# On *m*-ovoids of regular near polygons

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ABSTRACT. We generalise the work of Segre (1965), Cameron – Goethals – Seidel (1978), and Vanhove (2011) by showing that nontrivial *m*-ovoids of the dual polar spaces DQ(2d,q), DW(2d-1,q) and  $DH(2d-1,q^2)$  ( $d \ge 3$ ) are hemisystems. We also provide a more general result that holds for regular near polygons.

### 1. Introduction

Near polygons are a large class of point-line incidence geometries that contain the generalised 2d-gons introduced by J. Tits [17], and the dual polar spaces of P. J. Cameron [3]. A near polygon, as defined by E. Shult and A. Yanushka [15], is a point-line geometry such that for every point P and line  $\ell$ , there exists a unique point on  $\ell$  nearest to P. If d is the diameter of the collinearity graph of the near polygon, then we call the near polygon a near 2d-gon. A near 2d-gon is said to be regular if its collinearity graph is distance regular. Generalised 2d-gons are examples of near polygons, and the regular near 4-gons are precisely the finite generalised quadrangles (with an order). However, there exist regular near polygons that are not generalised 2d-gons; for example, every finite dual polar space (of rank at least 3) is an example of a regular near polygon.

An *m*-ovoid of a near 2*d*-gon is a set of points  $\mathcal{O}$  such that every line is incident with exactly *m* points of  $\mathcal{O}$ . The *trivial m*-ovoids are the empty set (m = 0) and the full set of points (m = s + 1; the number of points on a line). For dual polar spaces (that are not generalised quadrangles), the existence of 1-ovoids is mostly resolved, however, in rank 3, it is still not known whether  $\mathsf{DQ}^-(7,q)$  or  $\mathsf{DH}(6,q^2)$  can contain 1-ovoids. It follows from [13, 3.4.1] that there are no 1-ovoids of  $\mathsf{DW}(5,q)$  for q even, and the q odd case was settled by Thomas [16, Theorem 3.2] (see [5] and [8, Appendix] for alternative proofs). De Bruyn and Vanhove reproved this result [9, Corollary 3.14] and extended it to other regular near hexagons by showing that a finite generalised hexagon of order  $(s, s^3)$  with  $s \ge 2$  has no 1-ovoids [9, Corollary 3.19].

Another interesting case arises in the study of *m*-ovoids when *m* is exactly half of the number of points on a line. Such an *m*-ovoid is called a *hemisystem*. In 1965, Segre [14] showed that the only nontrivial *m*-ovoids of  $DH(3, q^2)$ , for *q* odd, are hemisystems. Cameron, Goethals and Seidel [4] extended Segre's result to all generalised quadrangles of order  $(q, q^2)$ , *q* odd. This was then extended further to regular near 2*d*-gons of order (s, t)by Vanhove [18], which also provided a generalisation of the so-called *Higman bound*: if

<sup>2010</sup> Mathematics Subject Classification. 05B25, 51E12, 51E20.

Key words and phrases. regular near polygon, dual polar space, hemisystem.

The first author acknowledges the support of the Australian Research Council (ARC) Future Fellowship FT120100036. The second author acknowledges the support of an Australian Postgraduate Award and a UWA Top-Up Scholarship. The third author acknowledges the support of a Hackett Postgraduate Research Scholarship.

s > 1 then the intersection number  $c_i$  for all  $i \in \{1, \ldots, d\}$  obeys the following inequality,

$$c_i \leqslant \frac{s^{2i} - 1}{s^2 - 1}.$$

Furthermore, if the bound is sharp for some  $c_i$  with  $i \in \{2, \ldots, d\}$  then any nontrivial *m*-ovoid is a hemisystem [18, Theorem 3].

Vanhove showed that for q odd if  $DH(2d-1,q^2)$  has a hemisystem then it induces a distance regular graph with classical parameters [18, Theorem 4]. Hence the question of the existence of hemisystems in  $DH(2d-1,q^2)$  is of great interest. Now, DW(2d-1,q) can be embedded in DH(2d-1,q), and lines in both geometries contain the same number of points. This implies that the intersection of an m-ovoid of DH(2d-1,q) with the points of DW(2d-1,q) is an m-ovoid of DW(2d-1,q). See also [7]. Therefore, the existence of a hemisystem in  $DH(2d-1,q^2)$  implies the existence of a hemisystem in DW(2d-1,q), and the existence question can be reframed for DW(2d-1,q). In this paper, we extend the work of Segre, Cameron – Goethals – Seidel, and Vanhove by showing that the only nontrivial m-ovoids of certain dual polar spaces are hemisystems.

THEOREM 1.1. The only nontrivial m-ovoids that exist in DQ(2d, q), DW(2d - 1, q)and  $DH(2d - 1, q^2)$ , for  $d \ge 3$ , are hemisystems (i.e., m = (q + 1)/2).

Theorem 1.1 follows from a more general, but perhaps more technical result, on m-ovoids of regular near polygons. Our main theorem is:

THEOREM 1.2. Let S be a regular near 2d-gon of order  $(s, t_2, t_3, \ldots, t_{d-1}, t)$  satisfying

$$t_i + 1 = \frac{(s^i + (-1)^i)(t_{i-1} + 1 + (-1)^i s^{i-2})}{s^{i-2} + (-1)^i}$$

for some  $3 \leq i \leq d$ . If a nontrivial m-ovoid of S exists, then it is a hemisystem.

De Bruyn and Vanhove [9, Theorem 3.2] prove that a regular near 2*d*-gon (with  $s, d \ge 2$ ) satisfies

(1) 
$$\frac{(s^{i}-1)(t_{i-1}+1-s^{i-2})}{s^{i-2}-1} \leqslant t_{i}+1 \leqslant \frac{(s^{i}+1)(t_{i-1}+1+s^{i-2})}{s^{i-2}+1}$$

for all  $i \in \{3, \ldots, d\}$ , and that a finite regular near 2*d*-gon with  $s \ge 2$  and  $d \ge 3$  which attains the lower bound for i = 3 is isomorphic to  $\mathsf{DQ}(2d, s)$ ,  $\mathsf{DW}(2d - 1, s)$  or  $\mathsf{DH}(2d - 1, s^2)$ , where s is a prime power [9, Theorem 3.5]. Note that the hypothesis of Theorem 1.2 is valid when the the upper bound is met for *i* even, or when the lower bound is met for *i* odd, in the De Bruyn–Vanhove bounds (1). Theorem 1.1 follows directly from [9, Theorem 3.5] and Theorem 1.2.

### 2. Background

This section contains information on some of the key facts about regular near 2d-gons and *m*-ovoids, which will be useful later in the paper. For greater depth, we refer the reader to Brouwer, Cohen and Neumaier's book [2].

Let  $\Gamma$  be a connected, undirected graph without loops. The *distance* between two vertices x and y, denoted d(x, y), is the shortest path length from x to y, and the maximum distance between any two given points is the *diameter* d of  $\Gamma$ . The set of all vertices at distance i from x is denoted by  $\Gamma_i(x)$ . A graph  $\Gamma$  of diameter d is said to be *distance* regular if there exist numbers  $b_i$  for  $i \in \{0, \ldots, d-1\}$  and  $c_i$  for  $i \in \{1, \ldots, d\}$  such that  $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$  and  $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$  for all x and y at distance i in  $\Gamma$ . If a graph is distance regular then there also exist constants  $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$  for all x and y at distance i with  $i \in \{1, \ldots, d-1\}$ . We call  $a_i$ ,  $b_i$  and  $c_i$  the intersection numbers of  $\Gamma$ .

Given x and y at distance l, there are  $p_{i,j}^l$  vertices that are at distance i from x and distance j from y. Furthermore,  $p_{1,i}^{i-1} = b_{i-1}$ ,  $p_{1,i}^i = a_i$ ,  $p_{1,i}^{i+1} = c_{i+1}$  and  $p_{i,j}^l = p_{j,i}^l$ . Therefore, combining [2, Lemma 4.1.7] and [2, §4.1 (10)], we may calculate  $p_{i+1,j}^l$ recursively using the following formula.

$$p_{i+1,j}^{l} = \frac{p_{i,j}^{l-1}c_l + p_{i,j}^{l}a_l + p_{i,j}^{l+1}b_l - p_{i-1,j}^{l}b_{i-1} - p_{i,j}^{l}a_i}{c_{i+1}}.$$

We also define the *i*-distance valencies of the graph,  $k_i := p_{i,i}^0$  for  $i \in \{0, 1, \ldots, d\}$  (and so  $k_1 = s(t+1)$ ).

Given a graph  $\Gamma$  of diameter d, for any distance i, the adjacency matrix  $A_i$  is the matrix indexed by the vertices of  $\Gamma$ , with entries

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } d(x,y) = i \\ 0 & otherwise. \end{cases}$$

The set of adjacency matrices  $\{A_0, A_1, \ldots, A_d\}$  forms a basis for the Bose-Mesner algebra for  $\Gamma$  which also has a unique basis of minimal idempotents  $\{E_0, E_1, \ldots, E_d\}$  (see [2, §2.6]). As a result, the Bose-Mesner algebra can be decomposed into mutually orthogonal subspaces corresponding to the image of each minimal idempotent. By convention,  $E_0$  has rank 1, that is,  $E_0 = \frac{1}{n}J$  where J is the 'all ones' matrix and n is the number of vertices of  $\Gamma$ . The dual degree set of a vector v is the set of indices of the minimal idempotents  $E_i$  such that  $vE_i \neq 0$  and  $i \neq 0$ . Two vectors are called design-orthogonal when their dual degree sets are disjoint. The following lemma about design-orthogonal vectors will be useful in the proof of the main theorem. It can be found in [10, Theorem 6.7], and is given here with a proof for completeness.

LEMMA 2.1. If f and g are design-orthogonal vectors, then  $f \cdot g = \frac{(f \cdot 1)(g \cdot 1)}{n}$ , where 1 is the 'all-ones' vector.

PROOF. Let  $\alpha = \frac{f \cdot \mathbb{1}}{\mathbb{1} \cdot \mathbb{1}}$  and  $\beta = \frac{g \cdot \mathbb{1}}{\mathbb{1} \cdot \mathbb{1}}$ . So  $(f - \alpha \mathbb{1}) \cdot \mathbb{1} = 0$  and  $(g - \beta \mathbb{1}) \cdot \mathbb{1} = 0$ . Since f and g are design-orthogonal,  $(f - \alpha \mathbb{1})$  and  $(g - \beta \mathbb{1})$  belong to a pair of direct sums of eigenspaces that intersect trivially and hence  $(f - \alpha \mathbb{1}) \cdot (g - \beta \mathbb{1}) = 0$ . Thus

$$f \cdot g = \alpha(g \cdot 1) + \beta(f \cdot 1) - \alpha\beta 1 \cdot 1$$
  
=  $\frac{(f \cdot 1)(g \cdot 1)}{1 \cdot 1} + \frac{(f \cdot 1)(g \cdot 1)}{1 \cdot 1} - \frac{(f \cdot 1)(g \cdot 1)}{1 \cdot 1}$   
=  $\frac{(f \cdot 1)(g \cdot 1)}{n}$ .

A near polygon, or near 2d-gon  $(d \ge 2)$  is an incidence geometry such that

- (1) every two points lie on at most one line,
- (2) any two points are at most at distance d in the collinearity graph, and
- (3) given a line  $\ell$  and a point P there is a unique point Q on  $\ell$  which is nearest to P with respect to distance in the collinearity graph.

A near polygon that has t + 1 lines on each point and s + 1 points on each line is said to have order (s,t). If in a near polygon of order (s,t) there also exist constants  $t_i$  for  $i \in \{0, \ldots, d\}$  such that there are  $t_i + 1$  lines on y containing a point at distance i - 1from x whenever two points x and y are at distance i, then such a near polygon is called regular, with parameters  $(s, t_2, t_3, \ldots, t_{d-1}, t)$ . Examples of regular near 2d-gons include the finite dual polar spaces; the point-line geometries obtained by taking the maximal totally isotropic subspaces of a finite polar space for the points, and the next-to-maximal subspaces for the lines. We refer the reader to [6, §1.9.5] for more on the definition of a dual polar space. In this paper, we will only be concerned with  $\mathsf{DW}(2d-1,s)$ ,  $\mathsf{DQ}(2d,s)$ , and  $\mathsf{DH}(2d-1,s^2)$ .

The finite regular near polygons are exactly the near polygons with distance regular collinearity graphs. Moreover, for all  $i \in \{0, \ldots, d\}$ 

$$a_i = (s-1)(t_i+1), \quad b_i = s(t-t_i), \quad c_i = t_i+1.$$

By definition,  $t_0 = -1$  and  $t_1 = 0$ . In a regular near 2*d*-gon with parameters  $(s, t_2, t_3, \ldots, t_{d-1}, d)$ , we have the following relations.

LEMMA 2.2. [2, §4.1 (7); (9); (1c)]  $k_i = k_{i-1} \frac{b_{i-1}}{c_i} = sk_{i-1} \frac{t - t_{i-1}}{t_i + 1} \quad (1 \le i \le d),$ 

$$p_{i,j}^{l}k_{l} = p_{l,j}^{i}k_{i} \quad (0 \leq i, j, \ell, d),$$
  
$$p_{1,i}^{i+1} = c_{i+1} \quad (1 \leq i \leq d).$$

Lemma 2.2 gives the following corollary.

COROLLARY 2.3. Let  $1 \leq i \leq d$ . Then

$$p_{i,i-1}^1 = \frac{k_i c_i}{k_1} = \frac{k_i (t_i + 1)}{s(t+1)}$$

The following lemma follows directly from the definition of an *m*-ovoid and the fact that there are s + 1 points on every line of a finite regular near 2*d*-gon S with parameters  $(s, t_2, \ldots, t_{d-1}, t)$ .

LEMMA 2.4. The complement of an m-ovoid of S is a (s+1-m)-ovoid.

LEMMA 2.5 ([18, Lemma 5]). If  $\mathcal{O}$  is an *m*-ovoid of  $\mathcal{S}$ , then for every  $i \in \{0, 1, \ldots, d\}$ and  $x \in \mathcal{O}$ ,

$$|\Gamma_i(x) \cap \mathcal{O}| = k_i \left(\frac{m}{s+1} + \left(-\frac{1}{s}\right)^i \left(1 - \frac{m}{s+1}\right)\right).$$

By Lemmas 2.4 and 2.5, we have the following:

COROLLARY 2.6. If  $\mathcal{O}$  is an *m*-ovoid of  $\mathcal{S}$ , then for every  $i \in \{0, 1, \ldots, d\}$  and  $x \notin \mathcal{O}$ ,

$$|\Gamma_i(x) \cap \mathcal{O}| = k_i \frac{m}{s+1} \left( 1 - \left(\frac{-1}{s}\right)^i \right).$$

## 3. Proof of the main result

We now prove Theorem 1.2. Recall that we are assuming that

$$c_i = t_i + 1 = \frac{(s^i + (-1)^i)(c_{i-1} + (-1)^i s^{i-2})}{s^{i-2} + (-1)^i}$$

for some  $3 \leq i \leq d$ .

PROOF. Let  $\mathcal{O}$  be a nontrivial *m*-ovoid of  $\mathcal{S}$ . Throughout this proof, we will let  $\chi_{\mathcal{O}}$  denote the *characteristic vector* of  $\mathcal{O}$  with respect to the set of points  $\mathcal{P}$ :

$$(\chi_{\mathcal{O}})_y = \begin{cases} 1 & \text{if } y \in \mathcal{O} \\ 0 & otherwise. \end{cases}$$

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A simple double counting argument shows that  $|\mathcal{O}|$  is equal to  $m|\mathcal{L}|/(t+1)$  where  $\mathcal{L}$  is the set of lines of  $\mathcal{S}$ . If we also count flags (i.e., point-line incident pairs), then  $|\mathcal{P}|(t+1) = |\mathcal{L}|(s+1)$  where  $\mathcal{P}$  is the set of points of  $\mathcal{S}$ , and hence

$$|\mathcal{O}| = \frac{mn}{s+1},$$

where  $n = |\mathcal{P}|$ .

Recall that  $3 \leq i \leq d$ . Now we fix an element  $x \notin \mathcal{O}$  and count pairs (y, z) of elements of  $\mathcal{O}$  such that d(x, y) = i and either d(y, z) = i - 1 and d(x, z) = 1, or d(y, z) = 1 and d(x, z) = i - 1.

Let x and y be two points at distance i. Define  $v_{x,y}$  as in [9, Theorem 3.2(b)],

$$v_{x,y} := s(c_{i-1} + (-1)^i s^{i-2})(\chi_x + \chi_y) + \chi_{\Gamma_1(x) \cap \Gamma_{i-1}(y)} + \chi_{\Gamma_{i-1}(x) \cap \Gamma_1(y)}.$$

Note that

$$v_{x,y} \cdot \mathbb{1} = 2(s(c_{i-1} + (-1)^i s^{i-2}) + p_{1,i-1}^i) = 2(s(c_{i-1} + (-1)^i s^{i-2}) + c_i)$$

and furthermore that  $v_{x,y}$  and  $\chi_{\mathcal{O}}$  are design-orthogonal [9, Theorem 3.2] and hence by Lemma 2.1,

$$\mu := v_{x,y} \cdot \chi_{\mathcal{O}} = 2(s(c_{i-1} + (-1)^i s^{i-2}) + c_i)m/(s+1).$$

Let  $\Gamma$  be the collinearity graph of S.

Counting first y and then z, the number of pairs is

$$\sum_{y \in \mathcal{O} \cap \Gamma_{i}(x)} (|\Gamma_{1}(x) \cap \Gamma_{i-1}(y) \cap \mathcal{O}| + |\Gamma_{i-1}(x) \cap \Gamma_{1}(y) \cap \mathcal{O}|)$$

$$= \sum_{y \in \mathcal{O} \cap \Gamma_{i}(x)} (v_{x,y} - s(c_{i-1} + (-1)^{i}s^{i-2})(\chi_{x} + \chi_{y})) \cdot \chi_{\mathcal{O}}$$

$$= |\mathcal{O} \cap \Gamma_{i}(x)|(\mu - s(c_{i-1} + (-1)^{i}s^{i-2}))$$

$$= |\mathcal{O} \cap \Gamma_{i}(x)| \left(\frac{2(s(c_{i-1} + (-1)^{i}s^{i-2}) + c_{i})m}{s+1} - s(c_{i-1} + (-1)^{i}s^{i-2})\right)$$

$$= |\mathcal{O} \cap \Gamma_{i}(x)| \left(\frac{2c_{i}ms - (s+1-2m)(c_{i-1}s^{2} + (-1)^{i}s^{i})}{s(s+1)}\right).$$

Now, by Corollary 2.6 and Lemma 2.2,

$$|\mathcal{O} \cap \Gamma_i(x)|c_i = k_i c_i \frac{m}{s+1} \left( 1 - \left(-\frac{1}{s}\right)^i \right) = sk_{i-1}(t-t_{i-1}) \frac{m}{s+1} \left( 1 - \left(-\frac{1}{s}\right)^i \right)$$

and hence the number of pairs (y, z) is

(2) 
$$\frac{mk_{i-1}(t-t_{i-1})}{s+1} \left(1 - \left(-\frac{1}{s}\right)^i\right) \frac{2c_i ms - (s+1-2m)\left(c_{i-1}s^2 + (-1)^i s^i\right)}{c_i(s+1)}.$$

Now we consider the pairs the opposite way, namely counting z then y. The number of pairs (z, y) is equal to

$$\sum_{z \in \mathcal{O} \cap \Gamma_1(x)} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| + \sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)} |\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}|$$

Suppose d(z, x) = i - 1. There are t + 1 lines on z, and the set of points incident with these lines, other than the point z itself, form  $\Gamma_1(z)$ . There are  $t_{i-1} + 1$  lines on z incident with a unique point at distance i - 2 from x, and the remaining points on these lines are at distance i - 1 from x. Moreover, z is the unique nearest point to x with distance i - 1 for the remaining  $t - t_{i-1}$  lines, and hence any other point on these lines must have

distance i from x. Since z is in  $\mathcal{O}$ , there are m-1 additional points of  $\mathcal{O}$  on each such line. Therefore,

$$|\Gamma_1(z) \cap \Gamma_i(x) \cap \mathcal{O}| = (t - t_{i-1})(m - 1).$$

Now suppose d(z, x) = 1. We will compute  $|\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}|$ . Take  $z \in \mathcal{O} \cap \Gamma_1(x)$ and consider a point  $w \in \Gamma_{i-2}(z) \cap \Gamma_{i-1}(x)$ . Note that any point y is collinear with some such point w, giving rise to the following equation:

$$(3) \sum_{y \in \Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}} |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \Gamma_{1}(y)| = \sum_{w \in \Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}} |\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O} \cap \Gamma_{1}(w)| + \sum_{w \in \Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}^{c}} |\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O} \cap \Gamma_{1}(w)|$$

where  $\mathcal{O}^c$  is the complement of  $\mathcal{O}$  within the set of points of  $\mathcal{S}$ .

Let  $\ell$  be a line through w. There is a point on  $\ell$  which is the unique closest point to x. If this point is w, then every other point must be at distance i from x. If this point is not w, then it must be distance i - 2 from x, and every other point on  $\ell$  is distance i - 1 from x. There are  $t_{i-1} + 1$  lines on w with a unique point at distance i - 2 from x. Hence there are  $t - t_{i-1}$  lines  $\ell'$  for which w is the unique nearest point to x and every other point on  $\ell'$  is at distance i from x. Moreover, note that if a point y is at distance i from x, then it cannot be distance i - 2 from z, since d(x, z) = 1, and thus any point other than w on any line  $\ell'$  is in  $\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \Gamma_1(w)$ . There are m - 1 such points in  $\mathcal{O}$  when  $w \in \mathcal{O}$ , otherwise there are m such points in  $\mathcal{O}$ .

There are  $t_{i-1}$  lines on any point y which have a unique point at distance i-2 from z, and hence also at distance i-1 from x. Recalling that  $|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x)| = p_{i-1,i-2}^1$ , our Equation (3) becomes:

$$\begin{aligned} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}|(t_{i-1}+1) &= |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|(t-t_{i-1})(m-1) \\ &+ |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}^c|(t-t_{i-1})m \\ &= |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|(t-t_{i-1})(m-1) \\ &+ (p_{i-1,i-2}^1 - |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|)(t-t_{i-1})m \\ &= p_{i-1,i-2}^1(t-t_{i-1})m - |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|(t-t_{i-1}). \end{aligned}$$

Hence we obtain an iterative formula,

$$|\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| = p_{i-1,i-2}^1 \frac{t - t_{i-1}}{t_{i-1} + 1} m - \frac{t - t_{i-1}}{t_{i-1} + 1} |\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}|,$$

which, with the help of Lemma 2.2 and Corollary 2.3, we can write as a recurrence relation

$$sf_i = m - f_{i-1}, \quad f_1 = 1$$

where  $f_i := \frac{1}{p_{i,i-1}^1} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}|$  for all  $i \ge 1$ . (Note:  $|\Gamma_0(z) \cap \Gamma_1(x) \cap \mathcal{O}| = 1$  and  $p_{1,0}^1 = 1$ ). Therefore, by the elementary theory of recurrence relations, we have

$$f_{i} = \frac{m - s\left(-\frac{1}{s}\right)^{i}\left(-m + s + 1\right)}{s + 1}$$

for all  $i \ge 1$ . Hence, by Corollary 2.3,

$$\begin{aligned} |\Gamma_{i-1}(z) \cap \Gamma_i(x) \cap \mathcal{O}| &= p_{i,i-1}^1 \left( \frac{m - s \left( -\frac{1}{s} \right)^i (-m + s + 1)}{s + 1} \right) \\ &= \frac{k_{i-1}(t - t_{i-1})}{t + 1} \left( \frac{m - s \left( -\frac{1}{s} \right)^i (-m + s + 1)}{s + 1} \right) \\ &= \frac{k_{i-1}(t - t_{i-1})}{s^{i-1}(t + 1)} \left( \frac{m}{s + 1} \left( s^{i-1} + (-1)^{i-2} \right) + (-1)^{i-1} \right). \end{aligned}$$

Now, making use of Corollary 2.6, we sum our two terms together:

$$\sum_{z \in \mathcal{O} \cap \Gamma_{1}(x)} |\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}| + \sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)} |\Gamma_{1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}|$$
  
= $|\mathcal{O} \cap \Gamma_{1}(x)| |\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}| + |\mathcal{O} \cap \Gamma_{i-1}(x)| |\Gamma_{1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}|$   
= $s(t+1)\frac{m}{s+1}\left(1+\frac{1}{s}\right)\frac{k_{i-1}(t-t_{i-1})}{s^{i-1}(t+1)}\left(\frac{m}{s+1}\left(s^{i-1}+(-1)^{i-2}\right)+(-1)^{i-1}\right)$   
 $+k_{i-1}\frac{m}{s+1}\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)(t-t_{i-1})(m-1)$   
= $\frac{mk_{i-1}(t-t_{i-1})}{s+1}\left(m\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)+\left(-\frac{1}{s}\right)^{i-1}(s+1)+(m-1)\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)\right)$ 

and therefore, the number of pairs (z, y) is

(4) 
$$\frac{mk_{i-1}(t-t_{i-1})}{s+1} \left(2m-1+\left(\frac{-1}{s}\right)^{i-1}(s-2m+2)\right).$$

Equating the two counts, (2) and (4) yields

$$\left(1 - \left(-\frac{1}{s}\right)^{i}\right) \frac{2c_{i}ms - (s+1-2m)\left(c_{i-1}s^{2} + (-1)^{i}s^{i}\right)}{c_{i}(s+1)}$$
$$= 2m - 1 + \left(\frac{-1}{s}\right)^{i-1}(s-2m+2).$$

Taking the difference of each side of the above equation and factoring gives

$$\frac{s^{-i}(s+1-2m)\left(c_i\left(s^i+(-1)^i(s+2)s\right)+\left((-1)^i-s^i\right)\left(c_{i-1}s^2+(-1)^is^i\right)\right)}{c_i(s+1)} = 0$$

and hence

(5) 
$$(s+1-2m)\left(c_i\left(s^i+(-1)^i(s+2)s\right)+\left((-1)^i-s^i\right)\left(c_{i-1}s^2+(-1)^is^i\right)\right)=0$$

Now by assumption,

$$c_{i-1}s^{2} + (-1)^{i}s^{i} = c_{i}\frac{s^{i} + (-1)^{i}s^{2}}{s^{i} + (-1)^{i}}$$

and hence

$$c_{i} \left(s^{i} + (-1)^{i}(s+2)s\right) + \left((-1)^{i} - s^{i}\right) \left(c_{i-1}s^{2} + (-1)^{i}s^{i}\right)$$
  
=  $c_{i} \left(s^{i} + (-1)^{i}(s+2)s + \left((-1)^{i} - s^{i}\right)\frac{s^{i} + (-1)^{i}s^{2}}{s^{i} + (-1)^{i}}\right)$   
=  $c_{i}\frac{2(-1)^{i}(s+1) \left(s^{i} + (-1)^{i}s\right)}{s^{i} + (-1)^{i}}.$ 

Since i > 1, we have  $s^i + (-1)^i s \neq 0$ , and therefore, Equation (5) becomes m = (s + 1)/2.

## 4. Further results and computation

Theorem 1.1 leaves open the natural question of whether there exist hemisystems of DQ(6,q), DW(5,q) and  $DH(5,q^2)$ . Firstly, De Bruyn and Vanhove announced in conference presentations that there are no hemisystems of DW(5,3), and that there is a unique example for DQ(6,3). We thank the referee and Michel Lavrauw for mentioning these results to us. For small values of (odd) q, we have found examples for DQ(6,q), and we have listed the known examples in Table 1. In particular, we could show by using the computer algebra system GAP [11], a package FINING [1], and the mixed-integer programming software GUROBI [12] that there is a unique example up to equivalence in DQ(6,3). For DQ(6,5), there were numerous examples found admitting an element of order 5 or 9, but we were unable to enumerate them all.

q	Stabiliser	Number up to equivalence
3	$2 \times A_5$	1
5	$D_{60}$	4
	$D_{20}$	16

TABLE 1. Some known examples of hemisystems of DQ(6, q), for small q.

For  $\mathsf{DW}(5,q)$ , it seems the situation is different, despite its combinatorial parameters being identical to those of  $\mathsf{DQ}(6,q)$ . By computer, we showed that there are no hemisystems of  $\mathsf{DW}(5,q)$  for  $q \in \{3,5\}$ . We make the following conjectures:

CONJECTURE 4.1. There are no hemisystems of DW(5,q), for all prime powers q.

If true, this would also imply that there are no hemisystems of  $DH(5, q^2)$ , for all prime powers q, answering a problem posed by Vanhove [19, Appendix B, Problem 7].

CONJECTURE 4.2. For each odd prime power q, there exists a hemisystem of DQ(6,q).

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