NEIGHBOUR-TRANSITIVE CODES IN JOHNSON GRAPHS

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ABSTRACT. The Johnson graph $J(v, k)$ has, as vertices, the k-subsets of a v-set $\mathcal{V}$ and as edges the pairs of k-subsets with intersection of size $k - 1$. We introduce the notion of a neighbour-transitive code in $J(v, k)$. This is a vertex subset Γ such that the subgroup G of graph automorphisms leaving Γ invariant is transitive on both the set Γ of 'codewords' and also the set of 'neighbours' of Γ, which are the non-codewords joined by an edge to some codeword. We classify all examples where the group G is a subgroup of the symmetric group $\mathrm{Sym}\,(\mathcal{V})$ and is intransitive or imprimitive on the underlying v-set $\mathcal{V}$. In the remaining case where $G \leq \mathrm{Sym}\,(\mathcal{V})$ and G is primitive on $\mathcal{V}$, we prove that, provided distinct codewords are at distance at least 3, then G is 2-transitive on $\mathcal{V}$. We examine many of the infinite families of finite 2-transitive permutation groups and construct surprisingly rich families of examples of neighbour-transitive codes. A major unresolved case remains.

Key-words: codes in graphs, Johnson graph, 2-transitive permutation group, neighbour-transitive.

Mathematics Subject Classification (2010): 05C25, 20B25, 94B60.

1. INTRODUCTION

In 1973, Philippe Delsarte [7] introduced the notion of a code in a distance regular graph, namely a vertex subset whose elements are the codewords and with distance between codewords being the natural distance in the graph. In particular he defined a special class of such codes, now called completely regular codes, 'which enjoy combinatorial (and often algebraic) symmetry akin to that observed for perfect codes' [17, page 1]. (Completely regular codes are defined in Subsection 2.1.) Disappointingly, not many completely regular codes with good error-correcting properties (large distance between distinct codewords) were found and, for such codes in binary Hamming graphs having at least three codewords, it has been conjectured that the minimum distance between distinct codewords is at most 8 (see [2, page

2]). In fact Neumaier [20] conjectured that the only completely regular code with minimum distance 8 in a binary Hamming graph is the extended binary Golay code. Even though Neumaier's conjecture was disproved by Borges, Rifa, and Zinoviev [3], there are very few codes known with these properties.

Delsarte's paper [7] posed explicitly the question of existence of completely regular codes in Johnson graphs, and our focus in this paper is on a related family of codes in these graphs which contains many completely regular examples. Completely regular codes in Johnson graphs have been studied by Meyerowitz [18, 19] and Martin [15, 16]. We relax the stringent regularity conditions imposed for complete regularity, and replace them with conditions involving only codewords and their immediate neighbours. On the other hand, we strengthen the regularity conditions for codewords and their neighbours to a local transitivity property. The codes we study are called neighbour-transitive codes. We construct surprisingly rich classes of examples arising from both combinatorial and geometric structures, including some famlies with unbounded minimum distance.

Some but not all of the examples we construct are completely regular, generalising the constructions and results in [15, 16, 18, 19]. Other constructions raise new questions about geometric configurations in projective and affine spaces, and spaces of binary quadratic forms. The last case, associated with the 2-transitive actions of symplectic groups on binary quadratic forms, gives rise to a significant open problem *(see below)*. Our work generalises also the as yet unpublished study in [12] by Godsil and the second author of completely transitive codes in Johnson graphs.

1.1. **Johnson graphs and neighbour-transitive codes.** The *Johnson graph* $J(v, k)$, based on a set $\mathcal{V}$ of v elements called *points*, is the graph whose vertex set is the set $\binom{\mathcal{V}}{k}$ of all k-subsets of $\mathcal{V}$, with edges being the unordered pairs $\{\gamma, \gamma'\}$ of k-subsets such that $|\gamma \cap \gamma'| = k - 1$.

Since $J(v, 1)$ and $J(v, v-1)$ are both the complete graph on $\mathcal{V}$, we assume that $2 \leq k \leq v-2$. Moreover, since $J(v, k) \cong J(v, v-k)$, we may sometimes, when convenient, restrict our analysis to the case $k \leq v/2$. This is discussed further in Subsection 1.4.

The graph $J(v, k)$ admits the symmetric group $\mathrm{Sym}(\mathcal{V})$ as a group of automorphisms, and if $k \neq v/2$ this is the full automorphism group. If $k = v/2$ then the complementation map τ that sends each k-subset γ to its complement $\overline{\gamma} := \mathcal{V} \setminus \gamma$ is also an automorphism of $J(v, k)$ and the full automorphism group is $\mathrm{Sym}(\mathcal{V}) \times \langle \tau \rangle \cong S_v \times Z_2$. This exceptional case is investigated in [22], and in this paper we consider subgroups of automorphisms contained in $\mathrm{Sym}(\mathcal{V})$.

The codes we study are proper subsets $\Gamma \subset \binom{\mathcal{V}}{k}$. The automorphism group $\mathrm{Aut}(\Gamma)$ of such a code Γ is the setwise stabiliser of Γ in the symmetric group $\mathrm{Sym}(\mathcal{V}) \cong S_v$ (or in $\mathrm{Sym}(\mathcal{V}) \times \langle \tau \rangle$ if $k = v/2$). By a *neighbour of* Γ we mean a k-subset γ_1 of $\mathcal{V}$ that is not a codeword but satisfies $|\gamma_1 \cap \gamma| = k - 1$ for some codeword $\gamma \in \Gamma$, that is to say, the distance $d(\gamma, \gamma_1)$ between γ and γ_1 in $J(v, k)$ is 1. By the *minimum distance* $\delta(\Gamma)$ of a code Γ, we mean the least distance in $J(v, k)$ between distinct codewords of Γ. Thus provided $\delta(\Gamma) > 1$, all vertices adjacent to a codeword are neighbours. We say that Γ is *code-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on Γ, and *neighbour-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on both Γ and the set Γ_1 of neighbours of Γ.

The concept of neighbour-transitivity for codes in $J(v, k)$ can be placed in a broader context by viewing a code Γ and its neighbour set Γ_1 as an incidence structure, with incidence between a codeword and a neighbour induced from adjacency in $J(v, k)$. (See Section 2 for more details.) This incidence structure, and also the code Γ, is called *G-incidence-transitive* if $G \leq \mathrm{Aut}(\Gamma)$ and G is transitive on codeword-neighbour pairs (γ, γ_1) with $\gamma \in \Gamma, \gamma_1 \in \Gamma_1$ and $d(\gamma, \gamma_1) = 1$. Each incidence transitive code is neighbour transitive (by definition), but if $\delta(\Gamma) \leq 2$, it is possible for Γ to be neighbour-transitive but not incidence-transitive (see Example 2.2), or for Γ to be code-transitive but not neighbour-transitive, or for $\mathrm{Aut}(\Gamma)$ to be transitive on Γ_1 but not transitive on Γ, and hence not neighbour-transitive (see Examples 2.3 and 2.4).

1.2. **Results and questions.** Neighbour transitivity may seem a rather restrictive condition. However examples range from the collection of all k-subsets of a fixed subset $\mathcal{U} \subseteq \mathcal{V}$ (Example 3.1), to the block set of the $5 - (12, 6, 1)$ Witt design associated with the Mathieu group M_{12} [21, Table 1], to the set of lines of a finite projective space (Example 7.3). Moreover the examples include the completely-transitive designs studied in [12] where transitivity is required not only on the code Γ and its neighbour set Γ_1, but also on each subset Γ_i of the distance partition determined by Γ (see Subsection 2.1). Many of the

constructions from [12] were mentioned in William J. Martin's papers [15, 16] on completely regular designs.

As a broad summary of the results of this paper, together with those of [10] and [21], for the case of minimum distance at least 3, we can announce that:

if $\Gamma \subset J(v, k)$ with $\delta(\Gamma) \geq 3$ such that $G := \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ is neighbour-transitive on Γ, then either Γ is known explicitly, or G is a symplectic group acting 2-transitively on a set $\mathcal{V}$ of quadratic forms.

Thus a major open problem remains, work on which is proceeding in the PhD project of Mark Ioppolo at the University of Western Australia. (Some examples are known in this case.)

Problem 1. *Classify the G-neighbour-transitive codes $\Gamma \subset J(v, k)$, where $G = \mathrm{Sp}(2n, 2)$ and $v = 2^{2n-1} \pm 2^{n-1}$.*

Our first result is a complete classification (proved in Sections 3 and 4) of the neighbour-transitive codes in $J(v, k)$ for which the automorphism group is intransitive, or transitive and imprimitive, on the point set $\mathcal{V}$. A transitive group A is imprimitive on $\mathcal{V}$ if it leaves invariant a non-trivial partition of $\mathcal{V}$.

Theorem 1.1. *If $\Gamma \subset \binom{\mathcal{V}}{k}$ is neighbour-transitive, and if $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ is intransitive on $\mathcal{V}$, or transitive and imprimitive on $\mathcal{V}$, then Γ is one of the codes in Example 3.1, 4.1, or 4.4.*

For a code Γ and group $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$, we say that Γ is *G-strongly incidence-transitive* if G is transitive on Γ and, for $\gamma \in \Gamma$, G_γ is transitive on the set of pairs $(\mathsf{u}, \mathsf{u}')$ with $\mathsf{u} \in \gamma, \mathsf{u}' \in \mathcal{V} \setminus \gamma$. It is not hard to see that each strongly incidence transitive code (which by definition is a proper subset of $\binom{\mathcal{V}}{k}$) is incidence transitive, and indeed there exist incidence transitive codes Γ which are not strongly incidence transitive, necessarily with $\delta(\Gamma) = 1$. Examples of such codes are given in Examples 3.1 and 4.4, see Lemmas 3.2 and 4.6, respectively. The next result Theorem 1.2 links the notions of incidence-transitivity, strong incidence-transitivity and neighbour-transitivity, and provides critical information about the case where G is primitive on $\mathcal{V}$. It is proved in Section 5.

Theorem 1.2. *Let $\Gamma \subset \binom{\mathcal{V}}{k}$ and $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$, where $2 \leq k \leq |\mathcal{V}| - 2$.*

  (a) *The code Γ is G-strongly incidence-transitive if and only if Γ is G-incidence-transitive and $\delta(\Gamma) \geq 2$.*
  (b) *If $\delta(\Gamma) \geq 3$ and Γ is G-neighbour-transitive, then Γ is G-strongly incidence-transitive.*
  (c) *If G is primitive on $\mathcal{V}$ and Γ is G-strongly incidence-transitive, then G is 2-transitive on $\mathcal{V}$.*

rank 1	the Suzuki, Ree and Unitary groups
affine	$G \leq \mathrm{A\Gamma L}(\mathcal{V})$ acting on $\mathcal{V} = \mathbb{F}_q^n$
linear	$\mathrm{PSL}(n,q) \leq G \leq \mathrm{P\Gamma L}(n,q)$ on $\mathrm{PG}(n-1,q)$

TABLE 1. Other types of 2-transitive permutation groups

In particular, if Γ is G-neighbour-transitive with $\delta(\Gamma) \geq 3$ and G is primitive on $\mathcal{V}$ then, by Theorem 1.2, G is 2-transitive on $\mathcal{V}$ and Γ is strongly G-incidence-transitive. Application of the classification of the finite 2-transitive permutation groups opens up the possibility of classifying such codes. Moreover, Theorem 1.2 suggests that the possibly larger class of G-strongly-incidence-transitive codes (with $\delta(\Gamma) \geq 2$) may also be analysed in this way.

To make progress with this analysis, we divide the finite 2-transitive permutation groups according to whether or not they lie in an infinite family of 2-transitive groups. Those which do not lie in an infinite family we call sporadic, and these cases are dealt with in [21], yielding 27 strongly-incidence-transitive (code, group) pairs [21, Table 1]. In the rest of this paper we focus on the infinite families of finite 2-transitive groups G. As mentioned above, we do not treat the 2-transitive actions of symplectic groups on quadratic forms, and indeed the open Problem 1 may be broadened to include the strongly-incidence-transitve case.

Problem 2. *Classify the G-strongly incidence-transitive codes $\Gamma \subset J(v,k)$, where $G = \mathrm{Sp}(2n,2)$ and $v = 2^{2n-1} \pm 2^{n-1}$.*

The other infinite families of 2-transitive groups may be subdivided coarsely as in Table 1. The 'rank 1 case' is completely analysed in Section 8, and we prove there the following classification result.

Theorem 1.3. *Suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$ is G-strongly incidence-transitive, where G is 2-transitive of rank 1 type on $\mathcal{V}$. Then either*

   (a) *$\mathrm{PSU}(3,q) \leq G \leq \mathrm{P\Gamma U}(3,q)$, and either Γ or $\overline{\Gamma}$ is the classical unital with $\delta(\Gamma) = q$, as in Example 8.1; or*

   (b) *$G = \mathrm{PSU}(3,3).2$, $k = 12$ or 16, and Γ or $\overline{\Gamma}$ is the set of 'bases' with $\delta(\Gamma) = 6$.*

The affine and linear cases are analysed in Sections 6 and 7. Propositions 6.1, 6.6, 7.2, and 7.4 of these sections yield the following information about the possible strongly incidence-transitive codes Γ in these cases. Here a codeword $\gamma \in \Gamma$ is a subset of points of an affine or projective space $\mathcal{V}$. We say that γ is of *class* $[m_1, m_2, m_3]_1$ if each (affine or projective) line meets γ in m_1, m_2 or m_3 points.

Theorem 1.4. *Suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$ is G-strongly incidence-transitive, where G is 2-transitive of affine or linear type on $\mathcal{V}$. Let $\gamma \in \Gamma$. Then*

v	$\min\{k, v-k\}$	$\delta(\Gamma)$	γ or $\overline{\gamma}$	G	Reference
			Affine		
q^n	q^s	$q^s - q^{s-1}$	s-subspace	$A\Gamma L(n,q)$	Example 6.5
16	4	3	Baer-subline	$A\Gamma L(1,16)$	Proposition 6.1
			Linear		
$\frac{q^n-1}{q-1}$	$\frac{q^s-1}{q-1}$	q^{s-1}	$(s-1)$-subspace	$P\Gamma L(n,q)$	Example 7.3
$q_0^2 + 1$	$q_0 + 1$	$q_0 - 1$	Baer-subline	$P\Gamma L(2, q_0^2)$	Example 7.1

TABLE 2. Strongly incidence-transitive codes for Theorem 1.4.

either G and γ or $\overline{\gamma}$ are as in one of the lines of Table 2, or one of the following holds.

(a) *$G = \mathrm{AGL}(n, 4)$ and $\mathcal{V} = \mathbb{F}_4^n$ with $n \geq 2$, γ is of class $[0, 2, 4]_1$, and $\frac{4^n+2}{3} \leq k \leq \frac{2(4^n-1)}{3}$; or*

(b) *$G = \mathrm{AGL}(n, 16)$ and $\mathcal{V} = \mathbb{F}_{16}^n$ with $n \geq 2$, and replacing γ by $\overline{\gamma}$ if necessary, γ is of class $[0, 4, 16]_1$, and $\frac{16^n+4}{5} \leq k \leq \frac{4(16^n-1)}{15}$; or*

(c) *$G = \mathrm{P\Gamma L}(n, q)$, $\mathcal{V} = \mathrm{PG}(n-1, q)$ of size $v = \frac{q^n-1}{q-1}$ with $n \geq 3$ and, replacing γ by $\overline{\gamma}$ if necessary, γ is of class $[0, x, q+1]_1$, where either*
　(i) *$x = 2$ and $\frac{v-1}{q} + 1 \leq k \leq \frac{2(v-1)}{q}$, or*
　(ii) *$x = q_0 + 1$ where $q = q_0^2$, and $\frac{v-1}{q_0} + 1 \leq k \leq \frac{v-1}{q_0} + \frac{v-1}{q}$.*

There are more examples of strongly incidence-transitive codes with affine or linear groups than the ones listed in Table 2. Example 6.7 gives another such code that satisfies Theorem 1.4 (a) with $n = 2$; in that example γ is the famous 6-point 2-transitive hyperoval. I asked about the possible structures of affine and projective point sets γ of classes $[0, x, q]_1$ or $[0, x, q+1]_1$ during a lecture I gave in 2012 in Ferrara at a Conference on Finite Geometry in honour of Frank De Clerck. Nicola Durante, who was present, harnessed the known results about such subsets and developed them a great deal further in [10]. He was able to classifiy all such sets with the relevant symmetry properties, and hence classify all strongly incidence-transitive codes in cases (a)–(c) of Theorem 1.4. We summarise his findings in Remark 1.5.

Remark 1.5. In the affine case, Durante classified geometrically all point subsets of $\mathrm{AG}(n, q)$ of class $[0, \sqrt{q}, q]_1$ in Propositions 2.3, 3.6, Corollary 2.4 and Theorems 3.13 and 3.15 of [10]. He used this information to classify in [10, Theorem 3.18] all such subsets γ satisfying the conditions in Theorem 1.4 (a) and (b). For $q = 4$, that is, for Theorem 1.4 (a), the examples for γ or $\overline{\gamma}$ are (i) cylinders with base the 2-transitive hyperoval or its complement in $\mathrm{AG}(2, 4)$, and (ii) unions of two parallel hyperplanes. For $q = 16$, that is, for Theorem 1.4 (b), the

examples for γ or $\overline{\gamma}$ are unions of four parallel hyperplanes of $\mathrm{AG}(n, 16)$ (so that four-point intersections of affine lines with the set are Baer sub-lines). In the projective case, that is, the case of Theorem 1.4 (c), Durante [10, Theorems 3.2 and 3.3] drew together known results about sets of projective points of class $[0, x, q+1]_1$, and proved that there are no examples satisfying the conditions in Theorem 1.4 (c).

We conclude this introductory section with a short commentary on the examples in Theorem 1.1, and the conditions in Theorem 1.2.

1.3. **Remarks on the examples in Theorem 1.1.** (a) The codes in Example 3.1 are precisely the completely regular codes of 'strength zero' classified by Meyerowitz [18, 19]. On the other hand, some, but not all, of the codes in Examples 4.1 and 4.4 are completely regular. Further, some, but not all, of the codes in Example 4.4 are examples of groupwise complete designs constructed by Martin [15]; and some of the codes in Example 4.2 were discovered as completely transitive designs in [12]. See Remarks 4.2 and 4.5 for more details.

(b) Most of the neighbour-transitive codes classified in Theorem 1.1 have minimum distance $\delta(\Gamma) = 1$, the exceptions being the codes consisting of a single codeword, the blocks of a partition, or 'blow-ups' of smaller neighbour-transitive codes. See Lemmas 3.2, 4.3 and 4.6.

1.4. **Remarks on the transitivity properties in Theorem 1.2.** (a) As we mentioned above, it is possible for a code to be incidence-transitive but not strongly incidence-transitive, so the condition on $\delta(\Gamma)$ in part (a) of Theorem 1.2 cannot be dropped.

(b) Strongly incidence-transitive codes exist with minimum distance as small as 2 (see Example 2.5), so the converse of part (b) of Theorem 1.2 is false.

(c) There exist G-imprimitive, G-strongly incidence transitive codes (see Lemma 4.6), so the primitivity condition on G cannot be dropped from part (c) of Theorem 1.2. In addition there exist neighbour-transitives codes which are neither incidence-transitive nor strongly incidence transitive, and for which the automorphism group is 2-transitive on $\mathcal{V}$ (see Example 2.2). Thus the converse of part (c) is false.

(d) Each code $\Gamma \subset \binom{\mathcal{V}}{k}$ has a kind of dual defined as follows. For a subset $\alpha \subset \mathcal{V}$, we write $\overline{\alpha} = \mathcal{V} \setminus \alpha$. The *complementary code* of Γ is $\overline{\Gamma} := \{\overline{\gamma} \mid \gamma \in \Gamma\}$. It is a code in $J(v, v-k)$ with neighbour set $\{\overline{\gamma} \mid \gamma \in \Gamma_1\}$. Moreover $\delta(\overline{\Gamma}) = \delta(\Gamma)$, and any of the properties of neighbour-transitivity, incidence-transitivity, or strong incidence-transitivity holds for Γ if and only if it holds for $\overline{\Gamma}$. Thus we may assume that $k \leq v/2$ for the proof of Theorem 1.2, and we do this also at various other stages of our investigation.

2. Preliminaries and examples

2.1. **Completely regular and completely transitive codes in graphs.** A code Γ in a connected graph Σ determines a distance partition $\{\Gamma_0, \ldots, \Gamma_{r-1}\}$ of the vertex set of Σ, where $\Gamma_0 = \Gamma$ and, for $i > 0$, Γ_i is the set of vertices which are at distance i from at least one codeword in Γ, and at distance at least i from each codeword. For the last non-empty set Γ_{r-1}, the parameter r is called the *covering index* of Γ. The code is *completely regular* if the partition $\{\Gamma_0, \ldots, \Gamma_{r-1}\}$ is equitable, that is to say, for any $i, j \in \{0, \ldots, r-1\}$ and any $\gamma \in \Gamma_i$, the number of vertices of Γ_j adjacent to γ is independent of the choice of γ in Γ_i, and depends only on i and j. Further, Γ is called *completely transitive* if the setwise stabiliser of Γ in $\mathrm{Aut}\,(\Sigma)$ (which automatically fixes each of the Γ_i setwise) is transitive on each Γ_i.

2.2. **Codes in Johnson graphs: notation and small examples.** It is useful to denote the arc set of the Johnson graph $J(v, k)$ by J, that is,

$$(1) \qquad J := \{(\alpha, \beta) \,|\, \alpha, \beta \in \binom{\mathcal{V}}{k}, |\alpha \cap \beta| = k - 1\}$$

and for $\alpha \in \binom{\mathcal{V}}{k}$, to write $J(\alpha) = \{\beta \,|\, (\alpha, \beta) \in J\}$. For a code $\Gamma \subset \binom{\mathcal{V}}{k}$ with neighbour set Γ_1, pairs $(\gamma, \gamma_1) \in \Gamma \times \Gamma_1$ with $d(\gamma, \gamma_1) = 1$ in $J(v, k)$ are called the *incidences* of Γ: the set of incidences is the subset $J \cap (\Gamma \times \Gamma_1)$ and, for $G \leq \mathrm{Aut}\,(\Gamma)$, Γ is *G-incidence-transitive* if G is transitive on $J \cap (\Gamma \times \Gamma_1)$.

For a subset $\alpha \subseteq \mathcal{V}$ we write $\overline{\alpha} := \mathcal{V} \setminus \alpha$, so that, for $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$, Γ is *G-strongly incidence-transitive* if G is transitive on Γ and, for $\gamma \in \Gamma$, G_γ acts transitively on $\gamma \times \overline{\gamma} = \{(\mathsf{u}, \mathsf{w}) \,|\, \mathsf{u} \in \gamma, \mathsf{w} \in \overline{\gamma}\}$. First we verify that this concept is indeed a strengthening of incidence-transitivity.

Lemma 2.1. *If Γ is G-strongly incidence-transitive, then Γ is G-incidence-transitive and $\delta(\Gamma) \geq 2$.*

Proof. Since $\Gamma \neq \binom{\mathcal{V}}{k}$ and the graph $J(v, k)$ is connected, the neighbour set Γ_1 is non-empty. Then since G is transitive on Γ, it follows that, for each $\gamma \in \Gamma$, $d(\gamma, \gamma_1) = 1$ for some $\gamma_1 \in \Gamma_1$. Moreover, since G_γ is transitive on $\gamma \times \overline{\gamma}$ it follows that G_γ is transitive on $J(\gamma)$, and hence $J(\gamma) \subseteq \Gamma_1$. Thus $\delta(\Gamma) \geq 2$, $\Gamma_1 = \cup_{\gamma \in \Gamma} J(\gamma)$, and G is transitive on $J \cap (\Gamma \times \Gamma_1)$, the set of incidences. $\square$

Next we give several examples that illustrate various differences between the transitivity concepts.

Example 2.2. Let $\mathcal{V}$ be the set of points of the projective line $\mathrm{PG}_1(q)$, where $q \equiv 1 \pmod 4$ and $q > 5$, and let $G = \mathrm{PSL}(2, q)$. Let Γ be one of the two G-orbits on 3-subsets of $\mathcal{V}$. Then Γ_1 is the other G-orbit on

3-subsets. Thus Γ is neighbour-transitive, but not incidence-transitive, and $\delta(\Gamma) = 1$.

Example 2.3. Let $|\mathcal{V}| = 9, k = 3$, and let $\mathcal{U} = \{U_1|U_2|U_3\}$ be a partition of $\mathcal{V}$ with 3 parts of size 3. Let $\Gamma = \Gamma' \cup \mathcal{U}$, where Γ' is the set of all 3-subsets that contain one point from each part of $\mathcal{U}$. Let Δ be the set of 3-subsets containing two points from one part of $\mathcal{U}$ and one point from a second part of $\mathcal{U}$. Then $\mathrm{Aut}\,(\Gamma) = \mathrm{Aut}\,(\Delta)$ is the stabiliser $S_3 \wr S_3$ of $\mathcal{U}$ in $\mathrm{Sym}\,(\mathcal{V})$, and $\mathcal{U}, \Gamma', \Delta$ are its three orbits in $\binom{\mathcal{V}}{3}$. The two codes Γ, Δ in $J(9,3)$ have neighbour sets Δ and Γ respectively. Thus Δ is code-transitive but not neighbour-transitive, while $\mathrm{Aut}\,(\Gamma)$ is transitive on $\Gamma_1 = \Delta$ but not on Γ. These codes have $\delta(\Delta) = \delta(\Gamma) = 1$.

The following is an incidence-transitive example which is not strongly incidence-transitive, for which the automorphism group has a natural proper subgroup with weaker transitivity properties than those of the full group, but is still transitive on the neighbours.

Example 2.4. Let $\mathcal{V}$ be the point set of the projective plane $\mathrm{PG}_2(q)$, where $q > 5$, let λ be a line, and let G be the setwise stabiliser of λ in $\mathrm{PGL}(3, q)$. Let Γ be the set of all 4-subsets of λ. Then Γ_1 consists of all 4-subsets γ_1 such that $|\gamma_1 \cap \lambda| = 3$, the distance $\delta(\Gamma) = 1$, and $\mathrm{Aut}\,(\Gamma) = S_{q^2} \times S_{q+1}$ is incidence-transitive, but not strongly incidence-transitive. On the other hand G is transitive on Γ_1, while Γ is a union of at least two G-orbits.

Finally we give an example from orthogonal geometry of a strongly incidence-transitive code with minimum distance 2, showing that such codes with minimum distance 2 do exist. An ovoid in a projective space $\mathrm{PG}(3, q)$, with $q \geq 3$, is a subset of $q^2 + 1$ points that meets each line in at most two points. (A modified definition is needed for $q = 2$.)

Example 2.5. Let $G = \mathrm{GO}_4^-(3)$ acting on an ovoid $\mathcal{V} \subset \mathrm{PG}_3(3)$. Then $|\mathcal{V}| = 10$. Let Γ be the set of 4-subsets of $\mathcal{V}$ (called 'circles') obtained as intersections of $\mathcal{V}$ with secant planes lying on external lines. Then $|\Gamma| = 30$, Γ is a G-orbit, and G is transitive on $\binom{\mathcal{V}}{4} \setminus \Gamma$. Thus $\Gamma_1 = \binom{\mathcal{V}}{4} \setminus \Gamma$. Now the subsets in Γ form the block set of a $3 - (10, 4, 1)$ design implying that $\delta(\Gamma) \geq 2$, and in fact $\delta(\Gamma) = 2$. For a circle $\gamma \in \Gamma$, $G_\gamma = \mathrm{PGL}_2(3) \cong S_4$ is transitive on γ, and for $\mathsf{u} \in \gamma$, $G_{\gamma,\mathsf{u}}$ is transitive on the 6 points of $\mathcal{V} \setminus \gamma$. Thus Γ is G-strongly incidence-transitive.

3. The intransitive neighbour-transitive codes

Several natural families of neighbour-transitive designs have an automorphism group in $\mathrm{Sym}\,(\mathcal{V})$ that is intransitive on the underlying point set $\mathcal{V}$. We describe these families and prove that they are the only exampes with automorphism groups intransitve on $\mathcal{V}$.

Example 3.1. Let $\mathcal{U}$ be a proper, non-empty subset of $\mathcal{V}$ and $2 \leq k \leq v - 2$, where $v = |\mathcal{V}|$. Define $\Gamma \subset \binom{\mathcal{V}}{k}$ as follows.

    (a) If $|\mathcal{U}| > k$, let Γ be the set of all k-subsets of $\mathcal{U}$.

    (a) If $|\mathcal{U}| = k$, let Γ be the singleton set $\{\mathcal{U}\}$.

    (c) If $|\mathcal{U}| < k$, let Γ be the set of all k-subsets of $\mathcal{V}$ containing $\mathcal{U}$.

As noted in Subsection 1.3, these are the completely regular codes of 'strength zero' classified by Meyerowitz [18, 19]. We examine neighbour-transitive subgroups of automorphisms of these designs. For $\mathcal{U} \subseteq \mathcal{V}$, we write $\overline{\mathcal{U}} := \mathcal{V} \setminus \mathcal{U}$, and we denote the setwise stabiliser of $\mathcal{U}$ in $\mathrm{Sym}\,(\mathcal{V})$ by $\mathrm{Stab}(\mathcal{U})$. For a positive integer k, a group of permutations of a set $\mathcal{U}$ is said to be *k-homogeneous* on $\mathcal{U}$ if it is transitive on the k-subsets of $\mathcal{U}$.

Lemma 3.2. *Let* $\mathcal{U}, k, \Gamma$ *be as in Example* 3.1*. Then* Γ *is neighbour-transitive,* $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V}) = \mathrm{Stab}(\mathcal{U})$ *is intransitive on* $\mathcal{V}$*, Also, if* $|\mathcal{U}| \neq k$ *then* $\delta(\Gamma) = 1$*. Moreover, if* $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$*, then* Γ *is* G*-neighbour-transitive if and only if* Γ *is* G*-incidence-transitive, if and only if*

    (a) $|\mathcal{U}| > k$*, and* G *is transitive on both* $\binom{\mathcal{U}}{k}$ *and* $\binom{\mathcal{U}}{k-1} \times \overline{\mathcal{U}}$*; or*

    (b) $|\mathcal{U}| = k$*, and* G *is transitive on* $\mathcal{U} \times \overline{\mathcal{U}}$*; or*

    (c) $|\mathcal{U}| < k$*, and* G *is transitive on both* $\binom{\overline{\mathcal{U}}}{k-|\mathcal{U}|}$ *and* $\mathcal{U} \times \binom{\overline{\mathcal{U}}}{k-|\mathcal{U}|+1}$*.*

Further, Γ *is* G*-strongly incidence transitive if and only if* G, k *are as in part (b).*

We remark that, in case (c) when $k \leq (v+1)/2$, transitivity on $\binom{\overline{\mathcal{U}}}{k-|\mathcal{U}|}$ follows from transitivity on $\mathcal{U} \times \binom{\overline{\mathcal{U}}}{k-|\mathcal{U}|+1}$ (see [9, Theorem 9.4A(ii)]).

Proof. It follows from the definition of Γ that $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V}) = \mathrm{Stab}(\mathcal{U})$ with orbits $\mathcal{U}$ and $\overline{\mathcal{U}}$ in $\mathcal{V}$, and also, if $k \neq v/2$ that $\delta(\Gamma) = 1$. Note that the neighbour set Γ_1 consists of all k-subsets γ_1 of $\mathcal{V}$ such that $|\gamma_1 \cap \mathcal{U}| = k - 1$ if $|\mathcal{U}| \geq k$, or $|\gamma_1 \cap \mathcal{U}| = |\mathcal{U}| - 1$ if $|\mathcal{U}| < k$. Let $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ and $\gamma_1 \in \Gamma_1$. If $|\mathcal{U}| = k$, then the criterion for neighbour-transitivity in (b) is clearly necessary and sufficient, and moreover it is equivalent to G-strong incidence transitivity.

Suppose next that $|\mathcal{U}| > k$. Then G is transitive on Γ if and only if G is k-homogeneous on $\mathcal{U}$. Here $|\gamma_1 \cap \mathcal{U}| = k - 1$, so G is transitive on Γ_1 if and only if G is $(k-1)$-homogeneous on $\mathcal{U}$ and $G_{\gamma_1 \cap \mathcal{U}}$ is transitive on $\overline{\mathcal{U}}$. Thus the conditions given in (a) are necessary and sufficient for G to be neighbour-transitive, and to be incidence-transitive. Here $\delta(\Gamma) = 1$, and hence by Lemma 2.1, Γ is not strongly incidence-transitive.

Suppose finally that $|\mathcal{U}| < k$. Then G is transitive on Γ if and only if G is $(k - |\mathcal{U}|)$-homogeneous on $\overline{\mathcal{U}}$. Here $|\gamma_1 \cap \mathcal{U}| = |\mathcal{U}| - 1$, so G is transitive on Γ_1 if and only if G is transitive on $\mathcal{U}$ and $G_{\gamma_1 \cap \mathcal{U}}$ (which is the stabiliser of the unique point of $\mathcal{U}$ not in γ_1) is $(k -$

$|\mathcal{U}|+1)$-homogeneous on $\overline{\mathcal{U}}$; this is equivalent to G being transitive on $\mathcal{U} \times \binom{\overline{\mathcal{U}}}{k-|\mathcal{U}|+1}$. Thus the condition in (c) is necessary and sufficient for neighbour transitivity, and for incidence-transitivity. Again $\delta(\Gamma) = 1$, and hence by Lemma 2.1, Γ is not strongly incidence-transitive. $\square$

Now we classify the codes Γ admitting a neighbour-transitive, intransitive group.

Proposition 3.3. *Suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$, where $2 \leq k \leq |\mathcal{V}| - 2$, and Γ admits a neighbour-transitive subgroup of $\mathrm{Sym}(\mathcal{V})$ that is intransitive on $\mathcal{V}$. Then Γ is as in Example 3.1.*

Proof. Let $G \leq \mathrm{Aut}(\Gamma) \cap \mathrm{Sym}(\mathcal{V})$ be neighbour-transitive on Γ and intransitive on $\mathcal{V}$. Suppose first that some codeword $\gamma \in \Gamma$ contains a G-orbit, and let $\mathcal{U}$ be the largest G-invariant subset of γ. Since G is transitive on Γ it follows that $\mathcal{U}$ is contained in each codeword of Γ. Now there exists some neighbour $\gamma_1 \in \Gamma_1$ that does not contain $\mathcal{U}$. If Γ did not contain every k-subset containing $\mathcal{U}$, then there would also be a neighbour $\gamma'_1 \in \Gamma_1$ containing $\mathcal{U}$. However, since G fixes $\mathcal{U}$ setwise, no element of G can map γ_1 to γ'_1, which is a contradiction. Hence Γ consists of all k-subsets that contain $\mathcal{U}$, as in Example 3.1.

Thus we may assume that a codeword $\gamma \in \Gamma$ does not contain any G-orbit in $\mathcal{V}$. Let $\mathcal{U}$ be a G-orbit that meets γ, and let $\mathsf{u} \in \gamma \cap \mathcal{U}$ and $\mathsf{u}' \in \mathcal{U} \setminus \gamma$. Set $k' := |\gamma \cap \mathcal{U}|$. Since G is transitive on Γ it follows that every codeword meets $\mathcal{U}$ in k' points. Since γ contains no G-orbit it follows that there exists a point $\mathsf{v} \in \overline{\mathcal{U}} \setminus \gamma$. The k-subset $\gamma_1 := (\gamma \setminus \{\mathsf{u}\}) \cup \{\mathsf{v}\}$ meets $\mathcal{U}$ in $k'-1$ points and hence does not lie in Γ. Since $d(\gamma, \gamma_1) = 1$ it follows that $\gamma_1 \in \Gamma_1$, and since G is transitive on Γ_1, all neighbours must meet $\mathcal{U}$ in $k'-1$ points. Suppose that $k' < k$ so that there exists $\mathsf{v}' \in \gamma \setminus \mathcal{U}$. Then the k-subset $\gamma' := (\gamma \setminus \{\mathsf{v}'\}) \cup \{\mathsf{u}'\}$ meets $\mathcal{U}$ in $k' + 1$ points and hence does not lie in either Γ or Γ_1, contradicting the fact that $d(\gamma, \gamma') = 1$. Hence $k' = k$, that is to say, $\gamma \subset \mathcal{U}$. Thus each codeword is contained in $\mathcal{U}$ and each neighbour meets $\mathcal{U}$ in $k - 1$ points. Now each k-subset $\gamma' \subset \mathcal{U}$ is connected to γ by a path in the Johnson graph $J(|\mathcal{U}|, k)$ based on $\mathcal{U}$, with each vertex on the path a k-subset of $\mathcal{U}$. It follows that each vertex on the path is a codeword, and in particular $\gamma' \in \Gamma$. Hence Γ consists of all k-subsets of $\mathcal{U}$, as in Example 3.1. $\square$

4. The imprimitive neighbour-transitive codes

Additional natural families of neighbour-transitive codes are based on a partition $\mathcal{U} = \{U_1 | U_2 | \ldots | U_b\}$ of the underlying set $\mathcal{V}$, with b equal-sized parts U_i of size a, where $v = ab, a > 1, b > 1$. We introduce the notion of the $\mathcal{U}$-*type* $\mathsf{t}_{\mathcal{U}}(\gamma)$ of a subset γ of $\mathcal{V}$ to describe how γ intersects the various parts of $\mathcal{U}$, namely $\mathsf{t}_{\mathcal{U}}(\gamma)$ is the multiset $\{1^{m_1}, 2^{m_2}, \ldots, a^{m_a}\}$ of size $b = \sum_{i=1}^{a} m_i$, where $\sum_{i=1}^{a} im_i = |\gamma|$, each $m_i \geq 0$ and exactly

Line	t	Conditions	$\mathcal{U}$-Type for Γ_1
1	$\{k\}$	$k \leq a$	$\{1, k-1\}$
2	$\{k-v+a, a^{b-1}\}$	$k \geq v-a$	$\{k-v+a+1, a-1, a^{b-2}\}$
3	$\{1^k\}$	$k \leq b$	$\{1^{k-2}, 2\}$
4	$\{(a-1)^{v-k}, a^{b-v+k}\}$	$k \geq v-b$	$\{a-2, (a-1)^{v-k-2},$ $a^{b-v+k+1}\}$
5	$\{c^b\}$	$k = cb$, with $1 < c < a$	$\{c-1, c^{b-2}, c+1\}$
6	$\{\frac{k-1}{2}, \frac{k+1}{2}\}$	k odd, with $a \geq 3$, $b = 2$	$\{\frac{k-3}{2}, \frac{k+3}{2}\}$
7	$\{1, 2^{(k-1)/2}\}$	k odd, with $a = 2$, $b \geq 3$	$\{1^3, 2^{(k-3)/2}\}$

TABLE 3. $\mathcal{U}$-Types for $\Gamma = \Gamma(a, b; \mathsf{t})$ and Γ_1 in Example 4.1.

m_i of the intersections $\gamma \cap U_1, \ldots, \gamma \cap U_b$ have size i. If some $m_i = 0$, we usually omit the entry i^{m_i} from $\mathsf{t}_{\mathcal{U}}(\gamma)$. For example, if $\gamma \subset U_b$ and $|\gamma| = k$, we write $\mathsf{t}_{\mathcal{U}}(\gamma)$ as $\{k\}$.

We give two constructions in Subsection 4.1 for codes admitting a group of automorphisms that is both neighbour-transitive on the code, and also transitive and imprimitive on $\mathcal{V}$. We prove in Subsection 4.2 that all such codes arise in one of these ways.

4.1. **Constructions of imprimitive codes.** Several families of codes are defined as the sets of all k-subsets of certain $\mathcal{U}$-types.

Example 4.1. Let $\mathcal{U} = \{U_1 | U_2 | \ldots | U_b\}$ be a partition of the v-set $\mathcal{V}$ with b parts of size a, where $v = ab, a > 1, b > 1$, and let $2 \leq k \leq v - 2$. For t as in one of the lines of Table 3, let $\Gamma(a, b; \mathsf{t})$ consist of all k-subsets γ of $\mathcal{V}$ with $\mathsf{t}_{\mathcal{U}}(\gamma) = \mathsf{t}$.

Remark 4.2. Some, but not all, of the neighbour-transitive codes in Example 4.1 are completely regular. A discussion is given in [15, Section 2]. Moreover the codes in Line 5 of Table 3 are completely transitive, not just neighbour-transitive. Also those in Line 1 of Table 3 are completely transitive if either $b = 2$ or $k = 3$. These were discovered as completely transitive codes in [12] (see also [15, page 181]).

We note that, because of the conditions given in Table 3, no code arises from more than one line. We denote the stabiliser in $\mathrm{Sym}(\mathcal{V})$ of the partition $\mathcal{U}$ by $\mathrm{Stab}(\mathcal{U}) \cong S_a \wr S_b$. It is transitive and imprimitive on $\mathcal{V}$, since $a > 1, b > 1$. We prove that each of these codes Γ is incidence-transitive and hence, in particular, is neighbour-transitive.

Lemma 4.3. *Let* $\mathcal{U}, k, \mathsf{t}, \Gamma = \Gamma(a, b; \mathsf{t})$, *be as in Example* 4.1. *Then* Γ *is* G-*incidence-transitive, where* $G = \mathrm{Aut}(\Gamma) \cap \mathrm{Sym}(\mathcal{V}) = \mathrm{Stab}(\mathcal{U})$ *is*

transitive and imprimitive on $\mathcal{V}$. Also $\delta(\Gamma) = 1$, except for Line 1 of Table 3 with $k = a$, and in this exceptional case $\delta(\Gamma) = k$.

Proof. Set $S = \mathrm{Stab}(\mathcal{U})$, let t be as in one of the lines of Table 3, and let $\Gamma = \Gamma(a, b; \mathsf{t})$. Clearly $\delta(\Gamma) = 1$ unless we are in Line 1 with $k = a$, and in this case $\Gamma = \mathcal{U}$ and $\delta(\Gamma) = k$. In all cases it is not difficult to prove that $S = \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$, that is $S = G$, and that S is transitive on Γ. Also it is not difficult to verify that in each case Γ_1 consists of all k-sets with $\mathcal{U}$-type as in Table 3. It then follows that Γ is G-incidence transitive. $\qquad\square$

The second set of examples involves a code Γ_0 in a smaller Johnson graph $J(b, k_0)$ based on a partition $\mathcal{U}$ of $\mathcal{V}$. We include the case $k_0 = 1$ for a uniform description and note that in this case the code in Example 4.4 also occurs in Example 4.1, namely in Line 1 of Table 3 with $k = a$. This is the only overlap between the two families of codes.

Example 4.4. *Let $\mathcal{U} = \{U_1 | U_2 | \ldots | U_b\}$ be a partition of the v-set $\mathcal{V}$ with b parts of size a, where $v = ab, a > 1, b \geq 4$, and let $k = ak_0$ where $1 \leq k_0 \leq b - 1$. For $\Gamma_0 \subseteq \binom{\mathcal{U}}{k_0}$, let $\Gamma(a, \Gamma_0)$ be the set of all k-subsets of $\mathcal{V}$ of the form $\cup_{U \in \gamma_0} U$, for some $\gamma_0 \in \Gamma_0$.*

Remark 4.5. These codes were studied by Martin [15] in the special case where $\Gamma_0 = \binom{\mathcal{U}}{k_0}$, and he called them *groupwise complete designs*. It follows from Lemma 4.6 that essentially all the strongly incidence-transitive codes $\Gamma(a, \Gamma_0)$ arising from Example 4.4 with $\delta(\Gamma_0) = 1$ are groupwise complete designs. Martin [15, Theorem 2.1] determined precisely which groupwise complete designs are completely regular codes. He proved further in [15, Theorem 3.1] that, if Γ is completely regular with minimum distance at least 2, and if Γ is a 1-design but not a 2-design[1], then Γ is a groupwise complete design. Thus it follows from Lemma 4.6 that most of the neighbour-transitive codes in Example 4.4 are *not* completely regular.

Lemma 4.6. *For $\Gamma = \Gamma(a, \Gamma_0)$ as in Example 4.4, $\mathrm{Aut}\,(\Gamma)$ contains $S_a \wr A$, where $A = \mathrm{Aut}\,(\Gamma_0) \cap \mathrm{Sym}\,(\mathcal{U})$, and $\delta(\Gamma) = a\delta(\Gamma_0)$.*

(a) *Moreover, if Γ_0 is A-strongly incidence-transitive, then Γ is $(S_a \wr A)$-strongly incidence-transitive, and either $\Gamma_0 = \binom{\mathcal{U}}{k_0}$ or $\delta(\Gamma_0) \geq 2$.*

(b) *Conversely, if $S_a \wr A$ is neighbour-transitive on Γ, then either Γ_0 is A-strongly incidence-transitive, or $a = 2$ and $\delta(\Gamma_0) = 1$.*

Proof. It follows from the definition of Γ that $\delta(\Gamma) = a\delta(\Gamma_0)$. Let $G := S_a \wr A$. Then, by definition, Γ is a G-invariant subset of $\binom{\mathcal{V}}{k}$, so $G \leq \mathrm{Aut}\,(\Gamma)$. Next suppose that Γ_0 is A-strongly incidence-transitive. Then G is transitive on Γ. Let $\gamma_0 \in \Gamma_0$ and $\gamma = \cup_{U \in \gamma_0} U$. Then

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[1]That is, each point of $\mathcal{V}$ lies in a constant number of codewords (k-subsets) in Γ, but some point pairs lie in different numbers of codewords.

$G_\gamma = S_a \wr A_{\gamma_0}$, and since A_{γ_0} is transitive on $\gamma_0 \times (\mathcal{U} \setminus \gamma_0)$, then also G_γ is transitive on $\gamma \times (\mathcal{V} \setminus \gamma)$. Thus Γ is G-strongly incidence-transitive. Suppose that $\delta(\Gamma_0) = 1$. Then we may choose γ_0 such that some adjacent k_0-subset $(\gamma_0 \setminus \{U\}) \cup \{W\} \in \Gamma_0$. Since Γ_0 is A-strongly incidence-transitive it follows that each k_0-subset of $\mathcal{U}$ adjacent to γ_0 in $J(b, k_0)$ also lies in Γ_0. Moreover, this property is independent of γ_0 since A is transitive on Γ_0. It follows that $\Gamma_0 = \binom{\mathcal{U}}{k_0}$, and (a) is proved.

Conversely suppose that G is neighbour-transitive on Γ. For $\gamma_0, \gamma_0' \in \Gamma_0$, let $\gamma = \cup_{U \in \gamma_0} U$ and $\gamma' = \cup_{U \in \gamma_0'} U$. Then some element $h \in G$ maps γ to γ', and hence the element of A induced by h maps γ_0 to γ_0'. Thus A is transitive on Γ_0. Now take $\gamma_0 = \gamma_0'$, let $U, U' \in \gamma_0$ and let W, W' belong to $\mathcal{U} \setminus \gamma_0$, where possibly $U = U'$ and/or $W = W'$. Let $\mathsf{u} \in U, \mathsf{u}' \in U', \mathsf{w} \in W, \mathsf{w}' \in W'$, and define $\gamma_1 = \gamma_1(\mathsf{u}, \mathsf{w}) = (\gamma \setminus \{\mathsf{u}\}) \cup \{\mathsf{w}\}$ and $\gamma_1' = \gamma_1'(\mathsf{u}', \mathsf{w}') = (\gamma \setminus \{\mathsf{u}'\}) \cup \{\mathsf{w}'\}$. Both are at distance 1 from γ and hence $\gamma_1, \gamma_1' \in \Gamma_1$ (recalling that $\delta(\Gamma) \geq a > 1$). Since G is transitive on Γ_1 there exists $g \in G$ such that $\gamma_1^g = \gamma_1'$. This element g maps $\gamma \setminus U$ to $\gamma \setminus U'$ (the unions of $\mathcal{U}$-classes contained in γ_1, γ_1' respectively), and maps $\gamma \cup W$ to $\gamma \cup W'$ (the unions of $\mathcal{U}$-classes meeting γ_1, γ_1' respectively). Hence g maps $\{U, W\}$ to $\{U', W'\}$. Suppose first that $a \geq 3$. Then U, U' are the only $\mathcal{U}$-classes such that $|\gamma_1 \cap U| = |\gamma_1' \cap U'| = a - 1$, and hence g maps U to U', and W to W'. Hence $g \in G_{\gamma_0}$ and we deduce that A_{γ_0} is transitive on $\gamma_0 \times (\mathcal{U} \setminus \gamma_0)$, so Γ_0 is A-strongly incidence-transitive.

Finally suppose that $a = 2$ and $\delta(\Gamma_0) \geq 2$. We showed that g maps $\eta := \gamma_0 \cup \{W\}$ to $\gamma_0 \cup \{W'\}$, $\nu := \gamma_0 \setminus \{U\}$ to $\gamma_0 \setminus \{U'\}$, and $\{U, W\}$ to $\{U', W'\}$. Suppose that $(U, W)^g = (W', U')$. Then $\gamma_0 \subset \eta^g = \gamma_0 \cup \{W'\}$ and $U' = W^g \notin \gamma_0^g$, so that $\gamma_0^g = (\gamma_0 \setminus \{U'\}) \cup \{W'\}$. This implies that $d(\gamma_0, \gamma_0^g) = 1$ which is a contradiction since $\gamma_0^g \in \Gamma_0$ and we assumed that $\delta(\Gamma_0) \geq 2$. Thus $(U, W)^g = (U', W')$.

Temporarily take $W' = W$. Then g fixes η and hence also γ_0, since $W^g = W' = W$. Since U, U' can be chosen arbitrarily in γ_0, it follows that A_{γ_0} is transitive on γ_0. Now instead take $U' = U$. Then g fixes ν, and hence also γ_0, since $U^g = U' = U$. Since W, W' can be chosen arbitrarily in $\mathcal{U} \setminus \gamma_0$ we conclude that $A_{\gamma_0, U}$ is transitive on $\mathcal{U} \setminus \gamma_0$. Hence Γ_0 is A-strongly incidence-transitive. $\square$

4.2. Classification of the imprimitive codes.

Now we classify the codes Γ admitting an imprimitive neighbour-transitive subgroup of $\mathrm{Sym}\,(V)$.

Proposition 4.7. *Suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$, where $2 \leq k \leq |\mathcal{V}| - 2$, and Γ admits a neighbour-transitive subgroup of $\mathrm{Sym}\,(\mathcal{V})$ that is transitive and imprimitive on $\mathcal{V}$. Then Γ is as in Example 4.1 or 4.4.*

Proof. Let $G \leq \mathrm{Aut}\,(\Gamma)$ be neighbour-transitive and imprimitive on $\mathcal{V}$, and let $\mathcal{U} = \{U_1 | U_2 | \ldots | U_b\}$ be a G-invariant partition of $\mathcal{V}$ with b parts

of size a, where $v = ab, a > 1, b > 1$. Choose $\gamma \in \Gamma$, set $e_i := |\gamma \cap U_i|$ for each i, and re-label the U_i so that $e_1 \geq e_2 \geq \cdots \geq e_b$. Then γ is of $\mathcal{U}$-type $\mathsf{t}_{\mathcal{U}}(\gamma) = \{e_1, e_2, \ldots, e_b\}$. We examine various k-subsets β of $\mathcal{V}$ such that $d(\gamma, \beta) = 1$. If $\mathsf{t}_{\mathcal{U}}(\beta) \neq \mathsf{t}_{\mathcal{U}}(\gamma)$, then $\beta \notin \Gamma$ since G is transitive on Γ and preserves the $\mathcal{U}$-types of k-subsets of $\mathcal{V}$. Hence $\beta \in \Gamma_1$. Moreover, since G is transitive on Γ_1, there do not exist two k-subsets β_1 and β_2 with $d(\gamma, \beta_1) = d(\gamma, \beta_2) = 1$ such that $\mathsf{t}_{\mathcal{U}}(\beta_1), \mathsf{t}_{\mathcal{U}}(\beta_2)$ and $\mathsf{t}_{\mathcal{U}}(\gamma)$ are pairwise distinct. We use the following notation. *For each i such that $e_i > 0$, u_i denotes a typical point of $\gamma \cap U_i$, and for each i such that $e_i < a$, w_i denotes a typical point of $U_i \setminus (\gamma \cap U_i)$.*

To simplify the analysis, replacing Γ by its complementary code $\overline{\gamma}$ in $J(v, v - k)$ if necessary, we may assume that $k \leq v/2$ (see remark (d) in Subsection 1.4). Note that in Table 3, the codes in lines 2, 4 are complementary to codes in lines 1, 3 respectively, while for the other lines the complementary code belongs to the same line.

Case $b = 2$.   Since $k \leq v/2 = a$, if $e_1 = a$, then $k = a$, $\gamma = U_1$, and as G is transitive on $\mathcal{U}$, we have $\Gamma = \mathcal{U}$ as in Example 4.1, Line 1 of Table 3, and in Example 4.4 with $k_0 = 1$. Suppose then that $e_1 < a$, and hence also $e_2 < a$, and set $\beta_1 = (\gamma \setminus \{\mathsf{u}_1\}) \cup \{\mathsf{w}_1\}$ and $\beta_2 = (\gamma \setminus \{\mathsf{u}_1\}) \cup \{\mathsf{w}_2\}$. Then $\mathsf{t}_{\mathcal{U}}(\beta_1) = \mathsf{t}_{\mathcal{U}}(\gamma)$ while $\mathsf{t}_{\mathcal{U}}(\beta_2) = \{e_1 - 1, e_2 + 1\}$. Suppose first that $\mathsf{t}_{\mathcal{U}}(\beta_2) = \mathsf{t}_{\mathcal{U}}(\gamma)$. Then k is odd, $\mathsf{t}_{\mathcal{U}}(\gamma) = \{\frac{k+1}{2}, \frac{k-1}{2}\}$, and $\beta_3 := (\gamma \setminus \{\mathsf{u}_2\}) \cup \{\mathsf{w}_1\}$ has $\mathsf{t}_{\mathcal{U}}(\beta_3) \neq \mathsf{t}_{\mathcal{U}}(\gamma)$. Thus in this case $\beta_3 \in \Gamma_1$ and $\beta_1, \beta_2 \in \Gamma$ for all choices of the $\mathsf{u}_i, \mathsf{w}_i$, so Γ consists of all k-subsets of type $\mathsf{t}_{\mathcal{U}}(\gamma)$, and $a \geq \frac{k+1}{2} + 1 \geq 3$, as in Example 4.1, Line 6 of Table 3. Now assume that $\mathsf{t}_{\mathcal{U}}(\beta_2) \neq \mathsf{t}_{\mathcal{U}}(\gamma)$ so that $\beta_2 \in \Gamma_1$. Then $\beta_1 \in \Gamma$ for all choices of $\mathsf{u}_1, \mathsf{w}_1$. Suppose also that $e_2 > 0$. Then $(\gamma \setminus \{\mathsf{u}_2\}) \cup \{\mathsf{w}_2\} \in \Gamma$ for all choices of $\mathsf{u}_2, \mathsf{w}_2$. Since G is transitive on $\mathcal{U}$ it follows in this case that Γ consists of all k-subsets of type $\mathsf{t}_{\mathcal{U}}(\gamma)$. If $e_1 = e_2$, then Γ is as in Example 4.1, Line 3 or 5 of Table 3. So assume that $e_1 > e_2 > 0$. Then $\beta_3 := (\gamma \setminus \{\mathsf{u}_2\}) \cup \{\mathsf{w}_1\}$ has $\mathsf{t}_{\mathcal{U}}(\beta_3) \neq \mathsf{t}_{\mathcal{U}}(\gamma), \mathsf{t}_{\mathcal{U}}(\beta_2)$, which is a contradiction. This leaves the possibility $e_2 = 0$, and here Γ is as in Example 4.1, Line 1 of Table 3.

Case $a = 2$, $b \geq 3$.   Here $\mathsf{t}_{\mathcal{U}}(\gamma) = \{2^c, 1^d\}$ for some c, d such that $k = 2c + d$. Suppose first that $c \geq 1$ and $d \geq 2$. Since $k \leq v/2 = b$, we have $e_b = 0$. Then $(\gamma \setminus \{\mathsf{u}_1\}) \cup \{\mathsf{w}_b\}$ has $\mathcal{U}$-type $\{2^{c-1}, 1^{d+2}\}$, and $(\gamma \setminus \{\mathsf{u}_{c+2}\}) \cup \{\mathsf{w}_{c+1}\}$ has $\mathcal{U}$-type $\{2^{c+1}, 1^{d-2}\}$, so both lie in Γ_1. Thus G is not transitive on Γ_1 and we have a contradiction. Therefore either $c = 0$ or $d \leq 1$. Suppose first that $c = 0$, and hence $d = k$. Then $\beta := (\gamma \setminus \{\mathsf{u}_2\}) \cup \{\mathsf{w}_1\} \notin \Gamma$, since it has $\mathcal{U}$-type $\{2, 1^{k-2}\}$, and so $\beta \in \Gamma_1$. Thus, for each $i \leq d$ and $j > d$, $(\gamma \setminus \{\mathsf{u}_i\}) \cup \{\mathsf{w}_j\}$ has $\mathcal{U}$-type $\mathsf{t}_{\mathcal{U}}(\gamma) = \{1^k\}$ and hence lies in Γ. It follows that Γ consists of all k-subsets of type $\{1^k\}$, as in Example 4.1, Line 3 of Table 3. Thus we may assume that $c > 0$ and $d \leq 1$.

Next take $d = 0$, so $c = k/2$. If $c = 1$ then Γ is as in Example 4.1, Line 1 of Table 3. so assume that $c \geq 2$. For $\gamma' \in \Gamma$ set $\gamma_0(\gamma') := \{U \in \mathcal{U} \mid U \subset \gamma'\}$, and let $\Gamma_0 := \{\gamma_0(\gamma') \mid \gamma' \in \Gamma\}$. Then $\Gamma_0 \subset \binom{\mathcal{U}}{c}$ with $2 \leq c \leq \frac{b}{2}$, so $\Gamma = \Gamma(2, \Gamma_0)$, as in Example 4.4.

Finally suppose that $d = 1$, so $c = (k-1)/2$. Let $i \leq c$ and $j > c+1$ (note that $c+1 < b$ since $k \leq \frac{v}{2} = b$). Then Γ_1 contains $\beta := (\gamma \setminus \{u_i\}) \cup \{w_j\}$ of $\mathcal{U}$-type $\{2^{c-1}, 1^3\}$, and since $\gamma' := (\gamma \setminus \{u_{c+1}\}) \cup \{w_j\}$ has $\mathcal{U}$-type $\{2^c, 1\}$, it lies in Γ. Letting w_j and j vary, we deduce that Γ contains all k-subsets that contain $\gamma \setminus \{u_{c+1}\} = U_1 \cup \ldots, \cup U_c$. Also the k-subset $\gamma'' := (\gamma' \setminus \{u_i\}) \cup \{w'_j\}$, where $U_j = \{w_j, w'_j\}$, has $\mathcal{U}$-type $\{2^c, 1\}$, and hence lies in Γ. Applying the previous argument to γ'' yields that Γ contains all k-subsets that contain $U_1 \cup \cdots \cup U_{i-1} \cup U_{i+1} \cup \cdots \cup U_c \cup U_j$, and this holds for all $i \leq c < j$. It follows that Γ consists of all k-subsets with $\mathcal{U}$-type $\{2^c, 1\}$ as in Example 4.1, Line 7 of Table 3.

Case $a \geq 3$ and $b \geq 3$.   We divide this remaining case into several subcases.

Subcase $e_1 - e_b \leq 1$.   Here $t_{\mathcal{U}}(\gamma) = \{c^i, (c-1)^{b-i}\}$, with $1 \leq c \leq a$ and $1 \leq i \leq b$, and $k = ci + (c-1)(b-i) \leq v/2$, so that $c < a$. Suppose first that both $i < b$ and $c \geq 2$. Then $\beta_1 := (\gamma \setminus \{u_{i+1}\}) \cup \{w_1\}$ has $\mathcal{U}$-type $t_{\mathcal{U}}(\beta_1) = \{c+1, c^{i-1}, (c-1)^{b-i-1}, c-2\} \neq t_{\mathcal{U}}(\gamma)$, and hence $\beta_1 \in \Gamma_1$. If $i \geq 2$, then $(\gamma \setminus \{u_2\}) \cup \{w_1\}$ has $\mathcal{U}$-type $\{c+1, c^{i-2}, (c-1)^{b-i+1}\} \neq t_{\mathcal{U}}(\gamma)$ or $t_{\mathcal{U}}(\beta_1)$ and we have a contradiction. Thus $i = 1$, but then $(\gamma \setminus \{u_3\}) \cup \{w_2\}$ has $\mathcal{U}$-type $\{c^2, (c-1)^{b-3}, c-2\}$ and again we have a contradiction. Therefore either $i = b$ or $c = 1$. Suppose first that $i = b$. Then, for distinct j, ℓ, the k-subset $\beta_2 := (\gamma \setminus \{u_j\}) \cup \{w_\ell\}$ has $\mathcal{U}$-type $t_{\mathcal{U}}(\beta_2) = \{c+1, c^{b-2}, c-1\}$, so $\beta_2 \in \Gamma_1$. This implies that $(\gamma \setminus \{u_j\}) \cup \{w_j\} \in \Gamma$, for all j and all choices of the points u_j, w_j, and it follows easily that Γ consists of all k-subsets of $\mathcal{U}$-type $\{c^b\}$ as in Example 4.1, Line 3 or 5 of Table 3. Suppose finally that $i < b$ and $c = 1$. Again it is easy to see that k-subsets in Γ_1 have $\mathcal{U}$-type $\{2, 1^{k-2}\}$ and that Γ consists of all k-subsets of $\mathcal{U}$-type $\{1^k\}$ as in Example 4.1, Line 3 of Table 3.

Thus we may assume that $e_1 \geq e_b + 2$. Let j be minimal such that $e_1 \geq e_j + 2$. Then $e_{j-1} = e_1$ or $e_1 - 1$, and $e_j \leq a - 2$. Define $\beta := (\gamma \setminus \{u_1\}) \cup \{w_j\}$ with $t_{\mathcal{U}}(\beta) = \{e_1 - 1, e_2, \ldots, e_{j-1}, e_j + 1, e_{j+1}, \ldots, e_b\}$ (possibly with the first and j^{th} entries out of order). In particular $t_{\mathcal{U}}(\beta) \neq t_{\mathcal{U}}(\gamma)$, so $\beta \in \Gamma_1$. Note that there is no entry of either $t_{\mathcal{U}}(\gamma)$ or $t_{\mathcal{U}}(\beta)$ greater than e_1.

Subcase $e_b + 2 \leq e_1 < a$.   If $e_2 > 0$, then $(\gamma \setminus \{u_2\}) \cup \{w_1\}$ has $\mathcal{U}$-type different from $t_{\mathcal{U}}(\gamma), t_{\mathcal{U}}(\beta)$, and we have a contradiction. Thus $e_2 = 0$, so $j = 2$, $k = e_1 < a$, $\gamma \subset U_1$, and so $t_{\mathcal{U}}(\beta) = \{e_1 - 1, 1\}$. Since G is transitive on Γ and Γ_1, it follows that Γ consists of all k-subsets of $\mathcal{U}$-type $\{k\}$, as in Example 4.1, Line 1 of Table 3.

Subcase $e_b + 2 \leq e_1 = a$. There exist $j_1 \geq 1, j_2 \geq 0$ such that $j_1 + j_2 = j - 1$ and $\mathsf{t}_{\mathcal{U}}(\gamma) = \{a^{j_1}, (a-1)^{j_2}, e_j, \ldots, e_b\}$. Then $\mathsf{t}_{\mathcal{U}}(\beta) = \{a^{j_1-1}, (a-1)^{j_2+1}, e_j + 1, e_{j+1}, \ldots, e_b\}$. Also $\beta' := (\gamma \setminus \{\mathsf{u}_1\}) \cup \{\mathsf{w}_b\}$ has $\mathcal{U}$-type $\{a^{j_1-1}, (a-1)^{j_2+1}, e_j, \ldots, e_{b-1}, e_b + 1\}$, different from $\mathsf{t}_{\mathcal{U}}(\gamma)$, and so $\beta' \in \Gamma_1$ and $\mathsf{t}_{\mathcal{U}}(\beta') = \mathsf{t}_{\mathcal{U}}(\beta)$. This implies that $e_j = e_b = c$, say, and $c \leq a - 2$, so that $\mathsf{t}_{\mathcal{U}}(\gamma) = \{a^{j_1}, (a-1)^{j_2}, c^{b-j+1}\}$ and $\mathsf{t}_{\mathcal{U}}(\beta) = \{a^{j_1-1}, (a-1)^{j_2+1}, c+1, c^{b-j}\}$.

Suppose that $j_2 > 0$. Then $\gamma' := (\gamma \setminus \{\mathsf{u}_{j_1+1}\}) \cup \{\mathsf{w}_j\}$ has $\mathcal{U}$-type $\{a^{j_1}, (a-1)^{j_2-1}, a-2, c+1, c^{b-j}\} \neq \mathsf{t}_{\mathcal{U}}(\beta)$, and so $\gamma' \in \Gamma$. Thus $\mathsf{t}_{\mathcal{U}}(\gamma') = \mathsf{t}_{\mathcal{U}}(\gamma)$ and it follows that $c = a - 2$. If $a \geq 4$ then $k > b \cdot \frac{a}{2} = \frac{v}{2}$, which is not so, and hence $a = 3$. Thus $k = b + 2j_1 + j_2 \leq \frac{v}{2} = \frac{3b}{2}$, so $2j_1 + j_2 \leq \frac{b}{2}$. In particular, the number of entries of $\mathsf{t}_{\mathcal{U}}(\gamma)$ equal to 1 is $b - j_1 - j_2 \geq \frac{b}{2} + j_1 \geq 2$. Hence the k-subset $(\gamma \setminus \{\mathsf{u}_b\}) \cup \{\mathsf{w}_j\}$ has $\mathcal{U}$-type different from $\mathsf{t}_{\mathcal{U}}(\gamma)$ and $\mathsf{t}_{\mathcal{U}}(\beta)$, and this is a contradiction.

Thus $j_2 = 0$, so $\mathsf{t}_{\mathcal{U}}(\gamma) = \{a^{j-1}, c^{b-j+1}\}$ and $\mathsf{t}_{\mathcal{U}}(\beta) = \{a^{j-2}, a-1, c+1, c^{b-j}\}$. Since $k \leq \frac{v}{2}$ we have $\frac{ab}{2} \leq v - k = (a-c)(b-j+1)$. In particular, $b - j + 1 > b/2 > 1$. If $c > 0$ then $(\gamma \setminus \{\mathsf{u}_b\}) \cup \{\mathsf{w}_j\}$ has $\mathcal{U}$-type $\{a^{j-1}, c+1, c^{b-j-1}, c-1\}$ different from $\mathsf{t}_{\mathcal{U}}(\gamma)$ and $\mathsf{t}_{\mathcal{U}}(\beta)$, and this is a contradiction. Thus $c = 0$ and $\mathsf{t}_{\mathcal{U}}(\gamma) = \{a^{j-1}\}$ with $k = a(j-1)$. If $j = 2$ then $\gamma = U_1$, $k = a$, and Γ is as in Example 4.1, Line 1 of Table 3. If $j \geq 3$, define $\Gamma_0 := \{\gamma_0(\gamma') | \gamma' \in \Gamma\}$, where for $\gamma' \in \Gamma$, $\gamma_0(\gamma') = \{U \in \mathcal{U} \mid U \subseteq \gamma'\}$. Since $2 \leq j - 1 \leq b/2 < b$, $\Gamma = \Gamma(a, \Gamma_0)$, as in Example 4.4. $\qquad\square$

The proof of Theorem 1.1 follows from Propositions 3.3 and 4.7.

5. PRIMITIVE NEIGHBOUR-TRANSITIVE CODES

For our analysis of G-neighbour-transitive codes in $J(v, k)$ with $G \leq \mathrm{Sym}\,(\mathcal{V})$, it remains to consider codes $\Gamma \subset \binom{\mathcal{V}}{k}$ such that $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ acts primitively on $\mathcal{V}$. Our first task is to prove Theorem 1.2. We use the notation introduced in Subsection 2.2.

Lemma 5.1. *If $\Gamma \subset \binom{\mathcal{V}}{k}$ and $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ is transitive on $\mathcal{V}$, then, for $\mathsf{u} \in \mathcal{V}$, the set $\Delta(\mathsf{u}) = \bigcap\{\gamma' \in \Gamma \mid \mathsf{u} \in \gamma'\}$ is a block of imprimitivity for $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ in $\mathcal{V}$. In particular, if $\mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ is primitive on $\mathcal{V}$, then $\Delta(\mathsf{u}) = \{\mathsf{u}\}$.*

Proof. Let $G = \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ and $g \in G$. Then $\Delta(\mathsf{u})^g = \Delta(\mathsf{u}^g)$, and hence, since G is transitive on $\mathcal{V}$, $|\Delta(\mathsf{u})|$ is independent of u. Suppose that $g \in G$ and $\mathsf{w} \in \Delta(\mathsf{u}) \cap \Delta(\mathsf{u})^g$. Now $\mathsf{w} \in \Delta(\mathsf{u})$ implies that $\Delta(\mathsf{u}) \supseteq \Delta(\mathsf{w})$, and consequently these two sets are equal. Similarly $\Delta(\mathsf{w}) = \Delta(\mathsf{u}^g)$, and hence $\Delta(\mathsf{u}) = \Delta(\mathsf{u})^g$. Thus $\Delta(\mathsf{u})$ is a block of imprimitivity for the action of G on $\mathcal{V}$. In particular, if G is primitive on $\mathcal{V}$ then, since $|\Delta(\mathsf{u})| \leq k < v$, we conclude that $\Delta(\mathsf{u}) = \{\mathsf{u}\}$. $\qquad\square$

5.1. **Proof of Theorem 1.2.** Suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$, $\gamma \in \Gamma$, $2 \leq k \leq v - 2$, and $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$. To prove part (a), suppose first that G is incidence-transitive on Γ and $\delta(\Gamma) \geq 2$. In particular G is transitive on Γ. Since $\delta(\Gamma) \geq 2$, it follows that $J(\gamma) \subseteq \Gamma_1$, and hence G_γ is transitive on $J(\gamma)$. This implies that G_γ is transitive on $\gamma \times \overline{\gamma}$. The converse assertion follows from Lemma 2.1.

(b)   Next suppose that $\delta(\Gamma) \geq 3$ and Γ is G-neighbour transitive. By part (a), it is sufficient to prove that Γ is G-incidence transitive. Let $(\gamma, \gamma_1), (\gamma', \gamma_1') \in \Gamma \times \Gamma_1$ be two incidences. Since G is transitive on Γ_1 there exists $g \in G$ such that $\gamma_1^g = \gamma_1'$. Then γ^g, γ' both lie in $J(\gamma_1')$ and hence $d(\gamma^g, \gamma') \leq 2$. Since $\delta(\Gamma) \geq 3$, this means that $\gamma^g = \gamma'$. Hence g maps (γ, γ_1) to (γ', γ_1'), so Γ is G-incidence-transitive.

(c)   Finally assume that G is primitive on $\mathcal{V}$ and that Γ is G-strongly incidence transitive. Let $\mathsf{u} \in \mathcal{V}$. Since $\overline{\Gamma}$ is also G-strongly incidence-transitive (see Subsection 1.4), we may assume that $2 \leq k \leq v/2$. By Lemma 5.1, the set $\Delta(\mathsf{u}) = \bigcap\{\gamma' \in \Gamma \,|\, \mathsf{u} \in \gamma'\}$ is equal to $\{\mathsf{u}\}$. Hence, for distinct points w, w' of $\mathcal{V} \setminus \{\mathsf{u}\}$, there are codewords $\gamma, \gamma' \in \Gamma$ containing u such that $\mathsf{w} \notin \gamma$ and $\mathsf{w}' \notin \gamma'$. By strong incidence-transitivity, $G_{\gamma,\mathsf{u}}$ is transitive on $\overline{\gamma}$ and $G_{\gamma',\mathsf{u}}$ is transitive on $\overline{\gamma}'$. Since $k \leq v/2$ and $\mathsf{u} \in \gamma \cap \gamma'$, it follows that $\overline{\gamma} \cap \overline{\gamma}' \neq \emptyset$, and hence the union $\overline{\gamma} \cup \overline{\gamma}'$ is contained in a single G_u-orbit. Thus w, w' lie in the same G_u-orbit, and since this holds for all w, w', it follows that G_u is transitive on $\mathcal{V} \setminus \{\mathsf{u}\}$. Hence G is 2-transitive on $\mathcal{V}$, completing the proof.

5.2. **Organising the 2-transitive classification.** From now on we suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$ with $\delta(\Gamma) \geq 2$, where $2 \leq k \leq |\mathcal{V}| - 2 = v - 2$, and that $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$ acts strongly incidence-transitively on Γ and 2-transitively on $\mathcal{V}$. Since $\delta(\Gamma) \geq 2$, Γ is a proper subset of $\binom{\mathcal{V}}{k}$, and in particular the group G is not k-homogeneous on $\mathcal{V}$, that is to say, G is not transitive on the k-subsets of $\mathcal{V}$. In particular $3 \leq k \leq v - 3$, and G does not contain the alternating group A_v. Note that, by Theorem 1.2, each G-neighbour transitive code Γ with $\delta(\Gamma) \geq 3$ is G-strongly incidence-transitive.

Comments on the strategy: Those 2-transitive groups G which do not lie in an infinite family of 2-transitive groups have been analysed completely in [21]. Thus we assume that G lies in one of the infinite families of 2-transitive groups as listed, for example, in [5, Chapter 7.3 and 7.4] or [9, Chapter 7.7]. As explained in the introduction, in this paper we address all families apart from the symplectic groups acting on quadratic forms. Thus we investigate the following cases.

   affine: $G \leq \mathrm{A\Gamma L}(\mathcal{V})$ acting on $\mathcal{V} = \mathbb{F}_q^n$;
   linear: $\mathrm{PSL}(n,q) \leq G \leq \mathrm{P\Gamma L}(n,q)$ on $\mathrm{PG}(n-1,q)$;
   rank 1:   the Suzuki, Ree and Unitary groups.

We treat the various infinite families of 2-transitive groups G separately. Let $\gamma \in \Gamma$. Since Γ is G-strongly incidence-transitive, G_γ is transitive on $\gamma \times \overline{\gamma}$. In particular $k(v - k)$ divides $|G_\gamma|$, and G is not k-homogeneous.

For each of these 2-transitive groups G, we need to determine all possibilities for the stabiliser G_γ (up to conjugacy). Note that, if $G_\gamma \leq H < G$ and H is intransitive on $\mathcal{V}$ then, since G_γ has only two orbits on $\mathcal{V}$, namely γ and $\overline{\gamma}$, it follows that the H-orbits are the sets γ and $\overline{\gamma}$, and hence $H = G_\gamma$. Thus G_γ is a proper subgroup of G which is maximal subject to having two orbits in $\mathcal{V}$. We make a small observation about the case of a transitive subgroup H.

Lemma 5.2. *Suppose that Γ is G-strongly incidence-transitive with $G \leq \mathrm{Aut}\,(\Gamma) \cap \mathrm{Sym}\,(\mathcal{V})$. Let $\gamma \in \Gamma$, and suppose that $G_\gamma < H < G$ with H transitive on $\mathcal{V}$ and leaving invariant a non-trivial partition Π of $\mathcal{V}$. Then γ is a union of some of the blocks of Π.*

Proof. Let $\pi \in \Pi$ be a block of Π containing a point u of γ. We claim that $\pi \subseteq \gamma$. Suppose to the contrary that $\pi \cap \overline{\gamma}$ contains a point w. Then $G_{\gamma,\mathsf{u}} \leq H_{\mathsf{u}} < H_\pi$, so π contains the $G_{\gamma,\mathsf{u}}$-orbit containing w, namely $\overline{\gamma}$. This implies that G_γ fixes the block π setwise, so π also contains the G_γ-orbit containing u, namely γ. Thus $\pi = \mathcal{V}$, a contradiction. Hence $\gamma \subseteq \pi$. $\square$

6. Affine groups

In this section we treat the 2-transitive affine groups. Here $\mathcal{V} = \mathbb{F}_q^n$ is an n-dimensional vector space over a field $\mathbb{F}_q$ of order $q = p^a$ and $v = q^n$, where p is a prime and $a, n \geq 1$. The group G is a semidirect product $N.L$, where N is the group of translations of $\mathcal{V}$ and L is a subgroup of the group $\Gamma\mathrm{L}(\mathcal{V})$ of semilinear transformations of $\mathcal{V}$, which is transitive on $\mathcal{V}^\# = \mathcal{V} \setminus \{0\}$. So G is a subgroup of $X = \mathrm{A}\Gamma\mathrm{L}(\mathcal{V})$, the full affine semilinear group. We view $\mathcal{V}$ as the point set of the affine geometry $\mathrm{AG}(n, q)$. We use the notation introduced in Section 5.2.

6.1. **One-dimensional affine groups.** Here $n = 1$ and we identify $\mathcal{V}$ with $\mathbb{F}_q$. For application in the case of arbitrary dimension, we only assume in this subsection that $1 \leq k \leq v - 1$, and we do not insist that Γ is a proper subset of $\binom{\mathcal{V}}{k}$. Let x be a primitive element of $\mathbb{F}_q$, so that $A := \langle x \rangle$ is the multiplicative group of $\mathbb{F}_q$. Let $B := \langle \sigma \rangle = \mathrm{Aut}\,(\mathbb{F}_q) \cong Z_a$. Then $N \cong Z_p^a$ and $X = N.(A.B) = \mathrm{A}\Gamma\mathrm{L}(1, q)$. The aim of this section is to prove Proposition 6.1.

Proposition 6.1. *Suppose that $n = 1$. Then, replacing γ by some G-image if necessary, one of the following holds.*

(i) *$k = 1$ or $q - 1$;*
(ii) *$v = q = 4$, $k = 2$, $G = \mathrm{A}\Gamma\mathrm{L}(1, 4)$, $G_\gamma = \mathrm{AGL}(1, 2).2 \cong Z_2^2$, γ or $\overline{\gamma}$ is $\mathbb{F}_2$, $\Gamma = \binom{\mathcal{V}}{2}$ and $\delta(\Gamma) = 1$;*

(iii) $v = q = 16$, $k = 4$ *or* 12, $G = A\Gamma L(1, 16)$, $G_\gamma = [2^2] . GL(1, 4).[4]$, γ *or* $\overline{\gamma}$ *is* $\mathbb{F}_4$ *(as k is 4 or 12 respectively), Γ is strongly incidence transitive and $\delta(\Gamma) = 3$.*

We note that only the example in Proposition 6.1 (iii) yields a strongly incidence-transitive code, since in case (ii), $\Gamma = \binom{\mathcal{V}}{2}$.

Set $M := G_\gamma \cap N$. A *primitive prime divisor* of $p^a - 1$ is a prime divisor r of $p^a - 1$ such that r does not divide $p^i - 1$ for any positive integer $i < a$. For such a prime r, p has order a modulo r and so a divides $r - 1$. In particular $r \geq a + 1$. By [25], such primes exist unless $(p, a) = (6, 2)$, or $a = 2$ and $p = 2^b - 1$ for some b.

Lemma 6.2. *The parameter k lies in $\{1, v - 1\}$ if and only if $M = 1$.*

Proof. If $k = 1$ or $v - 1$ then G_γ fixes an element of $\mathcal{V}$ and so $M = 1$. Conversely suppose that $M = 1$, and suppose that $2 \leq k \leq v - 2 = q - 2$. In particular $q \geq 4$. Then $|G_\gamma|$ is divisible by $k(v - k) = k(q - k) \geq 2(q - 2)$. Also $G_\gamma \cong G_\gamma N/N \leq X/N \cong AB$ so $|G_\gamma|$ divides $(p^a - 1)a$. In particular

$$2(p^a - 2) \leq k(p^a - k) \leq |G_\gamma| \leq (p^a - 1)a.$$

If $a = 1$, then these inequalities imply that $q \leq 3$ which is a contradiction. Thus $a \geq 2$. If $q = 4$ then $k = 2$, but then $k(q - k) = 4$ does not divide $|G_\gamma|$. Hence $q \geq 8$. If $a = 2$ then the displayed inequalities imply that k lies in $\{2, q - 2\}$, but then $k(q - k)$ does not divide $|G_\gamma|$. Hence $a \geq 3$. Next if $q = 64$, then $k(64 - k)$ divides $|G_\gamma|$ which divides 63.6, but there is no such k in the interval $[2, 62]$. Thus $q \neq 64$ and hence there exists a primitive prime divisor r of $p^a - 1$, and as we observed above, $r \geq a + 1 > 3$. Suppose that r does not divide $|G_\gamma|$. Then $2(p^a - 2) \leq (p^a - 1)a/r < p^a - 1$ which is a contradiction. Thus r divides $|G_\gamma|$. A Sylow r subgroup of AB is contained in A (since $r > a$) and hence is normal in AB. It follows that G_γ has a normal Sylow r-subgroup, say R. Without loss of generality we may assume that $R \leq A$. In particular R has a unique fixed point in $\mathcal{V}$ which therefore must be fixed by G_γ. This contradicts $2 \leq k \leq v - 2$. $\square$

Now we suppose that $M \neq 1$. Let $K = \mathbb{F}_p[G_\gamma \cap A]$ denote the subfield of $\mathbb{F}_q$ generated by $G_\gamma \cap A$.

Lemma 6.3. *If $M \neq 1$ then M is a K-vector space and K is a proper subfield of $\mathbb{F}_q$.*

Proof. The group M acts on $\mathcal{V}$ as a subgroup of translations. Thus for some $Y \subseteq \mathbb{F}_q$, $M = \{t_y \,|\, y \in Y\}$, where $t_y : v \mapsto v + y$ for $v \in \mathcal{V}$. Interchanging γ and $\overline{\gamma}$ if necessary, we may assume that $0 \in \gamma$. Then the M-orbit 0^M is equal to Y and is contained in γ. As M is a subgroup of N, the set Y is an $\mathbb{F}_p$-subspace of $\mathcal{V}$ (viewed as $\mathbb{F}_p^a$). Also $G_\gamma \cap A$ normalises M, and as A acts by multiplication on $\mathcal{V}$ it follows that, for each $z \in G_\gamma \cap A$ and $t_y \in M$, we have $z^{-1} t_y z = t_{yz} \in M$, that is to say

$Y(G_\gamma \cap A) \subseteq Y$. Since $K = \mathbb{F}_p[G_\gamma \cap A]$ it follows that Y is also closed under multiplication by arbitrary elements of K. Thus Y is a K-vector space, that we identify with M. Since M is intransitive on $\mathcal{V}$, K is a proper subfield of $\mathbb{F}_q$. $\square$

Lemma 6.4. *If $M \neq 1$ then (ii) or (iii) of Proposition* 6.1 *holds.*

Proof. Interchanging γ and $\overline{\gamma}$ if necessary, we may assume that $|\gamma| \leq v/2 = p^a/2$. (Note that each example for γ will provide an example for $\overline{\gamma}$.) Also since G is 2-transitive on $\mathcal{V}$, replacing γ by a G-image if necessary, we may assume that γ contains 0 and 1. Now we use arithmetic. We have $|M| = p^s$, where $1 \leq s < a$ by Lemma 6.3. As M is semiregular on $\mathcal{V}$, $k = |\gamma| = p^s m$ and $|\overline{\gamma}| = p^a - p^s m$, for some m such that $1 \leq m < p^{a-s}$. By assumption $p^s m(p^a - p^s m)$ divides $|G_\gamma|$ which divides $p^s(p^a - 1)a$. Thus $p^s m(p^{a-s} - m)$ divides $|G_\gamma|/p^s$ which divides $(p^a - 1)a$. In particular p^s divides a, so $a \geq p \geq 2$.

Suppose that $a = 2$. Then also $p^s = 2$ and $q = 4$. Since $2 \leq k \leq q/2$ we have $k = 2$, and since γ contains 0 and 1, we must have $\gamma = \{0, 1\} = \mathbb{F}_2$. Thus $G_\gamma = MB \cong Z_2^2$, and since G contains N and is 2-transitive on $\mathcal{V}$, it follows that $G = A\Gamma L(1, 4)$. Thus $\Gamma = \binom{\mathcal{V}}{2}$ and $\delta(\Gamma) = 1$, so (ii) of Proposition 6.1 holds. If $p^a = 2^6$ then since p^s divides $a = 6$, it follows that $p^s = 2$ and so $|M| = 2$. Since M is G_γ-invariant we have $G_\gamma \cap A = 1$. Thus $m(64 - 2m)$ divides $|G_\gamma|/2$ which divides 6, a contradiction. Hence we may assume that $a \geq 3$ and $p^a \neq 64$.

In particular $p^a - 1$ has a primitive prime divisor, say r, and as noted above, $r \geq a + 1$. If $|G_\gamma \cap A|$ were divisible by r then the subfield K would be equal to $\mathbb{F}_q$, contradicting Lemma 6.3. Hence r does not divide $|G_\gamma \cap A|$ and so $m(p^a - p^s m) \leq |G_\gamma|/p^s \leq \frac{p^a - 1}{r} a < p^a - 1$. The function $x(p^a - p^s x)$ is continuous on the interval $(1, p^{a-s})$ with a maximum at $x = p^{a-s}/2$. If $2 \leq m \leq p^{a-s} - 2$, then $p^a - 1 > m(p^a - p^s m) \geq 2(p^a - 2p^s)$ implying that $p^a < 4p^s - 1 < 4p^s$. Hence $p^{a-s} < 4$ so that $s = a - 1$, which contradicts the fact that $p^s = p^{a-1}$ divides a with $a \geq 3$. Thus $m = 1$ or $p^{a-s} - 1$, and since $|\gamma| \leq p^a/2$, this implies that $k = |\gamma| = p^s$, γ is the M-orbit containing $\{0, 1\}$, and $p^{a-s} - 1$ divides $\frac{p^a - 1}{r} \frac{a}{p^s}$. Since M is transitive on γ, we have $G_\gamma = MG_{\gamma, 0}$.

If $a = 3$ then its divisor p^s is also equal to 3, and the divisibility condition is that $p^{a-s} - 1 = 8$ divides $26/r$, which is impossible.

If $a = 4$ then $p^s = 2$ or 4, and the divisibility condition is '7 divides $30/r$' or '3 divides $15/r$' respectively. It follows that $k = p^s = 4$, $q = 16$, and 12 divides $|G_\gamma|/4$. Since $G_\gamma = MG_{\gamma, 0}$, the latter condition is equivalent to '12 divides $|G_{\gamma, 0}|$'. Then since $G_{\gamma, 0} \leq G_0 \leq \Gamma L(1, 16)$ and $k = 4$, it follows that $G_{\gamma, 0} \cap GL(1, 16) = \langle x^5 \rangle \cong Z_3$ and that γ coincides with the subfield $\mathbb{F}_4$. This implies that $G_\gamma = M(\langle x^5 \rangle . B)$, that the subfield $K = \mathbb{F}_4$, and that the two orbits of G_γ in $\mathcal{V}$ are $\gamma = \mathbb{F}_4$ and its complement $\overline{\gamma}$. Also, since G contains N and $\langle x^5 \rangle . B$, and is 2-transitive on $\mathcal{V}$, it follows that $G = A\Gamma L(1, 16)$. Moreover $G_{\gamma, 0} = \langle x^5 \rangle . B$ is

transitive on $\bar{\gamma}$, so Γ is strongly incidence transitive. Now G_γ induces a 2-transitive action on γ, and the stabiliser $G_{\gamma,\{0,1\}}$ has order 8. Since $|G_{\{0,1\}}| = |G|/\binom{16}{2} = 8$, it follows that $G_{\gamma,\{0,1\}} = G_{\{0,1\}}$, and so $\mathbb{F}_4$ is the only 4-element set containing $\{0,1\}$ and left invariant by $G_{\gamma,\{0,1\}}$. Hence γ is the only 'codeword' containing $\{0,1\}$, so $\delta(\Gamma) = 3$ and (iii) of Proposition 6.1 holds.

Thus we may assume that $a \geq 5$. The facts that p^s divides a and $a \geq 5$ together imply that $a - s \geq 4$. Hence either (i) $p = 2, a = 8, s = 2$ and $p^{a-s} - 1 = 2^6 - 1 = 63$, or (ii) $p^{a-s} - 1$ has a primitive prime divisor r'. Case (i) is impossible since 63 does not divide $a(p^a - 1)$. Thus we are in case (ii) and the prime r' divides $\frac{p^a-1}{r}\frac{a}{p^s}$. Suppose first that r' divides a/p^s. Then $a \geq r'p^s \geq (a - s + 1)p^s$, and hence

$$a \leq \frac{p^s(s-1)}{p^s - 1} = (s-1)(1 + \frac{1}{p^s - 1}) \leq 2(s-1) < 2^s \leq p^s \leq a$$

which is a contradiction. Hence r' divides $p^a - 1$. This implies that $a - s$ divides a, and hence $a - s \leq a/2$ so $s \geq a/2 \geq p^s/2 \geq 2^{s-1}$. It follows that either $a = p = 2s = 2$ or $a/2 = p = s = 2$, contradicting the assumption that $a \geq 5$. $\square$

Proposition 6.1 follows from Lemmas 6.2 and 6.4.

6.2. **General affine case.** Now suppose that $G = N.L \leq X = A\Gamma L(n,q)$ with $q = p^a$ and $n \geq 2$, acting on $\mathcal{V} = \mathbb{F}_q^n$ and that G_γ is transitive on $\gamma \times \bar{\gamma}$. The affine subspaces and their complements provide natural families of examples, since taking $G = X$ and γ or $\bar{\gamma}$ an affine subspace, the group G_γ is transitive on $\gamma \times \bar{\gamma}$.

Example 6.5. For any positive integer $s < n$, the set Γ of affine s-dimensional subspaces, and the set $\bar{\Gamma}$ of complements of these s-subspaces, are X-xtrongly incidence-transitive codes.

Our main result for the affine case shows that examples apart from those in Example 6.5 are very restricted. In particular, the codeword γ or its complement is a subset of class $[0, \sqrt{q}, q]_1$ (as defined before Theorem 1.4) and q must be 4 or 16.

Proposition 6.6. *Suppose that $\mathcal{V} = \mathbb{F}_q^n$ with $n \geq 2$, and $\Gamma \subset \binom{\mathcal{V}}{k}$ is G-strongly incidence-transitive, where $G \leq A\Gamma L(n,q)$ is 2-transitive on $\mathcal{V}$. Let $\gamma \in \Gamma$. Then one of the following holds.*

(i) *γ or $\bar{\gamma}$ is an affine subspace as in Example* 6.5, *or*
(ii) *$q = 4$ and each line of $AG(n,4)$ either lies in γ or $\bar{\gamma}$, or intersects γ in a Baer sub-line. Moreover $\frac{q^n+2}{3} \leq k \leq \frac{2(q^n-1)}{3}$.*
(iii) *$q = 16$ and, interchanging γ and $\bar{\gamma}$ if necessary, each line of $AG(n,q)$ either lies in γ or $\bar{\gamma}$, or intersects γ in a Baer sub-line of size 4. Moreover $\frac{q^n+4}{5} \leq k \leq \frac{4(q^n-1)}{15}$.*

Proof. Since G_γ is transitive on $\gamma \times \bar\gamma$, it follows that G_γ is transitive on the set $\mathcal{L}$ of lines of the affine space $\mathrm{AG}(n, q)$ that meet both γ and $\bar\gamma$. Thus for $\lambda \in \mathcal{L}$, $|\gamma \cap \lambda| = x$ is independent of the choice of λ, and $|\bar\gamma \cap \lambda| = q - x$ with $1 \le x < q$.

Moreover, the group induced on λ by $G_{\gamma,\lambda}$ is a subgroup of $\mathrm{A\Gamma L}(1, q)$. Let $\mathsf{v} \in \gamma \cap \lambda$. Then $G_{\gamma,\mathsf{v}}$ is transitive on $\bar\gamma$, and moreover the subset of lines of $\mathcal{L}$ containing v induces a $G_{\gamma,\mathsf{v}}$-invariant partition of $\bar\gamma$ into parts of size $q - x$. Hence $G_{\gamma,\mathsf{v},\lambda}$ is transitive on $\bar\gamma \cap \lambda$, and similarly, for $\mathsf{u} \in \bar\gamma \cap \lambda$, $G_{\gamma,\mathsf{u},\lambda}$ is transitive on $\gamma \cap \lambda$. Thus the subgroup of $\mathrm{A\Gamma L}(1, q)$ induced by $G_{\gamma,\lambda}$ on λ is transitive on $(\gamma \cap \lambda) \times (\bar\gamma \cap \lambda)$. It follows from Proposition 6.1 that one of $x \in \{1, q - 1\}$, or $q = 4$ with $x = 2$, or $q = 16$ with $x \in \{4, 12\}$.

Suppose first that $x \in \{1, q - 1\}$. Interchanging γ and $\bar\gamma$ if necessary, we may assume that $x = 1$. Then, for any pair of distinct points $\mathsf{v}, \mathsf{v}' \in \gamma$, the unique line λ containing v and v' lies entirely within γ. Thus γ (or the original $\bar\gamma$) is an affine subspace of $\mathrm{AG}(n, q)$, as in Example 6.5.

Now we consider the other possibilities. Interchanging γ and $\bar\gamma$ if necessary, we may assume that $q = x^2 \in \{4, 16\}$. Then the subset $\mathcal{L}'$ of $\mathcal{L}$ consisting of lines containing a given point $\mathsf{v} \in \gamma$ induces a partition of $\bar\gamma$ with $(q^n - k)/(q - x)$ parts of size $q - x$, and $G_{\gamma,\mathsf{v}}$ is transitive on $\mathcal{L}'$. Each line of $\mathcal{L}'$ intersects γ in a subset of size x which, by Proposition 6.1 is a Baer sub-line. The additional $(x - 1)(q^n - k)/(q - x)$ points of γ lying on these lines (apart from v) forms a $G_{\gamma,\mathsf{v}}$-orbit. Thus $k \ge 1 + (x - 1)(q^n - k)/(q - x)$, and rearranging gives $k \ge \frac{(x-1)q^n + q - x}{q - 1}$ which, since $q = x^2$, gives $k \ge \frac{q^n + x}{x + 1}$.

Similarly, the subset $\mathcal{L}''$ of $\mathcal{L}$ consisting of lines containing a given point $\mathsf{v}' \in \bar\gamma$ induces a partition of γ with k/x parts of size x, and $G_{\gamma,\mathsf{v}'}$ is transitive on $\mathcal{L}''$. Each line of $\mathcal{L}''$ intersects $\bar\gamma$ in a subset of size $q - x$. The additional $(q - x - 1)k/x$ points of $\bar\gamma$ lying on these lines (apart from v') form a $G_{\gamma,\mathsf{v}'}$-orbit. Thus $|\bar\gamma| = q^n - k \ge 1 + (q - x - 1)k/x$, and rearranging gives $k \le \frac{x(q^n - 1)}{q - 1}$. For $q = 4$ we therefore have $\frac{q^n + 2}{3} \le k \le \frac{2(q^n - 1)}{3}$ and for $q = 16$ we have $\frac{q^n + 4}{5} \le k \le \frac{4(q^n - 1)}{15}$, and parts (ii) and (iii) hold. $\qquad\square$

We note that the 2-transitive hyperoval in $\mathrm{PG}(2, 4)$ provides an example for case (ii) of Proposition 6.6.

Example 6.7. Let γ be a 2-transitive hyperoval in the projective plane $\mathrm{PG}(2, 4)$, and let λ be an external line of γ. Then $k = |\gamma| = 6$ and the complement of λ in the point set of $\mathrm{PG}(2, 4)$ is an affine space $\mathcal{V} = \mathrm{AG}(2, 4)$ containing γ. Let $\bar\gamma = \mathcal{V} \setminus \gamma$. Then the subgroup G_γ of $\mathrm{P\Gamma L}(3, 4)$ stabilising λ and γ setwise acts faithfully on $\mathcal{V}$ and is transitive on $\gamma \times \bar\gamma$.

To see this observe that $G_\gamma = \mathrm{SL}(2,4)\langle\sigma\rangle \cong S_5$ is 2-transitive on γ, and for $\mathsf{v} \in \gamma$, $G_{\gamma,\mathsf{v}}$ acts transitively on the five secant lines to γ containing v. Each of these secants contains two points of $\bar\gamma$ and one point of λ. Thus $G_{\gamma,\mathsf{v}}$ is transitive on $\bar\gamma$.

7. Linear case

In this section we investigate the 2-transitive projective linear groups. Here $\mathcal{V}$ is the point set of the projective geometry $\mathrm{PG}(n-1,q)$ of rank $n-1 \geq 1$ over a field $\mathbb{F}_q$ of order $q = p^a$, and $v = (q^n-1)/(q-1)$, where p is a prime and $a \geq 1$. Since the situation for affine 2-transitive groups was analysed in Section 6, we assume that $(n,q) \neq (2,2)$ or $(2,3)$. In general the group G satisfies $\mathrm{PSL}(n,q) \leq G \leq X := \mathrm{P\Gamma L}(n,q)$ and we often assume that $G = X$. We use the notation introduced in Section 5.2. Since G is 2-transitive and Γ is a proper subset of $\binom{\mathcal{V}}{k}$, we have in particular, $3 \leq k \leq v - 3$ and, for some k-subset $\gamma \subset \mathcal{V}$, the stabiliser G_γ is transitive on $\gamma \times \bar\gamma$.

7.1. **Rank 1 case.** Here $n = 2$, $q \geq 4$, and we identify $\mathcal{V} = \mathrm{PG}(1,q)$ with $\mathbb{F}_q \cup \{\infty\}$. We show that all examples arise from Baer sub-lines of $\mathcal{V}$. If $q = q_0^2$ then the subset $\mathbb{F}_{q_0} \cup \{\infty\}$, and its X-translates are the *Baer sub-lines.*

Example 7.1. Let $q = q_0^2$, and let $\gamma = \mathbb{F}_{q_0} \cup \{\infty\}$, so $k = |\gamma| = q_0 + 1$. Then the group $X_\gamma = N_X(\mathrm{PGL}(2,q_0))$ is transitive on $\gamma \times \bar\gamma$. Moreover, since any pair of Baer sub-lines intersects in at most two points, it follows that the corresponding strongly incidence-transitive code Γ has minimum distance $\delta(\Gamma) = q_0 - 1$.

Proposition 7.2. *If $n = 2$ and $3 \leq k \leq q - 2$, then $q = q_0^2$, $k = q_0 + 1$ or $q - q_0$, and γ or $\overline\gamma$ is a Baer sub-line, as in Example 7.1.*

Proof. We use the classification of the subgroups of $S := \mathrm{PSL}(2,q)$, [8, Chapter VII]. Replacing γ by $\overline\gamma$ if necessary, we may assume that $3 \leq k \leq v/2 = (q+1)/2$. It follows in particular that G_γ is not contained in a maximal parabolic subgroup, and $G_\gamma \cap \mathrm{PGL}(2,q) \not\leq D_{2(q-1)}$. Suppose that $G_\gamma \cap \mathrm{PGL}(2,q) \leq D_{2(q+1)}$. Since the Frobenius automorphism fixes the identity, the only way that G_γ can have two orbits is if $k = (q+1)/2$. Then $(q+1)^2/4$ divides $|G_\gamma|$, which divides $(q+1)a$. There are no possibilities with $q \geq 4$.

Now suppose that $G_\gamma \leq N_X(\mathrm{PSL}(2,q_0))$, where $q = q_0^b$ for some $b > 1$. Then G_γ fixes setwise a sub-line $\mathrm{PG}(1,q_0)$ of size $q_0 + 1 < v/2$. Since the only G_γ-orbit size at most $v/2$ is k, we have that γ is the set of points on a sub-line $\mathrm{PG}(1,q_0)$ with $k = q_0 + 1$, and hence $(q_0+1)(q-q_0)$ divides $|G_\gamma|$, which divides $q_0(q_0^2-1)a$. Thus $q_0^{b-1} - 1$ divides $(q_0-1)a$, which implies $b = 2$. Thus γ is a Baer sub-line, as in Example 7.1.

The remaining cases are those where $H := G_\gamma \cap \mathrm{PGL}(2,q) = A_5, S_4$ or A_4, and H is not contained in any 'subfield' subgroup $\mathrm{PGL}(2,q_0)$

with $q_0 < q$. Suppose first that $H = A_5$. Then $q \equiv \pm 1 \pmod{10}$, and since G_γ is not contained in a subfield subgroup, $a \le 2$. Since $3(q-2) \le k(v-k) \le |G_\gamma| \le 60a$, we have $q \in \{9, 11, 19\}$. The cases $q = 9$ and $q = 11$ are not possible since in these cases A_5 is transitive on $\mathcal{V}$. Thus $q = 19$. However there is no value of k in the interval $[3, 9]$ such that $k(20 - k)$ divides $|G_\gamma| = 60$.

This leaves $A_4 \le G_\gamma \le S_4.\langle \sigma \rangle$ with q odd. Since G_γ is not contained in a subfield subgroup, either $q = p$, or $q = p^2$ with $p \equiv \pm 3 \pmod{8}$. Since $3(q-2) \le k(v-k) \le |G_\gamma| \le 24a$, we have $q \in \{5, 7, 9\}$, and since $k(v-k)$ divides $|G_\gamma|$, it follows that $q = 9$ and $k = 4$. However in this case G_γ is contained in $N_X(\mathrm{PSL}(2, 3))$ which we assumed was not the case. $\qquad\qquad\square$

7.2. **Higher rank linear case.** Now we assume that $n \ge 3$. Here we have a family of examples arising from subspaces and their complements.

Example 7.3. Let $1 \le s < n$ and let $\gamma = \mathrm{PG}(s - 1, q)$ be an $(s - 1)$-dimensional subspace of $\mathcal{V}$, so $k = |\gamma| = (q^s - 1)/(q - 1)$. Then the subgroup X_γ is transitive on $\gamma \times \overline{\gamma}$. Thus the set Γ of $(s - 1)$-dimensional subspaces, and the set of their complements, form X-strongly incidence-transitive codes and each has minimum distance $\delta(\Gamma) = q^{s-1}$.

For $u \in \gamma, w \in \overline{\gamma}$, we call the line $\lambda(u, w)$ containing u and w a γ-*shared line*. Since G_γ is transitive on $\gamma \times \overline{\gamma}$, the γ-shared lines form a single G_γ-orbit on lines, and in particular they all meet γ in a constant number x of points, where $1 \le x \le q$. Thus γ is a subset of class $[0, x, q + 1]_1$. In Example 7.3, $x = 1$.

Proposition 7.4. *Suppose that $\mathcal{V} = \mathrm{PG}(n - 1, q)$, $\mathrm{PSL}(n, q) \le G \le \mathrm{P\Gamma L}(n, q)$, and $\Gamma \subset \binom{\mathcal{V}}{k}$ is G-strongly incidence-transitive, where $n \ge 3$ and $3 \le k \le |\mathcal{V}| - 3$. Let $\gamma \in \Gamma$. Then either γ or $\overline{\gamma}$ is a projective subspace as in Example 7.3 or, interchanging γ and $\overline{\gamma}$ if necessary, γ is a subset of class $[0, x, q + 1]_1$, where one of the following holds.*

(a) *$x = 2$ and $\frac{v-1}{q} + 1 \le k \le \frac{2(v-1)}{q}$, or*

(b) *$x = q_0 + 1$, $q = q_0^2$, $\frac{v-1}{q_0} + 1 \le k \le \frac{v-1}{q_0} + \frac{v-1}{q}$ and, for each γ-shared line λ, $\lambda \cap \gamma$ is a Baer sub-line.*

Remark 7.5. The parameters in part (a) suggest that γ might be a configuration similar to an oval or hyperoval in $\mathrm{PG}(2, q)$. For example, in $\mathrm{PG}(2, 4)$, the stabiliser in $\mathrm{PGL}(3, 4)$ of a hyperoval γ is $G_\gamma \cong S_6$, transitive on both γ and the complement $\overline{\gamma}$ with $|\gamma| = 6, |\overline{\gamma}| = 15$. However, for $u \in \gamma$, $G_{\gamma,u}$ has two orbits in $\overline{\gamma}$, one of them an external line to $\overline{\gamma}$. Thus this does not give rise to a strongly incidence-transitive code.

This argument about $\mathrm{PG}(2,4)$, together with Proposition 7.4, were used by Nico Durante in [10, Theorem 3.3] to prove that there are no additional examples in the linear case satisfying the conditions of Proposition 7.4 (a) or (b).

Proof. The group G_γ is transitive on the set $\mathcal{L}$ of γ-shared lines and, for $\lambda \in \mathcal{L}$, the group induced on λ by $G_{\gamma,\lambda}$ is a subgroup of $\mathrm{P\Gamma L}(2,q)$, independent of the choice of λ. Let $\mathsf{u} \in \lambda \cap \gamma$. Then $G_{\gamma,\mathsf{u}}$ is transitive on $\bar\gamma$ and moreover the subset of lines of $\mathcal{L}$ containing u induces a $G_{\gamma,\mathsf{u}}$-invariant partition of $\bar\gamma$ with parts of size $q + 1 - x$. Hence $G_{\gamma,\mathsf{u},\lambda}$ is transitive on $\lambda \cap \bar\gamma$. Similarly, if $\mathsf{u} \in \lambda \cap \bar\gamma$, then $G_{\gamma,\mathsf{u},\lambda}$ is transitive on $\lambda \cap \gamma$. Thus the subgroup of $\mathrm{P\Gamma L}(2,q)$ induced by $G_{\gamma,\lambda}$ on λ is transitive on $(\lambda \cap \gamma) \times (\lambda \cap \bar\gamma)$. It follows from Proposition 7.2 that, interchanging γ and $\bar\gamma$ if necessary, $(x, q + 1 - x) = (1, q)$, $(2, q - 1)$, or $(q_0 + 1, q - q_0)$, where in the third case, $q = q_0^2$ and $\lambda \cap \gamma$ is a Baer sub-line of λ.

Suppose first that $x = 1$. Then, for any pair of distinct points $\mathsf{u}, \mathsf{u}' \in \gamma$, the line λ containing u and u' lies entirely within γ. Thus γ is a subspace of $\mathcal{V}$ as in Example 7.3.

Now suppose that $x = 2$ or $x = q_0$. Then the subset $\mathcal{L}'$ of $\mathcal{L}$ consisting of lines containing a fixed point $\mathsf{u} \in \gamma$ induces a partition of $\bar\gamma$ with $(v - k)/(q + 1 - x)$ parts of size $q + 1 - x$, and $G_{\gamma,\mathsf{u}}$ is transitive on $\mathcal{L}'$. Each line of $\mathcal{L}'$ intersects γ in a set consisting of u and $x - 1$ further points. The $(x-1)(v-k)/(q+1-x)$ points of γ, distinct from u, lying on these lines forms a $G_{\gamma,\mathsf{u}}$-orbit contained in $\gamma \setminus \{\mathsf{u}\}$. Thus $k \geq 1 + \frac{(x-1)(v-k)}{q+1-x}$, and hence $k \geq \frac{(x-1)(v-1)}{q} + 1$.

Similarly, if $\mathsf{u} \in \bar\gamma$, then the subset $\mathcal{L}''$ of $\mathcal{L}$ consisting of lines containing u induces a partition of γ with k/x parts of size x, and $G_{\gamma,\mathsf{u}}$ is transitive on $\mathcal{L}''$. Each line of $\mathcal{L}''$ intersects $\bar\gamma$ in a set consisting of u and $q - x$ further points. The $(q - x)k/x$ points of $\bar\gamma$, distinct from u, lying on these lines forms a $G_{\gamma,\mathsf{u}}$-orbit. Thus $v - k \geq 1 + \frac{(q-x)k}{x}$, and hence $k \leq \frac{x(v-1)}{q}$. This yields $\frac{v-1}{q} + 1 \leq k \leq \frac{2(v-1)}{q}$ if $x = 2$ and $\frac{v-1}{q_0} + 1 \leq k \leq \frac{v-1}{q_0} + \frac{v-1}{q}$ if $x = q_0 + 1$. $\square$

8. Suzuki, Ree and rank 1 Unitary groups

In this section we treat the 2-transitive actions of Lie type groups G of rank 1 apart from the linear case which is handed in Subsection 7.1. Again we use the notation from Section 5.2: since G is not 3-transitive we assume that $3 \leq k \leq v - 3$. There is an infinite family of examples connected to the classical unitals in $\mathrm{PG}(2, q^2)$, (for information on these unitals see [1, 24]).

Let q be a prime power and $V = \mathbb{F}_{q^2}^3$. The involutory automorphism $x \to x^q$ of $\mathbb{F}_{q^2}$ allows us to define a Hermitian form $\varphi : V \times V \to \mathbb{F}_{q^2}$ as follows: for $\mathsf{x} = (x_1, x_2, x_3)$ and $\mathsf{y} = (y_1, y_2, y_3) \in V$, $\varphi(\mathsf{x}, \mathsf{y}) = x_1\bar{y}_3 + x_3\bar{y}_1 + x_2\bar{y}_2$, where we write $\bar{a} := a^q$ for $a \in \mathbb{F}_{q^2}$.

Example 8.1. The subgroup $G := \mathrm{P\Gamma U}(3, q)$ of $\mathrm{P\Gamma L}(3, q^2)$ preserving φ acts faithfully and 2-transitively on the set $\mathcal{V}$ of $v = q^3 + 1$ isotropic 1-spaces $\langle x \rangle$ of V (that is, $\varphi(x, x) = 0$). Each non-degenerate 2-space L of V, relative to φ, contains exactly $q + 1$ elements of $\mathcal{V}$, and we denote this $(q+1)$-subset of $\mathcal{V}$ by $L \cap \mathcal{V}$. The code $\Gamma \subset \binom{\mathcal{V}}{q+1}$ consisting of these $(q + 1)$-subsets, one for each non-degenerate 2-space L, is the classical unital. It is G-strongly incidence-transitive with minimum distance $\delta(\Gamma) = q$.

Lemma 8.2. *The claims made about Γ in Example* 8.1 *are valid.*

Proof. Let $\gamma = L \cap \mathcal{V}$ for some non-degenerate 2-space L. We prove that G_γ is transitive on $\gamma \times \bar{\gamma}$, where $G = \mathrm{GU}(3, q)$ (acting with kernel a subgroup of scalars of order $(3, q + 1)$). Denote by e_i the standard basis vector with 1 in the i-entry and other entries 0. Then e_1, e_3 are isotropic while $\varphi(e_2, e_2) = 1$. We take $\gamma = L \cap \mathcal{V}$, for L the φ-orthogonal complement of the non-isotropic vector e_2. It is straightforward to compute that

$$\gamma = \{\langle e_1 \rangle, \langle e_3 \rangle\} \cup \{\langle (x, 0, 1) \rangle \mid x^{q-1} = 1\}$$

of size $k = |\gamma| = q + 1$. The fact that $\mathrm{GU}(3, q)$ is transitive on incident point-block pairs of the unital follows from Witt's theorem, and hence G_γ is transitive on γ. By [9, pp.248–250], G contains (modulo scalars) the following elements

$$(2) \qquad t_{\alpha,\beta} = \begin{pmatrix} 1 & -\bar{\beta} & \alpha \\ 1 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}, \quad h_{\nu,\mu} = \begin{pmatrix} \nu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \bar{\nu}^{-1} \end{pmatrix}$$

for $\alpha, \beta, \nu, \mu \in \mathbb{F}_{q^2}$ such that $\alpha + \bar{\alpha} + \beta\bar{\beta} = 0$, $\nu \neq 0, \mu\bar{\mu} = 1$, and the stabiliser G_v in G of $v = \langle e_3 \rangle \in \gamma$ consists of the $q^3(q^2 - 1)(q + 1)$ products $h_{\nu,\mu}t_{\alpha,\beta}$. A straightforward computation shows that $G_{\gamma,v}$ has order $q(q^2 - 1)(q + 1)$, comprising those products with $\beta = 0$. For $x \in \mathbb{F}_{q^2}$ such that $x + \bar{x} + 1 = 0$ and $x \neq 1$, the vector $(x, 1, 1)$ is isotropic, so $u := \langle (x, 1, 1) \rangle \in \mathcal{V} \setminus \gamma = \bar{\gamma}$. The element $h_{\nu,\mu}t_{\alpha,0} \in G_{\gamma,v}$ maps $(x, 1, 1)$ to $(x\nu + \mu, \mu, x\nu\alpha + \bar{\nu}^{-1})$, and hence fixes u if and only if $x\nu + \mu = x\mu$ and $\mu = x\nu\alpha + \bar{\nu}^{-1}$; or equivalently, $\nu = \mu(x - 1)/x$ and $\alpha = (\mu - \bar{\nu}^{-1})/x\nu$ are determined by x and μ. Thus the $G_{\gamma,v}$-orbit containing u has length $q(q^2 - 1) = |\bar{\gamma}|$, whence G_γ is transitive on $\gamma \times \bar{\gamma}$ as claimed. Finally every two points of $\mathcal{V}$ lie in a unique codeword in Γ, and since G is 2-transitive on $\mathcal{V}$ the largest intersection of distinct codewords is 1, so the minimum distance of Γ is q. $\qquad\square$

We now prove Theorem 1.3, which deals with the 2-transitive groups G of rank 1, that is, groups with socle $T(q)$ of degree $v = |\mathcal{V}|$ as in one of the lines of Table 4.

$T(q)$	$q = p^a$	v	$\lvert T(q) \rvert$	$\lvert \mathrm{Out}\,(T) \rvert$
$\mathrm{Sz}(q)$	$2^{2c+1} > 2$	$q^2 + 1$	$q^2(q^2 + 1)(q - 1)$	a
$\mathrm{Ree}(q)$	$3^{2c+1} > 3$	$q^3 + 1$	$q^3(q^3 + 1)(q - 1)$	a
$\mathrm{Ree}(3)'$	3	$q^3 + 1$	$q^2(q^3 + 1)(q - 1)$	3
$\mathrm{PSU}(3, q)$	$q > 2$	$q^3 + 1$	$\frac{1}{d}q^3(q^3 + 1)(q^2 - 1)$	$2ad$
			where $d = (3, q + 1)$	

TABLE 4. Groups for Theorem 1.3

Proof of Theorem 1.3. Let $T \trianglelefteq G \leq \mathrm{Aut}\,(T)$ with $q = p^a$, $T = T(q)$ and $v = \lvert \mathcal{V} \rvert$ as in one of the lines of Table 4. We use the classification of the subgroups of G in [23, 13, 4] for the Suzuki, Ree and unitary groups, respectively. Suppose that $\Gamma \subset \binom{\mathcal{V}}{k}$ is G-strongly incidence transitive, and let $\gamma \in \Gamma$. Since $\Gamma \neq \binom{\mathcal{V}}{k}$ and G is 2-transitive on $\mathcal{V}$, we have $3 \leq k \leq v - 3$. Then G_γ has two orbits in $\mathcal{V}$, each of size at least 3, and it follows that G_γ is not contained in a parabolic subgroup. When G_γ is contained in other maximal subgroups we use the fact that $k(v - k)$ divides $\lvert G_\gamma \rvert$ and in particular that $k(v - k) \leq \lvert G_\gamma \rvert$.

If $T = \mathrm{Sz}(q)$, then by [23], the non-parabolic maximal subgroups of T have orders $2(q-1)$, or $4(q \pm r + 1)$, or $\lvert \mathrm{Sz}(q_0) \rvert$, where $2q = r^2$ and $q = q_0^b$ for an odd prime b. In each case $\lvert G_\gamma \rvert \leq a \lvert T_\gamma \rvert < 3(q^2 - 2) \leq k(v - k)$.

Suppose next that $T = \mathrm{Ree}(q)$, with $q > 3$, or $\mathrm{Ree}(3)' \cong \mathrm{PSL}(2, 8)$. Then by [13], the non-parabolic maximal subgroups of T have orders $6(q + 1)$, or $2\lvert \mathrm{PSL}(2, q) \rvert$ (with $q > 3$), or $6(q \pm r + 1)$ (with $3q = r^2$), or $\lvert \mathrm{Ree}(q_0) \rvert$ (with $q = q_0^b$ for an odd prime b). In each case $\lvert G_\gamma \rvert \leq \lvert \mathrm{Out}\,(T) \rvert . \lvert T_\gamma \rvert < 3(q^3 - 2) \leq k(v - k)$.

Thus $T = \mathrm{PSU}(3, q)$ with $q > 2$. We may assume that neither γ nor $\overline{\gamma}$ is as in Example 8.1. Then G_γ acts irreducibly on the underlying space $V = V(3, q^2)$, so G_γ is contained in an irreducible maximal subgroup H of G, and $H \cap T$ is contained in a maximal subgroup M of T. The list of maximal subgroups of T can be found in [4, Tables 8.5 and 8.6 on pp. 379], and we consider them in turn. First, however, we deal with the small cases where $q \in \{3, 4, 5\}$. For these groups, lists of maximal subgroups of T are available in [6], and for some properties we rely on computations in GAP [11] kindly done for us by Max Neunhöffer.

Case: $q = 3$. For $\mathsf{v} \in \mathcal{V}$ lying in a G_γ-orbit of length $\min\{k, v - k\} \leq v/2$, the subgroup $G_{\gamma, \mathsf{v}}$ has an orbit of length $\max\{k, v - k\} \geq v/2 = 14$. It follows that T_γ is not contained in the transitive maximal subgroup $\mathrm{PSL}(2, 7)$, and hence $T_\gamma = 4^2 : S_3$. A GAP computation confirms that this subgroup gives rise to an example with γ or $\overline{\gamma}$ of size 12 and $\delta(\Gamma) = 6$, and for the transitivity condition we need $G = T.2$. In this example, the codewords of size 12 are the 'bases' [6, page 14].

Case: $q = 4$. Since G_γ is irreducible and $\lvert G_\gamma \rvert$ is divisible by $k(65 - k)$, it follows that $k = 5$, $G = T.2$ or $T.4$, and $G_\gamma \cap T.2 = 5^2 : D_{12}$. However a GAP computation reveals that the subgroups $5^2 : D_{12}$ and $5^2 : (4 \times S_3)$

both have orbit lengths 15 and 50 in $\mathcal{V}$, and hence we get no example since $15 \cdot 50$ does not divide $|G_\gamma|$.

Case: $q = 5$. Since $|G_\gamma|$ is divisible by $k(126 - k)$ it follows that $k = 6$ and $G_\gamma \cap T = M_{10}$, which has two orbits in $\mathcal{V}$. However a GAP computation shows that these orbit lengths are 36 and 90, and $36 \cdot 90$ does not divide $|G_\gamma|$.

From now on we assume that $q \geq 7$.

Case: M preserves a direct decomposition of V. Then M is of type $(q + 1)^2 : S_3$, $M = H \cap T$, and M has order $6(q + 1)^2/(3, q + 1)$. Let $\mathsf{v} \in \mathcal{V}$ and note that $|T_\mathsf{v}| = q^3(q^2 - 1)/(3, q + 1)$. Let

$$b := (|M|, |T_\mathsf{v}|) = \frac{q + 1}{(3, q + 1)} \left(6(q + 1), q^3(q - 1)\right)$$

and note that $|M_\mathsf{v}|$ divides b, so that the orbit length $|\mathsf{v}^M|$ is divisible by $|M|/b$. We claim that $|\mathsf{v}^M| \geq \frac{q+1}{(2,q-1)}$ (so, since this holds for all v it implies that $k \geq \frac{q+1}{(2,q-1)}$). If $p = 2$ then $b \leq 6(q + 1)/(3, q + 1)$ and so $|\mathsf{v}^M| \geq q + 1$. Assume now that p is odd. If $q \equiv 0$ or $1 \pmod 3$, then $b = 3(q+1)(2(q+1), \frac{q^3(q-1)}{3}) \leq 12(q+1)$, so $|\mathsf{v}^M| \geq \frac{q+1}{2}$. Finally if $q \equiv 2 \pmod 3$, then $b \leq \frac{q+1}{3}(2(q + 1), q - 1) \leq 4(q + 1)/3$, so $|\mathsf{v}^M| \geq \frac{3(q+1)}{2}$, and the claim is proved. Thus $k \geq \frac{q+1}{(2,q-1)}$ and hence

$$\frac{q + 1}{(2, q - 1)}\left(q^3 + 1 - \frac{q + 1}{(2, q - 1)}\right) \leq k(q^3 + 1 - k) \leq |G_\gamma| \leq (q + 1)^2.6.2a$$

which implies that $q(q - 1)$ is at most $12a$ if q is even, or $24a$ if q is odd. Since $q \geq 7$, this means that $q = 8$. However if $q = 8$ then $b = 6$ and hence k is divisible by $|M|/6 = 27$, but then $k(q^3 + 1 - k)$ does not divide $|G_\gamma|$.

Case: M preserves an extension field structure on V. Here M is of type $(q^2 - q + 1) : 3$. However the cyclic group of order $q^2 - q + 1$ is semiregular on $\mathcal{V}$, and so, for $\mathsf{v} \in \mathcal{V}$, $|G_{\gamma,\mathsf{v}}|$ divides $6a$, which is less than $(q^3 + 1)/2$, so G_γ is not transitive on $\gamma \times \bar{\gamma}$.

Case: M is a subfield subgroup. Suppose first that q is odd and M is of type $\mathrm{SO}(3, q)$. Then $|H| \leq q(q^2 - 1)2a$ (see [14, Proposition 4.5.5]). Also H is intransitive on $\mathcal{V}$ and hence $G_\gamma = H$. Modulo scalars we can take $H \cap T$ to be the subgroup of matrices with entries in $\mathbb{F}_q$, so in particular H contains the subgroup H_0 consisting of the q matrices $t_{\alpha,\beta}$ in (2) with $\alpha, \beta \in \mathbb{F}_q$ and $2\alpha + \beta^2 = 0$. Consider the points $\mathsf{v} = \langle \mathsf{e}_1 \rangle$ and $\mathsf{u} = \langle (x, 1, 1) \rangle$ defined in the proof of Lemma 8.2, where here we choose $x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ as well as satisfying $x + \bar{x} + 1 = 0$. With this choice of x, the points v and u lie in different H-orbits (since $x \notin \mathbb{F}_q$), and a straightforward calculation shows that each of the H_0-orbits containing v and u has length q. Thus $k \geq q$ and hence

$$q^2(q^2 - 1) < q(q^3 + 1 - q) \leq k(q^3 + 1 - k) \leq |G_\gamma| \leq q(q^2 - 1).2a.$$

This implies that $q < 2a$, which is a contradiction.

Now suppose that M is of type $\mathrm{SU}(3, q_0)$ with $q = q_0^r$ and r an odd prime. Then arguing as above we have $G_\gamma = H$ and, modulo scalars, we may take $H \cap T$ to be the subgroup of matrices with entries in $\mathbb{F}_{q_0^2}$, so in particular H contains the subgroup H_0 consisting of the q_0^3 matrices $t_{\alpha,\beta}$ in (2) with $\alpha, \beta \in \mathbb{F}_{q_0^2}$ and $\alpha + \bar{\alpha} + \bar{\beta}\beta = 0$. The points v and u lie in different H-orbits, where this time we take the scalar $x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_{q_0^2}$, and the H_0-orbits containing these two points both have length q_0^3. Thus $k \geq q_0^3$ and so in this case, since $q \geq q_0^3$,

$$\frac{q_0^{3r+3}}{2} < q_0^3(q^3 - q_0^3) < |G_\gamma| \leq q_0^3(q_0^3 + 1)(q_0^2 - 1).2a < q_0^8.2a$$

and hence $4a > q_0^{3r-5} > q_0^r = q$ and we have a contradiction.

For each of the remaining groups M, we have $q = p \geq 7$.

Case: M is of symplectic type. The group M corresponds to a subgroup $3^{1+2} : Q_8.\frac{(q+1,9)}{3}$ of $\mathrm{SU}(3, q)$ and here $q \geq 11$. The order $|G_\gamma|$ is at most 1296 which is less than $k(q^3 + 1 - k)$.

Case: $M = \mathrm{PSL}(2, 7)$ with $q \equiv 3, 5, 6 \pmod 7$. We have $3(q^3 - 2) \leq 168 \cdot 2$ which is a contradiction for $q \geq 7$.

Case: $M = A_6$ with $q \equiv 11, 14 \pmod{15}$. We have $3(q^3 - 2) \leq 360 \cdot 2$ which is a contradiction for $q \geq 7$.

This completes the proof of Theorem 1.3.

References

[1] S. Barwick and G. Ebert, *Unitals in projective planes.* Springer Monographs in Mathematics. Springer, New York, 2008.

[2] J. Borges, J. Rifà, and V. A. Zinoviev, On completely regular binary codes and *t*-designs. In *Comb01–Euroconference on Combinatorics, Graph Theory and Applications*, 4 pp. (electronic), *Electron. Notes Discrete Math.* **10**, Elsevier, Amsterdam, 2001.

[3] J. Borges, J. Rifà, and V. A. Zinoviev, On non-antipodal binary completely regular codes. *Discrete Math.* **308** (2008), 3508–3525.

[4] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups.* LMS Lecture Note Series, Cambridge University Press, 2013.

[5] P. J. Cameron, *Permutation Group.* London Mathematical Society Student Texts 45, Cambridge University Press, Cambridge, 1999.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985.

[7] P. Delsarte, An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.* No. 10 (1973), vi+97 pp.

[8] L. E. Dickson, *Linear groups: With an exposition of the Galois field theory.* with an introduction by W. Magnus, Dover Publications, Inc., New York 1958.

[9] J. D. Dixon and B. Mortimer, *Permutation groups*, Springer-Verlag, New York, 1996.

[10] N. Durante, On sets with few intersection numbers in finite projective and affine spaces, submitted, 2013.

[11] The GAP Group, GAP – *Groups, Algorithms and Programming*, Version 4.4, 2004. http://www.gap-system.org

[12] C. D. Godsil and C. E. Praeger, Completely transitive designs, unpublished manuscript, 1997. Available at arXiv:1405.2176

[13] P. B. Kleidman, The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups $^2G_2(q)$, and their automorphism groups. *J. Algebra* **117** (1988), 30–71.

[14] P. Kleidman and M. Liebeck, *The subgroup structure of the classical groups*, Cambridge University Press, Cambridge 1990.

[15] W. J. Martin, Completely regular designs of strength one. *J. Algebraic Combin.* **3** (1994), 177–185.

[16] W. J. Martin, Completely regular designs. *J. Combin. Des.* **6** (1998), 261–273.

[17] W. J. Martin, Completely regular codes: a viewpoint and some problems In: *Proceedings of 2004 Com2MaC Workshop on Distance-Regular Graphs and Finite Geometry, July 24 - 26, 2004, Pusan, Korea.*

[18] A. Meyerowitz, Cycle-balanced partitions in distance-regular graphs. *J. Combin. Inform. System Sci.* **17** (1992), 39–42.

[19] A. Meyerowitz, Cycle-balance conditions for distance-regular graphs. In: The 2000 Com^{2}MaC Conference on Association Schemes, Codes and Designs (Pohang). *Discrete Math.* **264** (2003), 149–165.

[20] A. Neumaier, Completely regular codes. In: *A collection of contributions in honour of Jack van Lint, Discrete Math.* **106/107** (1992), 353–360.

[21] M. Neunhöffer and C. E. Praeger, Sporadic neighbour-transitive codes in Johnson graphs, *Designs, Codes and Crypt.* published on-line 14 July, 2013. doi:10.1007/s10623-013-9853-0

[22] M. Neunhöffer and C. E. Praeger, Complementary and self-complementary incidence-transitive codes in Johnson graphs. In preparation.

[23] M. Suzuki, On a class of doubly transitive groups. *Ann. of Math.* (2) **75** 1962, 105–145.

[24] D. E. Taylor, Unitary block designs. *J. Combinatorial Theory Ser. A* **16** (1974), 51–56.

[25] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. für Math. u. Phys.* **3**, (1892), 265–284.

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