-1}{p}\right)p$, or
- (b) $d \in \{-1, -2, 2\}$.

In case (a), we have $p \equiv 3 \pmod{4}$, since $\sigma_{-1}(\sqrt{p^*}) = \left(\frac{-1}{p}\right)\sqrt{p^*} = -\sqrt{p^*}$. In case (b), $d = 2$ does not occur, since $\sqrt{2} = \xi_8 + \xi_8^7$ is fixed by σ_{-1} . Hence we are only left with the three cases listed in this lemma. \square

Remark: When $p = 2$, the assertion $v \mid 2^s(k - \sqrt{n})$ in the above proof can be improved. Let N be a subgroup of order 2^{s-t} such that the Sylow 2-subgroup of $\bar{G} = G/N$ is elementary abelian. Let $rk_2(G)$ denote the maximum possible integer for such t . Then the same argument as above will give $\bar{D} = \sqrt{n} + \frac{k - \sqrt{n}}{v/2^{s-rk_2(G)}} \bar{G}$. It now follows $v \mid (2^{s-rk_2(G)}(k - \sqrt{n}))$.

When a is a pure imaginary number, we have the following result.

Lemma 3 *If $a + \bar{a} = 0$, then $d = -1$.*

Proof From $a + \bar{a} = 0$ and $a\bar{a} = n$, it follows that $a = \pm i\sqrt{n}$. So we obtain $Q(a) = Q(\sqrt{-1})$, then $d = -1$. \square

For convenience, we introduce some other notations: $\Delta = 2\sqrt{n} - a - \bar{a}$, $\Omega = (v(\sqrt{n} - a))/((a - \bar{a})\Delta)$, and $R = (k - \sqrt{n})/\Delta$. Write $D = \sum_{g \in G} d_g g$ and $D^{(-1)} = \sum_{g \in G} d'_g g$, with each of d_g, d'_g being 0 or 1. From the inversion formula, we have the following equations for each element $g \in G$:

$$\begin{aligned} vd_g &= ag^{-1}(U_a) + \bar{a}g^{-1}(U_b) + \sqrt{n}g^{-1}(U_c) + k; \\ vd'_g &= \bar{a}g^{-1}(U_a) + ag^{-1}(U_b) + \sqrt{n}g^{-1}(U_c) + k; \\ v\delta_g &= g^{-1}(U_a) + g^{-1}(U_b) + g^{-1}(U_c) + 1. \end{aligned}$$

Here $g^{-1}(U_z) = \sum_{\chi \in U_z} \chi(g^{-1})$ for $z = a, b, c$; $\delta_g = 1$ if $g = 1_G$ and 0 otherwise. Then we get

Table 1

| d_g | d'_g | $g^{-1}(U_a)$ | $g^{-1}(U_b)$ | $g^{-1}(U_c)$ |
|-------|--------|-------------------------|-------------------------|------------------------------|
| 1 | 1 | $-\frac{v}{\Delta} + R$ | $-\frac{v}{\Delta} + R$ | $\frac{2v}{\Delta} - 1 - 2R$ |
| 1 | 0 | $\Omega + R$ | $\bar{\Omega} + R$ | $\frac{v}{\Delta} - 1 - 2R$ |
| 0 | 1 | $\bar{\Omega} + R$ | $\Omega + R$ | $\frac{v}{\Delta} - 1 - 2R$ |
| 0 | 0 | R | R | $-1 - 2R$ |

$$g^{-1}(U_a) = \frac{(vd_g - k + \bar{a})(\sqrt{n} - a) - (vd'_g - k + a)(\sqrt{n} - \bar{a}) - \sqrt{n}\bar{a}v\delta_g + \sqrt{n}av\delta_g}{(a - \bar{a})\Delta},$$

$$g^{-1}(U_b) = \overline{g^{-1}(U_a)};$$

$$g^{-1}(U_c) = \frac{v(d_g + d'_g) - (a + \bar{a})(v\delta_g - 1) - 2k}{\Delta}.$$

Especially, when $g = 1_G$, the above equations give

$$|U_a| = |U_b| = \frac{v(\sqrt{n} - d_1)}{\Delta} + R, \text{ and}$$

$$|U_c| = \frac{v(2d_1 - a - \bar{a})}{\Delta} - 1 - 2R.$$

When $g \neq 1_G$, we divide it into four cases depending on the values of d_g, d'_g as listed in Table 1.

It is easy to see that $|D \cap D^{(-1)}|$ is just the coefficient of 1_G in D^2 . Using the inversion formula, we have

$$v|D \cap D^{(-1)}| = |U_a|(a^2 + \bar{a}^2) + |U_c|n + k^2,$$

which implies

$$\begin{aligned} |D \cap D^{(-1)}| &= \frac{|U_a|(a^2 + \bar{a}^2) + |U_c|n + k^2}{v} \\ &= \frac{1}{v} [|U_a|(-2n + a^2 + \bar{a}^2) + (2|U_a| + |U_c|)n + k^2] \\ &= \frac{1}{v} [-(v(\sqrt{n} - d_1) + k - \sqrt{n})(2\sqrt{n} + a + \bar{a}) + (v - 1)n + k^2] \\ &= k - (\sqrt{n} - d_1 + \frac{k - \sqrt{n}}{v})(2\sqrt{n} + a + \bar{a}) < k \end{aligned}$$

Next, we define the following sets, which form a partition of $G \setminus \{1_G\}$: $E_1 = D \cap D^{(-1)} \setminus \{1_G\}$, $E_2 = D \setminus D^{(-1)}$, $E_3 = D^{(-1)} \setminus D$, and $E_4 = G \setminus (D \cup D^{(-1)} \cup \{1_G\})$. Neither E_2 nor E_3 is empty, otherwise, $D = D^{(-1)} = D \cap D^{(-1)}$ implies that $\chi(D) = \chi(D^{(-1)}) = \chi(D)$, which is false for $\chi \in U_a$. Similarly, at least one of E_1, E_4 is not

empty, or else $D + D^{(-1)} = G - 1 + 2d_1$ and $\chi(D)$ takes only two character values when χ ranges over all nonprincipal characters. Therefore, at least three of $E_i, 1 \leq i \leq 4$ are not empty.

It is worthy to notice that when E_1 is empty but E_4 is not, we obtain a 3-class association scheme on \widehat{G} . Suppose \widehat{G} has conjugate classes $\{C_0, C_1, \dots, C_d\}$, where $C_0 = \{\chi_0\}$. Define the i -th relation R_i by $(x, y) \in R_i$ if and only if $yx^{-1} \in C_i$. It is well known that $\mathcal{X} = (\widehat{G}, \{R_i\}_{0 \leq i \leq d})$ is a d -class association scheme [3]. Set $\overline{C}_i = \sum_{x \in C_i} x$ for $0 \leq i \leq d$. Then $\{\overline{C}_0, \overline{C}_1, \dots, \overline{C}_d\}$ forms a Schur ring [14]. With the Bannai-Muzychuk criterion [1, 16] and the information provided in Table 1, we obtain a 3-class fusion scheme of \mathcal{X} whose first eigenmatrix is

$$P = \begin{matrix} & \chi_0 & U_a & U_b & U_c \\ \begin{matrix} 1 \\ E_2 \\ E_3 \\ E_4 \end{matrix} & \begin{pmatrix} 1 & \frac{v(n-d_1)}{\Delta} + R & \frac{v(n-d_1)}{\Delta} + R & \frac{v(2d_1-a-\bar{a})}{\Delta} - 1 - 2R \\ 1 & \Omega + R & \bar{\Omega} + R & \frac{v}{\Delta} - 1 - 2R \\ 1 & \bar{\Omega} + R & \Omega + R & \frac{v}{\Delta} - 1 - 2R \\ 1 & R & R & -1 - 2R \end{pmatrix} \end{matrix},$$

with

$$\det P = \frac{v^3}{(a-\bar{a})\Delta}.$$

In the following, we will derive some necessary conditions from the above discussions. We fix p to be the only prime divisor of Δ_d . Write $\sqrt{n} = p^x u$ for some nonnegative integer x and $(p, u) = 1$, i.e., $p^x \parallel \sqrt{n}$.

- (1) $\Delta | v, \Delta | (k - \sqrt{n}), (a - \bar{a})\Delta | v(\sqrt{n} - a), (a - \bar{a}) | v$.

These follow from the fact that all entries in Table 1 are algebraic integers, E_2 and E_3 are not empty, and at least one of E_1, E_4 is not empty. The last one comes from $\Omega - \bar{\Omega} = v/(a - \bar{a})$.

- (2) $v | (k - \sqrt{n})(2\sqrt{n} + a + \bar{a})$.

This is because $|D \cap D^{(-1)}|$ is an integer.

- (3) $w | (k - \sqrt{n}), \sqrt{n} + \frac{k - \sqrt{n}}{w} \leq p^s$.

These come from the proof of Lemma 2. If $p = 2$, we actually have $v | 2^{s-rk_2(G)}(k - \sqrt{n})$ and $\sqrt{n} + \frac{k - \sqrt{n}}{2^{rk_2(G)} w} \leq 2^{s-rk_2(G)}$.

- (4) $p | (2\sqrt{n} + a + \bar{a})$.

Suppose not, then we obtain $p^s | (k - \sqrt{n})$ from (2). Recall that $w | (k - \sqrt{n})$, so we have $v | (k - \sqrt{n})$ which is false.

- (5) $1 \leq 2d_1 - a - \bar{a}$.

This comes from $g^{-1}(U_c)$ is an integer not exceeding $|U_c|$.

Now, we give some information about other prime divisors of the order of G .

Proposition 2 *If q is another prime divisor of v , then we have $q|(a - \bar{a})$.*

Proof Assume that $q \nmid (a - \bar{a})$, then $q \nmid (\sqrt{n} - a)$, since otherwise $q | (\sqrt{n} - \bar{a})$, and their difference gives $q | (a - \bar{a})$. Let

$$\bar{G} = G_p \times \langle \alpha : \alpha^q = 1 \rangle$$

be a quotient group of G , and define \bar{D} as the image of D in \bar{G} .

When $d = -p$ where p is an odd prime, we have $\sigma_t(\sqrt{d}) = \sigma_t(\sqrt{p^*}) = (\frac{t}{p})\sqrt{p^*}$. Thus $T = \{t \in \mathbb{Z}_m^* | (\frac{t}{p}) = 1\}$. When $d = -1$, we have $p = 2$ and $4|m$. Since $\sqrt{-1} = \xi_4$, we see that $T = \{t \in \mathbb{Z}_m^* | t \equiv 1 \pmod{4}\}$. When $d = -2$, we have $p = 2$ and $8|m$. Since $\sqrt{-2} = \xi_8 + \xi_8^3$, we see that $T = \{t \in \mathbb{Z}_m^* | t \equiv 1, 3 \pmod{8}\}$. So whenever $d = -p, -1$ or -2 , we can find an integer $t \in T$ satisfying that $t \equiv 1 \pmod{p^s}$ and $t \equiv i \pmod{q}$ for any integer $i, 1 \leq i \leq q-1$, by the Chinese remainder theorem.

Because D is fixed by T , we can see that

$$\bar{D} = D_0 + D_1(\langle \alpha \rangle - 1)$$

with $D_0, D_1 \in \mathbb{Z}[G_p]$. Denote $\langle \alpha : \alpha^q = 1 \rangle$ by G_q , let $\chi_1 \times \chi_2$ be a character of $G_p \times G_q$. When χ_2 is principal on G_q , we have

$$\chi_1(D_0) + (q-1)\chi_1(D_1) = c_1, \quad (2)$$

otherwise,

$$\chi_1(D_0) - \chi_1(D_1) = c_2. \quad (3)$$

$c_1, c_2 \in \{a, \bar{a}, \sqrt{n}\}$ if χ_1 is nonprincipal on G_p , and $c_1 = k, c_2 = \sqrt{n}$ otherwise. When χ_1 is nonprincipal, by taking the difference of the above two equations, we obtain $q\chi_1(D_1) = c_1 - c_2$, and our assumption forces $c_1 = c_2$ and $\chi_1(D_1) = 0$. When χ_1 is principal, we have $\chi_1(D_1) = (k - \sqrt{n})/q$. Hence

$$D_1 = \frac{k - \sqrt{n}}{qp^s} G_p,$$

which gives $p^s | (k - \sqrt{n})$. Together with $w | (k - \sqrt{n})$, we obtain $v | (k - \sqrt{n})$ which is false. Therefore $q | (a - \bar{a})$. \square

In addition, we obtain some results about the multiplier of D .

Proposition 3 *If there is a prime q dividing n and coprime to v , then we have $D^{(q)} = D$.*

Proof Let Q be a prime ideal in $\mathbb{Z}[\xi_v]$ lying over q , then $Q|n$ since $q|n$. If $Q|a$ and $Q|\bar{a}$, then $Q|\Delta = 2\sqrt{n} - a - \bar{a}$, which gives $Q|v$ since $\Delta|v$. This contradicts to the fact that q is coprime to v . For $n = a\bar{a}$, we have Q divides exactly one of a, \bar{a} . Since $\sigma_q(Q) = Q$, $\{\sigma_q(a), \sigma_q(\bar{a})\} = \{a, \bar{a}\}$, we must have $\sigma_q(a) = a$ and $\sigma_q(\bar{a}) = \bar{a}$. Because $\chi(D^{(q)}) = \sigma_q(\chi(D))$, and $\chi(D)$ only takes the values k, \sqrt{n}, a , and \bar{a} , we immediately get that $\chi(D^{(q)}) = \chi(D)$ when χ ranges over all characters. It follows from the inversion formula that $D^{(q)} = D$. \square

Remark: (1) For such a prime q as in Proposition 3, from $\sigma_q(a) = a$, namely $\sigma_q(\sqrt{d}) = \sqrt{d}$, we have: $\left(\frac{q}{p}\right) = 1$ if $d = -p$ with p odd; $q \equiv 1, 3 \pmod{8}$ if $d = -2$; $q \equiv 1 \pmod{4}$ if $d = -1$.

(2) Let $q \neq p$ be a prime divisor of both n and v . Then by Proposition 2, we have $q|(a - \bar{a})$. From $4n = (a + \bar{a})^2 - (a - \bar{a})^2$, we get $q|(a + \bar{a})$. If q is odd, then $q|a$ and $q|\bar{a}$.

3 The case $d = -p$

We will first deal with the case $d = -p$ with p an odd prime in this section. Assume $d = -p$, where p is a prime $\equiv 3 \pmod{4}$.

From Lemma 3, we know $a + \bar{a} \neq 0$. Some new inequalities and necessary conditions about divisibility can be obtained under this assumption. We will use notation like (6') to mark the property which is obtained under the additional assumption that p is an odd prime. For an integer l , the greatest integer z such that p^z divides l is denoted by $\text{ord}_p(l)$.

(6') $p^x|(a + \bar{a}), p^x|(a - \bar{a}), p^x|k, s \geq x + 1$.

Since $d = -p \equiv 1 \pmod{4}$ and a is an algebraic integer in $\mathbb{Q}(\sqrt{-p})$, there exist integers e and f such that $a = e + f\frac{-1 + \sqrt{-p}}{2}$. Then $a - \bar{a} = f\sqrt{-p}$, so $\text{ord}_p((a - \bar{a})^2)$ is odd. The fact $p^x|(a + \bar{a})$ follows from $4n = (a + \bar{a})^2 - (a - \bar{a})^2$ and $\text{ord}_p((a - \bar{a})^2)$ is odd. Then we obtain $p^x|(a - \bar{a})$ and $p^x|\Delta = 2\sqrt{n} - a - \bar{a}$. Since $\Delta|(k - \sqrt{n})$, we have $p^x|(k - \sqrt{n})$ and hence $p^x|k$. From $w|(k - \sqrt{n})$, we have $v|(k - \sqrt{n})p^{s-x}$, which gives $s \geq x + 1$.

(7') $x \geq 1$.

Take a character $\chi \in U_a$, and let χ' be another character of G which coincides with χ on W and is principal on G_p . Thus we see $\chi(D) = a$ and $\chi'(D) = \sqrt{n}$. Since $(1 - \xi_{p^s})|(\chi(D) - \chi'(D))$, it follows that $(1 - \xi_{p^s})|(a - \sqrt{n})$. Similarly, we have $(1 - \xi_{p^s})|(\bar{a} - \sqrt{n})$. Recall that $p|(2\sqrt{n} + a + \bar{a})$, we have $(1 - \xi_{p^s})|4\sqrt{n}$. Therefore we get $p|4\sqrt{n}$, which implies $x \geq 1$.

(8') $p^x|k, p^x|(k - \sqrt{n}), p^s|(k + \sqrt{n}), \Delta|p^xw$.

From $k(k - 1) = \lambda(v - 1)$ and (1), we have $\text{ord}_p(\lambda) = \text{ord}_p(k) \geq x$. Then from $k^2 = n + \lambda v$ and $\text{ord}_p(n) = 2x < x + s \leq \text{ord}_p(\lambda v)$, we get $\text{ord}_p(k) = x$, i.e., $p^x|k$. Because $(k + \sqrt{n}) - (k - \sqrt{n}) = 2\sqrt{n}$, we readily verify that at least one of $\text{ord}_p(k + \sqrt{n}), \text{ord}_p(k - \sqrt{n})$ is x . Now $k^2 - n = (k + \sqrt{n})(k - \sqrt{n}) = \lambda v$ gives $\{\text{ord}_p(k + \sqrt{n}), \text{ord}_p(k - \sqrt{n})\} = \{s, x\}$. If $\text{ord}_p(k - \sqrt{n}) = s$, together with $w|(k - \sqrt{n})$ we get $v|(k - \sqrt{n})$ which is false. Hence we have $\text{ord}_p(k + \sqrt{n}) = s, \text{ord}_p(k - \sqrt{n}) = x$. The last one $\Delta|p^xw$ follows from $\Delta|v, \Delta|(k - \sqrt{n})$, and $\text{gcd}(v, (k - \sqrt{n})) = p^xw$.

Now define

$$\gamma = \frac{k - \sqrt{n}}{p^xw}.$$

From $w|(k - \sqrt{n})$ and $p^x|(k - \sqrt{n})$, we see that γ is an integer coprime to p . From

$$(k + \sqrt{n})(k - \sqrt{n}) = k^2 - n = \lambda v,$$

we have

$$\gamma(\lambda + n + \sqrt{n}) = p^{s-x}\lambda,$$

which gives

$$\begin{aligned} \lambda &= \frac{(n + \sqrt{n})\gamma}{p^{s-x} - \gamma} = \frac{(p^x u + 1)u}{p^{s-x} - \gamma} p^x \gamma, \\ k = n + \lambda &= \frac{p^{s-x}n + \sqrt{n}\gamma}{p^{s-x} - \gamma} = \frac{(p^s u + \gamma)u}{p^{s-x} - \gamma} p^x, \text{ and} \\ w &= \frac{k - \sqrt{n}}{p^x \gamma} = \frac{(n - \sqrt{n})p^{s-x} + 2\sqrt{n}\gamma}{\gamma p^x (p^{s-x} - \gamma)} = \frac{(p^x u - 1)u}{\gamma} + \frac{(p^x u + 1)u}{p^{s-x} - \gamma}. \end{aligned}$$

Write

$$\begin{aligned} a + \bar{a} &= -p^x \alpha, \\ a - \bar{a} &= \eta p^x \sqrt{-p} \end{aligned}$$

with $\alpha, \eta \in \mathbb{Z}$ and $\gcd(p, \alpha) = 1$. Because $1 \leq 2d_1 - a - \bar{a}$ and $a + \bar{a} \neq 0$, we must have $\alpha \geq 1$. From $4n = (a + \bar{a})^2 - (a - \bar{a})^2$, $4u^2 = \alpha^2 + p\eta^2$. From Proposition 2, we see that $\pi(w) = \pi(\eta) \setminus \{p\}$, where $\pi(w)$ denotes the set of prime divisors of w . From $\sqrt{n} + \frac{k - \sqrt{n}}{w} \leq p^s$ we get $u + \gamma \leq p^{s-x}$. After simplification, we reduce the above conditions to the following list:

$$\begin{array}{llll} \gamma|(p^x u - 1)u, & (p^{s-x} - \gamma)|(p^x u + 1)u, & 2u + \alpha|w, & p^{s-x}|p^x(2u - \alpha), \\ \eta|p^{s-x}w, & u + \gamma \leq p^{s-x}, & \pi(w) = \pi(\eta) \setminus \{p\}, & 4u^2 = \alpha^2 + p\eta^2, \\ s \geq x + 1, & x \geq 1, & \alpha \geq 1. & \end{array}$$

Remark: (1) From the expression of w , we can show that roughly $v \geq 4n$ with γ as a variable in $(0, p^{s-x})$, and minimized when $\gamma = \frac{p^{s-x}}{2}$. We have made no use of divisibility conditions involving a, \bar{a} . Notice that $w > 1$, since $(2u + \alpha)|w$ and $2u + \alpha \geq 3$, i.e., G can not be a p -group.

(2) From $|U_c| \geq 0$, we get $v \geq \frac{2k - 2d_1}{2d_1 + p^s \alpha} + 1$, which is trivial.

(3) Let $\widetilde{U}_c = \{\chi \in U_c \mid \chi \text{ is nonprincipal on } G_p\}$. Then we have $|\widetilde{U}_c| = |U_c| - (|W| - 1)$. We define a group action of $\mathbb{Z}_{p^s}^*$ on \widetilde{U}_c by $(t, \chi) \rightarrow \chi^t$, where $t \in \mathbb{Z}_{p^s}^*, \chi \in \widetilde{U}_c$. It is not hard to see that each orbit has length divisible by $p - 1$. Therefore we obtain $(p - 1)|(w(d_1 - \sqrt{n}) - (k - \sqrt{n}))$.

(4) Let $T_1 = \{t \in \mathbb{Z}_{p^s}^* \mid \binom{t}{p} = 1\}$. We can define a similar group action of T_1 on U_a as above. By an analogous argument, we have $\frac{p-1}{2} | (w\sqrt{n} - d_1 + (k - \sqrt{n}))$.

At this point, we may speculate all the known parameter sets. We have conducted a computer search for $p \in \{3, 5, 7, 11, 13, 17, 19\}$, $1 \leq x, s \leq 10$, $1 \leq \alpha, \eta \leq 10^4$, and failed to find a parameter set satisfying all the conditions listed in this section and Section 2. The examples below indicate that such parameter sets might exist.

Example 1 Take $p = 7$, $\alpha = 8$, $\eta = 24$, $u = 32$, $\gamma = 4$, $v = 2^3 \cdot 3^5 \cdot 7^3$, $k = 54656$, $n = 2^{10} \cdot 7^2$. In this case, $3|w$, $3|(k - \sqrt{n})$, and with $d_1 = 0$, all the conditions in this section are satisfied except $(\sqrt{n} - d_1 + \frac{k - \sqrt{n}}{v})(2\sqrt{n} + a + \bar{a}) \leq k$.

Example 2 Take $p = 11$, $\alpha = 30$, $\eta = 48$, $u = 81$, $\gamma = 980$, $v = 2^{10} \cdot 3 \cdot 11^5$, $k = 364287561$, $n = 3^8 \cdot 11^4$. In this case, $w \equiv 2 \pmod{5}$, $\sqrt{n} \equiv 1 \pmod{5}$, $5|(k - \sqrt{n})$, and with $d_1 = 1$, all the conditions in this section are satisfied except $\frac{p-1}{2} | (w\sqrt{n} - d_1 + (k - \sqrt{n}))$.

4 The case $d = -2$

Secondly, we move to the case $d = -2$ and $\Delta_d = -8$. Under such assumption, we have already known that $4|m$ in Section 2. Define

$$l = \min\{|N| \mid G/N \text{ has exponent strictly divisible by } 4\}.$$

So we may choose such a subgroup N of G , whose order is l and G/N has exponent strictly divisible by 4. Considering the homomorphic image of D in G/N , as in the proof of Lemma 2, we obtain

$$\sqrt{n} + \frac{k - \sqrt{n}}{v/l} \leq l.$$

Clearly l is a power of 2 and $l \leq 2^{s-2}$. Since $\sqrt{-2}$ lies in $Q(\xi_8)$ but not in $Q(\xi_4)$, we must have $8|\exp(G)$, hence $l \geq 2$. Write $k = \sqrt{n} + \gamma\Delta$ for some positive integer γ , and $a = u_1 + u_2\sqrt{-2}$, where u_1, u_2 are two integers satisfying $u_1^2 + 2u_2^2 = n$. We have run a computer search for $-10^4 \leq u_1 \leq 1$, $1 \leq u_2$, $\gamma \leq 10^4$ and found many parameter sets such that all the conditions in Section 2 are satisfied. Here we only give the following examples.

Example 3 When $a = 96(-1 + 2\sqrt{-2})t$, $\gamma = 216t$, $n = 2^{10} \cdot 3^4 \cdot t^2$, $v = 4n$, with $t \in \{2^i, 12 \cdot 2^i, 20 \cdot 2^i \mid i \text{ is a nonnegative integer}\}$, all the conditions in Section 2 are satisfied. If $t = 1$, we get $l \geq \frac{1}{(1 - \frac{k - \sqrt{n}}{v})} \sqrt{n} = 2\sqrt{n} = 2^6 \cdot 3^2$, so $l \geq 2^{10}$. This forces the Sylow 2-subgroup of G to be cyclic, but a result of Turyn says such difference sets do not exist, cf. [17, Theorem 2.4.11].

Example 4 When $a = 192(-7 + 4\sqrt{-2})t$, $\gamma = 972t$, $n = 2^{12} \cdot 3^6 \cdot t^2$, $v = 4n$, $k = 2n + \sqrt{n}$, with $t = 2^i$ for some nonnegative integer i , all the conditions in Section 2 are satisfied. If $t = 1$, $l \geq 2\sqrt{n} = 2^7 \cdot 3^3$, then $l \geq 2^{12}$. This is ruled out similarly as above.

5 The case $d = -1$

The analysis of the case $d = -1$ and $\Delta_d = -4$ is almost the same as that of the case $d = -2$. First define

$$l = \min\{|N| \mid G/N \text{ has exponent strictly divisible by } 2\}.$$

Similarly as above, we have

$$\sqrt{n} + \frac{k - \sqrt{n}}{v/l} \leq l, \quad (4)$$

where l is a power of 2, and $2 \leq l \leq 2^{s-1}$. Write $k = \sqrt{n} + \gamma\Delta$ for some positive integer γ , and $a = u_1 + u_2\sqrt{-1}$, where u_1, u_2 are two integers satisfying $u_1^2 + u_2^2 = n$. We have also run a computer search for $-10^4 \leq u_1 \leq 1, 1 \leq u_2, \gamma \leq 10^4$ and found many parameter sets such that all the conditions in Section 2 are satisfied. Here we give an example.

Example 5 When $a = 160(-3 + 4\sqrt{-1})t$, $\gamma = 500t$, $n = 2^{10} \cdot 5^4 \cdot t^2$, $v = 4n$, $k = 2n + \sqrt{n}$, with $t = 2^i$ for some nonnegative integer i , all the conditions in Section 2 are satisfied.

6 Special Cases

Three special cases will be considered in this section:

- (1) D is a Hadamard difference set with $a + \bar{a} = 0$;
- (2) G is a p -group;
- (3) $U_c \cup \{\chi_0\}$ is a subgroup of \widehat{G} .

6.1 D is a Hadamard difference set with $a + \bar{a} = 0$

Now let D be a Hadamard difference set with three nontrivial character values \sqrt{n} , a and \bar{a} . Assume that $a + \bar{a} = 0$, then by Lemma 3, we obtain $d = -1$ and $a = \pm i\sqrt{n}$. Since $1 \leq 2d_1 - a - \bar{a}$, we infer that $d_1 = 1$, i.e., $1_G \in D$. In the following, we split the discussion into two cases, according to the parameter of D .

If D is with parameter $(v, k, \lambda) = (4n, 2n + \sqrt{n}, n + \sqrt{n})$, then it will satisfy all the necessary conditions, with the only possible exception of (4), which becomes $l \geq 2\sqrt{n}$ here. Using formulas of Section 2, we have $\Delta = 2\sqrt{n}$, $R = \sqrt{n}$, $\Omega = -\sqrt{n} - i\sqrt{n}$, and $|D \cap D^{(-1)}| = 2\sqrt{n}$. Denote $H = D + D^{(-1)} - G$. There are $2\sqrt{n}$ ($= |D \cap D^{(-1)}|$) elements whose coefficients are 1 in H , and others have coefficients 0 or -1 . In addition, the sum of all coefficients of H is $2\sqrt{n}$ ($= 2k - v$). It now follows that H is a subset of G . Denote $M = U_c \cup \{\chi_0\}$, then we have

$$\chi(H) = \chi(D) + \chi(D^{(-1)}) - \chi(G) = \begin{cases} 2\sqrt{n}, & \text{if } \chi \in M; \\ 0, & \text{if } \chi \in \widehat{G} \setminus M. \end{cases}$$

Using inversion formula, we find $H^2 = 2\sqrt{n}H$ and $H = H^{(-1)}$, i.e., H is a subgroup of G with order $2\sqrt{n}$. We say a character of G annihilates a subgroup H of G if $\chi(h) = 1$ for all $h \in H$. The set of all characters of G annihilating H is called the annihilator of H in \widehat{G} , and denoted by H^\perp . We readily verify that $M = H^\perp$ and $|M| = |H^\perp| = |G/H| = 2\sqrt{n}$. As in the proof of Lemma 2, we find $G_2^\perp \leq M$ and $|G_2^\perp| = |G/G_2| = w$. This leads to $w|2\sqrt{n}$, which implies $w^2|4n$. We recall that $v = 4n = 2^s w$,

Table 2

| d_g | d'_g | $g^{-1}(U_a)$ | $g^{-1}(U_b)$ | $g^{-1}(U_c)$ |
|-------|--------|---------------|---------------|---------------|
| 1 | 1 | $-\sqrt{n}$ | $-\sqrt{n}$ | $2\sqrt{n}-1$ |
| 1 | 0 | $-i\sqrt{n}$ | $i\sqrt{n}$ | -1 |
| 0 | 1 | $i\sqrt{n}$ | $-i\sqrt{n}$ | -1 |

where $\gcd(w, 2) = 1$. Hence we find $w^2 | 2^s w$, which implies $w = 1$, i.e., G is a 2-group. It now follows that $n = 2^{s-2}$, and s must be even, since \sqrt{n} is an integer. Write $s = 2m$ for some nonnegative integer m , then D is with parameter $(v, k, \lambda) = (2^{2m}, 2^{2m-1} + 2^{m-1}, 2^{2m-2} + 2^{m-1})$. In this case, we may update Table 1 to get Table 2.

On the other hand, if D is with parameter $(v, k, \lambda) = (4n, 2n - \sqrt{n}, n - \sqrt{n})$, then it can be ruled out. First we find $|D \cap D^{(-1)}| = 1$ and $|U_c| = 1 + 2\sqrt{n}$. Denote $H = G + 1 - D - D^{(-1)}$. Clearly H is a subset of G with size $1 + 2\sqrt{n}$. We readily verify

$$\chi(H) = \begin{cases} 1 + 2\sqrt{n}, & \text{if } \chi = \chi_0, \\ 1 - 2\sqrt{n}, & \text{if } \chi \in U_c, \\ 1, & \text{if } \chi \in \widehat{G} \setminus M, \end{cases}$$

and

$$g^{-1}(U_c) = \begin{cases} 1 + 2\sqrt{n}, & \text{if } g = 1_G, \\ 1 - 2\sqrt{n}, & \text{if } g \in H, \\ 1, & \text{if } g \in G \setminus H, g \neq 1_G. \end{cases}$$

By the inversion formula, we obtain

$$H^2 = (2\sqrt{n} - 1) + (2 - 2\sqrt{n})H + 2G,$$

and

$$U_c^2 = (2\sqrt{n} - 1) + (2 - 2\sqrt{n})U_c + 2\widehat{G}.$$

Because the coefficients of H^2 are nonnegative, we have $2 - 2\sqrt{n} + 2 \geq 0$, which gives $\sqrt{n} \leq 2$. If $n = 1$, we have $D = \{1_G\}$, for $|D| = k = 1$ and D contains the identity element. But this contradicts to the fact that $\chi(D)$ takes three values. If $n = 4$, then we get $v = 16$ and $U_c^2 = 3 + 2(\widehat{G} - U_c)$. Hence for every element of U_c , its coefficient in U_c^2 is zero. By the result of Turyn [17, Theorem 2.4.11], G cannot be cyclic, so must be isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. It is a well known fact that $\widehat{G} \cong G$. In each case, there are exactly three elements of order 2 in \widehat{G} , and they all belong to U_c . Hence at least one element of order 2 must has positive coefficient in U_c^2 . This is a contradiction.

The following is a summary of our discussions.

Lemma 4 *Let D be a Hadamard difference set of order n in an abelian group G with three nontrivial character values \sqrt{n} , a and \bar{a} . If $a + \bar{a} = 0$, then D must be with parameter $(v, k, \lambda) = (2^{2m}, 2^{2m-1} + 2^{m-1}, 2^{2m-2} + 2^{m-1})$ for some nonnegative integer m . In particular, G is a 2-group. Let $H = D + D^{(-1)} - G$. Then one has that H is a subgroup of G , and $H^\perp = \chi_0 \cup \{\chi \in G \mid \chi(D) = \sqrt{n}\}$.*

6.2 G is a p -group

Recall that $w > 1$ if p is odd, hence we must have $p = 2$. By a result of Menon [15], the plausible difference set D is with parameter $(v, k, \lambda) = (4n, 2n \pm \sqrt{n}, n \pm \sqrt{n})$. Write $v = 4n = 2^s$ for some nonnegative integer s . Since $\Delta | v$, we can assume $\Delta = 2\sqrt{n} - a - \bar{a} = 2^u$ with u being a nonnegative integer. From $\Delta = 2\sqrt{n} - a - \bar{a} < 4\sqrt{n}$ and $-a - \bar{a} \geq 1 - 2d_1 \geq -1$, we get $2^{s/2} - 1 \leq 2^u < 2^{s/2+1}$. Consequently we find $2^u = 2^{s/2} = 2\sqrt{n}$, implying $a + \bar{a} = 0$. This implies that D satisfies the conditions of Lemma 4. Hence D must be with parameter $(v, k, \lambda) = (2^{2m}, 2^{2m-1} + 2^{m-1}, 2^{2m-2} + 2^{m-1})$ for some nonnegative integer m . Let $H = D + D^{(-1)} - G$. Then one has that H is a subgroup of G , and $H^\perp = \{\chi_0\} \cup \{\chi \in \widehat{G} \mid \chi(D) = \sqrt{n}\}$.

Let us further assume that the exponent of G is 4. From

$$\sqrt{n} + \frac{k - \sqrt{n}}{2^{rk_2(G)}} \leq 2^{s - rk_2(G)},$$

we get $m \geq rk_2(G)$. By the definition of $rk_2(G)$, we can verify that $m = rk_2(G)$, and $G \cong \mathbb{Z}_4^m$. Since $a = \pm i\sqrt{n}$, we find that each $\chi \in U_a \cup U_a^{(-1)}$ has order 4. Notice that $\widehat{G} \cong \mathbb{Z}_4^m$ has $2^m - 1 = 2\sqrt{n} - 1$ elements of order 2 and $|U_c| = 2\sqrt{n} - 1$, therefore U_c contains exactly all the characters of order 2. In other words, $H^\perp = \{\chi_0\} \cup \{\chi \in \widehat{G} \mid \chi(D) = \sqrt{n}\}$ is the unique maximal elementary abelian subgroup in \widehat{G} . Such difference sets have been constructed by Davis and Polhill [10].

6.3 $U_c \cup \{\chi_0\}$ is a subgroup of \widehat{G}

Let $M = U_c \cup \{\chi_0\}$. Since M is a subgroup, we see that $\psi(M)$ can only take two character values 0 and $|M|$, for each nonprincipal character ψ of \widehat{G} . Together with the fact

$$\psi(M) \in \left\{ -2R, \frac{v}{\Delta} - 2R, \frac{2v}{\Delta} - 2R \right\}$$

as listed in Table 1, we readily verify

$$\frac{v}{\Delta} - 2R = 0, \quad \frac{2v}{\Delta} - 2R = |M|,$$

since $-2R < 0$ and $\frac{v}{\Delta} - 2R < \frac{2v}{\Delta} - 2R$. We recall that $R = \frac{k - \sqrt{n}}{\Delta}$, therefore $v = 2(k - \sqrt{n})$ and $|M| = \frac{v}{\Delta}$. On the other hand, we have $k^2 = n + (k - n)v$. It now follows $v = 4n$, $k = 2n + \sqrt{n}$ and $\lambda = n + \sqrt{n}$, i.e., D is with parameter $(v, k, \lambda) = (4n, 2n + \sqrt{n}, n + \sqrt{n})$. Let $H = D + D^{(-1)} - G$. Then we have

$$\chi(H) = \begin{cases} 2\sqrt{n}, & \text{if } \chi \in M; \\ a + \bar{a}, & \text{if } \chi \in \widehat{G} \setminus M. \end{cases}$$

From the inversion formula, we find $H = a + \bar{a} + M^\perp$. Comparing the coefficients of 1_G on each side, it now follows

$$a + \bar{a} = \begin{cases} 0, & \text{if } d_1 = 1; \\ -2, & \text{if } d_1 = 0. \end{cases}$$

If $a + \bar{a} = -2$, then we have $\Delta = 2\sqrt{n} - a - \bar{a} = 2\sqrt{n} + 2$. From $\Delta | (k - \sqrt{n})$, we have $(\sqrt{n} + 1) | n$, which is false. Hence $a + \bar{a} = 0$. Therefore D also satisfies the conditions of Lemma 4.

Now we see that in either of the above two special cases, D satisfies the conditions of Lemma 4. We state the following as a summary of this section.

Theorem 1 *Let D be a difference set in an abelian group G with three nontrivial character values \sqrt{n} , a and \bar{a} . Denote $M = \{\chi_0\} \cup \{\chi \in \widehat{G} \mid \chi(D) = \sqrt{n}\}$. If*

- (1) D is a Hadamard difference set with $a + \bar{a} = 0$, or
- (2) G is a p -group, or
- (3) M is a subgroup of \widehat{G} ,

then D is with parameter $(v, k, \lambda) = (2^{2m}, 2^{2m-1} + 2^{m-1}, 2^{2m-2} + 2^{m-1})$ for some non-negative integer m . In particular, G is a 2-group. Let $H = D + D^{(-1)} - G$. Then one has that H is a subgroup of G and $H^\perp = M$.

7 Conclusion

In this paper, we make an attempt to find difference sets without the character divisibility property. Under the assumption that the difference sets have only three distinct nontrivial character values, we have derived some restrictions on the parameters. It turns out that their character values all lie in the field $Q(\sqrt{-d})$, where $d = 1$, $d = 2$ or d is an odd prime congruent to 3 modulo 4. We have conducted a computer search for plausible parameters satisfying all these conditions. When $d = 1$ or 2, we have found some plausible parameter sets satisfying all our conditions, some of which are listed in Examples 3-5. It is an interesting open problem to rule out these parameter sets or construct a difference set with these parameters. An affirmative answer to this problem will provide difference sets without the character divisibility property. On the other hand, we have not found any parameter set satisfying all the derived conditions when d is an odd prime.

At last, we take an initial step towards the case of $|X| = 4$. Similarly we introduce the following sets

$$U_z = \{\chi \in \widehat{G} \mid \chi(D) = z\}$$

for each $z \in \{a, b, c, d\}$. These four sets will form a partition of $\widehat{G} \setminus \{\chi_0\}$. According to the lengths of orbits of $Gal(Q(\xi_m)/Q)$ acting on a, b, c, d , we may divide it into the following four cases:

- (1) The Galois group $Gal(Q(\xi_m)/Q)$ acts transitively. For each prime $p | v$, there exists a character χ with $\text{ord}(\chi) = p$, so $\chi(D) \in Q(\xi_p)$. From the assumption that $Gal(Q(\xi_m)/Q)$ acts transitively on the set $\{a, b, c, d\}$, we have $\{a, b, c, d\} \subseteq$

$Q(\xi_p)$. If v has another prime divisor q different from p , then there exists a character ψ such that $\text{ord}(\psi) = q$, it follows as above that $\{a, b, c, d\} \subseteq Q(\xi_q)$. Since $Q(\xi_p) \cap Q(\xi_q) = Q$, we have $\{a, b, c, d\} \subseteq Q$, which is a contradiction. Therefore, the order of G must be a prime power, i.e., G is p -group.

- (2) There is one fixed point $\{a\}$, and an orbit of length three $\{b, c, d\}$. Then $a \in Q$. From $a\bar{a} = n$, we have $a = \sqrt{n}$ or $-\sqrt{n}$. Since $\{\sigma_{-1}(b), \sigma_{-1}(c), \sigma_{-1}(d)\} = \{b, c, d\}$, we may assume that $c = \bar{b}$ without loss of generality. Then $\sigma_{-1}(d) = d$, that is d is a real number. In addition, $d\bar{d} = n$, therefore $d = -a$. Then d will also be fixed by $\text{Gal}(Q(\xi_m)/Q)$, contradicting to our assumption.
- (3) There are two fixed points $\{a\}$, $\{b\}$ and an orbit of length two $\{c, d\}$. By the similar analysis as in Case (2), we have $a = \pm\sqrt{n}$, $b = -a$, and $d = \bar{c}$.
- (4) There are two orbits of length two $\{a, b\}$ and $\{c, d\}$. Then $b = \bar{a}$ and $d = \bar{c}$.

We leave this problem for our future considerations.

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