DIMENSIONAL DUAL HYPEROVALS IN CLASSICAL POLAR SPACES

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ABSTRACT. In this paper we show that n-dimensional dual hyperovals cannot exist in all but one classical polar space of rank n if n is *even*. This resolves a question posed by Yoshiara.

1. Definitions and preliminaries

An *n*-dimensional dual arc \mathcal{D} in a vector space V(N,q) over a finite field \mathbb{F}_q is a set of *n*-dimensional subspaces such that

- (1) each two intersect in exactly a one-dimensional space;
- (2) no three intersect non-trivially.

It is clear that $|\mathcal{D}| \leq \frac{q^n-1}{q-1} + 1$. For let S be any element of \mathcal{D} . Then the other elements of \mathcal{D} intersect S in distinct one-dimensional subspaces, of which there are $\frac{q^n-1}{q-1}$. If \mathcal{D} meets this bound, it is called an *n*-dimensional dual hyperoval. We will sometimes use the shorthand *n*-DA and *n*-DHO.

For background and a recent survey of known results and applications, we refer to [15]. Note that the definition therein are in terms of projective spaces, but here we use vector space terminology and notation, following [5]. In this paper we will mostly consider the case N = 2n. In [5], this is required in the definition, but we will not impose this restriction here.

It is known that *n*-dimensional dual hyperovals exist in V(2n, q) for all *n* and all *q* even, see for example [15]. It is an open problem whether any can exist when *q* is odd.

In this paper we will consider *n*-dimensional dual arcs in *polar spaces*, that is, where \mathcal{D} consists of maximum totally isotropic subspaces with respect to some nondegenerate form on V(N,q). Necessarily then we have that $N \in \{2n, 2n + 1, 2n + 2\}$.

It is known [15] that there exist *n*-dimensional dual hyperovals in the hyperbolic quadric $Q^+(2n-1,q)$ for all *n* odd and q = 2 (see Section 2 for notation). Furthermore, there exists a 3-dimensional dual hyperoval in the hermitian variety H(5,4), the *Mathieu dual hyperoval*.

In [15] Problem 4.7, the following (paraphrased) question is asked.

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JOHN SHEEKEY

Does the existence of an n-dimensional dual hyperoval in a polar space imply that n is odd?

Taniguchi [10] proved that *n*-dimensional "alternating doubly dual hyperovals" exist in V(2n, 2) if and only if *n* is odd. Dempwolff [4], showed that *n*-dimensional "symmetric doubly dual hyperovals" exist only if *n* is odd. We will see in Section 4 that the existence of such implies the existence of an *n*-dimensional dual hyperovals in the symplectic space W(2n - 1, q).

We respond now to these questions with the following theorem.

Theorem 1. Suppose \mathcal{D} is an n-dimensional dual hyperoval in a polar space \mathcal{P} of rank n. Then either n is odd or \mathcal{P} is an elliptic quadric.

The result is a simple application of a theorem of Vanhove.

2. Polar spaces

A classical polar space \mathcal{P} is the geometry of totally singular subspaces with respect to some non-degenerate quadratic form on V(N, q), or totally isotropic with respect to some non-degenerate symplectic or sesquilinear form on V(N, q). The rank of \mathcal{P} is the maximum (vector space) dimension of a subspace of \mathcal{P} . If every (n-1)dimensional space of a polar space of rank n is contained in precisely $q^e + 1$ totally isotropic n-dimensional spaces, then \mathcal{P} is said to have parameters (q, q^e) . See for example [2] for background. We tabulate the relevant polar spaces of rank n here.

Name	form	Notation	Ambient vector	Parameters	e
			space		
Hyperbolic quadric	quadratic	$Q^+(2n-1,q)$	V(2n,q)	(q,1)	0
Parabolic quadric	quadratic	Q(2n,q)	V(2n+1,q)	(q,q)	1
Elliptic quadric	quadratic	$Q^{-}(2n+1,q)$	V(2n+2,q)	(q,q^2)	2
Symplectic space	symplectic	W(2n-1,q)	V(2n,q)	(q,q)	1
Hermitian variety	sesquilinear	$H(2n-1,q^2)$	$V(2n,q^2)$	(q^2,q)	1/2
Hermitian variety	sesquilinear	$H(2n,q^2)$	$V(2n+1,q^2)$	(q^2, q^3)	3/2

Note that *n*-dimensional dual hyperovals in polar spaces defined by a quadratic form are often referred to as being of *orthogonal type*.

If q is even, W(2n-1,q) is isomorphic to Q(2n,q), and contains $Q^+(2n-1,q)$.

Example 1. Yoshiara defined in [14] the following *n*-dimensional dual hyperovals in V(2n, 2), and showed in [16] that they lie in $Q^+(2n-1, 2)$ (and hence W(2n-1, 2)) if and only if *n* is *odd*. Let *h* be an integer coprime to *n*, and for each $t \in \mathbb{F}_{2^n}$ define

$$S_t = \{ (x, x^{2^{-2h}}t + xt^{2^h}) : x \in \mathbb{F}_{2^n} \}.$$

Then $\mathcal{D} := \{S_t : t \in \mathbb{F}_{2^n}\}$ is an *n*-dimensional dual hyperoval in $Q^+(2n-1,2)$ (and W(2n-1,2)), where the quadratic form on V(2n,2) is

$$(a,b) \mapsto \operatorname{Tr}(ab^{2^n}),$$

and the associated symmetric (alternating) bilinear form on V(2n, 2) is

$$((a,b),(c,d)) \mapsto \operatorname{Tr}(ad^{2^n} - bc^{2^n})$$

 $\mathbf{2}$

Dempwolff and Kantor [5] gave a geometric construction leading to many inequivalent examples in $Q^+(2n-1,2)$. Dempwolff [3] gave further examples in W(2n-1,2) which cannot lie in $Q^+(2n-1,2)$.

Example 2. There exists a 3-dimensional dual hyperoval in V(6, 4) which lies in the polar space H(5, 4) known as the *Mathieu dual hyperoval*, see e.g. [6].

To the author's knowledge, no examples in other polar spaces are known. Del Fra [6] showed that the only 3-dimensional dual hyperovals in a polar space are the above examples.

Yoshiara [16] showed that *n*-dimensional dual hyperovals can exist in $Q^+(2n-1,q)$ only if *n* is odd.

3. DUAL POLAR GRAPHS AND MAIN RESULT

Given a polar space \mathcal{P} of rank n, we define the *dual polar graph* $\Gamma_{\mathcal{P}}$, whose vertices are the *n*-spaces of \mathcal{P} , and where two vertices are adjacent if their intersection has dimension n-1. Many properties of this graph are know, see for example [1], [12].

For a set \mathcal{D} of *n*-spaces of \mathcal{P} , the *inner distribution* is an (n + 1)-tuple of integers $a = (a_0, a_1, \ldots, a_n)$, where

$$a_i = \frac{\{(S,T) : S, T \in \mathcal{D} \mid \dim(S \cap T) = n - i\}}{|\mathcal{D}|}.$$

Equivalently, if we view \mathcal{D} as a subset of $\Gamma_{\mathcal{P}}$, and let d(S,T) denote the distance function on Γ , then

$$a_i = \frac{\{(S,T) : S, T \in \mathcal{D} \mid d(S,T) = i\}}{|\mathcal{D}|}.$$

In [13, Lemma 3.2], the following was proved.

Theorem 2 (Vanhove). Let \mathcal{P} be a classical polar space of rank n with parameters (q, q^e) , and let \mathcal{D} be a set of n-spaces in \mathcal{P} with inner distribution (a_0, a_1, \ldots, a_n) . Then

$$\sum_{i=0}^{n} \left(-\frac{1}{q^e}\right)^i a_i \ge 0.$$

Now suppose \mathcal{D} is a dimensional dual arc in \mathcal{P} . Then it is clear that

$$a_0 = 1$$

$$a_{n-1} = |\mathcal{D}| - 1$$

$$a_i = 0 \text{ otherwise.}$$

Hence we get that

$$1 + \left(-\frac{1}{q^e}\right)^{n-1} (|\mathcal{D}| - 1) \ge 0,$$

and so if n is even,

$$|\mathcal{D}| \le q^{(n-1)e} + 1.$$

Hence we get an upper bound for an n-dimensional dual arc in each classical polar space.

JOHN SHEEKEY

\mathcal{P}	Ambient vector	Parameters	e	$ \mathcal{D} \leq$	Size of DHO
	space				
$Q^+(2n-1,q)$	V(2n,q)	(q,1)	0	2	$\frac{q^n - 1}{q - 1} + 1$
Q(2n,q)	V(2n+1,q)	(q,q)	1	$q^{n-1} + 1$	$\frac{q^n - 1}{q - 1} + 1$
$Q^{-}(2n+1,q)$	V(2n+2,q)	(q,q^2)	2	$q^{2n-2} + 1$	$\frac{q^n - 1}{q - 1} + 1$
W(2n-1,q)	V(2n,q)	(q,q)	1	$q^{n-1} + 1$	$\frac{q^n-1}{q-1}+1$
$H(2n-1,q^2)$	$V(2n,q^2)$	(q^2,q)	1/2	$q^{n-1} + 1$	$\frac{q^{2n}-1}{q^2-1}+1$
$H(2n,q^2)$	$V(2n+1,q^2)$	(q^2, q^3)	3/2	$q^{3(n-1)/2} + 1$	$\frac{q^{2n}-1}{q^2-1} + 1$

Theorem 3. Suppose \mathcal{D} is an n-dimensional dual arc in \mathcal{P} , and suppose n is even. Then we have the following upper bounds on $|\mathcal{D}|$.

Proof of Theorem 1: This now now follows immediately by comparing the above upper bounds on n-dimensional dual arcs (fourth column) with the required size of an n-dimensional dual hyperoval (fifth column).

Remark 1. Note that an *n*-dimensional dual hyperoval is a special case of a constant-distance, constant-dimension subspace code [8], [9], or equivalently, a clique in the graph Γ_{n-1} , where Γ_i is the graph whose vertices are the vertices of Γ , and whose edges are between vertices at distance *i* in Γ . Note however that not every clique of the correct size in Γ_{n-1} gives rise to an *n*-dimensional dual hyperoval. As the proof of Theorem 3 does not use the fact that no three spaces intersect non-trivially, the same bounds hold for the relevant constant-distance subspace codes in each polar spaces.

This is the same method used by Vanhove in [11] to prove that the maximum size of a partial spread in H(2n-1,q) is $q^n + 1$ if n is odd.

Remark 2. This table does not imply any results for dimensional dual hyperovals in elliptic quadrics $Q^{-}(2n + 1, q)$. This problem seems to require a different approach. Note that such objects do not satisfy the definition of a dimensional dual hyperoval in [5].

4. Alternating and symmetric doubly dual hyperovals

An *n*-dimensional dual hyperoval \mathcal{D} in V(2n,q) is said to be "doubly dual" if $\mathcal{D}^{\perp} := \{S^{\perp} : S \in \mathcal{D}\}$ is also an *n*-dimensional dual hyperoval, where \perp is some nondegenerate polarity. Note that if \mathcal{D} lies in some polar space \mathcal{P} , it is doubly dual: we take \perp to be the polarity defined by the quadratic or sesquilinear form associated to \mathcal{P} , whence $S^{\perp} = S$ for all maximum subspaces S in \mathcal{P} . However, the converse is not necessarily true.

In [4] the concept of a (bilinear) symmetric doubly dual hyperoval was introduced, and it was proved that such objects can not exist in V(2n, q) for n even. We will now show that the existence of this implies the existence of an n-dimensional dual hyperoval in symplectic polar space.

Suppose there is some injective linear map $\beta : V(n,q) \to \operatorname{End}(V(n,q))$. Let us represent elements of V(2n,q) with elements of $V(n,q) \times V(n,q)$. For each $y \in V(n,q)$, define an *n*-dimensional subspace $S_y = \{(x,\beta(y)(x)) : x \in V(n,q)\}$ of V(2n,q), and define $\mathcal{D}_{\beta} = \{S_y : y \in V(n,q)\}$. If \mathcal{D}_{β} is an *n*-dimensional dual hyperoval, then it is called a *bilinear dual hyperoval*. Note that this can occur only if q = 2.

Define $\beta^{o}: V(n,q) \to \operatorname{End}(V(n,q))$ by $\beta^{o}(x)(y) = \beta(y)(x)$. If $\beta = \beta^{o}$, that is if $\beta(y)(x) = \beta(x)(y)$ for all $x, y \in V(n,q)$, then \mathcal{D}_{β} is called a symmetric dual hyperoval. If furthermore $\beta(x)(x) = 0$ for all x, then \mathcal{D}_{β} is called an alternating dual hyperoval.

Let $\langle, \rangle : V(n,q) \times V(n,q) \to \mathbb{F}_q$ be a nondegenerate symmetric bilinear form on V(n,q). Let t denote the adjoint operator with respect to this form, i.e. $\langle x, f(y) \rangle = \langle f^t(x), y \rangle$ for all $x, y \in V(n,q)$, and define $\beta^t : V(n,q) \to \operatorname{End}(V(n,q))$ by $\beta^t(x) = \beta(x)^t$.

Taniguchi [10] showed that alternating doubly dual hyperovals exist in V(2n, 2) if and only if n is odd. Dempwolff [4] improved this by showing that symmetric doubly dual hyperovals exist in V(2n, 2) only if n is odd. We now show that this result follows also from Theorem 1 of this paper.

The following lemma follows from [7], and from [3, Lemma 3.8].

Lemma 1. If there exists a symmetric bilinear n-dimensional doubly dual hyperoval \mathcal{D} in V(2n,2), then there exists an n-dimensional dual hyperoval in W(2n-1,2).

Combining this with Theorem 1 immediately gives us the following corollary.

Corollary 1. There exists a symmetric bilinear n-dimensional doubly dual hyperoval \mathcal{D} in V(2n, 2) only if n is odd.

Note that Theorem 1, applied to W(2n-1,q), does not require either bilinearity or that q = 2, and so this result is more general than the results of Taniguchi and Dempwolff.

Dempwolff further conjectured in [3] that *n*-dimensional doubly dual hyperovals over \mathbb{F}_2 exist only if *n* is odd. This remains an open problem.

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JOHN SHEEKEY

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 $\mathbf{6}$