Complete Weight Enumerators of a Family of Three-Weight Linear Codes

Shudi Yang · Zheng-An Yao

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Abstract Linear codes have been an interesting topic in both theory and practice for many years. In this paper, for an odd prime p, we present the explicit complete weight enumerator of a family of p-ary linear codes constructed with defining set. The weight enumerator is an immediate result of the complete weight enumerator, which shows that the codes proposed in this paper are three-weight linear codes. Additionally, all nonzero codewords are minimal and thus they are suitable for secret sharing.

Keywords Linear code \cdot Complete weight enumerator \cdot Gaussian period \cdot Gauss sum

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1 Introduction

Throughout this paper, let p be an odd prime and $r = p^m$ for a positive integer $m \ge 2$. Denote by \mathbb{F}_r a finite field with r elements. An $[n, \kappa, \delta]$ linear code C over \mathbb{F}_p is a κ -dimensional subspace of \mathbb{F}_p^n with minimum distance δ [7,28].

Let A_i denote the number of codewords with Hamming weight *i* in a linear code *C* of length *n*. The (ordinary) weight enumerator of *C* is defined by

$$A_0 + A_1 z + A_2 z^2 + \dots + A_n z^n,$$

S.D. Yang

E-mail: yangshd 3@mail2.sysu.edu.cn

Z.-A. Yao

Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, P.R. China E-mail: mcsyao@mail.sysu.edu.cn

Department of Mathematics, Sun Yat-sen University, Guangzhou 510275 and School of Mathematical Sciences, Qufu Normal University, Shandong 273165, P.R.China Tel.: +86-15602338023

where $A_0 = 1$. The sequence $(A_0, A_1, A_2, \dots, A_n)$ is called the (ordinary) weight distribution of the code C.

The complete weight enumerator of a code C over \mathbb{F}_p enumerates the codewords according to the number of symbols of each kind contained in each codeword. Denote elements of the field by $\mathbb{F}_p = \{w_0, w_1, \cdots, w_{p-1}\}$, where $w_0 = 0$. Also let \mathbb{F}_p^* denote $\mathbb{F}_p \setminus \{0\}$. For a codeword $\mathbf{c} = (c_0, c_1, \cdots, c_{n-1}) \in \mathbb{F}_p^n$, let $w[\mathbf{c}]$ be the complete weight enumerator of \mathbf{c} , which is defined as

$$w[\mathbf{c}] = w_0^{k_0} w_1^{k_1} \cdots w_{p-1}^{k_{p-1}},$$

where k_j is the number of components of **c** equal to w_j , $\sum_{j=0}^{p-1} k_j = n$. The complete weight enumerator of the code C is then

$$\mathrm{CWE}(C) = \sum_{\mathsf{c} \in C} w[\mathsf{c}].$$

The weight distribution of a linear code has attracted a lot of interests for many years and we refer the reader to [10,12,16,17,18,24,27,30,31,32, 34,35,36,37] and references therein for an overview of the related researches. It is not difficult to see that the complete weight enumerators are just the (ordinary) weight enumerators for binary linear codes. While for nonbinary linear codes, the weight enumerators can be obtained from their complete weight enumerators.

The information of the complete weight enumerator of a linear code is of vital use both in theory and in practice. For instance, Blake and Kith investigated the complete weight enumerator of Reed-Solomon codes and showed that they could be helpful in soft decision decoding [3,20]. In [19], the study of the monomial and quadratic bent functions was related to the complete weight enumerators of linear codes. It was illustrated by Ding *et al.* [9,11] that complete weight enumerators can be applied to calculate the deception probabilities of certain authentication codes. In [5,6,13], the authors studied the complete weight enumerators of some constant composition codes and presented some families of optimal constant composition codes.

However, it is extremely difficult to evaluate the complete weight enumerators of linear codes in general and there is little information on this topic in literature besides the above mentioned [3, 5, 6, 13, 20]. Kuzmin and Nechaev investigated the generalized Kerdock code and related linear codes over Galois rings and determined their complete weight enumerators in [21] and [22]. Very recently, the authors in [25] obtained the complete weight enumerators of some cyclic codes by using Gauss sums. Li *et al.* [23] treated some linear codes by using exponential sums and Galois theory. In this paper, we shall determine the complete weight enumerators of a class of linear codes over finite fields.

Set $\overline{D} = \{d_1, d_2, \cdots, d_n\} \subseteq \mathbb{F}_r$. Denote by Tr the trace function from \mathbb{F}_r to \mathbb{F}_p . A linear code associated with \overline{D} is defined by

$$C_{\overline{D}} = \{ (\operatorname{Tr}(ad_1), \operatorname{Tr}(ad_2), \cdots, \operatorname{Tr}(ad_n)) : a \in \mathbb{F}_r \}.$$

Then \overline{D} is called the defining set of this code $C_{\overline{D}}$ (see [8,14,15] for details).

It should be noted that the authors in [8,14] and [15] gave the definitions of the code $C_{\bar{D}}$ and the defining set \bar{D} . The authors in [14] established binary linear codes $C_{\bar{D}}$ with three weights. In [8], Ding proposed the general construction of the linear codes and determined their weights especially for three specific codes. The authors in [15] presented the defining set $\bar{D} = \{x \in \mathbb{F}_r^* : \operatorname{Tr}(x^2) = 0\}$ to construct a class of linear codes $C_{\bar{D}}$ with two or three weights and investigated their applications in secret sharing, and furthermore, their complete weight enumerators were established by Yang and Yao [33].

In this paper, let $r = p^m$. The defining set D is given by

$$D = \{ x \in \mathbb{F}_r^* : \operatorname{Tr}(x) = 0 \},$$
(1)

and let

$$C_D = \{ (\operatorname{Tr}(ax^2))_{x \in D} : a \in \mathbb{F}_r \}.$$

$$(2)$$

For this kind of linear codes, we will study their complete weight enumerators, and their weight enumerators as well. We should mention that the main idea of this paper indeed comes from [8,15]. As will be proved later, they are linear codes with three weights. In addition, each nonzero codeword of C_D constructed in this paper is minimal if $m \ge 4$, which indicates that the linear codes can be of use in secret sharing schemes [4] with nice access structures. The reader is referred to [15] for more information about this application.

The main results of this paper are given below.

Theorem 1 Let $N_a(\rho) = \#\{x \in \mathbb{F}_r^* : \operatorname{Tr}(x) = 0 \text{ and } \operatorname{Tr}(ax^2) = \rho\}$ with $\rho \in \mathbb{F}_p$ and $a \in \mathbb{F}_r$. Then the code C_D of (2) has parameters $[p^{m-1}-1,m]$ and its complete weight enumerator is given in Table 1 if m is even and Table 2 if m is odd.

Table 1 Complete weight enumerator of the code C_D if m is even

$N_a(0)$	$N_a(\rho)(\rho \neq 0)$	Frequency
$p^{m-2} - 1$ $p^{m-2} - 1$ $p^{m-2} - 1 - (p-1)p^{\frac{m-2}{2}}$ $p^{m-2} - 1 + (p-1)p^{\frac{m-2}{2}}$ $p^{m-1} - 1$	$ p^{m-2} + \bar{\eta}(\rho)p^{\frac{m-2}{2}} \\ p^{m-2} - \bar{\eta}(\rho)p^{\frac{m-2}{2}} \\ p^{m-2} + p^{\frac{m-2}{2}} \\ p^{m-2} - p^{\frac{m-2}{2}} \\ 0 $	$\frac{p^{m}-p^{m-1}}{p^{m}-p^{m-1}} \\ \frac{1}{2}(p^{\frac{m}{2}}+1)(p^{\frac{m-2}{2}}-1) \\ \frac{1}{2}(p^{\frac{m}{2}}-1)(p^{\frac{m-2}{2}}+1) \\ 1$

where $\bar{\eta}$ denotes the quadratic character of \mathbb{F}_p .

Theorem 2 The weight distribution of the code C_D of (2) is given in Table 3 if m is even and Table 4 if m is odd.

Example 1 (i) Let (p,m) = (3,4). Then by Theorems 1 and 2, the code C_D has parameters [26,4,12] with complete weight enumerator

$$w_0^{26} + 16w_0^{14}w_1^6w_2^6 + 27w_0^8w_1^6w_2^{12} + 27w_0^8w_1^{12}w_2^6 + 10w_0^2w_1^{12}w_2^{12} + 27w_0^8w_1^{12}w_2^{12} + 10w_0^2w_1^{12}w_2^{12} + 10w_0^2w_1^{12} + 10w_0^2w_1$$

$N_a(0)$	$N_a(\rho)(\rho \neq 0)$	Frequency
$\begin{array}{c} p^{m-2}-1\\ p^{m-2}-1\\ p^{m-2}+(p-1)p^{\frac{m-3}{2}}\\ p^{m-2}-(p-1)p^{\frac{m-3}{2}}\\ p^{m-1}-1 \end{array}$	$ p^{m-2} + \bar{\eta}(\rho)p^{\frac{m-1}{2}} \\ p^{m-2} - \bar{\eta}(\rho)p^{\frac{m-1}{2}} \\ p^{m-2} - p^{\frac{m-3}{2}} \\ p^{m-2} + p^{\frac{m-3}{2}} \\ 0 $	$\frac{p^{m-1}-1}{p^{m-1}-1}$ $\frac{p^{-1}}{2}\left(p^{m-1}+p^{\frac{m-1}{2}}\right)$ $\frac{p-1}{2}\left(p^{m-1}-p^{\frac{m-1}{2}}\right)$ 1

Table 2 Complete weight enumerator of the code C_D if m is odd

where $\bar{\eta}$ denotes the quadratic character of \mathbb{F}_p .

Table 3 The weight distribution of C_D if m is even

Weight i	Frequency A_i
$ \begin{array}{c} (p-1)p^{m-2} \\ (p-1)(p^{m-2}+p^{\frac{m-2}{2}}) \\ (p-1)(p^{m-2}-p^{\frac{m-2}{2}}) \\ 0 \end{array} $	$p^{m} - p^{m-1}$ $\frac{1}{2}(p^{\frac{m}{2}} + 1)(p^{\frac{m-2}{2}} - 1)$ $\frac{1}{2}(p^{\frac{m}{2}} - 1)(p^{\frac{m-2}{2}} + 1)$ 1

Table 4 The weight distribution of C_D if m is odd

Weight i	Frequency A_i
$(p-1)p^{m-2} (p-1)(p^{m-2}-p^{\frac{m-3}{2}}) (p-1)(p^{m-2}+p^{\frac{m-3}{2}}) 0$	$p^{m-1} - 1$ $\frac{p-1}{2} \left(p^{m-1} + p^{\frac{m-1}{2}} \right)$ $\frac{p-1}{2} \left(p^{m-1} - p^{\frac{m-1}{2}} \right)$ 1

and weight enumerator

$$1 + 16z^{12} + 54z^{18} + 10z^{24}$$

These results coincide with numerical computation by Magma.

(ii) Let (p,m) = (5,3). Then by Theorems 1 and 2, the code C_D has parameters [24,3,16] with complete weight enumerator

$$w_0^{24} + 60w_0^8 w_1^4 w_2^4 w_3^4 w_4^4 + 12w_0^4 w_1^{10} w_4^{10} + 12w_0^4 w_2^{10} w_3^{10} + 40w_1^6 w_2^6 w_3^6 w_4^6 w_4$$

and weight enumerator

$$1 + 60z^{16} + 24z^{20} + 40z^{24}$$

These results are confirmed by Magma.

The remainder of this paper is organized as follows. Section 2 recalls some definitions and results on Gaussian periods and Gauss sums over finite fields. Section 3 is devoted to the proof of Theorem 1. Section 4 shows that every nonzero codeword of the code is minimal. Section 5 concludes this paper.

2 Mathematical foundations

We start with cyclotomic classes and Gaussian periods over finite fields. Recall that $r = p^m$. Let α be a fixed primitive element of \mathbb{F}_r and r - 1 = sN, where s, N are two positive integers with s > 1 and N > 1. Define $C_i^{(N,r)} = \alpha^i \langle \alpha^N \rangle$ for $i = 0, 1, \dots, N - 1$, where $\langle \alpha^N \rangle$ denotes the subgroup of \mathbb{F}_r^* generated by α^N . The cosets $C_i^{(N,r)}$ are called the *cyclotomic classes* of order N in \mathbb{F}_r . Obviously, $\#C_i^{(N,r)} = \frac{r-1}{N}$.

Set $\zeta_p = \exp\left(\frac{2\pi\sqrt{-1}}{p}\right)$. The Gaussian periods of order N are defined by $\eta_i^{(N,r)} = \sum \chi_1(x).$

$$_{i} \qquad = \sum_{x \in C_{i}^{(N,r)}} \chi_{1}(x)$$

Here $\chi_1(x) = \zeta_p^{\operatorname{Tr}(x)}$ is the canonical additive character of \mathbb{F}_r , where Tr is the trace function from \mathbb{F}_r to \mathbb{F}_p defined by

$$\operatorname{Tr}(x) = \sum_{i=0}^{m-1} x^{p^i}, \quad x \in \mathbb{F}_r.$$

The orthogonal property of canonical additive character, which can be easily checked, is given by

$$\sum_{x \in \mathbb{F}_r} \chi_1(ax) = \begin{cases} r & \text{if } a = 0, \\ 0 & \text{if } a \in \mathbb{F}_r^* \end{cases}$$

Some results on Gaussian periods are given below [29].

Lemma 1 [29] Let $r = p^m$. When N = 2, the Gaussian periods are given by

$$\eta_0^{(2,r)} = \begin{cases} \frac{-1 + (-1)^{m-1} \sqrt{r}}{2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{-1 + (-1)^{m-1} (\sqrt{-1})^m \sqrt{r}}{2} & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

and $\eta_1^{(2,r)} = -1 - \eta_0^{(2,r)}$.

For later use, we introduce Gauss sums in the following. Let η be the quadratic character of \mathbb{F}_r [15]. The quadratic Gauss sum $G(\eta, \chi_1)$ over \mathbb{F}_r is defined by

$$G(\eta, \chi_1) = \sum_{x \in \mathbb{F}_r^*} \eta(x)\chi_1(x) = \sum_{x \in \mathbb{F}_r} \eta(x)\chi_1(x),$$

and the quadratic Gauss sum $G(\bar{\eta}, \bar{\chi}_1)$ over \mathbb{F}_p is defined by

$$G(\bar{\eta}, \bar{\chi}_1) = \sum_{x \in \mathbb{F}_p^*} \bar{\eta}(x) \bar{\chi}_1(x) = \sum_{x \in \mathbb{F}_p} \bar{\eta}(x) \bar{\chi}_1(x),$$

where $\bar{\eta}$ and $\bar{\chi}_1$ are the quadratic character and canonical additive character of \mathbb{F}_p , respectively.

The following lemmas will be needed in the sequel.

Lemma 2 (See Theorems 5.15 [26]) With the symbols and notation above, we have

$$G(\eta, \chi_1) = (-1)^{m-1} (\sqrt{-1})^{\frac{(p-1)^2}{4}m} \sqrt{r},$$

where $r = p^m$, and

$$G(\bar{\eta}, \bar{\chi}_1) = (\sqrt{-1})^{\frac{(p-1)^2}{4}} \sqrt{p}.$$

Lemma 3 (See Theorem 5.33 of [26]) Let $r = p^m$ and $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_r[x]$ with $a_2 \neq 0$. Then

$$\sum_{x \in \mathbb{F}_r} \chi_1(f(x)) = \chi_1(a_0 - a_1^2(4a_2)^{-1})\eta(a_2)G(\eta, \chi_1).$$

Lemma 4 (See Lemma 7 of [15]) If $m \ge 2$ is even, then $\eta(y) = 1$ for each $y \in \mathbb{F}_p^*$. If m is odd, then $\eta(y) = \overline{\eta}(y)$ for each $y \in \mathbb{F}_p$.

Lemma 5 ([15, 25]) For each $c \in \mathbb{F}_p$, let $t_c = \#\{a \in \mathbb{F}_r : \operatorname{Tr}(a^2) = c\}$. Then

$$t_{c} = \begin{cases} p^{m-1} & \text{if } m \text{ odd } and \ c = 0, \\ p^{m-1} + \bar{\eta}(c)(-1)^{\frac{p-1}{2}\frac{m-1}{2}}p^{\frac{m-1}{2}} & \text{if } m \text{ odd } and \ c \neq 0, \\ p^{m-1} - (-1)^{\frac{p-1}{2}\frac{m}{2}}(p-1)p^{\frac{m-2}{2}} & \text{if } m \text{ even } and \ c = 0 \\ p^{m-1} + (-1)^{\frac{p-1}{2}\frac{m}{2}}p^{\frac{m-2}{2}} & \text{if } m \text{ even } and \ c \neq 0 \end{cases}$$

3 The proofs of the main results

Let notation be as before. Our task of this section is to prove Theorem 1 depicted in Section 1, while Theorem 2 follows immediately from Theorem 1. Below we present some auxiliary results before proving the main results of this paper.

Lemma 6 Let $a \in \mathbb{F}_r^*$ and $\rho \in \mathbb{F}_p^*$. Then we have

$$\begin{split} &\sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_r} \zeta_p^{\operatorname{Tr}(azx^2 + yx) - z\rho} \\ &= \begin{cases} (-1)^{\frac{p-1}{2} \frac{m}{2}} \eta(a) p^{\frac{m+2}{2}} & \text{if } m \text{ even and } \operatorname{Tr}(a^{-1}) = 0, \\ -(-1)^{\frac{p-1}{2} \frac{m+2}{2}} \eta(a) \bar{\eta}(\operatorname{Tr}(a^{-1})) \bar{\eta}(\rho) p^{\frac{m+2}{2}} & \text{if } m \text{ even and } \operatorname{Tr}(a^{-1}) \neq 0, \\ (-1)^{\frac{p-1}{2} \frac{m-1}{2}} \eta(a) \bar{\eta}(\rho) p^{\frac{m+3}{2}} & \text{if } m \text{ odd and } \operatorname{Tr}(a^{-1}) = 0, \\ -(-1)^{\frac{p-1}{2} \frac{m-1}{2}} \eta(a) \bar{\eta}(\operatorname{Tr}(a^{-1})) p^{\frac{m+1}{2}} & \text{if } m \text{ odd and } \operatorname{Tr}(a^{-1}) \neq 0. \end{cases} \end{split}$$

 Proof It follows from Lemmas 3 and 4 that

$$\begin{split} &\sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_r} \zeta_p^{\text{Tr}(azx^2 + yx) - z\rho} \\ &= \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_r} \zeta_p^{\text{Tr}(azx^2 + yx)} \\ &= \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \sum_{y \in \mathbb{F}_p} \chi_1 \left(-\frac{y^2}{4az} \right) \eta(az) G(\eta, \chi_1) \\ &= \eta(a) G(\eta, \chi_1) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \eta(z) \sum_{y \in \mathbb{F}_p} \zeta_p^{-\frac{1}{4z} \operatorname{Tr}(a^{-1})y^2} \\ &= \begin{cases} p\eta(a) G(\eta, \chi_1) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \eta(z) & \text{if } \operatorname{Tr}(a^{-1}) = 0 \\ \eta(a) G(\eta, \chi_1) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \eta(z) \eta\left(-\frac{\operatorname{Tr}(a^{-1})}{4z} \right) G(\bar{\eta}, \bar{\chi}_1) & \text{if } \operatorname{Tr}(a^{-1}) \neq 0 \end{cases} \\ &= \begin{cases} p\eta(a) G(\eta, \chi_1) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \eta(z) & \text{if } \operatorname{Tr}(a^{-1}) = 0 \\ \eta(a) \bar{\eta}(-\operatorname{Tr}(a^{-1})) G(\eta, \chi_1) G(\bar{\eta}, \bar{\chi}_1) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \eta(z) \eta(z) & \text{if } \operatorname{Tr}(a^{-1}) = 0 \\ \eta(a) \bar{\eta}(-\operatorname{Tr}(a^{-1})) G(\eta, \chi_1) G(\bar{\eta}, \bar{\chi}_1) \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-z\rho} \eta(z) \bar{\eta}(z) & \text{if } \operatorname{Tr}(a^{-1}) = 0 \\ \eta(a) \bar{\eta}(-\operatorname{Tr}(a^{-1})) \bar{\eta}(-\rho) G(\eta, \chi_1) G(\bar{\eta}, \bar{\chi}_1)^2 & \text{if } m \text{ even and } \operatorname{Tr}(a^{-1}) = 0, \\ \eta(a) \bar{\eta}(-\operatorname{Pr}(a^{-1})) \bar{\eta}(-\rho) G(\eta, \chi_1) G(\bar{\eta}, \bar{\chi}_1) & \text{if } m \text{ odd and } \operatorname{Tr}(a^{-1}) = 0, \\ -\eta(a) \bar{\eta}(-\operatorname{Tr}(a^{-1})) G(\eta, \chi_1) G(\bar{\eta}, \bar{\chi}_1) & \text{if } m \text{ odd and } \operatorname{Tr}(a^{-1}) \neq 0. \end{cases} \end{cases}$$

From Lemma 2, we get the desired conclusions.

Lemma 7 For any $a \in \mathbb{F}_r^*$ and any $\rho \in \mathbb{F}_p^*$, let

$$N_a(\rho) = \#\{x \in \mathbb{F}_r^* : \operatorname{Tr}(x) = 0 \text{ and } \operatorname{Tr}(ax^2) = \rho\}.$$

Then we have

$$N_{a}(\rho) = \begin{cases} p^{m-2} + (-1)^{\frac{p-1}{2} \frac{m}{2}} \eta(a) p^{\frac{m-2}{2}} \\ if \ m \ even \ and \ \operatorname{Tr}(a^{-1}) = 0, \\ p^{m-2} - (-1)^{\frac{p-1}{2} \frac{m+2}{2}} \eta(a) \bar{\eta}(\operatorname{Tr}(a^{-1})) \bar{\eta}(\rho) p^{\frac{m-2}{2}} \\ if \ m \ even \ and \ \operatorname{Tr}(a^{-1}) \neq 0, \\ p^{m-2} + (-1)^{\frac{p-1}{2} \frac{m-1}{2}} \eta(a) \bar{\eta}(\rho) p^{\frac{m-1}{2}} \\ if \ m \ odd \ and \ \operatorname{Tr}(a^{-1}) = 0, \\ p^{m-2} - (-1)^{\frac{p-1}{2} \frac{m-1}{2}} \eta(a) \bar{\eta}(\operatorname{Tr}(a^{-1})) p^{\frac{m-3}{2}} \\ if \ m \ odd \ and \ \operatorname{Tr}(a^{-1}) \neq 0. \end{cases}$$

 Proof We point out that

$$N_a(\rho) = \#\{x \in \mathbb{F}_r : \operatorname{Tr}(x) = 0 \text{ and } \operatorname{Tr}(ax^2) = \rho\}$$

since $\rho \neq 0$. Hence, for any $a \in \mathbb{F}_r^*$ and any $\rho \in \mathbb{F}_p^*$, we have

$$N_{a}(\rho) = p^{-2} \sum_{x \in \mathbb{F}_{r}} \left(\sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{y \operatorname{Tr}(x)} \right) \left(\sum_{z \in \mathbb{F}_{p}} \zeta_{p}^{z(\operatorname{Tr}(ax^{2})-\rho)} \right)$$
$$= p^{-2} \sum_{y \in \mathbb{F}_{p}^{*}} \sum_{x \in \mathbb{F}_{r}} \zeta_{p}^{\operatorname{Tr}(yx)} + p^{-2} \sum_{z \in \mathbb{F}_{p}^{*}} \sum_{y \in \mathbb{F}_{p}} \sum_{x \in \mathbb{F}_{r}} \zeta_{p}^{\operatorname{Tr}(azx^{2}+yx)-z\rho} + p^{m-2}.$$

Note that

$$\sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_r} \zeta_p^{\mathrm{Tr}(yx)} = 0$$

since

$$\sum_{x \in \mathbb{F}_r} \zeta_p^{\mathrm{Tr}(yx)} = \begin{cases} r, & \text{if } y = 0, \\ 0, & \text{if } y \in \mathbb{F}_p^*. \end{cases}$$

The desired conclusions then follow from Lemma 6.

In order to calculate the frequency of the complete weight enumerator of codewords in C_D , we shall compute

$$\#\{a \in \mathbb{F}_r^* : \eta(a) = \pm 1 \text{ and } \operatorname{Tr}(a^{-1}) = 0\}$$

and

$$#\{a \in \mathbb{F}_r^* : \eta(a)\bar{\eta}(\operatorname{Tr}(a^{-1})) = \pm 1\},\$$

which are given in the following three lemmas.

Lemma 8 For any $a \in \mathbb{F}_r^*$, let

$$n_i = \#\{a \in \mathbb{F}_r^* : \eta(a) = i \text{ and } \operatorname{Tr}(a^{-1}) = 0\}, i \in \{1, -1\}.$$

Then, for m being odd, we have

$$n_1 = n_{-1} = \frac{p^{m-1} - 1}{2},$$

and for m being even, we have

$$\begin{cases} n_1 = \frac{r-1}{2p} + \frac{p-1}{p}\eta_0^{(2,r)}, \\ n_{-1} = \frac{r-1}{2p} + \frac{p-1}{p}\eta_1^{(2,r)}, \end{cases}$$

where $\eta_0^{(2,r)}$ and $\eta_1^{(2,r)}$ are given in Lemma 1.

Proof Observe that $n_{-1} = n - n_1 = p^{m-1} - 1 - n_1$. Thus, we only focus on n_1 . Let α be a fixed primitive element of \mathbb{F}_r . Then $\mathbb{F}_r^* = \langle \alpha \rangle$ and $\mathbb{F}_p^* = \langle \alpha^{\frac{r-1}{p-1}} \rangle$.

Note that $\eta(a) = 1$ if and only if $a \in C_0^{(2,r)}$. For the case of *m* being even, we have $\mathbb{F}_p^* \subseteq C_0^{(2,r)}$ since 2 divides $\frac{r-1}{p-1}$. Therefore, we obtain

$$n_{1} = \sum_{x \in C_{0}^{(2,r)}} \frac{1}{p} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{y \operatorname{Tr}(x)}$$
$$= \frac{r-1}{2p} + \frac{1}{p} \sum_{y \in \mathbb{F}_{p}^{*}} \sum_{x \in C_{0}^{(2,r)}} \zeta_{p}^{\operatorname{Tr}(yx)}$$
$$= \frac{r-1}{2p} + \frac{p-1}{p} \eta_{0}^{(2,r)},$$

where $\eta_0^{(2,r)}$ is given in Lemma 1. Similarly, for the case of *m* being odd, we have $\frac{r-1}{p-1} \equiv 1 \mod 2$, from which the asserted result follows.

Lemma 9 For any $a \in \mathbb{F}_r^*$, let

 $n_{i,j} = \#\{a \in \mathbb{F}_r^* : \eta(a) = i \text{ and } \bar{\eta}(\operatorname{Tr}(a^{-1})) = j\}, i, j \in \{1, -1\}.$

Then, for m being even, we have

$$n_{1,1} = n_{1,-1} = \frac{p-1}{4} \left(p^{m-1} + (-1)^{\frac{p-1}{2}\frac{m}{2}} p^{\frac{m-2}{2}} \right),$$

and for m being odd, we have

$$\begin{cases} n_{1,1} = \frac{p-1}{4} \left(p^{m-1} + (-1)^{\frac{p-1}{2} \frac{m-1}{2}} p^{\frac{m-1}{2}} \right), \\ n_{1,-1} = \frac{p-1}{4} \left(p^{m-1} - (-1)^{\frac{p-1}{2} \frac{m-1}{2}} p^{\frac{m-1}{2}} \right). \end{cases}$$

Proof We note the following fact:

$$n_{1,j} = \#\{a \in \mathbb{F}_r^* : a \in C_0^{(2,r)}, \bar{\eta}(\operatorname{Tr}(a^{-1})) = j\} \\ = \frac{1}{2} \#\{a \in \mathbb{F}_r^* : \bar{\eta}(\operatorname{Tr}(a^2)) = j\}, \ j \in \{1, -1\}.$$

Then, the desired results follow from Lemma 5.

Lemma 10 For any $a \in \mathbb{F}_r^*$, let

$$s_i = \#\{a \in \mathbb{F}_r^* : \eta(a)\bar{\eta}(\operatorname{Tr}(a^{-1})) = i\}, \ i \in \{1, -1\}.$$

Then, for m being even, we have

$$s_1 = s_{-1} = \frac{p^m - p^{m-1}}{2},$$

and for m being odd, we have

$$\begin{cases} s_1 = \frac{p-1}{2} \left(p^{m-1} + (-1)^{\frac{p-1}{2} \frac{m-1}{2}} p^{\frac{m-1}{2}} \right), \\ s_{-1} = \frac{p-1}{2} \left(p^{m-1} - (-1)^{\frac{p-1}{2} \frac{m-1}{2}} p^{\frac{m-1}{2}} \right). \end{cases}$$

Proof Since $\bar{\eta}$ and Tr are balanced, we have

$$\#\{a \in \mathbb{F}_r^* : \bar{\eta}(\operatorname{Tr}(a^{-1})) = -1\}$$

= $\frac{1}{2} \#\{a \in \mathbb{F}_r^* : \operatorname{Tr}(a^{-1}) \neq 0\}$
= $\frac{p^m - p^{m-1}}{2}.$

Following from the definition above, we deduce that

$$s_{1} = \#\{a \in \mathbb{F}_{r}^{*} : \eta(a)\bar{\eta}(\operatorname{Tr}(a^{-1})) = 1\}$$

= $n_{1,1} + n_{-1,-1}$
= $n_{1,1} + \#\{a \in \mathbb{F}_{r}^{*} : \bar{\eta}(\operatorname{Tr}(a^{-1})) = -1\} - n_{1,-1}$
= $n_{1,1} - n_{1,-1} + \frac{p^{m} - p^{m-1}}{2}.$

Similarly,

$$s_{-1} = \#\{a \in \mathbb{F}_r^* : \eta(a)\bar{\eta}(\operatorname{Tr}(a^{-1})) = -1\}$$
$$= n_{1,-1} - n_{1,1} + \frac{p^m - p^{m-1}}{2}.$$

The asserted results then follow from Lemma 9.

Remark 1 Lemma 10 determines s_1 and s_{-1} with the help of Lemma 9. In the following, we give another short proof for the case of m being even.

If *m* is even, Lemma 4 states that $\eta(y) = 1$ for each $y \in \mathbb{F}_p^*$. And there exists a $y' \in \mathbb{F}_p^*$ such that $\bar{\eta}(y') = -1$. So $\eta(y')\bar{\eta}(y') = -1$.

It follows that

$$s_{1} = \#\{a \in \mathbb{F}_{r}^{*} : \eta(a)\bar{\eta}(\operatorname{Tr}(a^{-1})) = 1\}$$

$$= \#\{a \in \mathbb{F}_{r}^{*} : \eta(a)\bar{\eta}(\operatorname{Tr}(a)) = 1\}$$

$$= \#\{y'a \in \mathbb{F}_{r}^{*} : \eta(y'a)\bar{\eta}(\operatorname{Tr}(y'a)) = 1\}$$

$$= \#\{y'a \in \mathbb{F}_{r}^{*} : \eta(y')\bar{\eta}(y')\eta(a)\bar{\eta}(\operatorname{Tr}(a)) = 1\}$$

$$= \#\{a \in \mathbb{F}_{r}^{*} : \eta(a)\bar{\eta}(\operatorname{Tr}(a)) = -1\} = s_{-1}.$$

Hence we have $s_1 = s_{-1} = \frac{p^m - p^{m-1}}{2}$ since

$$s_1 + s_{-1} = \#\{a \in \mathbb{F}_r^* : \operatorname{Tr}(a^{-1}) \neq 0\} = p^m - p^{m-1}.$$

3.1 The proof of Theorem 1

It is now sufficient to show the complete weight enumerator of C_D as stated in Theorem 1. Recall that

$$C_D = \{ (\operatorname{Tr}(ax^2))_{x \in D} : a \in \mathbb{F}_r \},\$$

where $D = \{x \in \mathbb{F}_r^* : \operatorname{Tr}(x) = 0\}.$

It is obvious that the code C_D has length $n = p^{m-1} - 1$ and dimension m. Observe that a = 0 gives the zero codeword and the contribution to the complete weight enumerator is w_0^n . Hence, we assume that $a \in \mathbb{F}_r^*$ for the rest of the proof.

For each codeword $(\operatorname{Tr}(ax^2))_{x\in D}$ of C_D and $\rho \in \mathbb{F}_p$, we should consider the number of solutions $x \in \mathbb{F}_r^*$ satisfying $\operatorname{Tr}(x) = 0$ and $\operatorname{Tr}(ax^2) = \rho$, i.e.,

$$N_a(\rho) = \#\{x \in \mathbb{F}_r^* : \operatorname{Tr}(x) = 0 \text{ and } \operatorname{Tr}(ax^2) = \rho\}.$$

Note that $N_a(0) = p^{m-1} - 1 - \sum_{\rho \in \mathbb{F}_p^*} N_a(\rho)$. Thus we only need to calculate $N_a(\rho)$ with $\rho \in \mathbb{F}_p^*$, which is shown in Lemma 7.

The desired conclusions of Theorem 1 then follow from Lemmas 7, 8 and 10. Note that for even m and $\rho \in \mathbb{F}_p^*$, the frequencies of the codewords with $N_a(\rho) = p^{m-2} + (-1)^{\frac{p-1}{2}\frac{m}{2}}p^{\frac{m-2}{2}}$ and $N_a(\rho) = p^{m-2} - (-1)^{\frac{p-1}{2}\frac{m}{2}}p^{\frac{m-2}{2}}$ are $\frac{p^{m-1}}{2p} + \frac{p-1}{p}\eta_0^{(2,r)}$ and $\frac{p^m-1}{2p} + \frac{p-1}{p}\eta_1^{(2,r)}$, respectively. The results are given in Table 1 according to Lemma 1.

4 Minimal codewords in C_D

In this section, we will show that each codeword of C_D given by (2) is minimal. Hence, we need to introduce some definitions [15].

The support of a vector $\mathbf{c} = (c_0, \cdots, c_{n-1}) \in \mathbb{F}_p^n$ is defined as

$$\{0 \le i \le n - 1 : c_i \ne 0\}.$$

We say that a vector \mathbf{x} covers a vector \mathbf{y} if the support of \mathbf{x} contains that of \mathbf{y} as a proper subset.

A minimal codeword of a linear code C is a nonzero codeword that does not cover any other nonzero codeword of C.

If each nonzero codeword of C is minimal, then the secret sharing scheme based on the dual code C^{\perp} may have nice access structure, see Theorem 12 of [15]. However, it is still very hard to construct such a linear code C. We list the following lemma for minimal codewords [1,2].

Lemma 11 Every nonzero codeword of a linear code C over \mathbb{F}_p is minimal, provided that

$$\frac{w_{min}}{w_{max}} > \frac{p-1}{p},$$

where w_{min} and w_{max} denote the minimum and maximum nonzero weights in C, respectively.

For the linear codes C_D of (2), if m is even and $m \ge 4$, we have

$$\frac{w_{min}}{w_{max}} = \frac{p^{m-2} - p^{\frac{m-2}{2}}}{p^{m-2} + p^{\frac{m-2}{2}}} > \frac{p-1}{p}.$$

The case of $m \ge 4$ being odd is proved similarly, see also [15].

Using Lemma 11, we conclude that all the nonzero codewords of C_D are minimal if $m \ge 4$. Therefore, we can construct secret sharing schemes based on the dual codes C_D^{\perp} with nice access structures. We omit the details here since it is similar to that of [15].

5 Concluding remarks

Inspired by [8,15], we constructed a family of three-weight linear codes. Their complete weight enumerators and weight enumerators were presented explicitly in this paper. We also showed that every nonzero codeword of C_D is minimal and thus the dual codes C_D^{\perp} can be applied to construct secret sharing schemes.

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