# The Graph Structure of Chebyshev Polynomials over Finite Fields and Applications 

Claudio Qureshi and Daniel Panario


#### Abstract

We completely describe the functional graph associated to iterations of Chebyshev polynomials over finite fields. Then, we use our structural results to obtain estimates for the average rho length, average number of connected components and the expected value for the period and preperiod of iterating Chebyshev polynomials.


## I. Introduction

The iteration of polynomials and rational functions over finite fields have recently become an active research topic. These dynamical systems have found applications in diverse areas, including cryptography, biology and physics. In cryptography, iterations of functions over finite fields were popularized by the Pollard rho algorithm for integer factorization [12]; its variant for computing discrete logarithms is considered the most efficient method against elliptic curve cryptography based on the discrete logarithm problem [13]. Other cryptographical applications of iterations of functions include pseudorandom bit generators [1], and integer factorization and primality tests [8], [9].

When we iterate functions over finite structures, there is an underlying natural functional graph. For a function $f$ over a finite field $\mathbb{F}_{q}$, this graph has $q$ nodes and a directed edge from vertex $a$ to vertex $b$ if and only if $f(a)=b$. It is well known, combinatorially, that functional graphs are sets of connected components, components are directed cycles of nodes, and each of these nodes is the root of a directed tree from leaves to its root; see, for example, [6].

Some functions over finite fields when iterated present strong symmetry properties. These symmetries allow mathematical proofs for some dynamical properties such as period and preperiod of a generic element, (average) "rho length" (number of iterations until cycling back), number of connected components,

[^0]cycle lengths, etc. In this paper we are interested on these kinds of properties for Chebyshev polynomials over finite fields, closely related to Dickson polynomials over finite fields. These polynomials, specially when they permute the elements of the field, have found applications in many areas including cryptography and coding theory. See [10] for a monograph on Dickson polynomials and their applications, including cryptography; for a more recent account on research in finite fields including Dickson polynomials, see [11].

Previous results for quadratic functions are in [17]; iterations of $x+x^{-1}$ have been dealt in [16] and iterations of Rédei functions over non-binary finite fields appeared in [14], [15]. Related to this paper, iterations of Chebyshev polynomials over finite fields have been treated in [7]. The graph and periodicity properties for Chebyshev polynomials over finite fields when the degree of the polynomial is a prime number are given in [7].

In this paper we study the action of Chebyshev functions of any degree over finite fields. We give a structural theorem for the functional graph from which it is not hard to derive many periodicity properties of these iterations. In the literature there are two kinds of Chebyshev polynomials: normalized and not normalized. We use the latter ones, generally known as Dickson polynomials of the first kind. In odd characteristic both kinds of Chebyshev polynomials are conjugates of each other, and so their functional graphs are isomorphic. However, this is not the case in even characteristic. Using the normalized version trivializes since we get $T_{n}(x)=1$ if $n$ is even, and $T_{n}(x)=x$ if $n$ is odd, where $T_{n}$ is the $n$th degree Chebyshev polynomial. As a consequence, we work with the non normalized version that is much richer in characteristic 2. Not much is known about Chebyshev polynomials over binary fields; see [5] for results over the 2 -adic integers.

In Section $\Pi$ we introduce relevant concepts for this paper like $v$-series and their associated trees. These trees play a central role in the description of the Chebyshev functional graph. Several results about a homomorphism of the Chebyshev functional graph, as well as a relevant covering notion, are given in Section [II. A decomposition of the Chebyshev's functional graph is given in Section IV. This decomposition leads naturally into three parts: the rational, the quadratic and the special component. Section V treats the rational and quadratic components. The special component is dealt in Section VI The main result of this paper (Theorem 4), a structural theorem for Chebyshev polynomials, is given in Section VII. We provide several examples to show applications of our main theorem. As a consequence of our main structural theorem, in this section we also obtain exact results for the parameters $N, C, T_{0}, T$ and $R$ for Chebyshev polynomials, where $N$ is the number of cycles (that is, the number of connected components), $T_{0}$ is the number of cyclic (periodic) points, $C$ is the expected value of the period, $T$ is the expected value of the preperiod, and $R$ is the expected rho length.

## II. Preliminaries

We denote by $\mathbb{F}_{q}$ a finite field with $q$ element, where $q$ is a prime power, and $\mathbb{Z}_{d}$ the ring of integers modulo $d$. Let $\mathbb{F}_{q}^{*}$ and $\mathbb{Z}_{d}^{*}$ denote the multiplicative group of inverse elements of $\mathbb{F}_{q}$ and $\mathbb{Z}_{d}$, respectively. Let $\bar{n}$ denote the equivalence class of $n$ modulo $d$. For $n, d \in \mathbb{Z}^{+}$with $\operatorname{gcd}(n, d)=1$, we denote by $o_{d}(n)$ and $\tilde{o}_{d}(n)$ the multiplicative order of $\bar{n}$ in $\mathbb{Z}_{d}^{*}$ and $\mathbb{Z}_{d}^{*} /\{1,-1\}$, respectively. It is easy to see that if $-\overline{1} \in\langle\bar{n}\rangle$ in $\mathbb{Z}_{d}^{*}$, then $\tilde{o}_{d}(n)=o_{d}(n) / 2$, otherwise $\tilde{o}_{d}(n)=o_{d}(n)$. For $m \in \mathbb{Z}^{+}$we denote by $\operatorname{rad}(m)$ the radical of $m$ which is defined as the product of the distinct primes divisors of $m$. We can decompose $m=v \omega$ where $\operatorname{rad}(v) \mid \operatorname{rad}(n)$ and $\operatorname{gcd}(\omega, n)=1$ which we refer as the $n$-decomposition of $m$. If $f: X \rightarrow X$ is a function defined over a finite set $X$, we denote by $\mathcal{G}(f / X)$ its functional graph.

The main object of study of this paper is the action of Chebyshev polynomials over finite fields $\mathbb{F}_{q}$. The Chebyshev polynomial of the first kind of degree $n$ is denoted by $T_{n}$. This is the only monic, degree- $n$ polynomial with integer coefficients verifying $T_{n}\left(x+x^{-1}\right)=x^{n}+x^{-n}$ for all $x \in \mathbb{Z}$. Table gives the first Chebyshev polynomials.

$$
\begin{aligned}
& T_{1}(x)=x \\
& T_{2}(x)=x^{2}-2 \\
& T_{3}(x)=x^{3}-3 x \\
& T_{4}(x)=x^{4}-4 x^{2}+2 \\
& T_{5}(x)=x^{5}-5 x^{3}+5 x \\
& T_{6}(x)=x^{6}-6 x^{4}+9 x^{2}-2 \\
& T_{7}(x)=x^{7}-7 x^{5}+14 x^{3}-7 x \\
& T_{8}(x)=x^{8}-8 x^{6}+20 x^{4}-16 x^{2}+2 \\
& T_{9}(x)=x^{9}-9 x^{7}+27 x^{5}-30 x^{3}+9 x \\
& T_{10}(x)=x^{10}-10 x^{8}+35 x^{6}-50 x^{4}+25 x^{2}-2
\end{aligned}
$$

TABLE I
First few Chebyshev polynomials $T_{n}(x)$ For $1 \leq n \leq 10$.

A remarkable property of these polynomials is that $T_{n} \circ T_{m}=T_{n m}$ for all $m, n \in \mathbb{Z}^{+}$. In particular, $T_{n}^{(k)}=T_{n^{k}}$, where $f^{(k)}$ denotes the composition of $f$ with itself $k$ times. Describing the dynamics of the Chebyshev polynomial $T_{n}$ acting on the finite field $\mathbb{F}_{q}$ is equivalent to describing the Chebyshev's graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$.

The case when $n=\ell$ is a prime number was dealt by Gassert; see [7] Theorem 2.3]. In this paper we extend these results for any positive integer $n$.

Example 1. For $n=30$ the corresponding Chebyshev polynomial is given by $T_{30}(x)=x^{30}-30 x^{28}+$
$405 x^{26}-3250 x^{24}+17250 x^{22}-63756 x^{20}+168245 x^{18}-319770 x^{16}+436050 x^{14}-419900 x^{12}+277134 x^{10}-$ $119340 x^{8}+30940 x^{6}-4200 x^{4}+225 x^{2}-2$. The graphs $\mathcal{G}\left(T_{30} / \mathbb{F}_{q}\right)$ for $q=19$ and $q=23$ are shown in Fig. 1$]$


Fig. 1. a) The Chebyshev's graph $\mathcal{G}\left(T_{30} / \mathbb{F}_{19}\right)$. b) The Chebyshev's graph $\mathcal{G}\left(T_{30} / \mathbb{F}_{23}\right)$.

Next we review some concepts from [14]. For $n$ and $v$ positive integers such that $\operatorname{rad}(v) \mid \operatorname{rad}(n)$, the $v$-series associated with $n$ is the finite sequence $v(n):=\left(v_{1}, \ldots, v_{D}\right)$ defined by the recurrence $v_{1}=\operatorname{gcd}(v, n), v_{k+1}=\operatorname{gcd}\left(\frac{v}{v_{1} v_{2} \cdots v_{k}}, n\right)$ for $1 \leq k<D$ and $v_{1} v_{2} \cdots v_{D}=v$ with $v_{D}>1$ if $v>1$, and $v(n)=(1)$ if $v=1$.

We write $A=\biguplus B_{i}$ to indicate that $A$ is the union of pairwise disjoint sets $B_{i}$. If $m \in \mathbb{Z}^{+}$and $T$ is a rooted tree, $\operatorname{Cyc}(m, T)$ denotes a graph with a unique directed cycle of length $m$, where every node in this cycle is the root of a tree isomorphic to $T$. We also consider the disjoint union of the graphs $G_{1}, \ldots, G_{k}$, denoted by $\bigoplus_{i=1}^{k} G_{i}$, and $k \times G=\bigoplus_{i=1}^{k} G$ for $k \in \mathbb{Z}^{+}$. If $T_{1}, \ldots, T_{k}$ are rooted trees, $\left\langle T_{1} \oplus \cdots \oplus T_{k}\right\rangle$ is a rooted tree such that its root has exactly $k$ predecessors $v_{1}, \ldots, v_{k}$, and $v_{i}$ is the root of a tree isomorphic to $T_{i}$ for $i=1, \ldots, k$. If $T$ is a tree that consists of a single node we simply write $T=\bullet$. In particular, $\operatorname{Cyc}(m, \bullet)$ denotes a directed cycle with $m$ nodes. The empty graph, denoted by $\emptyset$, is characterized by the properties: $\emptyset \oplus G=G$ for all graphs $G, k \times \emptyset=\emptyset$ for all $k \in \mathbb{Z}^{+}$and $\langle\emptyset\rangle=\bullet$.

We associate to each $v$-series $v(n)$ a rooted tree, denoted by $T_{\nu(n)}$, defined by the recurrence formula (see Fig. 2):

$$
\left\{\begin{array}{l}
T^{0}=\bullet  \tag{1}\\
T^{k}=\left\langle v_{k} \times T^{k-1} \oplus \bigoplus_{i=1}^{k-1}\left(v_{i}-v_{i+1}\right) \times T^{i-1}\right\rangle, 1 \leq i<D \\
T_{v(n)}=\left\langle\left(v_{D}-1\right) \times T^{D-1} \oplus \bigoplus_{i=1}^{D-1}\left(v_{i}-v_{i+1}\right) \times T^{i-1}\right\rangle
\end{array}\right.
$$

The tree $T_{\nu(n)}$ has $v$ vertices and depth $D$; see Proposition 2.14 and Theorem 3.16 of [14].
The following theorem is a direct consequence of Corollary 3.8 and Theorem 3.16 of [14]. As usual, $\varphi$ denotes Euler's totient function.

Theorem 1. Let $n \in \mathbb{Z}^{+}$and $m=v \omega$ be the $n$-decomposition of $m$. Denoting by $\mathcal{G}\left(n / \mathbb{Z}_{m}\right)$ the functional graph of the multiplication-by-n map on the cyclic group $\mathbb{Z}_{m}$, the following isomorphism holds:

$$
\mathcal{G}\left(n / \mathbb{Z}_{m}\right)=\bigoplus_{d \mid \omega} \frac{\varphi(d)}{o_{d}(n)} \times \operatorname{Cyc}\left(o_{d}(n), T_{v(n)}\right)
$$



Fig. 2. This figure (taken from [14]) illustrates the inductive definition of $T_{V}$ when $V$ is a $v$-series with four components $V=\left(v_{1}, v_{2}, v_{3}, v_{4}\right)$. A node $v$ labelled by a rooted tree $T$ indicates that $v$ is the root of a tree isomorphic to $T$.

A strategy to describe a functional graph $\mathcal{G}(f / X)$ of a function $f: X \rightarrow X$ is decomposing the set $X$ in $f$-invariant components. A subset $A \subseteq X$ is forward $f$-invariant when $f(A) \subseteq A$. In this case the graph $\mathcal{G}(f / A)$ is a subgraph of $\mathcal{G}(f / X)$. If $f^{-1}(A) \subseteq A$, the set $A$ is backward $f$-invariant. The set $A$ is $f$-invariant if it is both forward and backward $f$-invariant. In this case $\mathcal{G}(f / A)$ is not only a subgraph of $\mathcal{G}(f / X)$ but also a union of connected components and we can write $\mathcal{G}(f / X)=\mathcal{G}(f / A) \oplus \mathcal{G}\left(f / A^{c}\right)$, where $A^{c}=X \backslash A$. In this paper, we decompose the set $\mathbb{F}_{q}$ in $T_{n}$-invariant subsets $A_{1}, \ldots, A_{\kappa}$ such that each functional graph $\mathcal{G}\left(T_{n} / A_{i}\right)$ for $i=1, \ldots, \kappa$ is easier to describe than the general case and $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)=\bigoplus_{i=1}^{K} \mathcal{G}\left(T_{n} / A_{i}\right)$.

To describe a functional graph we need to describe not only the cyclic part but also the rooted trees attached to the periodic points. We introduce next some notation related to rooted trees (where the root is not necessarily a periodic point). Let $f: X \rightarrow X, x \in X$ and $N_{f}$ be the set of its non-periodic points. We define the set of predecessors of $x$ by

$$
\operatorname{Pred}_{x}(f / X)=\left\{y \in N_{f}: f^{(k)}(y)=x \text { for some } k \geq 1\right\} \cup\{x\} .
$$

We denote by $\operatorname{Tree}_{x}(f / X)$ the rooted tree with root $x$, vertex set $V=\operatorname{Pred}_{x}(f / X)$ and directed edges $(y, f(y))$ for $y \in V \backslash\{x\}$.

## III. RESULTS ON HOMOMORPHISM OF FUNCTIONAL GRAPHS

A directed graph is a pair $G=(V, E)$ where $V$ is the vertex set and $E \subseteq V \times V$ is the edge set. A homomorphism $\phi$ between two directed graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$, denoted by $\phi: G_{1} \rightarrow G_{2}$, is a function $\phi: V_{1} \rightarrow V_{2}$ such that if $\left(v, v^{\prime}\right) \in E_{1}$ then $\left(\phi(v), \phi\left(v^{\prime}\right)\right) \in E_{2}$. In the particular case of functional graphs, a homomorphism $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ is a function $\phi: X_{1} \rightarrow X_{2}$ satisfying $\phi \circ f_{1}=f_{2} \circ \phi$, or equivalently such that the following diagram commutes


It is easy to prove by induction that the relation $\phi \circ f_{1}=f_{2} \circ \phi$ implies $\phi \circ f_{1}^{(k)}=f_{2}^{(k)} \circ \phi$ for all $k \geq 1$, that is, $\phi: \mathcal{G}\left(f_{1}^{(k)} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2}^{(k)} / X_{2}\right)$ is also a homomorphism for all $k \geq 1$. If in addition $\phi$ is bijective (as function from $X_{1}$ to $\left.X_{2}\right)$ then $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ is an isomorphism of functional graphs. In this case the functional graphs are the same, up to the labelling of the vertices. The main result of this paper (Theorem 4) is an explicit description of $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$, the functional graph of the Chebyshev polynomial $T_{n}$ over a finite field $\mathbb{F}_{q}$.

In the first part of this section we introduce the concept of $\theta$-covering between two functional graphs and derive some properties. In the last part we apply these results to obtain some rooted tree isomorphism formulas which are used in the next sections.

## A. $\theta$-coverings

In our case of study (functional graph of Chebyshev polynomials) we consider the set $\tilde{\mathbb{F}}_{q}=\mathbb{F}_{q}^{*} \cup H$, where $H$ is the multiplicative subgroup of $\mathbb{F}_{q^{2}}^{*}$ of order $q+1$, and the following maps:

- The inversion map $i: \tilde{\mathbb{F}}_{q} \rightarrow \tilde{\mathbb{F}}_{q}$ given by $i(\alpha)=\alpha^{-1}$.
- The exponentiation map $r_{n}: \tilde{\mathbb{F}}_{q} \rightarrow \tilde{\mathbb{F}}_{q}$ given by $r_{n}(\alpha)=\alpha^{n}$.
- The map $\eta: \tilde{\mathbb{F}}_{q} \rightarrow \mathbb{F}_{q}$ given by $\eta(\alpha)=\alpha+\alpha^{-1}$.

A useful relationship between these maps and the Chebyshev map are $T_{n} \circ \eta=\eta \circ r_{n}$ and $r_{n} \circ i=i \circ r_{n}$. In other words we have the following commutative diagrams:

and


To describe the Chebyshev functional graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ it is helpful to consider the homomorphism $\eta: \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right) \rightarrow \mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ and to relate properties between these functional graphs. This homomorphism is not an isomorphism, but it has very nice properties that are captured in the next concept.

Definition 1. Let $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ be a homomorphism of functional graphs and $\theta: X_{1} \rightarrow X_{1}$ be a permutation (bijection) which commutes with $f_{1}$ (that is, $f_{1} \circ \theta=\theta \circ f_{1}$ ). Then $\phi$ is a $\theta$-covering if for every $a \in X_{2}$ there is $\alpha \in X_{1}$ such that $\phi^{-1}(a)=\left\{\theta^{(i)}(\alpha): i \in \mathbb{Z}\right\}$ (in other words, if the preimage of each point is a $\theta$-orbit). The homomorphism $\phi$ is a covering if it is a $\theta$-covering for some $\theta$ verifying the above properties.

We remark that a covering is necessarily onto and every isomorphism $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ is a covering (with respect to the identity map id : $X_{1} \rightarrow X_{1}, \operatorname{id}(x)=x$ ). We note that the condition of $\phi^{-1}(a)$ being a $\theta$-orbit for all $a \in X_{2}$ implies that $\phi \circ \theta=\phi$.

In [7] it is proved several properties of the map $\eta$. Namely $\eta$ is surjective, $\eta^{-1}(2)=\{1\}, \eta^{-1}(-2)=\{-1\}$, and for $a \in \mathbb{F}_{q}, \eta^{-1}(a)=\left\{\alpha, \alpha^{-1}\right\}$ where $\alpha$ and $\alpha^{-1}$ are the roots $\left(\right.$ in $\left.\mathbb{F}_{q^{2}}^{*}\right)$ of $x^{2}-a x+1=0$ which are distinct if $a \neq \pm 2$. In particular, with our notation, we have that $\eta: \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right) \rightarrow \mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ is a $i$-covering between these functional graphs.

Next we prove some general properties for coverings of functional graphs that are used in the next section for the particular case of the covering $\eta: \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right) \rightarrow \mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$. In the next propositions we denote by $P_{f}$ and $N_{f}$ the set of periodic and non-periodic points with respect to the map $f$, respectively. We note that if $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ is a homomorphism and $x \in P_{f_{1}}$ then there is a $k \geq 1$ such that $f_{1}^{(k)}(x)=x$. This implies $f_{2}^{(k)}(\phi(x))=\phi\left(f_{1}^{(k)}(x)\right)=\phi(x)$, thus $x \in \phi^{-1}\left(P_{f_{2}}\right)$ and we have $P_{f_{1}} \subseteq \phi^{-1}\left(P_{f_{2}}\right)$. The next proposition shows that when $\phi$ is a covering this inclusion is in fact an equality.

Proposition 1. Let $\theta: X_{1} \rightarrow X_{1}$ be a permutation satisfying $f_{1} \circ \theta=\theta \circ f_{1}$. If $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ is a $\theta$-covering then $\phi^{-1}\left(P_{f_{2}}\right)=P_{f_{1}}$.

Proof. Let $\ell$ be the order of $\theta$ (i.e. $\theta^{(\ell)}=$ id). It suffices to prove $\phi^{-1}\left(P_{f_{2}}\right) \subseteq P_{f_{1}}$. If $\alpha \in \phi^{-1}\left(P_{f_{2}}\right)$ then there is a $k \geq 1$ such that $f_{2}^{(k)}(\phi(\alpha))=\phi(\alpha)$. Since $f_{2}^{(k)}(\phi(\alpha))=\phi\left(f_{1}^{(k)}(\alpha)\right)$ we conclude that $f_{1}^{(k)}(\alpha)=\theta^{(i)}(\alpha)$ for some $i \in \mathbb{Z}$. Applying $f_{1}^{(k)}$ on both sides we obtain $f_{1}^{(2 k)}(\alpha)=f_{1}^{(k)}\left(\theta^{(i)}(\alpha)\right)=\theta^{(i)}\left(f_{1}^{(k)}(\alpha)\right)=\theta^{(2 i)}(\alpha)$. In the same way, applying $f_{1}^{(k)}$ several times, we have by induction that $f_{1}^{(m k)}(\alpha)=\theta^{(m i)}(\alpha)$ for all $m \geq 1$. With $m=\ell$ we obtain $f_{1}^{(\ell k)}(\alpha)=\theta^{(\ell i)}(\alpha)=\alpha$, thus $\alpha \in P_{f_{1}}$.

Remark 1. The equation $\phi^{-1}\left(P_{f_{2}}\right)=P_{f_{1}}$ is equivalent to $\phi^{-1}\left(N_{f_{2}}\right)=N_{f_{1}}$ since $\phi^{-1}\left(X^{c}\right)=\phi^{-1}(X)^{c}$.

Proposition 2. Let $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ be a homomorphism satisfying $\phi^{-1}\left(P_{f_{2}}\right)=P_{f_{1}}$ and $\alpha \in X_{1}$. We have $\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right) \subseteq \phi^{-1}\left(\operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)\right)$.

Proof. Let $\beta \in \operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right), \beta \neq \alpha$ (in particular $\left.\beta \in N_{f_{1}}\right)$. By definition, there is an integer $k \geq 1$ such that $f_{1}^{(k)}(\beta)=\alpha$. This implies $f_{2}^{(k)}(\theta(\beta))=\theta\left(f_{1}^{(k)}(\beta)\right)=\theta(\alpha)$. Since $\phi^{-1}\left(N_{f_{2}}\right)=N_{f_{1}}$ and $\beta \in N_{f_{1}}$ we have $\phi(\beta) \in N_{f_{2}}$, thus $\phi(\beta) \in \operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$.

Remark 2. If $\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right) \subseteq \phi^{-1}\left(\operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)\right)$ then $\phi\left(\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)\right) \subseteq \operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$ since $\phi$ is surjective.

Proposition 3. Let $\theta: X_{1} \rightarrow X_{1}$ be a permutation satisfying $f_{1} \circ \theta=\theta \circ f_{1}, \alpha \in X_{1}$ and $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow$ $\mathcal{G}\left(f_{2} / X_{2}\right)$ be a $\theta$-covering. The equality $\phi\left(\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)\right)=\operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$ holds.

Proof. The inclusion $\phi\left(\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)\right) \subseteq \operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$ follows from Propositions 1 and 2 (see also Remark 2). To prove the other inclusion we consider $b \in \operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$ with $b \neq \phi(\alpha)$ (in particular $b \in N_{f_{2}}$ ) and $\beta \in X_{1}$ such that $b=\phi(\beta)$ (this is possible because $\phi$ is surjective). We have to prove that there is a point $\beta^{\prime} \in \operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)$ such that $\phi\left(\beta^{\prime}\right)=b$. By definition there is an integer $k \geq 1$ such that $f_{2}^{(k)}(b)=\phi(\alpha)$ and we have $\phi\left(f_{1}^{(k)}(\beta)\right)=f_{2}^{(k)}(\phi(\beta))=\phi(\alpha)$. Since $\phi$ is a $\theta$-covering, from $\phi\left(f_{1}^{(k)}(\beta)\right)=\phi(\alpha)$ we have that $\alpha=\theta^{(i)}\left(f_{1}^{(k)}(\beta)\right)$ for some integer $i$ and define $\beta^{\prime}=\theta^{(i)}(\beta)$. Using that $\theta$ and $f_{1}$ commute we obtain $f_{1}^{(k)}\left(\beta^{\prime}\right)=\theta^{(i)}\left(f_{1}^{(k)}(\beta)\right)=\alpha$ and $\phi\left(\beta^{\prime}\right)=\phi\left(\theta^{(i)}(\beta)\right)=\phi(\beta)=b$ (because $\phi \circ \theta=\phi)$. To conclude the proof we have to show that $\beta^{\prime} \in \operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)$ and it suffices to prove that $\beta^{\prime} \in N_{f_{1}}$. Since $\phi\left(\beta^{\prime}\right)=b \in N_{f_{2}}$ we have $\beta^{\prime} \in \phi^{-1}\left(N_{f_{2}}\right)=N_{f_{1}}$ by Proposition 1 (see also Remark 1).

With the same notation and hypothesis of Proposition 3, if we denote by $P_{1}=\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)$ and $P_{2}=\operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$ we have that the restricted function $\left.\phi\right|_{P_{1}}: P_{1} \rightarrow P_{2}$ is onto. We want to find conditions to guarantee that $\left.\phi\right|_{P_{1}}: P_{1} \rightarrow P_{2}$ is a bijection. We recall that the order of a permutation $\theta: X_{1} \rightarrow X_{1}$ is the smallest positive integer $\ell \geq 1$ such that $\theta^{(\ell)}=$ id. This implies that the cardinality of the $\theta$-orbit of a point $\alpha \in X_{1}$, given by $\left\{\theta^{(i)}(\alpha): 0 \leq i<\ell\right\}$, is a divisor of $\ell$.

Definition 2. Let $\theta: X_{1} \rightarrow X_{1}$ be a permutation of order $\ell$. A point $\alpha \in X_{1}$ is $\theta$-maximal, if the sequence of iterates: $\alpha, \theta(\alpha), \theta^{(2)}(\alpha), \ldots, \theta^{(\ell-1)}(\alpha)$ are pairwise distinct (that is, if the $\theta$-orbit of $\alpha$ has exactly $\ell$ elements).

Remark 3. An important particular case is when $\theta: X_{1} \rightarrow X_{1}$ is the identity map. In this case every point $\alpha \in X_{1}$ is $\theta$-maximal.

Proposition 4. Let $\theta: X_{1} \rightarrow X_{1}$ be a permutation satisfying $f_{1} \circ \theta=\theta \circ f_{1}$, $\alpha$ be a $\theta$-maximal point of $X_{1}$ and $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ be a $\theta$-covering. We denote by $P_{1}=\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)$ and $P_{2}=$ $\operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$. Then the restricted map $\left.\phi\right|_{P_{1}}: P_{1} \rightarrow P_{2}$ is a bijection.

Proof. By Proposition 3 we have that $\left.\phi\right|_{P_{1}}: P_{1} \rightarrow P_{2}$ is onto. To prove that $\left.\phi\right|_{P_{1}}$ is 1-to-1 we consider $\beta_{1}, \beta_{2} \in P_{1}$ such that $\phi\left(\beta_{1}\right)=\phi\left(\beta_{2}\right)$. Then there is an integer $i \in \mathbb{Z}$ such that $\beta_{2}=\theta^{(i)}\left(\beta_{1}\right)$. If the order of the permutation $\theta$ is $\ell$, we can suppose that $0 \leq i<\ell$ and we also have $\beta_{1}=\theta^{(\ell-i)}\left(\beta_{2}\right)$. We consider the smallest integers $s_{1}, s_{2} \geq 0$ such that $f_{1}^{\left(s_{i}\right)}\left(\beta_{i}\right)=\alpha$ for $i=1,2$ (they exist because $\beta_{1}, \beta_{2} \in P_{1}$ ). We want to prove that $s_{1}=s_{2}$. Consider the smallest integer $t \geq 0$ such that $f_{1}^{(t)}(\alpha) \in P_{f_{1}}$. We have that $\theta: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{1} / X_{1}\right)$ is an isomorphism of a functional graph (since $\theta$ is bijective and $\theta \circ f_{1}=f_{1} \circ \theta$ ), thus, by Proposition 1, $\theta^{-1}\left(P_{f_{1}}\right)=P_{f_{1}}$. We have that $f_{1}^{\left(t+s_{2}\right)}\left(\beta_{1}\right)=\theta^{(\ell-i)}\left(f_{1}^{\left(t+s_{2}\right)}\left(\beta_{2}\right)\right)=\theta^{(\ell-i)}\left(f_{1}^{(t)}(\alpha)\right) \in$ $\theta^{(\ell-i)}\left(P_{f_{1}}\right)=P_{f_{1}}$ (in particular $t+s_{2} \geq s_{1}$ because $f_{1}^{\left(t+s_{2}\right)}\left(\beta_{1}\right) \in P_{f_{1}}$ and $\beta_{1}$ is a predecessor of $\alpha$ ). We have that $f_{1}^{\left(t+s_{2}-s_{1}\right)}(\alpha)=f_{1}^{\left(t+s_{2}-s_{1}\right)}\left(f_{1}^{s_{1}}\left(\beta_{1}\right)\right)=f_{1}^{\left(t+s_{2}\right)}\left(\beta_{1}\right) \in P_{f_{1}}$ and by the minimality of $t$ we conclude that $s_{2} \geq s_{1}$. In a similar way we prove the other inequality $s_{2} \leq s_{1}$ obtaining $s_{2}=s_{1}$; let us denote by $s=s_{1}=s_{2}$. We have $\alpha=f_{1}^{(s)}\left(\beta_{2}\right)=f_{i}^{(s)}\left(\theta^{(i)}\left(\beta_{1}\right)\right)=\theta^{(i)}\left(f_{1}^{(s)}\left(\beta_{1}\right)\right)=\theta^{(i)}(\alpha)$ with $0 \leq i<\ell$. Using that $\alpha$ is $\theta$-maximal we conclude that $i=0$ and $\beta_{1}=\beta_{2}$ as desired.

## B. Rooted tree isomorphism formulas

Let $\phi: \mathcal{G}\left(f_{1} / X_{1}\right) \rightarrow \mathcal{G}\left(f_{2} / X_{2}\right)$ be a homomorphism of functional graph. We consider a point $\alpha \in X_{1}$ and the sets $P_{1}=\operatorname{Pred}_{\alpha}\left(f_{1} / X_{1}\right)$ and $P_{2}=\operatorname{Pred}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$. When $\phi\left(P_{1}\right) \subseteq P_{2}$ and the restricted map $\left.\phi\right|_{P_{1}}$ : $P_{1} \rightarrow P_{2}$ is a bijection, this map determines an isomorphism between the rooted trees $T_{1}=\operatorname{Tree}{ }_{\alpha}\left(f_{1} / X_{1}\right)$ and $T_{2}=\operatorname{Tree}_{\phi(\alpha)}\left(f_{2} / X_{2}\right)$ (i.e. a bijection between the vertices preserving directed edges). In this case we say that $\left.\phi\right|_{P_{1}}: T_{1} \rightarrow T_{2}$ is a rooted tree isomorphism and the trees $T_{1}$ and $T_{2}$ are isomorphic which is denoted by $T_{1} \simeq T_{2}$. Sometimes, when the context is clear, we abuse notation and write $T_{1}=T_{2}$ when these trees are isomorphic.

The first result is about the trees attached to the map $r_{n}(\alpha)=\alpha^{n}$. Since $\mathbb{F}_{q}^{*}$ and $H$ are closed under multiplication we have $r_{n}\left(\mathbb{F}_{q}^{*}\right) \subseteq \mathbb{F}_{q}^{*}$ and $r_{n}(H) \subseteq H$.

Proposition 5. Let $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be the $n$-decomposition of $q-1$ and $q+1$, respectively. Let $\alpha \in \mathbb{F}_{q}^{*}$ and $\beta \in H$ be two $r_{n}$-periodic points. Then $\operatorname{Tree}_{\alpha}\left(r_{n} / \mathbb{F}_{q}^{*}\right)=T_{\nu_{0}(n)}$ and $\operatorname{Tree}_{\beta}\left(r_{n} / H\right)=T_{\nu_{1}(n)}$.

Proof. The sets $\mathbb{F}_{q}^{*}$ and $H$ are multiplicative cyclic groups of order $q-1$ and $q+1$, respectively. In general, if $G$ is a multiplicative cyclic group of order $m=v \omega$ with $\operatorname{rad}(v) \mid \operatorname{rad}(n), \operatorname{gcd}(n, \omega)=1$, and $r_{n}: G \rightarrow G$ is the map given by $r_{n}(g)=g^{n}$ we prove that $\operatorname{Tree}_{g_{0}}\left(r_{n} / G\right)=T_{\nu(n)}$. Indeed, if $\xi$ is a generator of $G$ and
$\phi: \mathbb{Z}_{m} \rightarrow G$ is the map given by $\phi(i)=\xi^{i}$, then $r_{n} \circ \phi(i)=\left(\xi^{i}\right)^{n}=\xi^{n i}=\phi \circ n(i)$ (where $n$ denotes the multiplication-by- $n$ map). This implies that $\phi: \mathcal{G}\left(n / \mathbb{Z}_{m}\right) \rightarrow \mathcal{G}\left(r_{n} / G\right)$ is an isomorphism of functional graphs. Since all the trees attached to periodic points in $\mathcal{G}\left(n / \mathbb{Z}_{m}\right)$ are isomorphic to $T_{\nu(n)}$ (Theorem 1 ) the same occurs for the trees attached to periodic points in $\mathcal{G}\left(r_{n} / G\right)$.

Proposition 6. If $n \geq 1$ is an odd integer and $a \in \mathbb{F}_{q}$, then $\operatorname{Tree}_{a}\left(T_{n} / \mathbb{F}_{q}\right)$ and Tree $e_{-a}\left(T_{n} / \mathbb{F}_{q}\right)$ are isomorphic.

Proof. Consider the map op : $\mathbb{F}_{q} \rightarrow \mathbb{F}_{q}$ given by op $(x)=-x$. Since $n$ is an odd integer, the Chebyshev polynomial is an odd function and we have op $\circ T_{n}=T_{n} \circ$ op. Thus op: $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right) \rightarrow \mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ is an isomorphism of functional graphs and the results follows from Proposition 4,

Proposition 7. Let $\alpha \in \tilde{\mathbb{F}}_{q}$. Then, Tree $_{\alpha}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ and Tree $_{\alpha^{-1}}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ are isomorphic.
Proof. We consider the isomorphism of functional graphs $i: \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right) \rightarrow \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ given by $i(x)=x^{-1}$ (it is an isomorphism because $i: \tilde{\mathbb{F}}_{q} \rightarrow \tilde{\mathbb{F}}_{q}$ is bijective and $i \circ r_{n}=r_{n} \circ i$ ). The results follows from Proposition 4.

Proposition 8. Let $\alpha \in \tilde{\mathbb{F}}_{q}$ with $\alpha \neq \pm 1$ and $a=\eta(\alpha)$. Then, Tree $e_{\alpha}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ and Tree ${ }_{a}\left(T_{n} / \mathbb{F}_{q}\right)$ are isomorphic.

Proof. We consider the homomorphism $\eta: \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right) \rightarrow \mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ (it is a homomorphism because $\eta \circ r_{n}=T_{n} \circ \eta$ ). This homomorphism is in fact a $i$-covering because $\eta^{-1}(a)=\left\{\alpha, i(\alpha)=\alpha^{-1}\right\}$ where $\alpha \in \tilde{\mathbb{F}}_{q}$ is a root of $x^{2}-a x+1=0$. We note that $\alpha \in \tilde{\mathbb{F}}_{q}$ is not $i$-maximal if and only if $\alpha=\alpha^{-1}$ since $i$ is a permutation of order 2 ; this is equivalent to $\alpha= \pm 1$. If $\alpha \neq \pm 1$, then $\alpha$ is $i$-maximal and the result follows from Proposition 4.

## IV. Splitting the functional graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ into uniform components

The most simple case of functional graph is when the trees attached to the periodic points are isomorphic. In this case describing the functional graph is equivalent to describing the cycle decomposition of the periodic points and the rooted tree attached to any periodic point. We start with a definition.

Definition 3. A functional graph $\mathcal{G}(f / X)$ is uniform if for every pair of periodic points $x, x^{\prime} \in X$ the


In this section we decompose the set $\mathbb{F}_{q}$ in three $T_{n}$-invariant sets: $R$ (the rational component), $Q$ (the quadratic component) and $S$ (the special component), obtaining a decomposition of the Chebyshev
functional graph

$$
\begin{equation*}
\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)=\mathcal{G}\left(T_{n} / R\right) \oplus \mathcal{G}\left(T_{n} / Q\right) \oplus \mathcal{G}\left(T_{n} / S\right) . \tag{2}
\end{equation*}
$$

Moreover, we prove that the functional graphs of the right hand side are uniform (Proposition 10). We describe each component separately.

Lemma 1. We have $X \subseteq \mathbb{F}_{q}$ is $T_{n}$-invariant if and only if $\eta^{-1}(X)$ is $r_{n}$-invariant.
Proof. $(\Rightarrow)$ Let $\alpha \in \eta^{-1}(X)$. We have $\eta(\alpha) \in X$ and $T_{n}(\eta(\alpha)) \in X$ (because $X$ is forward $T_{n}$-invariant). Therefore $\eta\left(r_{n}(\alpha)\right)=T_{n}(\eta(\alpha)) \in X$ and then $r_{n}(\alpha) \in \eta^{-1}(X)$. This proves that $\eta^{-1}(X)$ is forward $r_{n^{-}}$ invariant. Now we consider $\beta \in \tilde{\mathbb{F}}_{q}$ such that $r_{n}(\beta)=\alpha \in \eta^{-1}(X)$. Then $T_{n}(\eta(\beta))=\eta\left(r_{n}(\beta)\right) \in X$. Since $X$ is backward $T_{n}$-invariant $\eta(\beta) \in X$, thus $\beta \in \eta^{-1}(X)$. This proves that $\eta^{-1}(X)$ is backward $r_{n}$-invariant.
$(\Leftarrow)$ Let $x \in X$. Since $\eta$ is surjective we can write $x=\eta(\alpha)$ for some $\alpha \in \tilde{\mathbb{F}}_{q}$. We have $\alpha \in \eta^{-1}(X)$ and using that $\eta^{-1}(X)$ is forward $r_{n}$-invariant we also have $r_{n}(\alpha) \in \eta^{-1}(X)$. Thus $T_{n}(x)=T_{n}(\eta(\alpha))=\eta\left(r_{n}(\alpha)\right) \in$ $X$. This proves that $X$ is forward $T_{n}$-invariant. Now we consider $y \in \mathbb{F}_{q}$ such that $T_{n}(y)=x \in X$ and we can write $y=\eta(\beta)$ with $\beta \in \tilde{\mathbb{F}}_{q}$ since $\eta$ is surjective. We have that $T_{n}(y)=T_{n}(\eta(\beta))=\eta\left(r_{n}(\beta)\right) \in X$, thus $r_{n}(\beta) \in \eta^{-1}(X)$. Using that $\eta^{-1}(X)$ is backward $r_{n}$-invariant we conclude that $\beta \in \eta^{-1}(X)$. Therefore $y=\eta(\beta) \in X$ which proves that $X$ is backward $T_{n}$-invariant.

Using the characterizations $\mathbb{F}_{q}^{*}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{ord}(\alpha) \mid q-1\right\}$ and $H=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{ord}(\alpha) \mid q+1\right\}$, we obtain the following decomposition of $\tilde{\mathbb{F}}_{q}$ into $r_{n}$-invariant subsets.

Lemma 2. The subsets $\tilde{S}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \alpha^{n^{k}}= \pm 1\right.$ for some $\left.k \geq 0\right\}, \tilde{R}=\mathbb{F}_{q}^{*} \backslash \tilde{S}$ and $\tilde{Q}=H \backslash \tilde{S}$ form a partition of $\tilde{\mathbb{F}}_{q}$ in $r_{n}$-invariant subsets.

Proof. Since $( \pm 1)^{n} \subseteq\{ \pm 1\}$, the set $\tilde{S}$ is forward $r_{n}$-invariant. If $\alpha^{n} \in \tilde{S}$ there exists $k \geq 0$ such that $\left(\alpha^{n}\right)^{n^{k}}=\alpha^{n^{k+1}}= \pm 1$. Thus $\alpha \in \tilde{S}$ and $\tilde{S}$ is backward $r_{n}$-invariant. This proves that $\tilde{S}$ is $r_{n}$-invariant.

The proofs of the $r_{n}$-invariance of $\tilde{R}$ and $\tilde{Q}$ are similar. We only prove that $\tilde{R}$ is $r_{n}$-invariant. It is easy to prove that the complement of an $r_{n}$-invariant is $r_{n}$-invariant and the intersection of two $r_{n}$ invariant sets is also $r_{n}$-invariant. Since $R=\mathbb{F}_{q}^{*} \cap \tilde{S}^{c}$, it suffices to prove that $\mathbb{F}_{q}^{*}$ is $r_{n}$-invariant. It is clear that $\mathbb{F}_{q}^{*}$ is forward $r_{n}$-invariant. To prove that $\mathbb{F}_{q}^{*}$ is backward $r_{n}$-invariant we use the characterization $\mathbb{F}_{q}^{*}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{ord}(\alpha) \mid q-1\right\}$. We consider $\beta \in \tilde{\mathbb{F}}_{q}$ such that $r_{n}(\beta)=\beta^{n} \in \mathbb{F}_{q}^{*}$. The multiplicative order of $\beta^{n}$ is given by $\operatorname{ord}\left(\beta^{n}\right)=\operatorname{ord}(\beta) / d$ with $d=\operatorname{gcd}(\operatorname{ord}(\beta), n)$. In particular $\operatorname{ord}(\beta) \mid q-1$ (because $\operatorname{ord}(\beta) \mid \operatorname{ord}\left(\beta^{n}\right)$ and $\left.\operatorname{ord}\left(\beta^{n}\right) \mid q-1\right)$, therefore $\beta \in \mathbb{F}_{q}^{*}$ by the above characterization of $\mathbb{F}_{q}^{*}$.

Proposition 9. Let $R=\eta(\tilde{R}), Q=\eta(\tilde{Q})$ and $S=\eta(\tilde{S})$. The sets $R, Q$ and $S$ form a partition of $\mathbb{F}_{q}$ in $T_{n}$-invariant sets. In particular the decomposition of $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ given by (2) holds.

Proof. It is straightforward to check that $\tilde{R}, \tilde{Q}$ and $\tilde{S}$ are $i$-invariant from which we obtain $\eta^{-1}(R)=\tilde{R}$, $\eta^{-1}(Q)=\tilde{Q}$ and $\eta^{-1}(S)=\tilde{S}$. By Lemma 2 these sets are $r_{n}$-invariant, and by Lemma $1 R, Q$ and $S$ are $T_{n}$-invariant.

We finish this section proving that the functional graphs $\mathcal{G}\left(T_{n} / R\right), \mathcal{G}\left(T_{n} / Q\right)$ and $\mathcal{G}\left(T_{n} / S\right)$ are uniform.
Proposition 10. The functional graphs $\mathcal{G}\left(T_{n} / R\right), \mathcal{G}\left(T_{n} / Q\right)$ and $\mathcal{G}\left(T_{n} / S\right)$ are uniform. Moreover, every tree attached to a $T_{n}$-periodic point in $\mathcal{G}\left(T_{n} / R\right)$ is isomorphic to $T_{\nu_{0}(n)}$ and every tree attached to a $T_{n}$-periodic point in $\mathcal{G}\left(T_{n} / Q\right)$ is isomorphic to $T_{\nu_{1}(n)}$.

Proof. The easy case is to prove that $\mathcal{G}\left(T_{n} / S\right)$ is uniform, the other two cases are similar and we prove only that $\mathcal{G}\left(T_{n} / R\right)$ is uniform. If $n$ or $q$ is even, the only $T_{n}$-periodic point in $S$ is 2 and there is nothing to prove. If $n$ and $q$ are odd there are two $T_{n}$-periodic points in $S, 2$ and -2 , and the uniformity of $\mathcal{G}\left(T_{n} / S\right)$ follows from Proposition 6

We denote by $P_{f}$ the set of periodic points with respect to $f$ and consider $a \in R \cap P_{f}$. We can write $a=$ $\eta(\alpha)$ for some $\alpha \in \tilde{R}$ (in particular $a \in \mathbb{F}_{q}^{*}$ and $a \neq \pm 1$ ). By Proposition 8 . Tree ${ }_{a}\left(T_{n} / \mathbb{F}_{q}\right)$ and $\operatorname{Tree}_{\alpha}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ are isomorphic. Using that $\mathbb{F}_{q}^{*}$ is $r_{n}$-invariant and $a \in \mathbb{F}_{q}^{*}$ we have $\operatorname{Tree}_{\alpha}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)=\operatorname{Tree}_{\alpha}\left(r_{n} / \mathbb{F}_{q}^{*}\right)$ and by Proposition 1 (considering the $i$-covering $\eta: \mathcal{G}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right) \rightarrow \mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ ) we have that $\alpha$ is an $r_{n}$-periodic point. By Proposition 5 we have that $\operatorname{Tree}_{\alpha}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ is isomorphic to $T_{\nu_{0}(n)}$ and by transitivity $\operatorname{Tree}_{a}\left(T_{n} / \mathbb{F}_{q}\right)$ is also isomorphic to $T_{\nu_{0}(n)}$.

## V. The rational and quadratic components

In this section we describe the functional graphs $\mathcal{G}\left(T_{n} / R\right)$ and $\mathcal{G}\left(T_{n} / Q\right)$.
The following proposition is a simple generalization of Proposition 2.1 of [7] for the general $n$ case and is proved in a similar way.

Proposition 11. Let $a \in \mathbb{F}_{q}, \alpha \in \tilde{\mathbb{F}}_{q}$ such that $a=\alpha+\alpha^{-1}$ and $\operatorname{ord}(\alpha)=u d$ the $n$-decomposition of the (multiplicative) order of $\alpha$. Then $\operatorname{per}(a)=\tilde{o}_{d}(n)$ and $\operatorname{pper}(a)=\min \left\{k \geq 0: u \mid n^{k}\right\}$.

Proof. Let $\pi=\operatorname{per}(a)$ and $\rho=\operatorname{per}(a)$. Consider the following equivalences:

$$
\begin{aligned}
T_{n}^{\pi+\rho}(a)=T_{n}^{\rho}(a) & \Leftrightarrow T_{n^{\pi+\rho}}(a)=T_{n^{\rho}}(a) \\
& \Leftrightarrow \alpha^{n^{\pi+\rho}}+\alpha^{-n^{\pi+\rho}}=\alpha^{n^{\rho}}+\alpha^{-n^{\rho}} \\
& \Leftrightarrow\left(\alpha^{n^{\pi+\rho}}-\alpha^{n^{\rho}}\right)\left(\alpha^{n^{\pi+\rho}}-\alpha^{-n^{\rho}}\right)=0 \\
& \Leftrightarrow \alpha^{n^{\pi+\rho}}=\alpha^{n^{\rho}} \text { or } \alpha^{n^{\pi+\rho}}=\alpha^{-n^{\rho}} \\
& \Leftrightarrow n^{\pi+\rho} \equiv \pm n^{\rho}(\bmod u d) \\
& \Leftrightarrow n^{\pi} \equiv \pm 1(\bmod d) \text { and } u \mid n^{\rho} .
\end{aligned}
$$

By minimality, we conclude that $\pi=\tilde{o}_{d}(n)$ and $\rho=\min \left\{k \geq 0: u \mid n^{k}\right\}$.
Corollary 1. Let $\alpha \in \tilde{\mathbb{F}}_{q}$. The point $a=\alpha+\alpha^{-1} \in \mathbb{F}_{q}$ is $T_{n}$-periodic point if and only if the multiplicative order of $\alpha$ (as element of $\mathbb{F}_{q^{2}}^{*}$ ) is coprime with $n$.

Proof. Let $a=\alpha+\alpha^{-1} \in \mathbb{F}_{q}$ and $\operatorname{ord}(\alpha)=u d$ be the $n$-decomposition of the (multiplicative) order of $\alpha$. We have that $a$ is $T_{n}$-periodic point if and only if $\operatorname{pper}(a)=0$ and by Proposition 11 this happens if and only if $u \mid 1$, that is, if and only if $u=1$ and $\operatorname{gcd}(\operatorname{ord}(\alpha), n)=1$.

Corollary 2. Let $P_{T_{n}}$ be the set of $T_{n}$-periodic points, $\alpha \in \tilde{\mathbb{F}}_{q}$ and $a=\alpha+\alpha^{-1}$.

1. $a \in R \cap P_{T_{n}}$ if and only if $\operatorname{ord}(\alpha)>2$ and $\operatorname{ord}(\alpha) \mid \omega_{0}$;
2. $a \in Q \cap P_{T_{n}}$ if and only if $\operatorname{ord}(\alpha)>2$ and $\operatorname{ord}(\alpha) \mid \omega_{1}$;
3. $a \in S \cap P_{T_{n}}$ if and only if $\operatorname{ord}(\alpha) \leq 2$ and $\operatorname{gcd}(\operatorname{ord}(\alpha), n)=1$.

Proof. Since $\eta$ is surjective, $\eta\left(\eta^{-1}(X)\right)=X$ for all $X \subseteq \mathbb{F}_{q}$ (in particular $a \in X$ if and only if $\alpha \in \eta^{-1}(X)$ ). Denote $\tilde{P}_{T_{n}}:=\eta^{-1}\left(P_{T_{n}}\right)$. By Corollary 1, $\tilde{P}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{gcd}(\operatorname{ord}(\alpha), n)=1\right\}$. First we prove that $\tilde{P}_{T_{n}} \cap \tilde{S}=\tilde{P}_{T_{n}} \cap\{+1\}$. Indeed, if $\alpha \in \tilde{P}_{T_{n}} \cap \tilde{S}$, then $\operatorname{gcd}(\operatorname{ord}(\alpha), n)=1$ and $\alpha^{n^{k}}= \pm 1$ for some $k \geq 0$. Thus $\left.\operatorname{ord}(\alpha)=\frac{\operatorname{ord}(\alpha)}{\operatorname{gcd}\left(\operatorname{Ord}(\alpha), n^{k}\right)}=\operatorname{ord}\left(\alpha^{n^{k}}\right)=\operatorname{ord}( \pm 1) \right\rvert\, 2$ which implies $\alpha= \pm 1$. This proves that $\tilde{P}_{T_{n}} \cap \tilde{S} \subseteq$ $\tilde{P}_{T_{n}} \cap\{+1\}$ and the other inclusion is clear. We note that this is equivalent to $\tilde{P}_{T_{n}} \cap \tilde{S}^{c}=\tilde{P}_{T_{n}} \cap\{+1\}^{c}$. Now we prove the statements.

1. $a \in R \cap P_{T_{n}}$ if and only if $\alpha \in \tilde{R} \cap \tilde{P}_{T_{n}}=\mathbb{F}_{q}^{*} \cap \tilde{S}^{c} \cap \tilde{P}_{T_{n}}=\tilde{P}_{T_{n}} \cap \mathbb{F}_{q}^{*} \cap\{ \pm 1\}^{c}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{gcd}(\operatorname{ord}(\alpha), n)=\right.$ $1, \operatorname{ord}(\alpha) \mid q-1, \alpha \neq \pm 1\}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{ord}(\alpha) \mid \omega_{0}, \operatorname{ord}(\alpha)>2\right\}$.
2. This part is similar to 1 .; here we use $\alpha \in H$ if and only if $\operatorname{ord}(\alpha) \mid q+1$.
3. $a \in S \cap P_{T_{n}}$ if and only if $\alpha \in \tilde{S} \cap \tilde{P}_{T_{n}}=\tilde{P}_{T_{n}} \cap\{ \pm 1\}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{gcd}(\operatorname{ord}(\alpha), n)=1, \operatorname{ord}(\alpha) \leq 2\right\}$.

Next we obtain an isomorphism formula for the rational component and the quadratic component of $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$.

Theorem 2. Let $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be their $n$-decompositions. The rational component of the Chebyshev's graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ is given by:

$$
\mathcal{G}\left(T_{n} / R\right)=\bigoplus_{\substack{d \mid \omega_{0} \\ d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \times \operatorname{Cyc}\left(\tilde{o}_{d}(n), T_{\nu_{0}(n)}\right) ;
$$

the quadratic component is given by

$$
\mathcal{G}\left(T_{n} / Q\right)=\bigoplus_{\substack{d \mid \omega_{1} \\ d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \times \operatorname{Cyc}\left(\tilde{o}_{d}(n), T_{v_{1}(n)}\right) .
$$

Proof. We only prove the statement for the rational component since the proof for the quadratic component is similar. Let $P_{T_{n}}$ be the set of $T_{n}$-periodic points and $R_{d}=\left\{\alpha+\alpha^{-1}: \alpha \in \tilde{\mathbb{F}}_{q}\right.$, $\left.\operatorname{ord}(\alpha)=d\right\}$. By Corollary 2, $R \cap P_{T_{n}}$ is the disjoint union of $R_{d}$ with $d \mid \omega_{0}, d>2$. If $\operatorname{ord}(\alpha)=d \mid \omega_{0}$ we have that $\operatorname{gcd}(d, n)=1$ and $\operatorname{ord}\left(\alpha^{n}\right)=\operatorname{ord}(\alpha) / \operatorname{gcd}(\operatorname{ord}(\alpha), n)=\operatorname{ord}(\alpha)$. Then we have the following decomposition $\mathcal{G}\left(T_{n} / R \cap P_{T_{n}}\right)=\bigoplus_{\substack{d \mid \omega_{0} \\ d>2}} \mathcal{G}\left(T_{n} / R_{d}\right)$. By Proposition 11 , every point in $\mathcal{G}\left(T_{n} / R_{d}\right)$ belongs to a cycle of length $\tilde{o}_{d}(n)$. Thus,

$$
\begin{equation*}
\mathcal{G}\left(T_{n} / R \cap P_{T_{n}}\right)=\bigoplus_{\substack{d \mid \omega_{0} \\ d>2}} \frac{\# R_{d}}{\tilde{o}_{d}(n)} \times \operatorname{Cyc}\left(\tilde{o}_{d}(n), \bullet\right) \tag{3}
\end{equation*}
$$

For each $d \mid \omega_{0}, d>2$, we consider the set $\tilde{R}_{d}=\left\{\alpha \in \tilde{\mathbb{F}}_{q}: \operatorname{ord}(\alpha)=d\right\}$. By a standard counting argument $\# \tilde{R}_{d}=\varphi(d)$ and using that the restriction of $\eta$ to $\tilde{R}$ is a 2-to-1 map from $\tilde{R}$ onto $R$ we obtain $\# R=\# \tilde{R} / 2=\varphi(d) / 2$. Substituting this expression into Equation (3) and using the uniformity of $\mathcal{G}\left(T_{n} / R\right)$ (Proposition 10 we obtain $\mathcal{G}\left(T_{n} / R\right)=\bigoplus_{d \mid \omega_{0}}^{d>2} \frac{\varphi(d)}{2 \tilde{\sigma}_{d}(n)} \times \operatorname{Cyc}\left(\tilde{o}_{d}(n), T_{\nu_{0}(n)}\right)$.

## VI. The special component of $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$

In this section we describe the special component of the Chebyshev functional graph $\mathcal{G}\left(T_{n} / S\right)$ where $S=\left\{a \in \mathbb{F}_{q}: T_{n}(a)^{(k)}= \pm 2\right.$, for some $\left.k \geq 0\right\}$. If $n$ and $q$ are odd, $T_{n}(-2)=-2$ and $T_{n}(2)=2$ then the only periodic points of $T_{n}$ in $S$ are 2 and -2 . In this case the trees attached to the fixed points 2 and -2 are isomorphic (Proposition 10). If either $n$ is even or $q$ is even, $T_{n}(-2)=2=T_{n}(2)$ and the only periodic point of $T_{n}$ in $S$ is 2 (if $q$ is even this is true because $2=-2$ ). The next proposition summarizes the above discussion.

Proposition 12. Let $\mathcal{T}=\operatorname{Tree} e_{2}\left(T_{n} / \mathbb{F}_{q}\right)$ be the rooted tree attached to the fixed point 2 for the Chebyshev polynomial $T_{n}$ restricted to the set $S=\left\{a \in \mathbb{F}_{q}: T_{n}(a)^{(k)}= \pm 2\right.$, for some $\left.k \geq 0\right\}$. Then

$$
\mathcal{G}\left(T_{n} / S\right)= \begin{cases}2 \times \operatorname{Cyc}(1, T) & \text { if } n \text { is odd and } q \text { is odd; } \\ \operatorname{Cyc}(1, T) & \text { otherwise. }\end{cases}
$$

We remark that $\operatorname{Tree}_{2}\left(T_{n} / S\right)=\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)$, which is a consequence of $S$ being $T_{n}$-invariant (Proposition (9). By Proposition 12, to describe the special component it suffices to describe the tree $\mathcal{T}=$ $\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)$. If $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ is the $n$-decomposition of $q-1$ and $q+1$, respectively, the rooted trees attached to the periodic points are isomorphic to $T_{\nu_{0}(n)}$ in the rational component and isomorphic to $T_{\nu_{1}(n)}$ in the quadratic component (Proposition 10). In the case of the special component the situation is different, the tree $\mathcal{T}=\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)$ is not isomorphic to a tree associated to a $v$-series (that is, the trees associated to the multiplication by $n$ map over $\mathbb{Z}_{m}$ for some $m \in \mathbb{Z}^{+}$). However we show in this section that the tree $\mathcal{T}$ can be expressed as a "mean" of the trees $T_{\nu_{0}(n)}$ and $T_{\nu_{1} 0(n)}$. In the first part of this section we define the bisection of trees together some of their main properties. In the second part we deduce an isomorphism formula for the special component of the Chebyshev graph.

## A. Bisection of rooted trees

We start by defining the sum of rooted trees.
Definition 4. Let $T=\left\langle T_{1} \oplus T_{2} \oplus \cdots \oplus T_{r}\right\rangle$ and $T^{\prime}=\left\langle T_{1}^{\prime} \oplus T_{2}^{\prime} \oplus \cdots \oplus T_{s}^{\prime}\right\rangle$ be two rooted trees. We define their sum as $T+T^{\prime}=\left\langle T_{1} \oplus T_{2} \oplus \cdots \oplus T_{r} \oplus T_{1}^{\prime} \oplus T_{2}^{\prime} \oplus \cdots \oplus T_{s}^{\prime}\right\rangle$.

We remark that the tree consisting of a unique node $T=\bullet=\langle\emptyset\rangle$ is the neutral element of the sum. The tree $T-T^{\prime}$ denotes a tree such that $T=T^{\prime}+\left(T-T^{\prime}\right)$ in case this tree exists (if it exists, it is unique up to isomorphism). We note that $\left(T_{1}+T_{2}\right)-T^{\prime}$ is defined if and only if $T_{i}-T^{\prime}$ is defined for some $i=1$, 2. If $T_{1}-T^{\prime}$ is defined then $\left(T_{1}+T_{2}\right)-T^{\prime}=\left(T_{1}-T^{\prime}\right)+T_{2}$ and if $T_{2}-T^{\prime}$ is defined then $\left(T_{1}+T_{2}\right)-T^{\prime}=T_{1}+\left(T_{2}-T^{\prime}\right)$. Therefore when $\left(T_{1}+T_{2}\right)-T^{\prime}$ is defined we can write this tree as $T_{1}+T_{2}-T^{\prime}$ without ambiguity.

A forest is a graph that can be expressed as a disjoint union of rooted trees. A tree $T$ is even if it can be expressed as $T=\langle 2 \times F\rangle$ for some forest $F$ and it is quasi-even if it can be expressed as $T=\left\langle 2 \times F \oplus T^{\prime}\right\rangle$ for some forest $F$ and some even tree $T^{\prime}$ (i.e. $T^{\prime}=\left\langle 2 \times F^{\prime}\right\rangle$ for some forest $F^{\prime}$ ). In particular the tree $T=\bullet$ is even because $T=\langle 2 \times \emptyset\rangle$. For these classes of trees we define the bisection as follows.

Definition 5. If $T=\langle 2 \times F\rangle$ is an even tree, its bisection is the tree $\frac{1}{2} T=\langle F\rangle$. If $T=\left\langle 2 \times F \oplus\left\langle 2 \times F^{\prime}\right\rangle\right\rangle$ is a quasi-even tree its bisection is defined as the tree $\frac{1}{2} T=\left\langle F \oplus\left\langle F^{\prime}\right\rangle\right\rangle$.

Example 2. The tree associated with the v-series $18(30)=(6,3)$ is given by $T_{(6,3)}=\left\langle 2 \times T \oplus 3 \times T^{\prime}\right\rangle$ where $T=\langle 6 \times \bullet\rangle$ and $T^{\prime}=\bullet$. Thus $T_{(6,3)}$ is quasi-even since it can be written as $T_{(6,3)}=\left\langle 2 \times F \oplus T^{\prime}\right\rangle$ with $F=T \oplus T^{\prime}$ and $T^{\prime}=\langle 2 \times \emptyset\rangle$ is even. The bisection of this tree is given by $\frac{1}{2} T_{(6,3)}=\langle F \oplus\langle\emptyset\rangle\rangle=\left\langle T \oplus 2 \times T^{\prime}\right\rangle$.

Even and quasi-even trees are very restricted classes of trees, however they contain all trees associated with $v$-series as stated in the following proposition.

Proposition 13. If $T_{v(n)}$ is the tree associated with $v(n)=\left(v_{1}, \ldots, v_{D}\right)$, then $T_{\nu(n)}$ is even when $v$ is odd and quasi-even when $v$ is even.

Proof. By Equation 11 we have $T_{\nu(n)}=\left\langle\left(v_{D}-1\right) \times T^{D-1} \oplus \bigoplus_{i=1}^{D-1}\left(v_{i}-v_{i+1}\right) \times T^{i-1}\right\rangle$, where the $T_{i}$ are pairwise non-isomorphic rooted trees. When $v$ is odd, $v_{i}$ is odd for $1 \leq i \leq D$. Then, $v_{D}-1$ and $v_{i}-v_{i+1}$ are even for $1 \leq i \leq D-1$ and the tree $T_{\nu(n)}$ is even. When $v$ is even, we have that $v_{1}, \ldots, v_{k}$ are even and $v_{k+1}, \ldots, v_{D}$ are odd for some $k, 1 \leq k \leq D$. If $k=D$, then $v_{D}-1$ is odd and $v_{i}-v_{i+1}$ are even for $1 \leq i \leq D-1$ and the tree $T_{\nu(n)}$ is quasi-even. If $k<D$, then $v_{D}-1$ and $v_{i}-v_{i+1}$ are even for $1 \leq i \leq k-1$ and $k+1 \leq i \leq D$, and $v_{k}-v_{k+1}$ is odd. Thus, $T_{\nu(n)}$ is also quasi-even.

We note that the if $T_{1}$ and $T_{2}$ are rooted trees, then $\left|T_{1}+T_{2}\right|=\left|T_{1}\right|+\left|T_{2}\right|-1$ where, as usual, $|T|$ denotes the number of nodes of $T$. The next proposition establishes a relation between $|T|$ and $\left|\frac{1}{2} T\right|$.

Proposition 14. Let $T$ be a rooted tree with $|T|=N$ nodes. We have

$$
|1 / 2 \cdot T|= \begin{cases}\frac{N+1}{2} & \text { if } T \text { is even } \\ \frac{N+2}{2} & \text { if } T \text { is quasi-even. }\end{cases}
$$

Proof. If $T$ is even, there is a forest $S$ with $s$ nodes such that $T=\langle 2 \times S\rangle$. We have $N=|T|=1+2 s$ from which we obtain $s=\frac{N-1}{2}$. Since $\frac{1}{2} T=\langle S\rangle,\left|\frac{1}{2} T\right|=s+1=\frac{N-1}{2}+1=\frac{N+1}{2}$. If $T$ is quasi-even, there is a pair of forests $S$ and $R$ with $s$ and $r$ nodes, respectively, such that $T=\langle 2 \times S \oplus\langle 2 \times R\rangle\rangle$. We have $N=|T|=1+2 s+1+2 r=2(r+s+1)$ from which we obtain $r+s+1=\frac{N}{2}$. Since $\frac{1}{2} T=\langle S \oplus\langle R\rangle\rangle$, $\left|\frac{1}{2} T\right|=1+s+1+r=1+\frac{N}{2}=\frac{N+2}{2}$.

## B. The tree $\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)$

The next theorem describe the rooted tree attached to the fixed point 2 for the Chebyshev polynomial $T_{n}: \mathbb{F}_{q} \rightarrow \mathbb{F}_{q}$. We require the following lemma.

Lemma 3. Let $n>1$ be an even integer, $\mathbb{F}_{q}$ be an odd characteristic finite field and $H$ be the multiplicative subgroup of $\mathbb{F}_{q^{2}}^{*}$ with order $q+1$.
(i) If $q \equiv 3(\bmod 4)$, the equation $x^{n}=-1$ has no solution in $\mathbb{F}_{q}^{*}$.
(ii) If $q \equiv 1(\bmod 4)$, the equation $x^{n}=-1$ has no solution in $H$.

Proof. Let $\alpha \in \tilde{\mathbb{F}}_{q}$ be a solution of $x^{n}=-1$. From the relations $\operatorname{ord}\left(\alpha^{n}\right)=\operatorname{ord}(\alpha) / \operatorname{gcd}(\operatorname{ord}(\alpha), n)$ and $\operatorname{ord}(-1)=2$, we conclude that if $n$ is even, then $4 \mid \operatorname{ord}(\alpha)$. By Lagrange theorem, $\alpha \in \mathbb{F}_{q}^{*}$ implies $4 \mid q-1$ and $q \not \equiv 3(\bmod 4) ;$ and $\alpha \in H$ implies $4 \mid q-3$ and $q \not \equiv 1(\bmod 4)$.

Theorem 3. Let $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be their $n$-decompositions. The rooted tree associated with the fixed point 2 is described as follows:

$$
\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)= \begin{cases}1 / 2 \cdot T_{v_{0}(n)}+1 / 2 \cdot T_{v_{1}(n)} & \text { if } n \text { is odd or } q \text { is even; } \\ 1 / 2 \cdot T_{v_{0}(n)}+1 / 2 \cdot T_{v_{1}(n)}-\langle\bullet\rangle & \text { if } n \text { is even and } q \text { is odd. }\end{cases}
$$

Proof. The isomorphism formula is obtained after relating $\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)$ and $\operatorname{Tree}_{1}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$. First we consider the case when $n$ is odd or $q$ is even. In this case $r_{n}(-1)=-1$ or $-1=1$, in both cases we have that the predecessors of 1 in $\operatorname{Tree}_{1}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ are in $\mathbb{F}_{q}^{*}$ or in $H$ (but not in both). Since the sets $\mathbb{F}_{q}^{*}$ and $H$ are backward $r_{n}$-invariant (Lemma 2), we have

$$
\operatorname{Tree}_{1}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)=\operatorname{Tree}_{1}\left(r_{n} / \mathbb{F}_{q}^{*}\right)+\operatorname{Tree}_{1}\left(r_{n} / H\right)=T_{v_{0}(n)}+T_{\nu_{1}(n)},
$$

where in the last equality we use Proposition 5. Now, we write $r_{n}^{-1}(1) \cap \mathbb{F}_{q}^{*}=\left\{\alpha_{1}, \ldots, \alpha_{2 s}, 1\right\}$ with $\alpha_{s+i}=\alpha_{i}^{-1}, \alpha_{i} \neq \pm 1$, for all $i: 1 \leq i \leq s$ and $r_{n}^{-1}(1) \cap H=\left\{\beta_{1}, \ldots, \beta_{2 t}, 1\right\}$ with $\beta_{t+j}=\beta_{j}^{-1}, \beta_{j} \neq \pm 1$, for all $j: 1 \leq j \leq t$. Denote by $\tilde{T}\left(\alpha_{i}\right):=\operatorname{Tree}_{\alpha_{i}}\left(r_{n} / \mathbb{F}_{q}^{*}\right)$ for $1 \leq i \leq 2 s$ and $\tilde{T}\left(\beta_{j}\right):=\operatorname{Tree}_{\beta_{j}}\left(r_{n} / H\right)$ for $1 \leq j \leq 2 t$. Using Proposition 7 we have that $T_{\nu_{0}(n)}=\operatorname{Tree}_{1}\left(r_{n} / \mathbb{F}_{q}^{*}\right)=\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{2 s}\right)\right\rangle=$ $\left\langle 2 \times\left(\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right)\right)\right\rangle$, from which we obtain

$$
1 / 2 \cdot T_{\nu_{0}(n)}=\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right)\right\rangle .
$$

In the same way we obtain

$$
1 / 2 \cdot T_{\nu_{1}(n)}=\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right)\right\rangle
$$

Let $a_{i}=\eta\left(\alpha_{i}\right), T\left(a_{i}\right)=\operatorname{Tree}_{a_{i}}\left(T_{n} / \mathbb{F}_{q}\right), b_{j}=\eta\left(\alpha_{j}\right)$ and $T\left(b_{j}\right)=\operatorname{Tree}_{b_{j}}\left(T_{n} / \mathbb{F}_{q}\right)$ for $1 \leq i \leq s, 1 \leq j \leq t$. We have $T_{n}^{-1}(2)=\left\{a_{1}, \ldots, a_{s}, b_{1}, \ldots, b_{t}, 2\right\}$ and

$$
\begin{aligned}
\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right) & =\left\langle T\left(a_{1}\right) \oplus \cdots \oplus T\left(a_{s}\right) \oplus T\left(b_{1}\right) \oplus \cdots \oplus T\left(b_{t}\right)\right\rangle \\
& =\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right) \oplus \tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right)\right\rangle \quad \text { (by Prop. 8) } \\
& =\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right)\right\rangle+\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right)\right\rangle \\
& =1 / 2 \cdot T_{v_{0}(n)}+1 / 2 \cdot T_{v_{1}(n)} .
\end{aligned}
$$

Now we consider the case when $n$ is even and $q$ is odd. Here we can write $r_{n}^{-1}(1) \cap \mathbb{F}_{q}^{*}=\left\{\alpha_{1}, \ldots, \alpha_{2 s},-1,1\right\}$ with $\alpha_{s+i}=\alpha_{i}^{-1}, \alpha_{i} \neq \pm 1$, for all $i: 1 \leq i \leq s, r_{n}^{-1}(1) \cap H=\left\{\beta_{1}, \ldots, \beta_{2 t},-1,1\right\}$ with $\beta_{t+j}=\beta_{j}^{-1}, \beta_{j} \neq \pm 1$, for all $j: 1 \leq j \leq t$ and $r_{n}^{-1}(-1)=\left\{\gamma_{1}, \ldots, \gamma_{2 r}\right\}$ with $\gamma_{r+k}=\gamma_{k}^{-1}, \gamma_{k} \neq \pm 1$, for all $k: 1 \leq k \leq r$.

Denote by $\tilde{T}\left(\alpha_{i}\right):=\operatorname{Tree}_{\alpha_{i}}\left(r_{n} / \mathbb{F}_{q}^{*}\right)$ for $1 \leq i \leq 2 s, \tilde{T}\left(\beta_{j}\right):=\operatorname{Tree}_{\beta_{j}}\left(r_{n} / H\right)$ for $1 \leq j \leq 2 t, \tilde{T}\left(\gamma_{k}\right):=$ $\operatorname{Tree}_{\gamma_{k}}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$ for $1 \leq k \leq 2 r$ and $\tilde{T}(-1):=\operatorname{Tree}_{-1}\left(r_{n} / \tilde{\mathbb{F}}_{q}\right)$. In this case we have, by Proposition 7 , $\tilde{T}(-1)=\left\langle\tilde{T}\left(\gamma_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\gamma_{2 r}\right)\right\rangle=\left\langle 2 \times\left(\tilde{T}\left(\gamma_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\gamma_{r}\right)\right)\right\rangle$, thus

$$
\begin{equation*}
1 / 2 \cdot \tilde{T}(-1)=\left\langle\tilde{T}\left(\gamma_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\gamma_{r}\right)\right\rangle \tag{4}
\end{equation*}
$$

Let $a_{i}=\eta\left(\alpha_{i}\right), T\left(a_{i}\right)=\operatorname{Tree}_{a_{i}}\left(T_{n} / \mathbb{F}_{q}\right), b_{j}=\eta\left(\alpha_{j}\right), T\left(b_{j}\right)=\operatorname{Tree}_{b_{j}}\left(T_{n} / \mathbb{F}_{q}\right), c_{k}=\eta\left(\gamma_{k}\right), T\left(c_{k}\right)=\operatorname{Tree}_{c_{k}}\left(T_{n} / \mathbb{F}_{q}\right)$ for $1 \leq i \leq s, 1 \leq j \leq t, 1 \leq k \leq r$ and $T(-2)=\operatorname{Tree}_{-2}\left(T_{n} / \mathbb{F}_{q}\right)$. We have $T_{n}^{-1}(2)=\left\{a_{1}, \ldots, a_{s}, b_{1}, \ldots, b_{t},-2,2\right\}$, $T_{n}^{-1}(-2)=\left\{c_{1}, \ldots, c_{r}\right\}$. By Proposition 7 and Equation (4) we have $T(-2)=\left\langle T\left(c_{1}\right) \oplus \cdots \oplus T\left(c_{r}\right)\right\rangle=$ $\left\langle\tilde{T}\left(\gamma_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\gamma_{r}\right)\right\rangle=1 / 2 \cdot \tilde{T}(-1)$, thus

$$
\begin{gather*}
\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)=\left\langle T\left(a_{1}\right) \oplus \cdots \oplus T\left(a_{s}\right) \oplus T\left(b_{1}\right) \oplus \cdots \oplus T\left(b_{t}\right) \oplus T(-2)\right\rangle \\
=\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right) \oplus \tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right) \oplus 1 / 2 \cdot \tilde{T}(-1)\right\rangle . \tag{5}
\end{gather*}
$$

Now we consider two subcases: $q \equiv 1(\bmod 4)$ and $q \equiv 3(\bmod 4)$. First we consider the subcase $q \equiv 1$ $(\bmod 4)$. By Lemma 3 we have $r_{n}^{-1}(-1) \cap H=\emptyset$ and $r_{n}^{-1}(-1) \subseteq \mathbb{F}_{q}^{*}$. Thus $\tilde{T}(-1)=\operatorname{Tree}_{-1}\left(T_{n} / \mathbb{F}_{q}^{*}\right)$ and we have, by Propositions 5 and 7. $T_{\nu_{0}(n)}=\operatorname{Tree}_{1}\left(r_{n} / \mathbb{F}_{q}^{*}\right)=\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{2 s}\right) \oplus \tilde{T}(-1)\right\rangle=$ $\left\langle 2 \times\left(\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right)\right) \oplus \tilde{T}(-1)\right\rangle$. Therefore

$$
\begin{equation*}
1 / 2 \cdot T_{\nu_{0}(n)}=\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right) \oplus 1 / 2 \cdot \tilde{T}(-1)\right\rangle \tag{6}
\end{equation*}
$$

Since $r_{n}^{-1}(-1) \cap H=\emptyset$, we have $T_{\nu_{1}(n)}=\operatorname{Tree}_{1}\left(r_{n} / H\right)=\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{2 t}\right) \oplus \bullet\right\rangle=\left\langle 2 \times\left(\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus\right.\right.$ $\left.\left.\tilde{T}\left(\beta_{t}\right)\right) \oplus \bullet\right\rangle$ and $1 / 2 \cdot T_{\nu_{1}(n)}=\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right) \oplus \bullet\right\rangle=\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right)\right\rangle+\langle\bullet\rangle$; from which we obtain

$$
\begin{equation*}
1 / 2 \cdot T_{\nu_{1}(n)}-\langle\bullet\rangle=\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right)\right\rangle \tag{7}
\end{equation*}
$$

Substituting Equations (6) and (7) in Equation (5) we have

$$
\begin{aligned}
\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right) & =\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right) \oplus \tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right) \oplus 1 / 2 \cdot \tilde{T}(-1)\right\rangle \\
& =\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right) \oplus 1 / 2 \cdot \tilde{T}(-1)\right\rangle+\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right)\right\rangle \\
& =1 / 2 \cdot T_{v_{0}(n)}+1 / 2 \cdot T_{\nu_{1}(n)}-\langle\bullet\rangle .
\end{aligned}
$$

The proof of the subcase $q \equiv 3(\bmod 4)$ is similar. In this case applying Lemma 3 we obtain $\tilde{T}(-1)=$ $\operatorname{Tree}_{-1}\left(T_{n} / H\right)$ and using the same arguments used for the subcase $q \equiv 1(\bmod 4)$ we obtain

$$
\begin{equation*}
1 / 2 \cdot T_{\nu_{1}(n)}=1 / 2 \cdot \operatorname{Tree}_{1}\left(r_{n} / H\right)=\left\langle\tilde{T}\left(\beta_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\beta_{t}\right) \oplus 1 / 2 \cdot \tilde{T}(-1)\right\rangle \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
1 / 2 \cdot T_{\nu_{0}(n)}-\langle\bullet\rangle=\left\langle\tilde{T}\left(\alpha_{1}\right) \oplus \cdots \oplus \tilde{T}\left(\alpha_{s}\right)\right\rangle \tag{9}
\end{equation*}
$$

Using Equations (5), (8) and (9) we have $\operatorname{Tree}_{2}\left(T_{n} / \mathbb{F}_{q}\right)=1 / 2 \cdot T_{\nu_{0}(n)}+1 / 2 \cdot T_{\nu_{1}(n)}-\langle\bullet\rangle$.

## VII. Structure theorem for Chebyshev polynomial and consequences

A. Isomorphism formula for $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$

We summarize all the information in the following main theorem of this paper, which follows from Theorems 2 and 3 and Proposition 12.

Theorem 4. Let $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be the $n$-decomposition of $q-1$ and $q+1$, respectively. The Chebyshev graph admits a decomposition of the form $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)=\mathcal{G}^{R} \oplus \mathcal{G}^{Q} \oplus \mathcal{G}^{S}$ where the rational component $\mathcal{G}^{R}$ is given by

$$
\mathcal{G}^{R}=\bigoplus_{\substack{d \mid \omega_{0} \\ d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \times \operatorname{Cyc}\left(\tilde{o}_{d}(n), T_{\nu_{0}(n)}\right) ;
$$

the quadratic component $\mathcal{G}^{Q}$ is given by

$$
\mathcal{G}^{Q}=\bigoplus_{\substack{d \mid \omega_{1} \\ d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \times \operatorname{Cyc}\left(\tilde{o}_{d}(n), T_{v_{1}(n)}\right)
$$

and the special component $\mathcal{G}^{S}$ is given by

$$
\mathcal{G}^{S}= \begin{cases}\operatorname{Cyc}\left(1,1 / 2 \cdot T_{\nu_{0}(n)}+1 / 2 \cdot T_{\nu_{1}(n)}-\langle\bullet\rangle\right) & \text { if } n \text { is even and } q \text { is odd; } \\ 2 \times \operatorname{Cyc}\left(1,1 / 2 \cdot T_{\nu_{0}(n)}+1 / 2 \cdot T_{\nu_{1}(n)}\right) & \text { if } n \text { is odd and } q \text { is odd; } \\ \operatorname{Cyc}\left(1,1 / 2 \cdot T_{\nu_{0}(n)}+1 / 2 \cdot T_{\nu_{1}(n)}\right) & \text { if } q \text { is even. }\end{cases}
$$

## B. Examples

We provide a series of examples showing our main result.
Example 3. We consider the Chebyshev polynomial $T_{30}$ over $\mathbb{F}_{19}$ (see Figure $\mathbb{1}$ ). We have $19-1=18=$ $v_{0} \omega_{0}$ with $v_{0}=18, \omega_{0}=1$ and $19+1=20=v_{1} \omega_{1}$ with $v_{1}=20, \omega_{1}=1$. Since $\omega_{0}, \omega_{1} \leq 2$ both the rational and the quadratic components of $\mathcal{G}\left(T_{30} / \mathbb{F}_{19}\right)$ are empty. We calculate the $v$-series $18(30)=(6,3)$ and $20(30)=(10,2)$ obtaining

$$
\mathcal{G}\left(T_{30} / \mathbb{F}_{19}\right)=C y c\left(1, \frac{1}{2} T_{(6,3)}+\frac{1}{2} T_{(10,2)}-\langle\bullet\rangle\right) .
$$

Thus, the graph $\mathcal{G}\left(T_{30} / \mathbb{F}_{19}\right)$ consist of a loop corresponding to the fix point 2 and a tree $T=\frac{1}{2} T_{(6,3)}+$ $\frac{1}{2} T_{(10,2)}-\langle\bullet\rangle$ attached to this point; see Figure 3.


Fig. 3. Construction of the tree $T=\frac{1}{2} T_{(6,3)}+\frac{1}{2} T_{(10,2)}-\langle\bullet\rangle$.

Example 4. Now we consider again the Chebyshev polynomial $T_{30}$ but this time over $\mathbb{F}_{23}$ (see Figure Tb). We have $23-1=22=v_{0} \omega_{0}$ with $v_{0}=2, \omega_{0}=11$ and $23+1=24=v_{1} \omega_{1}$ with $v_{1}=24, \omega_{1}=1$. In this case the quadratic component of $\mathcal{G}\left(T_{30} / \mathbb{F}_{23}\right)$ is empty and the rational component is $\frac{\varphi(11)}{2 \tilde{o}_{11}(30)} \times$ Cyc $\left(\tilde{o}_{11}(30), T_{2(30)}\right)$. Since $\varphi(11)=10, \tilde{o}_{11}(30)=5$ and $T_{2(30)}=T_{(2)}=\langle\bullet\rangle$, it is given by Cyc $(5,\langle\bullet\rangle)$. We calculate the $v$-series $24(30)=(6,2,2)$. Then, the Chebyshev's graph of $T_{30}$ over $\mathbb{F}_{23}$ is given by:

$$
\mathcal{G}\left(T_{30} / \mathbb{F}_{23}\right)=\operatorname{Cyc}(5,\langle\bullet\rangle) \oplus \operatorname{Cyc}\left(1, \frac{1}{2} T_{(2)}+\frac{1}{2} T_{(6,2,2)}-\langle\bullet\rangle\right) .
$$

We have $\frac{1}{2} T_{(2)}=\frac{1}{2}\langle\langle 2 \times \emptyset\rangle\rangle=\langle\langle\emptyset\rangle\rangle=\langle\bullet\rangle$ (i.e. $T_{(2)}$ is invariant under bisection), and after simplifying we obtain $\mathcal{G}\left(T_{30} / \mathbb{F}_{23}\right)=C y c(5,\langle\bullet\rangle) \oplus C y c\left(1, \frac{1}{2} T_{(6,2,2)}\right)$. To obtain a more explicit formula we calculate the bisection of $T_{(6,2,2)}$. Using the recursive formula (7), we obtain $T_{(6,2,2)}=\langle 4 \times \bullet \oplus T\rangle$ where $T=$ $\langle 4 \times \bullet \oplus 2 \times\langle 6 \times \bullet\rangle\rangle$, then $T_{(6,2,2)}$ is quasi-even and $T$ is even. Since $\frac{1}{2} T=\langle 2 \times \bullet \oplus\langle 6 \times \bullet\rangle\rangle$, we have $\frac{1}{2} T_{(6,2,2)}=\left\langle 2 \times \bullet \oplus \frac{1}{2} T\right\rangle=\langle 2 \times \bullet \oplus\langle 2 \times \bullet \oplus\langle 6 \times \bullet\rangle\rangle$.

Example 5. We consider again the Chebyshev polynomial $T_{30}$, this time over the reasonably large finite field $\mathbb{F}_{739}$ where the symmetries can be better appreciated; see Figure 4 We calculate the 30 -decomposition of $738=18 \cdot 41\left(v_{0}=18, \omega_{0}=41\right)$ and $740=120 \cdot 37\left(v_{1}=20, \omega_{1}=37\right)$. Since $\varphi(41)=40, \tilde{o}_{41}(30)=20$, $\varphi(37)=36, \tilde{o}_{37}(30)=9$, the rational component $\mathcal{G}^{R}$ and the quadratic component $\mathcal{G}^{Q}$ are given by $\mathcal{G}^{R}=\operatorname{Cyc}\left(20, T_{18(30)}\right)$ and $\mathcal{G}^{Q}=2 \times \operatorname{Cyc}\left(9, T_{20(30)}\right)$. We have $18(30)=(6,3)$ and $20(30)=(10,2)$. Thus the special component is $\mathcal{G}^{S}=\operatorname{Cyc}\left(1, \frac{1}{2} T_{(6,3)}+\frac{1}{2} T_{(10,2)}-\langle\bullet\rangle\right)$ and the structure of the whole graph is given by $\mathcal{G}\left(T_{30} / \mathbb{F}_{739}\right)=\operatorname{Cyc}\left(20, T_{(6,3)}\right) \oplus 2 \times \operatorname{Cyc}\left(9, T_{(10,2)}\right) \oplus \operatorname{Cyc}\left(1, \frac{1}{2} T_{(6,3)}+\frac{1}{2} T_{(10,2)}-\langle\bullet\rangle\right)$.


Fig. 4. Structure of the functional graph $\mathcal{G}\left(T_{30} / \mathbb{F}_{739}\right)$.

| $n$ | $\mathcal{G}^{R}$ | $\mathcal{G}^{Q}$ | $\mathcal{G}^{S}$ |
| :---: | :---: | :---: | :---: |
| 2 | $\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$ | $2 \times \operatorname{Cyc}(4, \bullet)$ | $\operatorname{Cyc}(1, \bullet)$ |
| 3 | $\operatorname{Cyc}(2,\langle 2 \times \bullet\rangle)$ | $\operatorname{Cyc}(8, \bullet)$ | $\operatorname{Cyc}(1,\langle\bullet\rangle)$ |
| 4 | $3 \times \operatorname{Cyc}(1, \bullet) \oplus 2 \times \operatorname{Cyc}(2, \bullet)$ | $4 \times \operatorname{Cyc}(2, \bullet)$ | $\operatorname{Cyc}(1, \bullet)$ |
| 5 | $\operatorname{Cyc}(1,\langle 4 \times \bullet\rangle)$ | $\operatorname{Cyc}(8, \bullet)$ | $\operatorname{Cyc}(1,\langle 2 \times \bullet\rangle)$ |
| 6 | $2 \times \operatorname{Cyc}(1,\langle 2 \times \bullet\rangle)$ | $\operatorname{Cyc}(8, \bullet)$ | $\operatorname{Cyc}(1,\langle\bullet\rangle)$ |
| 7 | $\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$ | $\operatorname{Cyc}(8, \bullet)$ | $\operatorname{Cyc}(1, \bullet)$ |
| 8 | $\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$ | $2 \times \operatorname{Cyc}(4, \bullet)$ | $\operatorname{Cyc}(1, \bullet)$ |
| 9 | $2 \times \operatorname{Cyc}(1,\langle 2 \times \bullet\rangle)$ | $2 \times \operatorname{Cyc}(4, \bullet)$ | $\operatorname{Cyc}(1,\langle\bullet\rangle)$ |
| 10 | $\operatorname{Cyc}(1,\langle 4 \times \bullet\rangle)$ | $\operatorname{Cyc}(8, \bullet)$ | $\operatorname{Cyc}(1,\langle 2 \times \bullet\rangle)$ |
| 15 | $\emptyset$ | $2 \times \operatorname{Cyc}(4, \bullet)$ | $\operatorname{Cyc}(1,\langle 7 \times \bullet\rangle)$ |
| 17 | $\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$ | $\emptyset$ | $\operatorname{Cyc}(1,\langle 8 \times \bullet\rangle)$ |
| 34 | $3 \times \operatorname{Cyc}(1, \bullet) \oplus 2 \times \operatorname{Cyc}(2, \bullet)$ | $\emptyset$ | $\operatorname{Cyc}(1,\langle 8 \times \bullet\rangle)$ |
| 255 | $\emptyset$ | $\emptyset$ | $\operatorname{Cyc}(1,\langle 15 \times \bullet\rangle)$ |

Graph structure for Chebyshev polynomials $T_{n}$ over the binary field $\mathbb{F}_{16}$. We recall that $T=\langle m \times \bullet\rangle$ denotes a tree consisting of a root with $m$ PREDECESSORS.

Example 6. We consider the action of Chebyshev polynomials over the binary field $\mathbb{F}_{16}$. Using Theorem 4 we obtain the structure of the rational component $\mathcal{G}^{R}$, the quadratic component $\mathcal{G}^{Q}$ and the special component $\mathcal{G}^{S}$ of the Chebyshev graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{16}\right)$ for $2 \leq n \leq 10$ and $n=15,17,34$ and 255; see Table III

## C. Chebyshev involutions and permutations

It is well known that the Chebyshev polynomial $T_{n}$ is a permutation polynomial over $\mathbb{F}_{q}$ if and only if $\operatorname{gcd}\left(q^{2}-1, n\right)=1$. Using that $T_{\nu(n)}=\bullet$ if and only if $v=1$, this condition can be obtained as a direct corollary of Theorem 4 together with the decomposition into disjoint cycles.

Corollary 3. The Chebyshev polynomial $T_{n}$ is a permutation polynomial over $\mathbb{F}_{q}$ if and only if $\operatorname{gcd}\left(q^{2}-\right.$ $1, n)=1$. In this case, if $q-1=v_{0} \omega_{0}$ and $q+1=\nu_{1} \omega_{1}$ are their $n$-decompositions, we have the following decomposition of $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ into disjoint cycles:

$$
\bigoplus_{\substack{d \mid \omega_{0} \\ d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \times C y c\left(\tilde{o}_{d}(n), \bullet\right) \oplus \bigoplus_{\substack{d \mid \omega_{1} \\ d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \times C y c\left(\tilde{o}_{d}(n), \bullet\right) \oplus k \times C y c(1, \bullet),
$$

where $k=2$ if nq is odd, and $k=1$ otherwise.

A particular case of cryptographic interest is permutation polynomials that are involutions [2], [3], that is, when the composition with itself is the identity map. For Chebyshev polynomials we obtain the following characterization.

Corollary 4. Let $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be the $n$-decomposition of $q-1$ and $q+1$, respectively. The Chebyshev polynomial $T_{n}$ is an involution over $\mathbb{F}_{q}$ if and only if $v_{0}=v_{1}=1, n^{2} \equiv \pm 1\left(\bmod \omega_{1}\right)$ and $n^{2} \equiv \pm 1\left(\bmod \omega_{2}\right)$.

Proof. The condition $\nu_{0}=v_{1}=1$ is equivalent to $\operatorname{gcd}\left(q^{2}-1, n\right)=1$ which is equivalent to $T_{n}$ being a permutation by Corollary 3. If this condition is satisfied, $T_{n}$ is an involution if and only if $\tilde{o}_{d}(n) \in\{1,2\}$ for all $d$ such that $d \mid \omega_{0}$ or $d \mid \omega_{1}$, if and only if $n^{2} \equiv \pm 1$ for all $d$ with $d \mid \omega_{0}$ or $d \mid \omega_{1}$, if and only if $n^{2} \equiv \pm 1\left(\bmod \omega_{1}\right)$ and $n^{2} \equiv \pm 1\left(\bmod \omega_{2}\right)$.

Example 7. Consider the Chebyshev polynomial $T_{31}$ over $\mathbb{F}_{25}$. Here $n=31, q=25, v_{0}=v_{1}=1, \omega_{1}=$ $24, \omega_{1}=26$. Since $31^{2} \equiv 1(\bmod 24)$ and $31^{2} \equiv-1(\bmod 26)$, the polynomial $T_{31}$ is an involution over $\mathbb{F}_{25}$.
D. Explicit formulas for $N, T_{0}, C, T$ and $R$

Let $\mathcal{G}=\mathcal{G}(f / X)$ be a functional graph where $X$ is a finite set. Given $x_{0} \in X$ there are integers $c \geq 1$ and $t \geq 0$ such that $x_{0}^{c+t}=x_{0}^{t}$. The smallest integers with this property are denoted by $\operatorname{per}\left(x_{0}\right):=c$ (the period of $x_{0}$ ) and $\operatorname{pper}\left(x_{0}\right):=t$ (the preperiod of $\left.x_{0}\right)$. The rho length of $x_{0}$ is $\operatorname{rho}\left(x_{0}\right):=\operatorname{per}\left(x_{0}\right)+\operatorname{pper}\left(x_{0}\right)$. We also consider the parameters $N, T_{0}, C, T$ and $R$ where

- $N(\mathcal{G})$ is the number of connected component of $\mathcal{G}$;
- $T_{0}(\mathcal{G})$ is the number of periodic points;
- $C(\mathcal{G})=\frac{1}{|X|} \sum_{x \in X} \operatorname{per}(x)$ is the expected value of the period;
- $T(\mathcal{G})=\frac{1}{|X|} \sum_{x \in X} \operatorname{pper}(x)$ is the expected value of the preperiod and
- $R(\mathcal{G})=\frac{1}{|X|} \sum_{x \in X} \operatorname{rho}(x)$ is the expected value of the rho length.

We apply our structural theorem to deduce explicit formulas for the parameters $N, T_{0}, C$ and $T$ for Chebyshev polynomials over $\mathbb{F}_{q}$ (the average rho length can be obtained from $R=C+T$ ). These parameters were studied in [4] for the exponentiation map and in [15] for Rédei functions.

We remark that the above parameters are invariant under isomorphism (i.e. isomorphic functional graphs have the same value). Related to $C$ and $T$ we consider the parameters $\widehat{C}$ and $\widehat{T}$ defined as the sum of the values of the periods and preperiods, respectively, from which we can easily obtain $C$ and $T$. The advantage of working with these parameters instead of $C$ and $T$ is that they are additive (i.e. $\widehat{C}\left(\mathcal{G}_{1} \oplus \mathcal{G}_{2}\right)=\widehat{C}\left(\mathcal{G}_{1}\right)+\widehat{C}\left(\mathcal{G}_{2}\right)$ and $\left.\widehat{T}\left(\mathcal{G}_{1} \oplus \mathcal{G}_{2}\right)=\widehat{T}\left(\mathcal{G}_{1}\right)+\widehat{T}\left(\mathcal{G}_{2}\right)\right)$ as well as the parameters $N$ and $T_{0}$. For additive parameters it suffices to know their values on each connected component. In the case of Chebyshev polynomials over finite fields, each connected component of its functional graph is uniform. It is immediate to check that if $\mathcal{G}=\operatorname{Cyc}(m, T)$ where $T$ is a rooted tree with depth $D$, then $N(\mathcal{G})=1$; $T_{0}(\mathcal{G})=m ; \widehat{C}(\mathcal{G})=m^{2}|T|$ and $\widehat{T}(\mathcal{G})=m \sum_{j=1}^{D} j h(j)$ where $h(j)$ denotes the number of nodes in $T$ at $\operatorname{depth} j$. When the rooted tree $T$ is the tree attached to a $v$-series $T=T_{\nu(n)}$ we have the following formulas, whose proof is the same as the given one in [15] for Rédei functions.

Lemma 4 ([15], Proposition 2.2.). Let $n, v, m$ be positive integers with $\operatorname{rad}(v) \mid \operatorname{rad}(n)$. Consider $v(n)=$ $\left(v_{1}, v_{2}, \ldots, v_{D}\right)$ and $\mathcal{G}=\operatorname{Cyc}\left(m, T_{v(n)}\right)$. Then $N(\mathcal{G})=1, T_{0}(\mathcal{G})=m, \widehat{C}(\mathcal{G})=m^{2} v$ and $\widehat{T}(\mathcal{G})=m \sum_{j=1}^{D-1} v_{1} \cdots v_{j}$.

The next lemma shows how the parameter $\widehat{T}$ behaves regarding to addition and bisection of trees.
Lemma 5. The following statements hold.

1) If $\mathcal{G}_{1}=\operatorname{Cyc}\left(1, T_{1}\right), \mathcal{G}_{2}=\operatorname{Cyc}\left(1, T_{2}\right)$ and $\mathcal{G}=\operatorname{Cyc}\left(1, T_{1}+T_{2}\right)$, then $\widehat{T}(\mathcal{G})=\widehat{T}\left(\mathcal{G}_{1}\right)+\widehat{T}\left(\mathcal{G}_{2}\right)$.
2) If $\mathcal{G}=\operatorname{Cyc}(1, T)$ where $T$ is an even or quasi-even rooted tree and $\mathcal{G}^{\prime}=\operatorname{Cyc}\left(1, \frac{1}{2} T\right)$, then $\widehat{T}\left(\mathcal{G}^{\prime}\right)=$ $\begin{cases}\frac{\widehat{T}(\mathcal{G})}{2} & \text { if } T \text { is even; } \\ \frac{\widehat{T}(\mathcal{G})+1}{2} & \text { if } T \text { is quasi-even. }\end{cases}$

Proof. 1. Denote by $h_{1}(j), h_{2}(j)$ and $h(j)$ the number of nodes at depth $j$ in $T_{1}, T_{2}$ and $T_{1}+T_{2}$, respectively. Clearly we have $h(0)=1$ and $h(j)=h_{1}(j)+h_{2}(j)$ for $j \geq 1$, from which we obtain $\widehat{T}(\mathcal{G})=\sum j h(j)=$ $\sum j h_{1}(j)+\sum j h_{2}(j)=\widehat{T}\left(\mathcal{G}_{1}\right)+\widehat{T}\left(\mathcal{G}_{2}\right)$.

[^1]2. First we consider the case when $T$ is even. We can write $T=\langle 2 \times S\rangle$ for some forest $S$. We denote by $h_{S}(j)$ the number of nodes at depth $j$ in $\frac{1}{2} T=\langle S\rangle$. We have that $\widehat{T}(\mathcal{G})=\sum j \cdot 2 h_{S}(j)=2 \sum j h_{S}(j)=$ $2 \widehat{T}\left(\mathcal{G}^{\prime}\right)$. Now we consider the case when $T$ is quasi-even. We can write $T=\langle 2 \times S \oplus\langle 2 \times R\rangle\rangle$. We denote by $h_{S}(j)$ and $h_{R}(j)$ the number of nodes at depth $j$ in $\langle S\rangle$ and $\langle R\rangle$, respectively. We have that $\widehat{T}(\mathcal{G})=\left(\sum j \cdot 2 h_{R}(j)\right)+1+\sum(j+1) \cdot 2 h_{S}(j)$ and $\widehat{T}\left(\mathcal{G}^{\prime}\right)=\left(\sum j h_{R}(j)\right)+1+\sum(j+1) h_{S}(j)$. Thus $2 \widehat{T}\left(\mathcal{G}^{\prime}\right)=\widehat{T}(\mathcal{G})+1$.

Next we calculate formulas for $\widehat{C}$ and $\widehat{T}$ for the special component $\mathcal{G}^{S}$ of the Chebyshev functional graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$.

Lemma 6. Let $n$ be a positive integer, $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be the $n$-decompositions of $q-1$ and $q+1$, respectively. Let $v_{0}(n)=\left(a_{1}, \ldots, a_{D}\right), v_{1}(n)=\left(b_{0}, \ldots, b_{D^{\prime}}\right), A=\sum_{i=1}^{D-1} a_{1} \cdots a_{i}$ and $B=\sum_{i=1}^{D^{\prime}-1} b_{1} \cdots b_{i}$. Denote by $\mathcal{G}^{S}$ the special component of the Chebyshev graph $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$. The following formulas for $\widehat{C}$ and $\widehat{T}$ hold.

$$
\widehat{C}\left(\mathcal{G}^{S}\right)=\left\{\begin{array}{ll}
v_{0}+v_{1}, & \text { if } n q \text { is odd; } \\
\frac{v_{0}+\nu_{1}}{2}, & \text { otherwise. }
\end{array} \quad \text { and } \widehat{T}\left(\mathcal{G}^{S}\right)= \begin{cases}A+B, & \text { if nq is odd; } \\
\frac{A+B}{2}, & \text { otherwise. }\end{cases}\right.
$$

Proof. First we consider the case when $q n$ is odd. In this case both $v_{0}$ and $v_{1}$ are odd and, by Proposition 13. both rooted trees $T_{\nu_{0}(n)}$ and $T_{\nu_{1}(n)}$ are even. From Proposition 14. Theorem 4 and the fact that $\left|T_{\nu(n)}\right|=v$ (see Equation 11 and the following paragraph), we have $\widehat{C}\left(\mathcal{G}^{S}\right)=2\left|\frac{1}{2} T_{v_{0}(n)}+\frac{1}{2} T_{v_{1}(n)}\right|=$ $2\left(\frac{v_{0}+1}{2}+\frac{v_{1}+1}{2}-1\right)=v_{0}+v_{1}$. Applying Lemmas 4 and 5 we obtain $\widehat{T}\left(\mathcal{G}^{S}\right)=2 \cdot\left(\frac{A}{2}+\frac{B}{2}\right)=A+B$.

Now we consider the case when $q$ is even. In this case again both $v_{0}$ and $v_{1}$ are odd and consequently both rooted trees $T_{v_{0}(n)}$ and $T_{\nu_{1}(n)}$ are even. By Proposition 14 and Theorem 4 , we have $\widehat{C}\left(\mathcal{G}^{S}\right)=\left\lvert\, \frac{1}{2} T_{\nu_{0}(n)}+\right.$ $\frac{1}{2} T_{\nu_{0}(n)} \left\lvert\,=\frac{v_{0}+1}{2}+\frac{\nu_{1}+1}{2}-1=\frac{\nu_{0}+\nu_{1}}{2}\right.$. Applying Lemmas 4 and 5 we obtain $\widehat{T}\left(\mathcal{G}^{S}\right)=\frac{A}{2}+\frac{B}{2}=\frac{A+B}{2}$.

The remainder case is when $n$ is even and $q$ is odd. In this case both $v_{0}$ and $v_{1}$ are even. By Proposition 13 both $T_{\nu_{0}(n)}$ and $T_{\nu_{1}(n)}$ are quasi-even. By Proposition 14 and Theorem 4 we have $\widehat{C}\left(\mathcal{G}^{S}\right)=\left\lvert\, \frac{1}{2} T_{\nu_{0}(n)}+\right.$ $\frac{1}{2} T_{\nu_{0}(n)}-\langle\bullet\rangle \left\lvert\,=\frac{v_{0}+2}{2}+\frac{\nu_{1}+2}{2}-1-1=\frac{v_{0}+\nu_{1}}{2}\right.$. Applying Lemmas 4 and 5 we obtain $\widehat{T}\left(\mathcal{G}^{S}\right)=\frac{A+1}{2}+\frac{B+1}{2}-1=$ $\frac{A+B}{2}$.

Theorem 5. Let $n$ be a positive integer. Let $q-1=v_{0} \omega_{0}$ and $q+1=v_{1} \omega_{1}$ be the $n$-decompositions of $q-1$ and $q+1$, respectively. Let $v_{0}(n)=\left(a_{1}, \ldots, a_{D}\right)$ and $v_{1}(n)=\left(b_{0}, \ldots, b_{D^{\prime}}\right)$. Then, the following holds for $\mathcal{G}=\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ :

- the number of cycles in $\mathcal{G}\left(T_{n} / \mathbb{F}_{q}\right)$ is $N(\mathcal{G})=\frac{1}{2}\left(\sum_{d \mid \omega_{0}} \frac{\varphi(d)}{\bar{\sigma}_{d}(n)}+\sum_{d \mid \omega_{1}} \frac{\varphi(d)}{\bar{\sigma}_{d}(n)}\right)$;
- the number of periodic points is given by $T_{0}(\mathcal{G})=\frac{\omega_{0}+\omega_{1}}{2}$;
- the expected value of $\operatorname{per}(a)$ where a runs over the elements of $\mathbb{F}_{q}$ is

$$
C(\mathcal{G})=\frac{q-1}{2 q}\left(\frac{1}{\omega_{0}} \sum_{d \mid \omega_{0}} \varphi(d) \tilde{o}_{d}(n)\right)+\frac{q+1}{2 q}\left(\frac{1}{\omega_{1}} \sum_{d \mid \omega_{1}} \varphi(d) \tilde{o}_{d}(n)\right) ;
$$

- the expected value of $\operatorname{pper}(a)$ where a runs over the elements of $\mathbb{F}_{q}$ is

$$
T(\mathcal{G})=\frac{q-1}{2 q}\left(\frac{1}{\nu_{0}} \sum_{i=1}^{D-1} a_{1} \ldots a_{i}\right)+\frac{q+1}{2 q}\left(\frac{1}{\nu_{1}} \sum_{i=1}^{D^{\prime}-1} b_{1} \ldots b_{i}\right) .
$$

Proof. Applying Theorem 4 we have

$$
\begin{align*}
N(\mathcal{G}) & =N\left(\mathcal{G}^{R}\right)+N\left(\mathcal{G}^{Q}\right)+N\left(\mathcal{G}^{S}\right) \\
& =\sum_{\substack{d \mid \omega_{0} \\
d>2}} \frac{\varphi(d)}{2 \tilde{\sigma}_{d}(n)}+\sum_{\substack{d \mid \omega_{1} \\
d>2}} \frac{\varphi(d)}{2 \tilde{\sigma}_{d}(n)}+\left\{\begin{aligned}
2, & \text { if } n q \text { is odd; } \\
1, & \text { if } n q \text { is even. }
\end{aligned}\right. \tag{10}
\end{align*}
$$

Since both $\omega_{0}$ and $\omega_{1}$ are even when $n q$ is odd and both $\omega_{0}$ and $\omega_{1}$ are odd when $n q$ is even, we have

$$
\sum_{\substack{d \mid \omega_{0}  \tag{11}\\ d \leq 2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)}+\sum_{\substack{d \mid \omega_{1} \\ d \leq 2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)}= \begin{cases}1+1=2, & \text { if } n q \text { is odd } \\ \frac{1}{2}+\frac{1}{2}=1, & \text { if } n q \text { is even. }\end{cases}
$$

By Equations 10, and 11 we have $N(\mathcal{G})=\sum_{d \mid \omega_{0}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)}+\sum_{d \mid \omega_{1}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)}$.
Applying Theorem 4 we have

$$
\begin{align*}
T_{0}(\mathcal{G}) & =T_{0}\left(\mathcal{G}^{R}\right)+T_{0}\left(\mathcal{G}^{Q}\right)+T_{0}\left(\mathcal{G}^{S}\right) \\
& =\sum_{\substack{d \mid \omega_{0} \\
d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n)+\sum_{\substack{d \mid \omega_{1} \\
d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n)+T_{0}\left(\mathcal{G}^{S}\right) \\
& =\sum_{\substack{d \mid \omega_{0} \\
d>2}} \frac{\varphi(d)}{2}+\sum_{\substack{d \mid \omega_{1} \\
d>2}} \frac{\varphi(d)}{2}+ \begin{cases}2, & \text { if } n q \text { is odd; } \\
1, & \text { if } n q \text { is even. }\end{cases} \tag{12}
\end{align*}
$$

Since $\omega_{0}$ and $\omega_{1}$ are even when $n q$ is odd and $\omega_{0}$ and $\omega_{1}$ are odd otherwise, we have

$$
\sum_{\substack{d \mid \omega_{0}  \tag{13}\\ d \leq 2}} \frac{\varphi(d)}{2}+\sum_{\substack{d \mid \omega_{1} \\ d \leq 2}} \frac{\varphi(d)}{2}= \begin{cases}1+1=2, & \text { if } n q \text { is odd } \\ \frac{1}{2}+\frac{1}{2}=1, & \text { if } n q \text { is even }\end{cases}
$$

By Equations (12) and (13) we have

$$
T_{0}(\mathcal{G})=\sum_{d \mid \omega_{0}} \frac{\varphi(d)}{2}+\sum_{d \mid \omega_{1}} \frac{\varphi(d)}{2}=\frac{\omega_{0}}{2}+\frac{\omega_{1}}{2}=\frac{\omega_{0}+\omega_{1}}{2} .
$$

Using that $\operatorname{Cyc}\left(m, T_{v(n)}\right)$ has exactly $m v$ nodes (see Equation (1) and the following paragraph) and applying Theorem 4 and Lemma 6 we have

$$
\begin{align*}
\widehat{C}(\mathcal{G}) & =\widehat{C}\left(\mathcal{G}^{R}\right)+\widehat{C}\left(\mathcal{G}^{Q}\right)+\widehat{C}\left(\mathcal{G}^{S}\right) \\
& =\sum_{\substack{d \mid \omega_{0} \\
d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n) \cdot \tilde{o}_{d}(n) v_{0}+\sum_{\substack{d \mid \omega_{1} \\
d>2}} \frac{\varphi(d)}{\tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n) \cdot \tilde{o}_{d}(n) v_{1}+\widehat{C}\left(\mathcal{G}^{S}\right) \\
& =\frac{v_{0}}{2} \sum_{\substack{d \mid \omega_{0} \\
d>2}} \varphi(d) \tilde{o}_{d}(n)+\frac{v_{1}}{2} \sum_{\substack{d \mid \omega_{1} \\
d>2}} \varphi(d) \tilde{o}_{d}(n)+ \begin{cases}v_{0}+v_{1}, & \text { if } n q \text { is odd; } \\
\frac{v_{0}+v_{1}}{2}, & \text { otherwise. }\end{cases} \tag{14}
\end{align*}
$$

Since $\omega_{0}$ and $\omega_{1}$ are even when $n q$ is odd and $\omega_{0}$ and $\omega_{1}$ are odd otherwise, we have

$$
\frac{v_{0}}{2} \sum_{\substack{d \mid \omega_{0}  \tag{15}\\ d \leq 2}} \varphi(d) \tilde{o}_{d}(n)+\frac{v_{1}}{2} \sum_{\substack{d \mid \omega_{1} \\ d \leq 2}} \varphi(d) \tilde{\sigma}_{d}(n)= \begin{cases}v_{0}+v_{1}, & \text { if } n q \text { is odd; } \\ \frac{v_{0}+\nu_{1}}{2}, & \text { otherwise } .\end{cases}
$$

By Equations (14) and (15) we have

$$
\begin{aligned}
\widehat{C}(\mathcal{G}) & =\frac{\nu_{0}}{2} \sum_{d \mid \omega_{0}} \varphi(d) \tilde{o}_{d}(n)+\frac{\nu_{1}}{2} \sum_{d \mid \omega_{1}} \varphi(d) \tilde{o}_{d}(n) \\
& =\frac{q-1}{2}\left(\frac{1}{\omega_{0}} \sum_{d \mid \omega_{0}} \varphi(d) \tilde{o}_{d}(n)\right)+\frac{q+1}{2}\left(\frac{1}{\omega_{1}} \sum_{d \mid \omega_{1}} \varphi(d) \tilde{o}_{d}(n)\right) .
\end{aligned}
$$

Dividing both sides by $q$ we obtain the formula for $C(\mathcal{G})$.
Now we deduce the formula for $T$. Denote $A=\sum_{i=1}^{D-1} a_{1} \cdots a_{i}$ and $B=\sum_{i=1}^{D^{\prime}-1} B_{1} \cdots B_{i}$. Using Lemmas 4 and 6 and Theorem 4 we obtain

$$
\begin{align*}
\widehat{T}(\mathcal{G}) & =\widehat{T}\left(\mathcal{G}^{R}\right)+\widehat{T}\left(\mathcal{G}^{Q}\right)+\widehat{T}\left(\mathcal{G}^{S}\right) \\
& =\sum_{\substack{d \mid \omega_{0} \\
d>2}} \frac{\varphi(d)}{2 \tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n) A+\sum_{\substack{d \mid \omega_{1} \\
d>2}} \frac{\varphi(d)}{\tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n) B+\widehat{T}\left(\mathcal{G}^{S}\right) \\
& =\frac{A}{2} \sum_{\substack{d \mid \omega_{0} \\
d>2}} \varphi(d)+\frac{B}{2} \sum_{\substack{d \mid \omega_{1} \\
d>2}} \varphi(d)+ \begin{cases}A+B, & \text { if } n q \text { is odd; } \\
\frac{A+B}{2}, & \text { otherwise. }\end{cases} \tag{16}
\end{align*}
$$

Since $\omega_{0}$ and $\omega_{1}$ are even when $n q$ is odd and $\omega_{0}$ and $\omega_{1}$ are odd otherwise, we have

$$
\frac{A}{2} \sum_{\substack{d \mid \omega_{0}  \tag{17}\\ d \leq 2}} \varphi(d)+\frac{B}{2} \sum_{\substack{d \mid \omega_{1} \\ d \leq 2}} \varphi(d)= \begin{cases}A+B, & \text { if } n q \text { is odd } \\ \frac{A+B}{2}, & \text { otherwise }\end{cases}
$$

By Equations (16) and (17) we have

$$
\widehat{T}(\mathcal{G})=\frac{A}{2} \sum_{d \mid \omega_{0}} \varphi(d)+\frac{B}{2} \sum_{d \mid \omega_{1}} \varphi(d)=\frac{A \omega_{0}}{2}+\frac{B \omega_{1}}{2} .=\frac{q-1}{2} \cdot \frac{A}{v_{0}}+\frac{q+1}{2} \cdot \frac{B}{\nu_{1}} .
$$

Dividing both sides by $q$ we obtain the formula for $T(\mathcal{G})$.

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[^0]:    Claudio Qureshi is with the Institute of Mathematics, Statistics and Computing Science of the University of Campinas, SP , Brazil (email: cqureshi@ime.unicamp.br) and Daniel Panario is with School of Mathematics and Statistics, Carleton University, Canada (email: daniel@math.carleton.ca)

[^1]:    ${ }^{1}$ The depth of a node $x$ in a rooted tree $T$ with root $r$ is the length of the smallest path connecting $x$ to $r$. If $T$ is a rooted tree attached to a cyclic node in a functional graph, the depth of a node is the same as its preperiod.

