# Symmetric Disjunctive List-Decoding Codes 

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#### Abstract

In this paper, we consider symmetric disjunctive list-decoding (SLD) codes, which are a class of binary codes based on a symmetric disjunctive sum (SDS) of binary symbols. By definition, the SDS takes values from the ternary alphabet $\{0,1, *\}$, where the symbol $*$ denotes "erasure". Namely: SDS is equal to 0 (1) if all its binary symbols are equal to 0 (1), otherwise SDS is equal to $*$. The main purpose of this work is to obtain bounds on the rate of these codes.


Index terms. Symmetric disjunctive codes, random coding bounds, nonadaptive symmetric group testing.

## 1 Statement of Problem and Results

### 1.1 Notations and Definitions

Let $N, t, s$, and $L$ be integers, where $2 \leq s<t, 1 \leq L \leq t-s$. Let $\triangleq$ denote the equality by definition, $|A|$ - the size of the set $A$ and $[N] \triangleq\{1,2, \ldots, N\}$ - the set of integers from 1 to $N$. The standard symbol $\lfloor a\rfloor$ will be used to denote the largest integer $\leq a$.

A binary $(N \times t)$-matrix

$$
X=\left\|x_{i}(j)\right\|, \quad x_{i}(j)=0,1, \quad \boldsymbol{x}_{i} \triangleq\left(x_{i}(1), \ldots, x_{i}(t)\right), \quad \boldsymbol{x}(j) \triangleq\left(x_{1}(j), \ldots, x_{N}(j)\right),
$$

$i \in[N], j \in[t]$, with $N$ rows $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N}$ and $t$ columns $\boldsymbol{x}(1), \ldots, \boldsymbol{x}(t)$ (codewords) is called a binary code of length $N$ and size $t=\left\lfloor 2^{R N}\right\rfloor$, where a fixed parameter $R>0$ is called a rate of the code $X$. The number of 1's in the codeword $x(j)$, i.e., $|\boldsymbol{x}(j)| \triangleq \sum_{i=1}^{N} x_{i}(j)$, is called a weight of $x(j), j \in[t]$. A code $X$ is called a constant weight binary code of weight $w, 1 \leq w<N$, if for any $j \in[t]$, the weight $|\boldsymbol{x}(j)|=w$.

Let $\boldsymbol{u} \bigvee \boldsymbol{v}$ denote the disjunctive sum of binary columns $\boldsymbol{u}, \boldsymbol{v} \in\{0,1\}^{N}$. If $\boldsymbol{x}, \boldsymbol{y} \in\{0,1, *\}^{N}$ are arbitrary ternary columns with components from the alphabet $\{0,1, *\}$, then the ternary column $\boldsymbol{z}=\left(z_{1}, z_{2}, \ldots, z_{N}\right) \in\{0,1, *\}^{N}$,

$$
z_{i} \triangleq \begin{cases}0, & \text { if } \quad x_{i}=y_{i}=0 \\ 1, & \text { if } \quad x_{i}=y_{i}=1 \\ *, & \text { otherwise }\end{cases}
$$

is called a symmetric disjunctive sum [1] of $\boldsymbol{x}$ and $\boldsymbol{y}$. This operation will be denoted by $\nabla$, that is $\boldsymbol{z}=\boldsymbol{x} \nabla \boldsymbol{y}$. We say that a binary column $\boldsymbol{u}$ covers a column $\boldsymbol{v}(\boldsymbol{u} \succeq \boldsymbol{v})$ if $\boldsymbol{u} \bigvee \boldsymbol{v}=\boldsymbol{u}$, and a ternary column $\boldsymbol{u}$ symmetrically covers a column $\boldsymbol{v}(\boldsymbol{u} \unrhd \boldsymbol{v})$ if $\boldsymbol{u} \nabla \boldsymbol{v}=\boldsymbol{u}$.

### 1.2 Symmetric Disjunctive List-Decoding Codes (SLD $s_{L}$-codes)

Definition 1. [2, 3]. A binary code $X$ is said to be a disjunctive list-decoding code of strength $s$ with list size $L$ (LD $s_{L}$-code) if the disjunctive sum of any $s$ codewords of $X$ covers not more than $L-1$ other codewords of $X$ that are not components of the given sum. In other words, for any two disjoint sets $\mathcal{S}, \mathcal{L} \subset[t],|\mathcal{S}|=s,|\mathcal{L}|=L, \mathcal{S} \cap \mathcal{L}=\varnothing$, there exist a row $\boldsymbol{x}_{i}, i \in[N]$, and a column $\boldsymbol{x}(j), j \in \mathcal{L}$, such that

$$
x_{i}(k)=0 \quad \forall k \in \mathcal{S} \quad \text { and } \quad x_{i}(j)=1
$$

Denote by $t_{l d}(N, s, L)$ the maximal size of LD $s_{L^{-}}$-codes of length $N$ and by $N_{l d}(t, s, L)$ the minimal length of LD $s_{L}$-codes of size $t$. Define the rate of LD $s_{L}$-codes:

$$
\begin{equation*}
R_{L}(s) \triangleq \varlimsup_{N \rightarrow \infty} \frac{\log _{2} t_{l d}(N, s, L)}{N}=\varlimsup_{t \rightarrow \infty} \frac{\log _{2} t}{N_{l d}(t, s, L)} \tag{1}
\end{equation*}
$$

Definition 2. $[4,5,6]$. A binary code $X$ is said to be a symmetric disjunctive list-decoding code of strength $s$ with list size $L$ (SLD $s_{L}$-code) if the symmetric disjunctive sum of any $s$ codewords of $X$ symmetrically covers not more than $L-1$ other codewords of $X$ that are not components of the given sum. In other words, for any two disjoint sets $\mathcal{S}, \mathcal{L} \subset[t],|\mathcal{S}|=s$, $|\mathcal{L}|=L, \mathcal{S} \cap \mathcal{L}=\varnothing$, there exist a row $\boldsymbol{x}_{i}, i \in[N]$, and a column $\boldsymbol{x}(j), j \in \mathcal{L}$, such that

$$
\begin{array}{llll}
x_{i}(k)=0 & \forall k \in \mathcal{S} & \text { and } & x_{i}(j)=1, \quad \text { or } \\
x_{i}(k)=1 & \forall k \in \mathcal{S} & \text { and } & x_{i}(j)=0
\end{array}
$$

Denote by $t_{s l d}(N, s, L)$ the maximal size of $\mathrm{SLD} s_{L}$-codes of length $N$ and by $N_{s l d}(t, s, L)$ the minimal length of SLD $s_{L}$-codes of size $t$. Define the rate of SLD $s_{L}$-codes:

$$
\begin{equation*}
R_{L}^{*}(s) \triangleq \varlimsup_{N \rightarrow \infty} \frac{\log _{2} t_{s l d}(N, s, L)}{N}=\varlimsup_{t \rightarrow \infty} \frac{\log _{2} t}{N_{s l d}(t, s, L)} \tag{2}
\end{equation*}
$$

Remark 1. An SLD $s_{1}$-code is the special case of separating codes [7]. More specifically, for $L=1$, Definition 2 coincides with the definition of $(s, 1)$-separating code with the alphabet size $q=2$. Some results and applications of $(s, 1)$-separating binary codes are presented in the survey [8].

Theorem 1. (Monotonicity properties). The rate of $S L D s_{L}$-codes satisfies the following inequalities

$$
\begin{equation*}
R_{L}^{*}(s+1) \leq R_{L}^{*}(s) \leq R_{L+1}^{*}(s) \tag{3}
\end{equation*}
$$

Proof of Theorem 1. It immediately follows from Definition 2 that every SLD $(s+1)_{L^{-}}$ code is the corresponding SLD $s_{L}$-code, so the left inequality in (3) takes place. Simultaneously, every SLD $s_{L}$-code is $\operatorname{SLD} s_{L+1}$-code, therefore the right inequality in $(3)$ is true.

### 1.3 Applications of Symmetric Disjunctive Codes

Applications of SLD $s_{L}$-codes relate to the non-adaptive symmetric group testing which is based on the symmetric disjunctive sum of binary symbols ${ }^{1}$. Group testing deals with identification of defective units in a given pool. We use symmetric group tests, i.e., take a subset of the

[^0]pool and check it. The outcome of a symmetric group test belongs to the ternary alphabet. It is equal to 0,1 or $*$, if all tested units are not defective, all units are defective or at least one unit is defective and at least another one is not defective, respectively. The symmetric group testing was motivated by applications [1] in electrical devices testing (a) and chemical analysis (b).
(a). Consider the situation, where one need to test electrical devices such as conductors (not light bulbs that give a visual result) [1]. These conductors are connected both in parallel and in series and the results for these two arrangements are obtained separately by throwing a switch. If we get current for the series configuration then all are good. If we get no current for the parallel configuration then all are defective. In the one remaining case (no current for the series configuration and current for the parallel configuration), we have at least one good unit and at least 1 defective unit. Hence for our purposes, this compound test to determine which of these three situations holds is to be regarded as a single test and we wish to minimize the number of such tests.
(b). The second possible application is in the chemical analysis of several specimens [1], where it is known a priori that each specimen contains either $A$ or $B$ but not both, which are two specific substances of interest. Suppose a mixture of several specimens is formed and then we split the result into 2 aliquot parts. By using reagent $\alpha$, which precipitates $A$ and does not react with $B$, we can detect "no $A$ " by no precipitate in one of the 2 aliquot parts. Similarly, by using reagent $\beta$, which precipitates $B$ and does not react with $A$, we can detect "no $B$ " by no precipitate in the other of the 2 aliquot parts. "Some $A$ and some $B$ " is indicated if both reagents cause precipitation. Regarding this compound test as a single test, we want to classify the specimens as containing $A$ or containing $B$ in the smallest number of tests.

Suppose the size of the pool equals $t$ and the number of defected units does not exceed $s$. As is the case with LD $s_{L}$-codes [9], SLD $s_{L}$-codes can be considered in connection with the problem of constructing two-stage non-adaptive symmetric group testing procedures. In the first stage, one does $N$ tests that can be depicted as an binary $(N \times t)$-matrix $X=\left\|x_{i}(j)\right\|$, where a column $\boldsymbol{x}(j)$ corresponds to the $j$-th unit, a row $\boldsymbol{x}_{i}$ corresponds to the $i$-th test and $x_{i}(j) \triangleq 1$ if and only if the $j$-th unit is included into the $i$-th testing group. Then the ternary column $y$ of the test results equals the symmetric disjunctive sum of the columns which correspond to the defective units. Let $X$ be SLD $s_{L}$-code, after decoding of the result column $y$, i.e. search of codewords which are symmetrically covered by $y$, a set of $\leq s+L-1$ elements is selected. These units are separately tested in the second stage. Note that for $s \geq 2$ the rate $R_{L}^{*}(s)$ of SLD $s_{L}$-codes is a monotonically nondecreasing function of $L \geq 1$, and its limit

$$
R_{\infty}^{*}(s)=\lim _{L \rightarrow \infty} R_{L}^{*}(s)
$$

can be interpreted as the maximum rate of two-stage non-adaptive symmetric group testing procedures in a search for $\leq s$ defects with the use of SLD $s_{L}$-codes.

In papers [4, 5], we suggested another application of SLD codes called reference communication system. Let a system contain $M$ terminal stations $\mathrm{S}_{1}, \mathrm{~S}_{2}, \ldots, \mathrm{~S}_{M}$ and let a multiple-access channel (MAC) connect these $M$ stations to a central station (CS). Each terminal station has a source. In every time interval, the source can produce a binary information packet of length $K$. Introduce $t \triangleq 2^{K}$ and enumerate all $2^{K}$ possible information packets by integers from 1 to $t$. The packets are encoded into binary sequences of length $N$ by a code $X=(\boldsymbol{x}(i), i \in[t])$, where the codeword $\boldsymbol{x}(i), i \in[t]$, is the encoded sequence corresponding to the information packet number $i$. Denote by $\mathcal{S}$ the set of numbers of generated packets and suppose $|\mathcal{S}| \leq s$.

The CS is interested only in the contents of the received packet and not in the senders. Using a feedback broadcast channel (FBC) the CS answers all $M$ stations to all requests. The model of MAC corresponds to the frequency modulation, i.e., the output ternary sequence $\boldsymbol{y}$ is the symmetric disjunctive sum of the inputs. The scheme of reference communication system is represented on Figure 1.


Figure 1: Reference communication system
Let the terminal stations use an SLD $s_{L}$-code $X$. Since the number of information packets produced by the terminal stations in the same time interval is not more than $s$, the CS is able to recover at most $s+L-1$ packets, which contain $s$ transmitted packets.

Note that the model of MAC can also correspond to the impulse modulation, i.e., the output binary sequence is the disjunctive sum of the inputs. In this case, it is convenient to use LD $s_{L}$-codes for encoding and decoding information packets. The case of impulse modulation was considered in [5].

Another application of SLD $s_{1}$-codes concerns with undetermined data [10, 11]. Given an alphabet $A=\left\{a_{1}, a_{2}, \ldots, a_{t}\right\}$ of basic symbols, to every nonempty subset $T \subseteq[t]$, assign a symbol $a_{T}$, which is called undetermined. Its specification is any basic symbol $a_{i}, i \in T$. By a specification of a sequence of undetermined symbols we mean the result of replacing all its symbols by some of its specifications. The symbol $a_{[t]}$ that can be specified by any basic symbol is called indefinite and is denoted by $*$. Let $\mathcal{T}$ be a system of subsets $T \subseteq[t]$ and let $A^{*}=A_{\mathcal{T}}^{*}=\left\{a_{T} \mid T \in \mathcal{T}\right\}$ be an undetermined alphabet associated with the system.

Consider a problem of coding of undetermined sequences such that the original undetermined sequence can be completely reconstructed from the encoded sequence. One coding method refers to a binary representation $[10,11]$ of undetermined alphabet, which is defined as a pair $\left(X, X^{*}\right)$ of $(N \times t)$-matrix $X$ with columns $\boldsymbol{x}(i) \in\{0,1\}^{N}, i \in[t]$, and $(N \times|\mathcal{T}|)$-matrix $X^{*}$ with columns $\boldsymbol{x}(T) \in\{0,1, *\}^{N}, T \in \mathcal{T}$, where $\boldsymbol{x}(i)$ specifies $\boldsymbol{x}(T)$ in undetermined alphabet $\{0,1, *\}$ if and only if $i \in T$. Advantages of such method are linear in $t$ complexity of the symbol reconstruction and the fact that the mentioned condition allows to know only a small matrix $X$ for reconstruction of the original undetermined sequence while the matrix $X^{*}$ may contain up to $2^{t}$ columns. Obviously, an SLD $s_{1}$-code $X=(\boldsymbol{x}(i), i \in[t])$ and the matrix $X^{*}=\left(\nabla_{i \in T} \boldsymbol{x}(i), T \in \mathcal{T}\right)$ give the fairly compact binary representation of undetermined alphabet associated with the system $\mathcal{T}=[t] \cup\{T \subset[t]| | T \mid \leq s\}[11]$.

### 1.4 Relations Between Parameters of LD $s_{L}$-Codes and SLD $s_{L}$-Codes

The following evident propositions from [4, 5, 6] associate the rate of LD $s_{L}$-codes (1) with the rate of SLD $s_{L}$-codes (2).

Proposition 1. $[4,5,6]$. Any $L D s_{L}$-code is the corresponding $S L D s_{L}$-code.
Proposition 2. [4, 5, 6]. Let $X=\left\|x_{i}(j)\right\|$ be an SLD $s_{L}$-code of length $N$ and size $t$. Consider $(N \times t)$-matrix $X^{\prime}=\left\|x_{i}^{\prime}(j)\right\|$ with elements

$$
x_{i}^{\prime}(j) \triangleq \begin{cases}1, & \text { if } x_{i}(j)=0 \\ 0, & \text { if } x_{i}(j)=1\end{cases}
$$

Then the code of length $2 N$ and size $t$ composed of all rows of the codes $X$ and $X^{\prime}$ is an $L D$ $s_{L}$-code.

Corollary 1. $[4,5,6]$. The rates of $L D s_{L}$-codes and $S L D$ s $s_{L}$-codes satisfy inequalities:

$$
\begin{equation*}
R_{L}(s) \leq R_{L}^{*}(s) \leq 2 R_{L}(s) \tag{4}
\end{equation*}
$$

The next obvious proposition allows us to get another upper bound on the rate of SLD $s_{L}$-codes.

Proposition 3. Let $X$ be an $L D s_{L}$-code of length $N$ and size $t$ with a codeword $\boldsymbol{x}\left(j_{0}\right)$ of weight $w$. Then the code $X^{\prime \prime}$ of length $N-w$ and size $t-1$ constructed from the code $X$ by removing the codeword $\boldsymbol{x}\left(j_{0}\right)$ and all rows $x_{i}$, for which $x_{i}\left(j_{0}\right)=1$, is an $L D(s-1)_{L}$-code.

Corollary 2. The rate of SLD $s_{L}$-codes has the following upper bound:

$$
\begin{equation*}
R_{L}^{*}(s) \leq R_{L}(s-1) . \tag{5}
\end{equation*}
$$

Proof of Corollary 2. Let $X$ be an arbitrary SLD $s_{L}$-code of length $N$ and size $t$. The code $X_{1}$ obtained in Proposition 2 from the code $X$ is a constant weight LD $s_{L}$-code of length $2 N$, size $t$ and weight $N$. Then the code $X_{2}$ obtained in Proposition 3 from the code $X_{1}$ is an $\mathrm{LD}(s-1)_{L}$-code of length $N$ and size $t-1$. Hence as $N \rightarrow \infty$ the inequality

$$
\frac{\log _{2}[t-1]}{N} \leq R_{L}(s-1)(1+o(1))
$$

holds. It means correctness of (5).
The best presently known lower and upper bounds on the rate $R_{L}(s)$ were recently obtained in [12, 13]. The use of the inequalities (4) and (5), the lower bound $\underline{R}_{L}(s)[12]$ and the upper bound $\bar{R}_{L}(s)$ [12] on the rate of LD $s_{L}$-codes yields the results below.

Theorem 2. (Relationship between $R_{L}^{*}(s)$ and $R_{L}(s)$ )
The following three statements hold.

1. For any fixed $s \geq 2$ and $L \geq 1$ the rates $R_{L}^{*}(s)$ and $R_{L}(s)$ have relationship

$$
R_{L}(s) \leq R_{L}^{*}(s) \leq \min \left\{2 R_{L}(s), R_{L}(s-1)\right\} .
$$

2. For any fixed $L \geq 1$ and $s \rightarrow \infty$

$$
R_{L}^{*}(s)=R_{L}(s)(1+o(1))
$$

3. For any fixed $s \geq 2$ and $L \geq 1$ the rate of an $S L D s_{L}$-code satisfies the inequality

$$
\underline{R}_{L}(s) \leq R_{L}^{*}(s) \leq \bar{R}_{L}^{*}(s) \triangleq \min \left\{2 \bar{R}_{L}(s), \bar{R}_{L}(s-1)\right\} .
$$

### 1.5 Random Coding Bounds on the Rate of SLD $s_{L}$-codes

In the given paper, we develop a random coding method based on the ensemble of constantweight codes and establish new lower random coding bounds on the rate of SLD $s_{L}$-codes. Some of the methods which are used in the proof of the next theorem are presented in [12, 13].

Theorem 3. (Lower random coding bound $\underline{R}_{L}^{*}(s)$ ).
The following three statements hold.

1. For any fixed $L \geq 1$ and $s \geq 2$ we have the inequality

$$
\begin{equation*}
R_{L}^{*}(s) \geq \underline{R}_{L}^{*}(s) \triangleq \max _{0<Q \leq 1 / 2}\left(h(Q)+\frac{B_{L}(s, Q)}{s+L-1}\right), \tag{6}
\end{equation*}
$$

where

$$
\begin{align*}
h(Q) & \triangleq-Q \log _{2} Q-(1-Q) \log _{2}[1-Q] \\
B_{L}(s, Q) & \triangleq Q \log _{2}\left[\frac{p(1-z)}{p(1-z)+q(1-z)}\right]+(1-Q) \log _{2}\left[\frac{p(z)}{p(z)+q(z)}\right]  \tag{7}\\
p(z) & \triangleq z^{s}\left(z-z^{s}\right)^{L} \\
q(z) & \triangleq\left(z-z^{s}\right)\left(1-z^{s}-(1-z)^{s}\right)^{L}
\end{align*}
$$

and $z$ is the unique root of the equation

$$
\begin{equation*}
Q(p(z)+q(z))=(1-Q)(p(1-z)+q(1-z)) . \tag{8}
\end{equation*}
$$

2. For fixed $L=1,2, \ldots$ and $s \rightarrow \infty$

$$
\begin{equation*}
\underline{R}_{L}^{*}(s) \geq \frac{L}{s^{2} \log _{2} e}(1+o(1)) . \tag{9}
\end{equation*}
$$

3. For fixed $s=2,3, \ldots$ there exists a limit

$$
\begin{equation*}
\underline{R}_{\infty}^{*}(s) \triangleq \lim _{L \rightarrow \infty} \underline{R}_{L}^{*}(s)=\log _{2}\left[\frac{(s-1)^{s-1}}{s^{s}}+1\right] . \tag{10}
\end{equation*}
$$

If $s \rightarrow \infty$, then

$$
\underline{R}_{\infty}^{*}(s)=\frac{\log _{2} e}{e s}(1+o(1))=\frac{0.5307 \ldots}{s}(1+o(1)) .
$$

The numerical values of the lower bound (6)-(8) are shown in Table 1, where the argument of maximum in (6) is denoted by $Q_{L}^{*}(s)$. Note that the lower bound (6)-(8) improves the random coding bound obtained in [14] using the ensemble with independent binary symbols of codewords. In addition one can see that for small values of $s \geq 2$ and $L \geq 1$, the lower bounds (6)-(8) are greater than the lower bounds $\underline{R}_{L}(s)$ on the rate of LD $s_{L}$-codes from [12].

Note that, for $s \rightarrow \infty$, the asymptotic lower bound of $\underline{R}_{L}^{*}(s)(9)$ coincides with the asymptotic behavior of the random coding bound on the rate of LD $s_{L}$-codes [12]. In addition, for $L \rightarrow \infty$, the asymptotics of $\underline{R}_{L}^{*}(s)(10)$ coincides with the asymptotic behavior of the mentioned above bound from [12].

Table 1: Numerical values of the lower bound $\underline{R}_{L}^{*}(s)$

| $s_{L}$ | $2_{1}$ | $2_{2}$ | $2_{3}$ | $2_{4}$ | $2_{5}$ | $2_{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\underline{R}_{L}^{*}(s)$ | 0.2075 | 0.2457 | 0.2635 | 0.2744 | 0.2819 | 0.2874 |
| $Q_{L}^{*}(s)$ | 0.5000 | 0.2764 | 0.2432 | 0.2297 | 0.2228 | 0.2180 |
| $s_{L}$ | $3_{1}$ | $3_{2}$ | $3_{3}$ | $3_{4}$ | $3_{5}$ | $3_{6}$ |
| $\underline{R}_{L}^{*}(s)$ | 0.0800 | 0.1153 | 0.1348 | 0.1470 | 0.1552 | 0.1611 |
| $Q_{L}^{*}(s)$ | 0.2000 | 0.1794 | 0.1686 | 0.1613 | 0.1561 | 0.1524 |
| $s_{L}$ | $4_{1}$ | $4_{2}$ | $4_{3}$ | $4_{4}$ | $4_{5}$ | $4_{6}$ |
| $\underline{R}_{L}^{*}(s)$ | 0.0439 | 0.0684 | 0.0838 | 0.0941 | 0.1014 | 0.1068 |
| $Q_{L}^{*}(s)$ | 0.1479 | 0.1391 | 0.1326 | 0.1275 | 0.1234 | 0.1201 |
| $s_{L}$ | $5_{1}$ | $5_{2}$ | $5_{3}$ | $5_{4}$ | $5_{5}$ | $5_{6}$ |
| $\underline{R}_{L}^{*}(s)$ | 0.0279 | 0.0456 | 0.0575 | 0.0660 | 0.0723 | 0.0771 |
| $Q_{L}^{*}(s)$ | 0.1209 | 0.1150 | 0.1103 | 0.1064 | 0.1030 | 0.1003 |
| $s_{L}$ | $6_{1}$ | $6_{2}$ | $6_{3}$ | $6_{4}$ | $6_{5}$ | $6_{6}$ |
| $\underline{R}_{L}^{*}(s)$ | 0.0194 | 0.0325 | 0.0420 | 0.0490 | 0.0544 | 0.0587 |
| $Q_{L}^{*}(s)$ | 0.1027 | 0.0983 | 0.0947 | 0.0915 | 0.0889 | 0.0865 |

## 2 Proof of Theorem 3

This Section contains five lemmas that are only stated. The proofs of Lemma 1-5 are presented in Appendix.

Proof of Statement 1. Fix $L \geq 1, s \geq 2$ and a parameter $Q, 0<Q \leq 1 / 2$. The bound (6)-(8) is obtained by the method of random coding over the ensemble of binary constantweight codes [15] defined as the ensemble $E(N, t, Q)$ of binary codes $X$ of length $N$ and size $t$, where the codewords are chosen independently and equiprobably from the set consisting of all $\binom{N}{(Q N\rfloor}$ codewords of a fixed weight $\lfloor Q N\rfloor$. A pair of sets $(\mathcal{S}, \mathcal{L}),|\mathcal{S}|=s,|\mathcal{L}|=L, \mathcal{S} \cap \mathcal{L}=\varnothing$, we call an $\left(s_{L}^{*}\right)$-bad pair if

$$
\nabla_{i \in \mathcal{S}} x(i) \unrhd \nabla_{j \in \mathcal{L}} x(j)
$$

For the ensemble $E(N, t, Q)$, denote by $P(N, Q, s, L)$ the probability of the event "the pair $(\mathcal{S}, \mathcal{L})$ is $\left(s_{L}^{*}\right)$-bad". Note that the absence of $\left(s_{L}^{*}\right)$-bad pair of subsets in the code is the criterion of SLD $s_{L}$-code. Hence, similarly to the arguments in the proof of the lower random coding bound on the rate $R_{L}(s)(1)$ in [12], the rate $R_{L}^{*}(s)(2)$ satisfies the inequality

$$
\begin{align*}
& R_{L}^{*}(s) \geq \underline{R}_{L}^{*}(s) \triangleq \frac{1}{s+L-1} \max _{0<Q<1} A_{L}^{*}(s, Q), \\
& A_{L}^{*}(s, Q) \triangleq \varlimsup_{N \rightarrow \infty} \frac{-\log _{2} P(N, Q, s, L)}{N} \tag{11}
\end{align*}
$$

Note that the set of all $s_{L}^{*}$-bad pairs of any codeword weight is invariant under the binary negation operation, it implies the equality $P(N, Q, s, L)=P(N, 1-Q, s, L)$. Therefore, it is enough to consider only $0<Q \leq 1 / 2$.

To complete the proof of the theorem, it is sufficient to compute the function $A_{L}^{*}(s, Q)(11)$.
Lemma 1. If there exists a solution $z, 0<z<1$, of the equation (8), then the function
$A_{L}^{*}(s, Q)$ (11) equals

$$
\begin{equation*}
(s+L-1) h(Q)+(1-Q) \log _{2}\left[\frac{p(z)}{p(z)+q(z)}\right]+Q \log _{2}\left[\frac{p(1-z)}{p(1-z)+q(1-z)}\right] \tag{12}
\end{equation*}
$$

where the functions $h(\cdot), p(\cdot)$ and $q(\cdot)$ are determined by (7).
Lemma 2. The function

$$
\begin{equation*}
\rho(z) \triangleq \frac{p(z)+q(z)}{p(1-z)+q(1-z)}, \quad 0<z<1 \tag{13}
\end{equation*}
$$

continuously maps the interval $(0,1)$ into the interval $(0,+\infty)$ and strictly increases.
By Lemma 2 the equation (8) has the unique solution. Thus, the condition of Lemma 1 is clear, it means that the bound (6)-(8) is proved.

Proof of Statement 2. For fixed $s \geq 2$ and $L \geq 1$, let us interpret equation (8) as a function $Q_{L}(s, z)$ of the argument $z, 0<z<1$, i.e.,

$$
\begin{equation*}
Q_{L}(s, z) \triangleq \frac{p(1-z)+q(1-z)}{p(1-z)+q(1-z)+p(z)+q(z)}, \tag{14}
\end{equation*}
$$

where the functions $p(\cdot)$ and $q(\cdot)$ are determined in (7).
Due to existence and uniqueness of the root of the equation (8), continuity and monotonicity of the function (14) (by Lemma 2), one can rewrite the definition of the random coding bound (6)-(8) as

$$
\begin{equation*}
\underline{R}_{L}^{*}(s) \triangleq \max _{1 / 2 \leq z<1} T_{L}(s, z) \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{L}(s, z) \triangleq h\left(Q_{L}(s, z)\right)+B_{L}\left(s, Q_{L}(s, z)\right) . \tag{16}
\end{equation*}
$$

Let $L \geq 1$ be fixed and $s \rightarrow \infty$. If in definition (16) we put $z=1-\lambda / s$, where the parameter $\lambda=\lambda_{L}$ is independent of $s$, then (15) means that

$$
\begin{equation*}
\underline{R}_{L}^{*}(s) \geq T_{L}\left(s, 1-\frac{\lambda}{s}\right) . \tag{17}
\end{equation*}
$$

Lemma 3. For a fixed $L \geq 1$ and $s \rightarrow \infty$, the next asymptotic equality holds:

$$
\begin{equation*}
T_{L}\left(s, 1-\frac{\lambda}{s}\right)=\frac{L}{s^{2}}\left(-\lambda \log _{2}\left[1-e^{-\lambda}\right]\right)(1+o(1)) . \tag{18}
\end{equation*}
$$

Taking derivative one can check that at $\lambda=\frac{1}{\log _{2} e}$ the maximum

$$
\begin{equation*}
\max _{\lambda>0}\left\{-\lambda \log _{2}\left[1-e^{-\lambda}\right]\right\}=\frac{1}{\log _{2} e} \tag{19}
\end{equation*}
$$

is attained. Therefore, (17) and (19) imply for the random coding bound (6)-(8) the asymptotic inequality (9).

Proof of Statement 3. For fixed $s \geq 2$ and $L \geq 1$, let us introduce the following function

$$
\begin{equation*}
g(z) \triangleq g_{L}(s, z)=\frac{z-z^{s}}{1-z-(1-z)^{s}}, \quad \frac{1}{2} \leq z<1 . \tag{20}
\end{equation*}
$$

It is clear that $g(z)(20)$ monotonically increases in the interval $[1 / 2,1)$, attains 1 at the point $z=\frac{1}{2}$ and has the left limit $s-1$ as $z \rightarrow 1$.

For large enough parameter $L$ and a fixed parameter $c>0$ independent of $L$, one can see that the root of equation

$$
\begin{equation*}
\left(\frac{g(z)}{1+g(z)}\right)^{L}=c(1-z), \quad \frac{1}{2} \leq z<1 \tag{21}
\end{equation*}
$$

exists and is unique, since the left-hand side of (21) monotonically increases and the right-hand side of (21) strictly decreases. Denote this root by $z_{L}(s, c)$.

Let $s \geq 2$ be fixed and $L \rightarrow \infty$.
Lemma 4. The substitution of $z=z_{L}(s, c)$ into the function (16) yields

$$
\begin{align*}
T_{L}\left(s, z_{L}(s, c)\right) \cdot(1+o(1))=\log _{2}[s+c]-\frac{s+c-1}{s+c} & \log _{2}[s+c-1]+ \\
& +\frac{1}{s+c} \log _{2}\left[\frac{(s-1)^{s-1}}{s^{s}}\right], \quad L \rightarrow \infty . \tag{22}
\end{align*}
$$

The definition (15) means that

$$
\begin{equation*}
\underline{R}_{L}^{*}(s) \geq T_{L}\left(s, z_{L}(s, c)\right)(1+o(1)), \quad L \rightarrow \infty, \quad \forall c=c(s)>0 . \tag{23}
\end{equation*}
$$

Calculating the derivative in $c$, one can check that maximum of the right-hand side of (22) is attained at the point $c=c(s)=\frac{s^{s}-(s-1)^{s}}{(s-1)^{s-1}}$. If we substitute this value $c=c(s)$ into (22), then the use of (23) establishes for the random coding bound (6)-(8) the inequality

$$
\begin{equation*}
\underline{R}_{L}^{*}(s) \geq \log _{2}\left[\frac{(s-1)^{s-1}}{s^{s}}+1\right](1+o(1)), \quad L \rightarrow \infty \tag{24}
\end{equation*}
$$

Lemma 5. The asymptotic inequality (24) is an equality.
Statement 3 of Theorem 3 is proved.

## A Proofs of Lemma 1-5

Proof of Lemma 1. Let us use the terminology of types [16]. Consider an arbitrary set of size $s$ consisting of binary codewords of length $N$ and weight $\lfloor Q N\rfloor:(\boldsymbol{x}(1), \ldots, \boldsymbol{x}(s))$, where $\boldsymbol{x}(i) \in\{0,1\}^{N}, \forall i \in[s]$. The set forms $(N \times s)$-matrix $X_{s}$. Let $\boldsymbol{a} \triangleq\left(a_{1}, \ldots, a_{s}\right) \in\{0,1\}^{s}$. Denote a type of the matrix $X_{s}$ by $\{n(\boldsymbol{a})\}$, where $n(\boldsymbol{a}), 0 \leq n(\boldsymbol{a}) \leq N$ is the number of $\boldsymbol{a}$-rows in the matrix $X_{s}$. Obviously, for any matrix $X_{s}$ we have

$$
\sum_{a} n(\boldsymbol{a})=N .
$$

By $n(\boldsymbol{O})(n(\boldsymbol{1}))$ denote the number of the rows in $X_{s}$ consisting of all zeros (ones). It allows to represent $P(N, Q, s, L)$ as

$$
\begin{equation*}
P(N, Q, s, L)=\sum_{\{n(\boldsymbol{a})\} \in \mathcal{N}} \frac{N!}{\prod_{a} n(\boldsymbol{a})!}\binom{N-n(\boldsymbol{O})-n(\boldsymbol{1})}{\lfloor Q N\rfloor-n(\mathbf{1})}^{L}\binom{N}{\lfloor Q N\rfloor}^{-s-L}, \tag{25}
\end{equation*}
$$

where the set $\mathcal{N}$ consists of all possible types $n(\boldsymbol{a}), \boldsymbol{a} \in\{0,1\}^{s}$, such that:

$$
\begin{gather*}
0 \leq n(\boldsymbol{a}) \leq N \quad \forall \boldsymbol{a} \in\{0,1\}^{s}, \quad n(\boldsymbol{0}) \leq N-\lfloor Q N\rfloor, \quad n(\boldsymbol{1}) \leq\lfloor Q N\rfloor, \\
\sum_{a} n(\boldsymbol{a})=N, \quad \sum_{a: a_{i}=1} n(\boldsymbol{a})=\lfloor Q N\rfloor \quad \forall i \in[s\rfloor . \tag{26}
\end{gather*}
$$

Let $N \rightarrow \infty$. For every type $n(\boldsymbol{a}), \boldsymbol{a} \in\{0,1\}^{s}$, let us consider the corresponding distribution $\tau \triangleq\{\tau(\boldsymbol{a})\}: \tau(\boldsymbol{a})=\frac{n(\boldsymbol{a})}{N}$. Thus, for $N \rightarrow \infty$, the set $\mathcal{N}$ accords with the set $\mathcal{T}$ consisting of the distributions with the following properties induced by (26):

$$
\tau \in \mathcal{T} \Longleftrightarrow\left\{\begin{array}{c}
0 \leq \tau(\boldsymbol{a}) \leq 1 \quad \forall \boldsymbol{a} \in\{0,1\}^{s}, \quad \tau(\boldsymbol{0}) \leq 1-Q, \quad \tau(\mathbf{1}) \leq Q,  \tag{27}\\
\sum_{\boldsymbol{a} \in\{0,1\}^{s}} \tau(\boldsymbol{a})=1, \quad \sum_{a: a_{i}=1} \tau(\boldsymbol{a})=Q \quad \forall i \in[s] .
\end{array}\right\}
$$

Applying the Stirling approximation, we obtain the following logarithmic asymptotic behavior of the summand in the sum (25) for $\tau \in \mathcal{T}$ :

$$
\begin{align*}
& -\log _{2} \sum_{\{n(\boldsymbol{a})\} \in \mathcal{N}} \frac{N!}{\prod_{a} n(\boldsymbol{a})!}\binom{N-n(\boldsymbol{0})-n(\mathbf{1})}{\lfloor Q N\rfloor-n(\mathbf{1})}^{L}\binom{N}{\lfloor Q N\rfloor}^{-s-L}= \\
& \quad=N F(\tau, Q)(1+o(1)), \quad \text { where }, \\
& F(\tau, Q) \triangleq \sum_{a} \tau(\boldsymbol{a}) \log _{2}[\tau(\boldsymbol{a})]-(1-\tau(\boldsymbol{0})-\tau(\mathbf{1})) \operatorname{Lh}\left(\frac{Q-\tau(\mathbf{1})}{1-\tau(\boldsymbol{0})-\tau(\mathbf{1})}\right)+  \tag{28}\\
& +(s+L) h(Q) .
\end{align*}
$$

For the given $Q$, let the minimum of the function $F(\tau, Q)$ be attained at $\tau_{Q}=\left\{\tau_{Q}(\boldsymbol{a})\right\}$, then

$$
\begin{equation*}
A_{L}^{*}(s, Q) \triangleq \varlimsup_{N \rightarrow \infty} \frac{-\log _{2} P(s, L, Q, N)}{N}=F\left(\tau_{Q}, Q\right)=\min _{\tau \in \mathcal{T}} F(\tau, Q) . \tag{29}
\end{equation*}
$$

Since $F$ is continuous in the admissible compact space $\mathcal{T}$, finding the minimum of $F$ under constraints (27) with excluded boundaries is sufficient to calculate (29). Let us write the minimization problem: $F \rightarrow$ min,

Search domain $\mathbb{T}$ :

$$
\begin{equation*}
0<\tau(\boldsymbol{a})<1 \quad \forall \boldsymbol{a} \in\{0,1\}^{s}, \quad \tau(\boldsymbol{1})<Q, \quad \tau(\boldsymbol{0})<1-Q \tag{30}
\end{equation*}
$$

$$
\begin{align*}
\text { Restrictions: } & \begin{cases}\sum_{\boldsymbol{a} \in\{0,1\}^{s}} \tau(\boldsymbol{a})=1, \\
\sum_{a: a_{i}=1} \tau(\boldsymbol{a})=Q \quad \forall i \in[s],\end{cases}  \tag{31}\\
\text { Main Function: } \quad & F(\tau, Q)=(28): \mathbb{T} \rightarrow \mathbb{R} . \tag{32}
\end{align*}
$$

To find the extremal distribution $\tau_{Q}$ we apply the standard Lagrange multipliers method. Consider the Lagrangian:

$$
\begin{equation*}
\Lambda \triangleq F(\tau, Q)+\lambda_{0}\left(\sum_{\boldsymbol{a} \in\{0,1\}^{s}} \tau(\boldsymbol{a})-1\right)+\sum_{i=1}^{s} \lambda_{i}\left(\sum_{\boldsymbol{a}: a_{i}=1} \tau(\boldsymbol{a})-Q\right) . \tag{33}
\end{equation*}
$$

The necessary conditions for the extremal distribution $\tau_{Q}$ are:

Let us show that the matrix of second derivatives of the Lagrangian is positive definite. Indeed, we have

$$
\begin{aligned}
\frac{\partial^{2} \Lambda}{\partial(\tau(\boldsymbol{a}))^{2}} & =\frac{\log _{2} e}{\tau(\boldsymbol{a})}>0, \quad \forall \boldsymbol{a} \in\{0,1\}^{s} \backslash\{\boldsymbol{O}, \boldsymbol{1}\}, \\
\frac{\partial^{2} \Lambda}{\partial(\tau(\boldsymbol{0}))^{2}} & =\frac{\log _{2} e}{\tau(\boldsymbol{0})}+L \log _{2} e \frac{Q-\tau(\boldsymbol{1})}{(1-\tau(\boldsymbol{O})-\tau(\mathbf{1}))(1-Q-\tau(\boldsymbol{O}))}>0, \\
\frac{\partial^{2} \Lambda}{\partial(\tau(\mathbf{1}))^{2}} & =\frac{\log _{2} e}{\tau(\mathbf{1})}+L \log _{2} e \frac{1-Q-\tau(\boldsymbol{O})}{(1-\tau(\boldsymbol{O})-\tau(\mathbf{1}))(Q-\tau(\mathbf{1}))}>0, \\
\frac{\partial^{2} \Lambda}{\partial(\tau(\boldsymbol{0})) \partial(\tau(\mathbf{1}))} & =-L \log _{2} e \frac{1}{1-\tau(\boldsymbol{O})-\tau(\mathbf{1})}<0,
\end{aligned}
$$

and the other elements of the matrix are zeros. That is why, this matrix is positive definite. Note that the matrix of second derivatives of the function $F(\tau, Q)$ coincides with the above matrix. Therefore [17], $F$ is strictly $\cup$-convex in the domain $\mathbb{T}$. Moreover, the constraint equations (31) define an affine subspace $\mathbb{G}$ in $\mathbb{R}^{2^{s}}$ of dimension $\left(2^{s}-s-1\right)$, that is why $F$ is strictly $\cup$-convex in $\mathbb{T} \cap \mathbb{G}$. Hence a local minimum of $F$ in $\mathbb{T} \cap \mathbb{G}$ is global and unique. Due to the Karush-Kuhn-Tacker theorem [17], it is clear that each solution satisfying the system (34) and the constraints (31) is unique and gives the desired minimum distribution $\tau_{Q}$ for $F(\tau, Q)$.

Note that the symmetry of the problem yields equality: $\nu \triangleq \lambda_{1}=\lambda_{2}=\ldots=\lambda_{s}$. To prove this, we need to check that $\lambda_{i}=\lambda_{j}$ for $i \neq j$. Let $\overline{\boldsymbol{a}}_{i} \triangleq(0, \ldots, 1, \ldots, 0)$ be a row of length $s$, which has 1 at the $i$-th position and $0^{\prime} s$ at the other positions. A permutation of indices $i$ and $j$ leads to an equivalent problem. Hence, if $\tau_{Q}^{1}$ is a solution, then $\tau_{Q}^{2}$ is also a solution, where $\tau_{Q}^{2}(\boldsymbol{a}) \triangleq \tau_{Q}^{1}(\tilde{\boldsymbol{a}})$ and $\tilde{\boldsymbol{a}}$ is a row, obtained by permutation of indices $i$ and $j$ from the row $\boldsymbol{a}$. The uniqueness of the solution $\tau_{Q}$ implies that the distribution $\tau_{Q}^{1}$ coincides with the distribution $\tau_{Q}^{2}$. In particular, $\tau_{Q}^{1}\left(\overline{\boldsymbol{a}}_{i}\right)=\tau_{Q}^{2}\left(\overline{\boldsymbol{a}}_{i}\right)=\tau_{Q}^{1}\left(\overline{\boldsymbol{a}}_{j}\right)$. From the first equation of (34), it follows that $\lambda_{i}=\lambda_{j}$.

Introduce a parameter $\mu \triangleq e 2^{\lambda_{0}}$. Then the equations (34) have the form:

$$
\left\{\begin{array}{l}
\log _{2} \mu+\log _{2}[\tau(\boldsymbol{a})]+\nu \sum_{i=1}^{s} a_{i}=0,  \tag{35}\\
\log _{2} \mu+\log _{2}[\tau(\boldsymbol{O})]+L \log _{2}\left[\frac{1-\tau(\boldsymbol{O})-\tau(\mathbf{1})}{1-Q-\tau \boldsymbol{O})}\right]=0, \\
\log _{2} \mu+\log _{2}[\tau(\mathbf{1})]+L \log _{2}\left[\frac{1-\tau(\boldsymbol{O})-\tau(\mathbf{1})}{Q-\tau(\mathbf{1})}\right]+s \nu=0 .
\end{array}\right.
$$

After substitution $z \triangleq \frac{1}{1+2^{-\nu}}, 0<z<1$, the first equation of (35) gives

$$
\begin{equation*}
\tau(\boldsymbol{a})=\frac{2^{-\nu \sum a_{i}}}{\mu}=\frac{1}{\mu z^{s}}(1-z)^{\sum a_{i}} z^{s-\sum a_{i}} \quad \forall \boldsymbol{a} \in\{0,1\}^{s} \backslash\{\boldsymbol{0}, \boldsymbol{1}\} . \tag{36}
\end{equation*}
$$

Substitution (36) into the first and the second equations of the system (31) leads to

$$
\begin{align*}
& 1=\frac{1}{\mu z^{s}} \sum_{i=1}^{s-1}\binom{s}{i} z^{i}(1-z)^{s-i}+\tau(\boldsymbol{O})+\tau(\mathbf{1})=\frac{1-z^{s}-(1-z)^{s}}{\mu z^{s}}+\tau(\boldsymbol{0})+\tau(\mathbf{1}),  \tag{37}\\
& Q=\frac{1}{\mu z^{s}} \sum_{i=1}^{s-1}\binom{s-1}{i} z^{i}(1-z)^{s-i}+\tau(\boldsymbol{1})=\frac{1-z-(1-z)^{s}}{\mu z^{s}}+\tau(\mathbf{1}) \tag{38}
\end{align*}
$$

correspondingly. Subtraction (38) from (37) yields

$$
\begin{equation*}
1-Q=\frac{z-z^{s}}{\mu z^{s}}+\tau(\boldsymbol{O}) \tag{39}
\end{equation*}
$$

Due to (37)-(39) the second and third equations of the system (35) are equivalent to

$$
\begin{gather*}
\mu\left(1-Q-\frac{z-z^{s}}{\mu z^{s}}\right)\left(\frac{1-z^{s}-(1-z)^{s}}{z-z^{s}}\right)^{L}=1 \\
\mu\left(Q-\frac{1-z-(1-z)^{s}}{\mu z^{s}}\right)\left(\frac{1-z^{s}-(1-z)^{s}}{1-z-(1-z)^{s}}\right)^{L}=1 \tag{40}
\end{gather*}
$$

respectively.
To shorten the formulas let us introduce the functions of the parameters $s, L$ and $z$ :

$$
\begin{align*}
& p(z) \triangleq p_{L}(s, z)=z^{s}\left(z-z^{s}\right)^{L} \\
& q(z) \triangleq q_{L}(s, z)=\left(z-z^{s}\right)\left(1-z^{s}-(1-z)^{s}\right)^{L}  \tag{41}\\
& r(z) \triangleq r_{L}(s, z)=z^{s}\left(1-z^{s}-(1-z)^{s}\right)^{L}
\end{align*}
$$

The use of such notations yields the following expressions of $\mu$ from the both equations (40):

$$
\begin{align*}
& \mu=\frac{1}{1-Q} \frac{p(z)+q(z)}{r(z)},  \tag{42}\\
& \mu=\frac{1}{Q} \frac{p(1-z)+q(1-z)}{r(z)} . \tag{43}
\end{align*}
$$

Equating of (42) and (43) leads to the equation on the parameter $z$ :

$$
Q(p(z)+q(z))=(1-Q)(p(1-z)+q(1-z))
$$

which coincides with the equation (8).
The substitutions (42) into (39) and (43) into (38) give:

$$
\begin{align*}
\tau(\boldsymbol{O}) & =(1-Q) \frac{p(z)}{p(z)+q(z)}  \tag{44}\\
\tau(\mathbf{1}) & =Q \frac{p(1-z)}{p(1-z)+q(1-z)}
\end{align*}
$$

So, let us calculate the value of $F(\tau, Q)$ (28), where the distribution $\tau$ is specified by (36) and (44). At the beginning, we compute the following sum:

$$
\begin{align*}
& \sum_{a:} \tau(\boldsymbol{a} \neq \boldsymbol{a}, \boldsymbol{1} \\
& \log _{2}[\tau(\boldsymbol{a})]=\{\text { by }(36)\}= \\
&= \sum_{i=1}^{s-1}\binom{s}{i} \frac{1}{\mu z^{s}}(1-z)^{s-i} z^{i}\left(\log _{2}\left[\frac{1}{\mu z^{s}}\right]+i \log _{2} z+(s-i) \log _{2}[1-z]\right)= \\
&= \frac{1-z^{s}-(1-z)^{s}}{\mu z^{s}} \log _{2}\left[\frac{1}{\mu z^{s}}\right]+\frac{z-z^{s}}{\mu z^{s}} \log _{2}\left[z^{s}\right]+\frac{1-z-(1-z)^{s}}{\mu z^{s}} \log _{2}\left[(1-z)^{s}\right]= \\
&=\{\text { by }(37),(39) \text { and }(38)\}= \\
&=(1-\tau(\boldsymbol{O})-\tau(\boldsymbol{1})) \log _{2}\left[\frac{1}{\mu z^{s}}\right]+(1-Q-\tau(\boldsymbol{O})) \log _{2}\left[z^{s}\right]+(Q-\tau(\boldsymbol{1})) \log _{2}\left[(1-z)^{s}\right]=  \tag{45}\\
&=(1-Q-\tau(\boldsymbol{O})) \log _{2}\left[\frac{1}{\mu}\right]+(Q-\tau(\boldsymbol{1})) \log _{2}\left[\frac{(1-z)^{s}}{\mu z^{s}}\right]
\end{align*}
$$

Further, the use of (45) implies

$$
\begin{align*}
\sum_{\boldsymbol{a}: \boldsymbol{a} \neq \boldsymbol{0}, \boldsymbol{1}} & \tau(\boldsymbol{a}) \log _{2}[\tau(\boldsymbol{a})]-(1-\tau(\boldsymbol{O})-\tau(\boldsymbol{1})) L h\left(\frac{Q-\tau(\mathbf{1})}{1-\tau(\boldsymbol{0}) \tau(\mathbf{1})}\right)= \\
& =(1-Q-\tau(\boldsymbol{0}))\left(-\log _{2} \mu-L \log _{2}\left[\frac{1-\tau(\boldsymbol{O})-\tau(\boldsymbol{1})}{1-Q-\tau(\boldsymbol{0})}\right]\right)+ \\
& +(Q-\tau(\mathbf{1}))\left(-\log _{2} \mu-\log _{2}\left[\frac{z^{s}}{(1-z)^{s}}\right]-L \log _{2}\left[\frac{1-\tau(\boldsymbol{0})-\tau(\boldsymbol{1})}{Q-\tau(\boldsymbol{1})}\right]\right)= \\
& =\{\text { by }(35)\}= \\
& =(1-Q-\tau(\boldsymbol{0})) \log _{2}[\tau(\boldsymbol{0})]+(Q-\tau(\mathbf{1})) \log _{2} \tau(\mathbf{1}) \tag{46}
\end{align*}
$$

Finally, the use of (46) and (44) leads to

$$
F(\tau, Q)=(s+L) h(Q)+(1-Q) \log _{2}[\tau(\boldsymbol{O})]+Q \log _{2}[\tau(\boldsymbol{1})]=(12)
$$

Thus, Lemma 1 is proved.
Proof of Lemma 2. Let us rewrite the formula (13) using the monotonically increasing function $g(z)(20)$ :

$$
\begin{equation*}
\rho(z)=\frac{z^{s}(g(z))^{L}+\left(z-z^{s}\right)(1+g(z))^{L}}{(1-z)^{s}+\left(1-z-(1-z)^{s}\right)(1+g(z))^{L}} \tag{47}
\end{equation*}
$$

The devision of the numerator and the denominator of $(47)$ by $\left(z-z^{s}\right)(1+g(z))^{L}$ leads to

$$
\rho(z)=\frac{\left(\frac{g(z)}{1+g(z)}\right)^{L} \cdot \frac{z^{s}}{z-z^{s}}+1}{\frac{(1-z)^{s}}{z-z^{s}} \cdot \frac{1}{(1+g(z))^{L}}+\frac{1}{g(z)}}
$$

where the function $\frac{z^{s}}{z-z^{s}}$ is strictly increasing and the function $\frac{(1-z)^{s}}{z-z^{s}}$ is strictly decreasing. Thus, it is clear that $\rho(z)$ is strictly increasing.

Note that $g(z) \rightarrow \frac{1}{s-1}$ as $z \rightarrow 0$ and $g(z) \rightarrow s-1$ as $z \rightarrow 1$. Therefore, by (47) the following limits are true:

$$
\begin{aligned}
\lim _{z \rightarrow 0+0} \rho(z) & =0, \\
\lim _{z \rightarrow 1-0} \rho(z) & =+\infty .
\end{aligned}
$$

Lemma 2 is proved.
Proof of Lemma 3. Let us introduce the following notations:

$$
\begin{align*}
& U_{L}(s, z) \triangleq \frac{p(1-z)}{p(1-z)+q(1-z)}  \tag{48}\\
& V_{L}(s, z) \triangleq \frac{p(z)}{p(z)+q(z)}
\end{align*}
$$

Then the function (16) can be represented as

$$
\begin{equation*}
T_{L}(s, z)=-Q \log _{2} Q-(1-Q) \log _{2}[1-Q]+\frac{1}{s+L-1}\left(Q \log _{2} U+(1-Q) \log _{2} V\right) \tag{49}
\end{equation*}
$$

where the shorthands $Q=Q_{L}(s, z), U=U_{L}(s, z)$ and $V=V_{L}(s, z)$ are used.
Computation of two first terms of asymptotic expansions of $p(z), q(z), p(1-z), q(1-z)$ for $z=1-\lambda / s$ and $s \rightarrow \infty$ leads to the equalities

$$
\begin{align*}
& p(1-z)=p\left(\frac{\lambda}{s}\right)=\left(\frac{\lambda}{s}\right)^{s+L}-\left(\frac{\lambda}{s}\right)^{s(L+1)}, \\
& q(1-z)=q\left(\frac{\lambda}{s}\right)=\frac{\lambda\left(1-e^{-\lambda}\right)^{L}}{s}+\frac{L \lambda^{3} e^{-\lambda}\left(1-e^{-\lambda}\right)^{L-1}}{2 s^{2}}+o\left(\frac{1}{s^{2}}\right),  \tag{50}\\
& p(z)=p\left(1-\frac{\lambda}{s}\right)=e^{\lambda}\left(1-e^{-\lambda}\right)^{L}+\frac{\lambda e^{-\lambda}\left(1-e^{-\lambda}\right)^{L}\left(\lambda+L \lambda-2 L e^{\lambda}-\lambda e^{\lambda}\right)}{2\left(e^{\lambda}-1\right) s}+o\left(\frac{1}{s}\right), \\
& q(z)=q\left(1-\frac{\lambda}{s}\right)=\left(1-e^{-\lambda}\right)^{L}+\frac{\lambda e^{-\lambda}\left(1-e^{-\lambda}\right)^{L}\left(\lambda+L \lambda-2 e^{\lambda}\right)}{2 s}+o\left(\frac{1}{s}\right) .
\end{align*}
$$

Using (50), one can obtain the following asymptotic equalities for the expressions (14),(48)

$$
\begin{align*}
Q_{L}\left(s, 1-\frac{\lambda}{s}\right) & =\frac{\lambda}{s}+\frac{L \lambda^{2}}{\left(e^{\lambda}-1\right) s^{2}}+o\left(\frac{1}{s^{2}}\right) \\
U_{L}\left(s, 1-\frac{\lambda}{s}\right) & =\left(\frac{\lambda}{s}\right)^{s+L-1}\left(1-e^{-\lambda}\right)^{-L}\left(1+o\left(\frac{1}{s}\right)\right),  \tag{51}\\
V_{L}\left(s, 1-\frac{\lambda}{s}\right) & =e^{-\lambda}\left(1+\frac{\lambda-L \lambda-\lambda^{2} / 2}{s}+o\left(\frac{1}{s}\right)\right) .
\end{align*}
$$

Finally, equalities (51) yield the asymptotic behavior of (49) that coincides with (18).
Proof of Lemma 4. Let $s \geq 2$ be fixed and $L \rightarrow \infty$. It is obvious that

$$
\begin{align*}
z_{L}(s, c) & =1+o(1), \quad \text { and hence, } \\
g\left(z_{L}(s, c)\right) & =(s-1)(1+o(1)) . \tag{52}
\end{align*}
$$

The use of definitions (7) and division of upper and lower parts of fractions (14),(48) by $\left(1-z-(1-z)^{s}\right)$ allow us to rewrite expressions $Q, U$ and $V(14),(48)$ in a more convenient form

$$
\begin{align*}
Q_{L}(s, z) & =\frac{(1-z)^{s}+\left(1-z-(1-z)^{s}\right)(1+g(z))^{L}}{(1-z)^{s}+\left(1-z^{s}-(1-z)^{s}\right)(1+g(z))^{L}+z^{s}(g(z))^{L}} \\
U_{L}(s, z) & =\frac{(1-z)^{s}}{(1-z)^{s}+\left(1-z-(1-z)^{s}\right)(1+g(z))^{L}}  \tag{53}\\
V_{L}(s, z) & =\frac{z^{s}(g(z))^{L}}{z^{s}(g(z))^{L}\left(z-z^{s}\right)(1+g(z))^{L}}
\end{align*}
$$

The equalities (52)-(53) imply the following asymptotics

$$
\begin{align*}
Q_{L}\left(s, z_{L}(s, c)\right) & =\frac{1}{s+c}(1+o(1)) \\
U_{L}\left(s, z_{L}(s, c)\right) & =\left(\frac{(s-1)^{s-1}}{s^{s}}\right)^{L}(1+o(1))  \tag{54}\\
V_{L}\left(s, z_{L}(s, c)\right) & =\frac{1}{1+\frac{s}{c}}(1+o(1))
\end{align*}
$$

Next, the substitution (54) into the expression (49) involves (22).
Proof of Lemma 5. To prove the equality sign in (24), let us denote arbitrary sequence of argument of maximum (15) by $z=z_{L}(s), 1 / 2 \leq z_{L}(s)<1$. We will consider some cases and find a contradictions with (24). First, suggest that the sequence $z_{L}(s)$ is bounded by a constant $d<1$, i.e., $1 / 2 \leq z_{L}(s) \leq d<1$. Then due to (53) the asymptotic equalities

$$
\begin{align*}
Q_{L}\left(s, z_{L}(s)\right) & =\frac{1}{1+g(z)}(1+o(1)) \\
U_{L}\left(s, z_{L}(s)\right) & =\frac{(1-z)^{s}}{1-z-(1-z)^{s}} \frac{1}{(1+g(z))^{L}}(1+o(1))  \tag{55}\\
V_{L}\left(s, z_{L}(s)\right) & =\frac{z^{s}}{z-z^{s}}\left(\frac{g(z)}{1+g(z)}\right)^{L}(1+o(1)), \quad L \rightarrow \infty
\end{align*}
$$

hold. However, the computation of asymptotic behavior of $T_{L}\left(s, z_{L}(s)\right)$ (49), using (55), yields $\underline{R}_{L}^{*}(s)=T_{L}\left(s, z_{L}(s)\right) \rightarrow 0$ as $L \rightarrow \infty$. The current case involves the contradiction with (24). Hence, it is clear without less of generality that $z_{L}(s) \rightarrow 1$ ((52) holds).

Further, let us assume that

$$
\begin{equation*}
\left(\frac{g(z)}{1+g(z)}\right)^{L} \frac{1}{1-z} \rightarrow 0, \quad L \rightarrow \infty \tag{56}
\end{equation*}
$$

Then using (52) and (56) one can achive the following asymptotic behaviors of (53)

$$
\begin{align*}
Q_{L}\left(s, z_{L}(s)\right) & =\frac{1}{s}(1+o(1)), \\
U_{L}\left(s, z_{L}(s)\right) & =\frac{(1-z)^{s-1}}{(1+g(z))^{L}}(1+o(1)),  \tag{57}\\
V_{L}\left(s, z_{L}(s)\right) & =\frac{1}{s}\left(\frac{g(z)}{1+g(z)}\right)^{L} \frac{1}{1-z}(1+o(1)), \quad L \rightarrow \infty .
\end{align*}
$$

Nevertheless, the equalities (52) and (57) leads to $\underline{R}_{L}^{*}(s)=T_{L}\left(s, z_{L}(s)\right) \rightarrow 0$ as $L \rightarrow \infty$. So, the current case has the contradiction with (24).

Next, let us assume that

$$
\begin{equation*}
\left(\frac{g(z)}{1+g(z)}\right)^{L} \frac{1}{1-z} \rightarrow \infty, \quad L \rightarrow \infty . \tag{58}
\end{equation*}
$$

The use of (52) and (58) leads to the following asymptotic behavior of (53)

$$
\begin{align*}
Q_{L}\left(s, z_{L}(s)\right) & =\left(\frac{1+g(z)}{g(z)}\right)^{L}(1-z)(1+o(1)) \\
U_{L}\left(s, z_{L}(s)\right) & =\frac{(1-z)^{s-1}}{(1+(z))^{L}}(1+o(1))  \tag{59}\\
V_{L}\left(s, z_{L}(s)\right) & =1+o(1), \quad L \rightarrow \infty
\end{align*}
$$

It is obvious that the equalities (52) and (59) yield

$$
\begin{equation*}
T_{L}\left(s, z_{L}(s)\right)=\frac{Q(s-1)}{s+L-1} \log _{2}[1-z]+o(1) \tag{60}
\end{equation*}
$$

One can see that from the first equality in (59) it follows that

$$
Q=O(1-z) .
$$

Therefore, the asymptotic equality (60) implies $\underline{R}_{L}^{*}(s)=T_{L}\left(s, z_{L}(s)\right) \rightarrow 0$ as $L \rightarrow \infty$. Therefore, the current case has the contradiction with (24).

Without loss of generality we can conclude that

$$
\begin{equation*}
\left(\frac{g(z)}{1+g(z)}\right)^{L}=c(1-z)(1+o(1)) . \tag{61}
\end{equation*}
$$

Note that (61) is similar to (21). Finally, using (52) and (61) one can obtain the equalities (54). And we get the formula (22) again.

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[^0]:    ${ }^{1}$ The adaptive symmetric group testing for the search of binomial sample was considered in [1].

