# On $m$-ovoids of regular near polygons 

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#### Abstract

We generalise the work of Segre (1965), Cameron - Goethals - Seidel (1978), and Vanhove (2011) by showing that nontrivial $m$-ovoids of the dual polar spaces $\operatorname{DQ}(2 d, q), \operatorname{DW}(2 d-1, q)$ and $\operatorname{DH}\left(2 d-1, q^{2}\right)(d \geqslant 3)$ are hemisystems. We also provide a more general result that holds for regular near polygons.


## 1. Introduction

Near polygons are a large class of point-line incidence geometries that contain the generalised $2 d$-gons introduced by J. Tits [17], and the dual polar spaces of P. J. Cameron [3]. A near polygon, as defined by E. Shult and A. Yanushka [15], is a point-line geometry such that for every point $P$ and line $\ell$, there exists a unique point on $\ell$ nearest to $P$. If $d$ is the diameter of the collinearity graph of the near polygon, then we call the near polygon a near $2 d$-gon. A near $2 d$-gon is said to be regular if its collinearity graph is distance regular. Generalised $2 d$-gons are examples of near polygons, and the regular near 4 -gons are precisely the finite generalised quadrangles (with an order). However, there exist regular near polygons that are not generalised $2 d$-gons; for example, every finite dual polar space (of rank at least 3) is an example of a regular near polygon.

An $m$-ovoid of a near $2 d$-gon is a set of points $\mathcal{O}$ such that every line is incident with exactly $m$ points of $\mathcal{O}$. The trivial $m$-ovoids are the empty set $(m=0)$ and the full set of points ( $m=s+1$; the number of points on a line). For dual polar spaces (that are not generalised quadrangles), the existence of 1 -ovoids is mostly resolved, however, in rank 3 , it is still not known whether $\mathrm{DQ}^{-}(7, q)$ or $\mathrm{DH}\left(6, q^{2}\right)$ can contain 1-ovoids. It follows from $[13,3.4 .1]$ that there are no 1 -ovoids of $\operatorname{DW}(5, q)$ for $q$ even, and the $q$ odd case was settled by Thomas [16, Theorem 3.2] (see [5] and [8, Appendix] for alternative proofs). De Bruyn and Vanhove reproved this result [9, Corollary 3.14] and extended it to other regular near hexagons by showing that a finite generalised hexagon of order $\left(s, s^{3}\right)$ with $s \geqslant 2$ has no 1 -ovoids [9, Corollary 3.19].

Another interesting case arises in the study of $m$-ovoids when $m$ is exactly half of the number of points on a line. Such an m-ovoid is called a hemisystem. In 1965, Segre [14] showed that the only nontrivial $m$-ovoids of $\mathrm{DH}\left(3, q^{2}\right)$, for $q$ odd, are hemisystems. Cameron, Goethals and Seidel [4] extended Segre's result to all generalised quadrangles of order $\left(q, q^{2}\right), q$ odd. This was then extended further to regular near $2 d$-gons of order $(s, t)$ by Vanhove [18], which also provided a generalisation of the so-called Higman bound: if

[^0]$s>1$ then the intersection number $c_{i}$ for all $i \in\{1, \ldots, d\}$ obeys the following inequality,
$$
c_{i} \leqslant \frac{s^{2 i}-1}{s^{2}-1}
$$

Furthermore, if the bound is sharp for some $c_{i}$ with $i \in\{2, \ldots, d\}$ then any nontrivial $m$-ovoid is a hemisystem [18, Theorem 3].

Vanhove showed that for $q$ odd if $\mathrm{DH}\left(2 d-1, q^{2}\right)$ has a hemisystem then it induces a distance regular graph with classical parameters [18, Theorem 4]. Hence the question of the existence of hemisystems in $\mathrm{DH}\left(2 d-1, q^{2}\right)$ is of great interest. Now, $\operatorname{DW}(2 d-1, q)$ can be embedded in $\mathrm{DH}(2 d-1, q)$, and lines in both geometries contain the same number of points. This implies that the intersection of an $m$-ovoid of $\mathrm{DH}(2 d-1, q)$ with the points of $\operatorname{DW}(2 d-1, q)$ is an $m$-ovoid of $\operatorname{DW}(2 d-1, q)$. See also $[7]$. Therefore, the existence of a hemisystem in $\operatorname{DH}\left(2 d-1, q^{2}\right)$ implies the existence of a hemisystem in $\operatorname{DW}(2 d-1, q)$, and the existence question can be reframed for $\operatorname{DW}(2 d-1, q)$. In this paper, we extend the work of Segre, Cameron - Goethals - Seidel, and Vanhove by showing that the only nontrivial $m$-ovoids of certain dual polar spaces are hemisystems.

Theorem 1.1. The only nontrivial m-ovoids that exist in $\operatorname{DQ}(2 d, q)$, $\operatorname{DW}(2 d-1, q)$ and $\mathrm{DH}\left(2 d-1, q^{2}\right)$, for $d \geqslant 3$, are hemisystems (i.e., $\left.m=(q+1) / 2\right)$.

Theorem 1.1 follows from a more general, but perhaps more technical result, on $m$ ovoids of regular near polygons. Our main theorem is:

Theorem 1.2. Let $\mathcal{S}$ be a regular near $2 d$-gon of order $\left(s, t_{2}, t_{3}, \ldots, t_{d-1}, t\right)$ satisfying

$$
t_{i}+1=\frac{\left(s^{i}+(-1)^{i}\right)\left(t_{i-1}+1+(-1)^{i} s^{i-2}\right)}{s^{i-2}+(-1)^{i}}
$$

for some $3 \leqslant i \leqslant d$. If a nontrivial $m$-ovoid of $\mathcal{S}$ exists, then it is a hemisystem.
De Bruyn and Vanhove [9, Theorem 3.2] prove that a regular near $2 d$-gon (with $s, d \geqslant 2$ ) satisfies

$$
\begin{equation*}
\frac{\left(s^{i}-1\right)\left(t_{i-1}+1-s^{i-2}\right)}{s^{i-2}-1} \leqslant t_{i}+1 \leqslant \frac{\left(s^{i}+1\right)\left(t_{i-1}+1+s^{i-2}\right)}{s^{i-2}+1} \tag{1}
\end{equation*}
$$

for all $i \in\{3, \ldots, d\}$, and that a finite regular near $2 d$-gon with $s \geqslant 2$ and $d \geqslant 3$ which attains the lower bound for $i=3$ is isomorphic to $\mathrm{DQ}(2 d, s)$, $\mathrm{DW}(2 d-1, s)$ or $\mathrm{DH}\left(2 d-1, s^{2}\right)$, where $s$ is a prime power [9, Theorem 3.5]. Note that the hypothesis of Theorem 1.2 is valid when the the upper bound is met for $i$ even, or when the lower bound is met for $i$ odd, in the De Bruyn-Vanhove bounds (1). Theorem 1.1 follows directly from [9, Theorem 3.5] and Theorem 1.2.

## 2. Background

This section contains information on some of the key facts about regular near $2 d$-gons and $m$-ovoids, which will be useful later in the paper. For greater depth, we refer the reader to Brouwer, Cohen and Neumaier's book [2].

Let $\Gamma$ be a connected, undirected graph without loops. The distance between two vertices $x$ and $y$, denoted $\mathrm{d}(x, y)$, is the shortest path length from $x$ to $y$, and the maximum distance between any two given points is the diameter $d$ of $\Gamma$. The set of all vertices at distance $i$ from $x$ is denoted by $\Gamma_{i}(x)$. A graph $\Gamma$ of diameter $d$ is said to be distance regular if there exist numbers $b_{i}$ for $i \in\{0, \ldots, d-1\}$ and $c_{i}$ for $i \in\{1, \ldots, d\}$ such that $b_{i}=\left|\Gamma_{i+1}(x) \cap \Gamma_{1}(y)\right|$ and $c_{i}=\left|\Gamma_{i-1}(x) \cap \Gamma_{1}(y)\right|$ for all $x$ and $y$ at distance $i$ in $\Gamma$. If a graph is distance regular then there also exist constants $a_{i}=\left|\Gamma_{i}(x) \cap \Gamma_{1}(y)\right|$ for all $x$ and
$y$ at distance $i$ with $i \in\{1, \ldots, d-1\}$. We call $a_{i}, b_{i}$ and $c_{i}$ the intersection numbers of $\Gamma$.

Given $x$ and $y$ at distance $l$, there are $p_{i, j}^{l}$ vertices that are at distance $i$ from $x$ and distance $j$ from $y$. Furthermore, $p_{1, i}^{i-1}=b_{i-1}, p_{1, i}^{i}=a_{i}, p_{1, i}^{i+1}=c_{i+1}$ and $p_{i, j}^{l}=$ $p_{j, i}^{l}$. Therefore, combining [2, Lemma 4.1.7] and [2, §4.1 (10)], we may calculate $p_{i+1, j}^{l}$ recursively using the following formula.

$$
p_{i+1, j}^{l}=\frac{p_{i, j}^{l-1} c_{l}+p_{i, j}^{l} a_{l}+p_{i, j}^{l+1} b_{l}-p_{i-1, j}^{l} b_{i-1}-p_{i, j}^{l} a_{i}}{c_{i+1}}
$$

We also define the $i$-distance valencies of the graph, $k_{i}:=p_{i, i}^{0}$ for $i \in\{0,1, \ldots, d\}$ (and so $k_{1}=s(t+1)$ ).

Given a graph $\Gamma$ of diameter $d$, for any distance $i$, the adjacency matrix $A_{i}$ is the matrix indexed by the vertices of $\Gamma$, with entries

$$
\left(A_{i}\right)_{x y}= \begin{cases}1 & \text { if } \mathrm{d}(x, y)=i \\ 0 & \text { otherwise }\end{cases}
$$

The set of adjacency matrices $\left\{A_{0}, A_{1}, \ldots, A_{d}\right\}$ forms a basis for the Bose-Mesner algebra for $\Gamma$ which also has a unique basis of minimal idempotents $\left\{E_{0}, E_{1}, \ldots, E_{d}\right\}$ (see [2, $\S 2.6]$ ). As a result, the Bose-Mesner algebra can be decomposed into mutually orthogonal subspaces corresponding to the image of each minimal idempotent. By convention, $E_{0}$ has rank 1, that is, $E_{0}=\frac{1}{n} J$ where $J$ is the 'all ones' matrix and $n$ is the number of vertices of $\Gamma$. The dual degree set of a vector $v$ is the set of indices of the minimal idempotents $E_{i}$ such that $v E_{i} \neq 0$ and $i \neq 0$. Two vectors are called design-orthogonal when their dual degree sets are disjoint. The following lemma about design-orthogonal vectors will be useful in the proof of the main theorem. It can be found in [10, Theorem 6.7], and is given here with a proof for completeness.

Lemma 2.1. If $f$ and $g$ are design-orthogonal vectors, then $f \cdot g=\frac{(f \cdot 1)(g \cdot 1)}{n}$, where $\mathbb{1}$ is the 'all-ones' vector.

Proof. Let $\alpha=\frac{f \cdot \mathbb{1}}{1 \cdot 1}$ and $\beta=\frac{g \cdot \mathbb{1}}{1 \cdot \mathbb{1}}$. So $(f-\alpha \mathbb{1}) \cdot \mathbb{1}=0$ and $(g-\beta \mathbb{1}) \cdot \mathbb{1}=0$. Since $f$ and $g$ are design-orthogonal, $(f-\alpha \mathbb{1})$ and $(g-\beta \mathbb{1})$ belong to a pair of direct sums of eigenspaces that intersect trivially and hence $(f-\alpha \mathbb{1}) \cdot(g-\beta \mathbb{1})=0$. Thus

$$
\begin{aligned}
f \cdot g & =\alpha(g \cdot \mathbb{1})+\beta(f \cdot \mathbb{1})-\alpha \beta \mathbb{1} \cdot \mathbb{1} \\
& =\frac{(f \cdot \mathbb{1})(g \cdot \mathbb{1})}{\mathbb{1} \cdot \mathbb{1}}+\frac{(f \cdot \mathbb{1})(g \cdot \mathbb{1})}{\mathbb{1} \cdot \mathbb{1}}-\frac{(f \cdot \mathbb{1})(g \cdot \mathbb{1})}{\mathbb{1} \cdot \mathbb{1}} \\
& =\frac{(f \cdot \mathbb{1})(g \cdot \mathbb{1})}{n} .
\end{aligned}
$$

A near polygon, or near $2 d$-gon $(d \geqslant 2)$ is an incidence geometry such that
(1) every two points lie on at most one line,
(2) any two points are at most at distance $d$ in the collinearity graph, and
(3) given a line $\ell$ and a point $P$ there is a unique point $Q$ on $\ell$ which is nearest to $P$ with respect to distance in the collinearity graph.
A near polygon that has $t+1$ lines on each point and $s+1$ points on each line is said to have order $(s, t)$. If in a near polygon of order $(s, t)$ there also exist constants $t_{i}$ for $i \in\{0, \ldots, d\}$ such that there are $t_{i}+1$ lines on $y$ containing a point at distance $i-1$ from $x$ whenever two points $x$ and $y$ are at distance $i$, then such a near polygon is called regular, with parameters $\left(s, t_{2}, t_{3}, \ldots, t_{d-1}, t\right)$. Examples of regular near $2 d$-gons include the finite dual polar spaces; the point-line geometries obtained by taking the maximal
totally isotropic subspaces of a finite polar space for the points, and the next-to-maximal subspaces for the lines. We refer the reader to $[6, \S 1.9 .5]$ for more on the definition of a dual polar space. In this paper, we will only be concerned with $\operatorname{DW}(2 d-1, s), \mathrm{DQ}(2 d, s)$, and $\mathrm{DH}\left(2 d-1, s^{2}\right)$.

The finite regular near polygons are exactly the near polygons with distance regular collinearity graphs. Moreover, for all $i \in\{0, \ldots, d\}$

$$
a_{i}=(s-1)\left(t_{i}+1\right), \quad b_{i}=s\left(t-t_{i}\right), \quad c_{i}=t_{i}+1
$$

By definition, $t_{0}=-1$ and $t_{1}=0$. In a regular near $2 d$-gon with parameters $\left(s, t_{2}, t_{3}, \ldots, t_{d-1}, d\right)$, we have the following relations.

Lemma 2.2. [2, §4.1 (7); (9); (1c)]

$$
\begin{aligned}
k_{i} & =k_{i-1} \frac{b_{i-1}}{c_{i}}=s k_{i-1} \frac{t-t_{i-1}}{t_{i}+1} \quad(1 \leqslant i \leqslant d), \\
p_{i, j}^{l} k_{l} & =p_{l, j}^{i} k_{i} \quad(0 \leqslant i, j, \ell, d), \\
p_{1, i}^{i+1} & =c_{i+1} \quad(1 \leqslant i \leqslant d) .
\end{aligned}
$$

Lemma 2.2 gives the following corollary.
Corollary 2.3. Let $1 \leqslant i \leqslant d$. Then

$$
p_{i, i-1}^{1}=\frac{k_{i} c_{i}}{k_{1}}=\frac{k_{i}\left(t_{i}+1\right)}{s(t+1)} .
$$

The following lemma follows directly from the definition of an $m$-ovoid and the fact that there are $s+1$ points on every line of a finite regular near $2 d$-gon $\mathcal{S}$ with parameters $\left(s, t_{2}, \ldots, t_{d-1}, t\right)$.

Lemma 2.4. The complement of an $m$-ovoid of $\mathcal{S}$ is a $(s+1-m)$-ovoid.
Lemma 2.5 ([18, Lemma 5]). If $\mathcal{O}$ is an $m$-ovoid of $\mathcal{S}$, then for every $i \in\{0,1, \ldots, d\}$ and $x \in \mathcal{O}$,

$$
\left|\Gamma_{i}(x) \cap \mathcal{O}\right|=k_{i}\left(\frac{m}{s+1}+\left(-\frac{1}{s}\right)^{i}\left(1-\frac{m}{s+1}\right)\right) .
$$

By Lemmas 2.4 and 2.5, we have the following:
Corollary 2.6. If $\mathcal{O}$ is an m-ovoid of $\mathcal{S}$, then for every $i \in\{0,1, \ldots, d\}$ and $x \notin \mathcal{O}$,

$$
\left|\Gamma_{i}(x) \cap \mathcal{O}\right|=k_{i} \frac{m}{s+1}\left(1-\left(\frac{-1}{s}\right)^{i}\right)
$$

## 3. Proof of the main result

We now prove Theorem 1.2. Recall that we are assuming that

$$
c_{i}=t_{i}+1=\frac{\left(s^{i}+(-1)^{i}\right)\left(c_{i-1}+(-1)^{i} s^{i-2}\right)}{s^{i-2}+(-1)^{i}}
$$

for some $3 \leqslant i \leqslant d$.
Proof. Let $\mathcal{O}$ be a nontrivial $m$-ovoid of $\mathcal{S}$. Throughout this proof, we will let $\chi_{\mathcal{O}}$ denote the characteristic vector of $\mathcal{O}$ with respect to the set of points $\mathcal{P}$ :

$$
\left(\chi_{\mathcal{O}}\right)_{y}= \begin{cases}1 & \text { if } y \in \mathcal{O} \\ 0 & \text { otherwise }\end{cases}
$$

A simple double counting argument shows that $|\mathcal{O}|$ is equal to $m|\mathcal{L}| /(t+1)$ where $\mathcal{L}$ is the set of lines of $\mathcal{S}$. If we also count flags (i.e., point-line incident pairs), then $|\mathcal{P}|(t+1)=|\mathcal{L}|(s+1)$ where $\mathcal{P}$ is the set of points of $\mathcal{S}$, and hence

$$
|\mathcal{O}|=\frac{m n}{s+1},
$$

where $n=|\mathcal{P}|$.
Recall that $3 \leqslant i \leqslant d$. Now we fix an element $x \notin \mathcal{O}$ and count pairs $(y, z)$ of elements of $\mathcal{O}$ such that $\mathrm{d}(x, y)=i$ and either $\mathrm{d}(y, z)=i-1$ and $\mathrm{d}(x, z)=1$, or $\mathrm{d}(y, z)=1$ and $\mathrm{d}(x, z)=i-1$.

Let $x$ and $y$ be two points at distance $i$. Define $v_{x, y}$ as in [9, Theorem 3.2(b)],

$$
v_{x, y}:=s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)\left(\chi_{x}+\chi_{y}\right)+\chi_{\Gamma_{1}(x) \cap \Gamma_{i-1}(y)}+\chi_{\Gamma_{i-1}(x) \cap \Gamma_{1}(y)} .
$$

Note that

$$
v_{x, y} \cdot \mathbb{1}=2\left(s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)+p_{1, i-1}^{i}\right)=2\left(s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)+c_{i}\right)
$$

and furthermore that $v_{x, y}$ and $\chi_{\mathcal{O}}$ are design-orthogonal [9, Theorem 3.2] and hence by Lemma 2.1,

$$
\mu:=v_{x, y} \cdot \chi_{\mathcal{O}}=2\left(s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)+c_{i}\right) m /(s+1) .
$$

Let $\Gamma$ be the collinearity graph of $\mathcal{S}$.
Counting first $y$ and then $z$, the number of pairs is

$$
\begin{aligned}
& \sum_{y \in \mathcal{O} \cap \Gamma_{i}(x)}\left(\left|\Gamma_{1}(x) \cap \Gamma_{i-1}(y) \cap \mathcal{O}\right|+\left|\Gamma_{i-1}(x) \cap \Gamma_{1}(y) \cap \mathcal{O}\right|\right) \\
= & \sum_{y \in \mathcal{O} \cap \Gamma_{i}(x)}\left(v_{x, y}-s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)\left(\chi_{x}+\chi_{y}\right)\right) \cdot \chi_{\mathcal{O}} \\
= & \left|\mathcal{O} \cap \Gamma_{i}(x)\right|\left(\mu-s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)\right) \\
= & \left|\mathcal{O} \cap \Gamma_{i}(x)\right|\left(\frac{2\left(s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)+c_{i}\right) m}{s+1}-s\left(c_{i-1}+(-1)^{i} s^{i-2}\right)\right) \\
= & \left|\mathcal{O} \cap \Gamma_{i}(x)\right|\left(\frac{2 c_{i} m s-(s+1-2 m)\left(c_{i-1} s^{2}+(-1)^{i} s^{i}\right)}{s(s+1)}\right) .
\end{aligned}
$$

Now, by Corollary 2.6 and Lemma 2.2,

$$
\left|\mathcal{O} \cap \Gamma_{i}(x)\right| c_{i}=k_{i} c_{i} \frac{m}{s+1}\left(1-\left(-\frac{1}{s}\right)^{i}\right)=s k_{i-1}\left(t-t_{i-1}\right) \frac{m}{s+1}\left(1-\left(-\frac{1}{s}\right)^{i}\right)
$$

and hence the number of pairs $(y, z)$ is

$$
\begin{equation*}
\frac{m k_{i-1}\left(t-t_{i-1}\right)}{s+1}\left(1-\left(-\frac{1}{s}\right)^{i}\right) \frac{2 c_{i} m s-(s+1-2 m)\left(c_{i-1} s^{2}+(-1)^{i} s^{i}\right)}{c_{i}(s+1)} \tag{2}
\end{equation*}
$$

Now we consider the pairs the opposite way, namely counting $z$ then $y$. The number of pairs $(z, y)$ is equal to

$$
\sum_{z \in \mathcal{O} \cap \Gamma_{1}(x)}\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|+\sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)}\left|\Gamma_{1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right| .
$$

Suppose $\mathrm{d}(z, x)=i-1$. There are $t+1$ lines on $z$, and the set of points incident with these lines, other than the point $z$ itself, form $\Gamma_{1}(z)$. There are $t_{i-1}+1$ lines on $z$ incident with a unique point at distance $i-2$ from $x$, and the remaining points on these lines are at distance $i-1$ from $x$. Moreover, $z$ is the unique nearest point to $x$ with distance $i-1$ for the remaining $t-t_{i-1}$ lines, and hence any other point on these lines must have
distance $i$ from $x$. Since $z$ is in $\mathcal{O}$, there are $m-1$ additional points of $\mathcal{O}$ on each such line. Therefore,

$$
\left|\Gamma_{1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|=\left(t-t_{i-1}\right)(m-1) .
$$

Now suppose $\mathrm{d}(z, x)=1$. We will compute $\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|$. Take $z \in \mathcal{O} \cap \Gamma_{1}(x)$ and consider a point $w \in \Gamma_{i-2}(z) \cap \Gamma_{i-1}(x)$. Note that any point $y$ is collinear with some such point $w$, giving rise to the following equation:

$$
\begin{equation*}
\sum_{y \in \Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}}\left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \Gamma_{1}(y)\right|=\sum_{w \in \Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}}\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O} \cap \Gamma_{1}(w)\right| \tag{3}
\end{equation*}
$$

where $\mathcal{O}^{c}$ is the complement of $\mathcal{O}$ within the set of points of $\mathcal{S}$.
Let $\ell$ be a line through $w$. There is a point on $\ell$ which is the unique closest point to $x$. If this point is $w$, then every other point must be at distance $i$ from $x$. If this point is not $w$, then it must be distance $i-2$ from $x$, and every other point on $\ell$ is distance $i-1$ from $x$. There are $t_{i-1}+1$ lines on $w$ with a unique point at distance $i-2$ from $x$. Hence there are $t-t_{i-1}$ lines $\ell^{\prime}$ for which $w$ is the unique nearest point to $x$ and every other point on $\ell^{\prime}$ is at distance $i$ from $x$. Moreover, note that if a point $y$ is at distance $i$ from $x$, then it cannot be distance $i-2$ from $z$, since $\mathrm{d}(x, z)=1$, and thus any point other than $w$ on any line $\ell^{\prime}$ is in $\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \Gamma_{1}(w)$. There are $m-1$ such points in $\mathcal{O}$ when $w \in \mathcal{O}$, otherwise there are $m$ such points in $\mathcal{O}$.

There are $t_{i-1}$ lines on any point $y$ which have a unique point at distance $i-2$ from $z$, and hence also at distance $i-1$ from $x$. Recalling that $\left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x)\right|=p_{i-1, i-2}^{1}$, our Equation (3) becomes:

$$
\begin{aligned}
\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|\left(t_{i-1}+1\right)= & \left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}\right|\left(t-t_{i-1}\right)(m-1) \\
& \quad+\left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}^{c}\right|\left(t-t_{i-1}\right) m \\
= & \left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}\right|\left(t-t_{i-1}\right)(m-1) \\
& \quad+\left(p_{i-1, i-2}^{1}-\left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}\right|\right)\left(t-t_{i-1}\right) m \\
= & p_{i-1, i-2}^{1}\left(t-t_{i-1}\right) m-\left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}\right|\left(t-t_{i-1}\right) .
\end{aligned}
$$

Hence we obtain an iterative formula,

$$
\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|=p_{i-1, i-2}^{1} \frac{t-t_{i-1}}{t_{i-1}+1} m-\frac{t-t_{i-1}}{t_{i-1}+1}\left|\Gamma_{i-2}(z) \cap \Gamma_{i-1}(x) \cap \mathcal{O}\right|
$$

which, with the help of Lemma 2.2 and Corollary 2.3, we can write as a recurrence relation

$$
s f_{i}=m-f_{i-1}, \quad f_{1}=1
$$

where $f_{i}:=\frac{1}{p_{i, i-1}^{1}}\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|$ for all $i \geqslant 1$. (Note: $\left|\Gamma_{0}(z) \cap \Gamma_{1}(x) \cap \mathcal{O}\right|=1$ and $p_{1,0}^{1}=1$ ). Therefore, by the elementary theory of recurrence relations, we have

$$
f_{i}=\frac{m-s\left(-\frac{1}{s}\right)^{i}(-m+s+1)}{s+1}
$$

for all $i \geqslant 1$. Hence, by Corollary 2.3,

$$
\begin{aligned}
\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right| & =p_{i, i-1}^{1}\left(\frac{m-s\left(-\frac{1}{s}\right)^{i}(-m+s+1)}{s+1}\right) \\
& =\frac{k_{i-1}\left(t-t_{i-1}\right)}{t+1}\left(\frac{m-s\left(-\frac{1}{s}\right)^{i}(-m+s+1)}{s+1}\right) \\
& =\frac{k_{i-1}\left(t-t_{i-1}\right)}{s^{i-1}(t+1)}\left(\frac{m}{s+1}\left(s^{i-1}+(-1)^{i-2}\right)+(-1)^{i-1}\right)
\end{aligned}
$$

Now, making use of Corollary 2.6, we sum our two terms together:

$$
\begin{aligned}
& \quad \sum_{z \in \mathcal{O} \cap \Gamma_{1}(x)}\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|+\sum_{z \in \mathcal{O} \cap \Gamma_{i-1}(x)}\left|\Gamma_{1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right| \\
& =\left|\mathcal{O} \cap \Gamma_{1}(x)\right|\left|\Gamma_{i-1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right|+\left|\mathcal{O} \cap \Gamma_{i-1}(x)\right|\left|\Gamma_{1}(z) \cap \Gamma_{i}(x) \cap \mathcal{O}\right| \\
& =s(t+1) \frac{m}{s+1}\left(1+\frac{1}{s}\right) \frac{k_{i-1}\left(t-t_{i-1}\right)}{s^{i-1}(t+1)}\left(\frac{m}{s+1}\left(s^{i-1}+(-1)^{i-2}\right)+(-1)^{i-1}\right) \\
& \quad+k_{i-1} \frac{m}{s+1}\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)\left(t-t_{i-1}\right)(m-1) \\
& = \\
& =\frac{m k_{i-1}\left(t-t_{i-1}\right)}{s+1}\left(m\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)+\left(\frac{-1}{s}\right)^{i-1}(s+1)+(m-1)\left(1-\left(-\frac{1}{s}\right)^{i-1}\right)\right)
\end{aligned}
$$

and therefore, the number of pairs $(z, y)$ is

$$
\begin{equation*}
\frac{m k_{i-1}\left(t-t_{i-1}\right)}{s+1}\left(2 m-1+\left(\frac{-1}{s}\right)^{i-1}(s-2 m+2)\right) \tag{4}
\end{equation*}
$$

Equating the two counts, (2) and (4) yields

$$
\begin{aligned}
& \left(1-\left(-\frac{1}{s}\right)^{i}\right) \frac{2 c_{i} m s-(s+1-2 m)\left(c_{i-1} s^{2}+(-1)^{i} s^{i}\right)}{c_{i}(s+1)} \\
& =2 m-1+\left(\frac{-1}{s}\right)^{i-1}(s-2 m+2)
\end{aligned}
$$

Taking the difference of each side of the above equation and factoring gives

$$
\frac{s^{-i}(s+1-2 m)\left(c_{i}\left(s^{i}+(-1)^{i}(s+2) s\right)+\left((-1)^{i}-s^{i}\right)\left(c_{i-1} s^{2}+(-1)^{i} s^{i}\right)\right)}{c_{i}(s+1)}=0
$$

and hence

$$
\begin{equation*}
(s+1-2 m)\left(c_{i}\left(s^{i}+(-1)^{i}(s+2) s\right)+\left((-1)^{i}-s^{i}\right)\left(c_{i-1} s^{2}+(-1)^{i} s^{i}\right)\right)=0 \tag{5}
\end{equation*}
$$

Now by assumption,

$$
c_{i-1} s^{2}+(-1)^{i} s^{i}=c_{i} \frac{s^{i}+(-1)^{i} s^{2}}{s^{i}+(-1)^{i}}
$$

and hence

$$
\begin{aligned}
& c_{i}\left(s^{i}+(-1)^{i}(s+2) s\right)+\left((-1)^{i}-s^{i}\right)\left(c_{i-1} s^{2}+(-1)^{i} s^{i}\right) \\
& =c_{i}\left(s^{i}+(-1)^{i}(s+2) s+\left((-1)^{i}-s^{i}\right) \frac{s^{i}+(-1)^{i} s^{2}}{s^{i}+(-1)^{i}}\right) \\
& =c_{i} \frac{2(-1)^{i}(s+1)\left(s^{i}+(-1)^{i} s\right)}{s^{i}+(-1)^{i}} .
\end{aligned}
$$

Since $i>1$, we have $s^{i}+(-1)^{i} s \neq 0$, and therefore, Equation (5) becomes $m=(s+$ $1) / 2$.

## 4. Further results and computation

Theorem 1.1 leaves open the natural question of whether there exist hemisystems of $\mathrm{DQ}(6, q), \operatorname{DW}(5, q)$ and $\mathrm{DH}\left(5, q^{2}\right)$. Firstly, De Bruyn and Vanhove announced in conference presentations that there are no hemisystems of $\operatorname{DW}(5,3)$, and that there is a unique example for $\operatorname{DQ}(6,3)$. We thank the referee and Michel Lavrauw for mentioning these results to us. For small values of (odd) $q$, we have found examples for $\operatorname{DQ}(6, q)$, and we have listed the known examples in Table 1. In particular, we could show by using the computer algebra system GAP [11], a package FinInG [1], and the mixed-integer programming software Gurobi [12] that there is a unique example up to equivalence in $\mathrm{DQ}(6,3)$. For $\mathrm{DQ}(6,5)$, there were numerous examples found admitting an element of order 5 or 9 , but we were unable to enumerate them all.

| $q$ | Stabiliser | Number up to equivalence |
| :---: | :---: | :---: |
| 3 | $2 \times A_{5}$ | 1 |
| 5 | $D_{60}$ | 4 |
|  | $D_{20}$ | 16 |

Table 1. Some known examples of hemisystems of $\operatorname{DQ}(6, q)$, for small $q$.

For $\operatorname{DW}(5, q)$, it seems the situation is different, despite its combinatorial parameters being identical to those of $\mathrm{DQ}(6, q)$. By computer, we showed that there are no hemisystems of $\operatorname{DW}(5, q)$ for $q \in\{3,5\}$. We make the following conjectures:

Conjecture 4.1. There are no hemisystems of $\operatorname{DW}(5, q)$, for all prime powers $q$.
If true, this would also imply that there are no hemisystems of $\mathrm{DH}\left(5, q^{2}\right)$, for all prime powers $q$, answering a problem posed by Vanhove [19, Appendix B, Problem 7].

Conjecture 4.2. For each odd prime power $q$, there exists a hemisystem of $\operatorname{DQ}(6, q)$.

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