The Graph Structure of Chebyshev Polynomials over Finite Fields and Applications

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Abstract

We completely describe the functional graph associated to iterations of Chebyshev polynomials over finite fields. Then, we use our structural results to obtain estimates for the average rho length, average number of connected components and the expected value for the period and preperiod of iterating Chebyshev polynomials.

I. INTRODUCTION

The iteration of polynomials and rational functions over finite fields have recently become an active research topic. These dynamical systems have found applications in diverse areas, including cryptography, biology and physics. In cryptography, iterations of functions over finite fields were popularized by the Pollard rho algorithm for integer factorization [12]; its variant for computing discrete logarithms is considered the most efficient method against elliptic curve cryptography based on the discrete logarithm problem [13]. Other cryptographical applications of iterations of functions include pseudorandom bit generators [1], and integer factorization and primality tests [8], [9].

When we iterate functions over finite structures, there is an underlying natural functional graph. For a function f over a finite field \mathbb{F}_q , this graph has q nodes and a directed edge from vertex a to vertex b if and only if f(a) = b. It is well known, combinatorially, that functional graphs are sets of connected components, components are directed cycles of nodes, and each of these nodes is the root of a directed tree from leaves to its root; see, for example, [6].

Some functions over finite fields when iterated present strong symmetry properties. These symmetries allow mathematical proofs for some dynamical properties such as period and preperiod of a generic element, (average) "rho length" (number of iterations until cycling back), number of connected components,

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cycle lengths, etc. In this paper we are interested on these kinds of properties for Chebyshev polynomials over finite fields, closely related to Dickson polynomials over finite fields. These polynomials, specially when they permute the elements of the field, have found applications in many areas including cryptography and coding theory. See [10] for a monograph on Dickson polynomials and their applications, including cryptography; for a more recent account on research in finite fields including Dickson polynomials, see [11].

Previous results for quadratic functions are in [17]; iterations of $x + x^{-1}$ have been dealt in [16] and iterations of Rédei functions over non-binary finite fields appeared in [14], [15]. Related to this paper, iterations of Chebyshev polynomials over finite fields have been treated in [7]. The graph and periodicity properties for Chebyshev polynomials over finite fields when the degree of the polynomial is a prime number are given in [7].

In this paper we study the action of Chebyshev functions of *any* degree over finite fields. We give a structural theorem for the functional graph from which it is not hard to derive many periodicity properties of these iterations. In the literature there are two kinds of Chebyshev polynomials: normalized and not normalized. We use the latter ones, generally known as Dickson polynomials of the first kind. In odd characteristic both kinds of Chebyshev polynomials are conjugates of each other, and so their functional graphs are isomorphic. However, this is not the case in even characteristic. Using the normalized version trivializes since we get $T_n(x) = 1$ if *n* is even, and $T_n(x) = x$ if *n* is odd, where T_n is the *n*th degree Chebyshev polynomial. As a consequence, we work with the non normalized version that is much richer in characteristic 2. Not much is known about Chebyshev polynomials over binary fields; see [5] for results over the 2-adic integers.

In Section II we introduce relevant concepts for this paper like ν -series and their associated trees. These trees play a central role in the description of the Chebyshev functional graph. Several results about a homomorphism of the Chebyshev functional graph, as well as a relevant covering notion, are given in Section III. A decomposition of the Chebyshev's functional graph is given in Section IV. This decomposition leads naturally into three parts: the rational, the quadratic and the special component. Section V treats the rational and quadratic components. The special component is dealt in Section VI. The main result of this paper (Theorem 4), a structural theorem for Chebyshev polynomials, is given in Section VII. We provide several examples to show applications of our main theorem. As a consequence of our main structural theorem, in this section we also obtain exact results for the parameters N, C, T_0, T and R for Chebyshev polynomials, where N is the number of cycles (that is, the number of connected components), T_0 is the number of cyclic (periodic) points, C is the expected value of the period, T is the expected rho length.

II. PRELIMINARIES

We denote by \mathbb{F}_q a finite field with q element, where q is a prime power, and \mathbb{Z}_d the ring of integers modulo d. Let \mathbb{F}_q^* and \mathbb{Z}_d^* denote the multiplicative group of inverse elements of \mathbb{F}_q and \mathbb{Z}_d , respectively. Let \overline{n} denote the equivalence class of n modulo d. For $n, d \in \mathbb{Z}^+$ with gcd(n, d) = 1, we denote by $o_d(n)$ and $\tilde{o}_d(n)$ the multiplicative order of \overline{n} in \mathbb{Z}_d^* and $\mathbb{Z}_d^*/\{1, -1\}$, respectively. It is easy to see that if $-\overline{1} \in \langle \overline{n} \rangle$ in \mathbb{Z}_d^* , then $\tilde{o}_d(n) = o_d(n)/2$, otherwise $\tilde{o}_d(n) = o_d(n)$. For $m \in \mathbb{Z}^+$ we denote by rad(m) the radical of mwhich is defined as the product of the distinct primes divisors of m. We can decompose $m = v\omega$ where $rad(v) \mid rad(n)$ and $gcd(\omega, n) = 1$ which we refer as the *n*-decomposition of m. If $f : X \to X$ is a function defined over a finite set X, we denote by $\mathcal{G}(f/X)$ its functional graph.

The main object of study of this paper is the action of Chebyshev polynomials over finite fields \mathbb{F}_q . The Chebyshev polynomial of the first kind of degree *n* is denoted by T_n . This is the only monic, degree-*n* polynomial with integer coefficients verifying $T_n(x + x^{-1}) = x^n + x^{-n}$ for all $x \in \mathbb{Z}$. Table I gives the first Chebyshev polynomials.

$$T_{1}(x) = x$$

$$T_{2}(x) = x^{2} - 2$$

$$T_{3}(x) = x^{3} - 3x$$

$$T_{4}(x) = x^{4} - 4x^{2} + 2$$

$$T_{5}(x) = x^{5} - 5x^{3} + 5x$$

$$T_{6}(x) = x^{6} - 6x^{4} + 9x^{2} - 2$$

$$T_{7}(x) = x^{7} - 7x^{5} + 14x^{3} - 7x$$

$$T_{8}(x) = x^{8} - 8x^{6} + 20x^{4} - 16x^{2} + 2$$

$$T_{9}(x) = x^{9} - 9x^{7} + 27x^{5} - 30x^{3} + 9x$$

$$T_{10}(x) = x^{10} - 10x^{8} + 35x^{6} - 50x^{4} + 25x^{2} - 2$$

TABLE I

FIRST FEW CHEBYSHEV POLYNOMIALS $T_n(x)$ for $1 \le n \le 10$.

A remarkable property of these polynomials is that $T_n \circ T_m = T_{nm}$ for all $m, n \in \mathbb{Z}^+$. In particular, $T_n^{(k)} = T_{n^k}$, where $f^{(k)}$ denotes the composition of f with itself k times. Describing the dynamics of the Chebyshev polynomial T_n acting on the finite field \mathbb{F}_q is equivalent to describing the Chebyshev's graph $\mathcal{G}(T_n/\mathbb{F}_q)$.

The case when $n = \ell$ is a prime number was dealt by Gassert; see [7, Theorem 2.3]. In this paper we extend these results for any positive integer *n*.

Example 1. For n = 30 the corresponding Chebyshev polynomial is given by $T_{30}(x) = x^{30} - 30x^{28} + 30x^{28} +$

 $405x^{26} - 3250x^{24} + 17250x^{22} - 63756x^{20} + 168245x^{18} - 319770x^{16} + 436050x^{14} - 419900x^{12} + 277134x^{10} - 119340x^8 + 30940x^6 - 4200x^4 + 225x^2 - 2$. The graphs $\mathcal{G}(T_{30}/\mathbb{F}_q)$ for q = 19 and q = 23 are shown in Fig. 1.

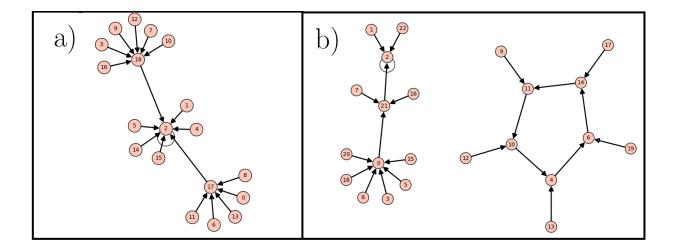


Fig. 1. a) The Chebyshev's graph $\mathcal{G}(T_{30}/\mathbb{F}_{19})$. b) The Chebyshev's graph $\mathcal{G}(T_{30}/\mathbb{F}_{23})$.

Next we review some concepts from [14]. For *n* and *v* positive integers such that rad(v) | rad(n), the *v*-series associated with *n* is the finite sequence $v(n) := (v_1, \ldots, v_D)$ defined by the recurrence $v_1 = gcd(v, n), v_{k+1} = gcd\left(\frac{v}{v_1v_2\cdots v_k}, n\right)$ for $1 \le k < D$ and $v_1v_2\cdots v_D = v$ with $v_D > 1$ if v > 1, and v(n) = (1) if v = 1.

We write $A = \bigcup B_i$ to indicate that A is the union of pairwise disjoint sets B_i . If $m \in \mathbb{Z}^+$ and T is a rooted tree, Cyc(m, T) denotes a graph with a unique directed cycle of length m, where every node in this cycle is the root of a tree isomorphic to T. We also consider the disjoint union of the graphs G_1, \ldots, G_k , denoted by $\bigoplus_{i=1}^k G_i$, and $k \times G = \bigoplus_{i=1}^k G$ for $k \in \mathbb{Z}^+$. If T_1, \ldots, T_k are rooted trees, $\langle T_1 \oplus \cdots \oplus T_k \rangle$ is a rooted tree such that its root has exactly k predecessors v_1, \ldots, v_k , and v_i is the root of a tree isomorphic to T_i for $i = 1, \ldots, k$. If T is a tree that consists of a single node we simply write $T = \bullet$. In particular, $Cyc(m, \bullet)$ denotes a directed cycle with m nodes. The empty graph, denoted by \emptyset , is characterized by the properties: $\emptyset \oplus G = G$ for all graphs $G, k \times \emptyset = \emptyset$ for all $k \in \mathbb{Z}^+$ and $\langle \emptyset \rangle = \bullet$.

We associate to each *v*-series v(n) a rooted tree, denoted by $T_{v(n)}$, defined by the recurrence formula (see Fig. 2):

$$\begin{cases} T^{0} = \bullet, \\ T^{k} = \langle v_{k} \times T^{k-1} \oplus \bigoplus_{i=1}^{k-1} (v_{i} - v_{i+1}) \times T^{i-1} \rangle, 1 \le i < D, \\ T_{\nu(n)} = \langle (v_{D} - 1) \times T^{D-1} \oplus \bigoplus_{i=1}^{D-1} (v_{i} - v_{i+1}) \times T^{i-1} \rangle. \end{cases}$$
(1)

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The tree $T_{\nu(n)}$ has ν vertices and depth D; see Proposition 2.14 and Theorem 3.16 of [14].

The following theorem is a direct consequence of Corollary 3.8 and Theorem 3.16 of [14]. As usual, φ denotes Euler's totient function.

Theorem 1. Let $n \in \mathbb{Z}^+$ and $m = v\omega$ be the n-decomposition of m. Denoting by $\mathcal{G}(n/\mathbb{Z}_m)$ the functional graph of the multiplication-by-n map on the cyclic group \mathbb{Z}_m , the following isomorphism holds:

$$\mathcal{G}(n/\mathbb{Z}_m) = \bigoplus_{d\mid\omega} \frac{\varphi(d)}{o_d(n)} \times Cyc\left(o_d(n), T_{\nu(n)}\right).$$

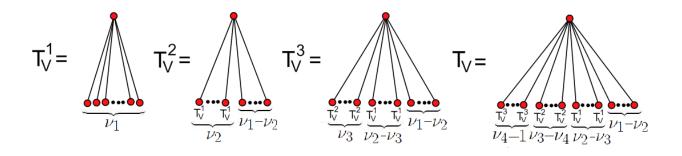


Fig. 2. This figure (taken from [14]) illustrates the inductive definition of T_V when V is a v-series with four components $V = (v_1, v_2, v_3, v_4)$. A node v labelled by a rooted tree T indicates that v is the root of a tree isomorphic to T.

A strategy to describe a functional graph $\mathcal{G}(f/X)$ of a function $f: X \to X$ is decomposing the set X in f-invariant components. A subset $A \subseteq X$ is forward f-invariant when $f(A) \subseteq A$. In this case the graph $\mathcal{G}(f/A)$ is a subgraph of $\mathcal{G}(f/X)$. If $f^{-1}(A) \subseteq A$, the set A is backward f-invariant. The set A is f-invariant if it is both forward and backward f-invariant. In this case $\mathcal{G}(f/A)$ is not only a subgraph of $\mathcal{G}(f/X)$ but also a union of connected components and we can write $\mathcal{G}(f/X) = \mathcal{G}(f/A) \oplus \mathcal{G}(f/A^c)$, where $A^c = X \setminus A$. In this paper, we decompose the set \mathbb{F}_q in T_n -invariant subsets A_1, \ldots, A_k such that each functional graph $\mathcal{G}(T_n/A_i)$ for $i = 1, \ldots, \kappa$ is easier to describe than the general case and $\mathcal{G}(T_n/\mathbb{F}_q) = \bigoplus_{i=1}^{\kappa} \mathcal{G}(T_n/A_i)$.

To describe a functional graph we need to describe not only the cyclic part but also the rooted trees attached to the periodic points. We introduce next some notation related to rooted trees (where the root is not necessarily a periodic point). Let $f : X \to X$, $x \in X$ and N_f be the set of its non-periodic points. We define the set of predecessors of x by

$$\operatorname{Pred}_{x}(f/X) = \{y \in N_{f} : f^{(k)}(y) = x \text{ for some } k \ge 1\} \cup \{x\}.$$

We denote by $\operatorname{Tree}_x(f/X)$ the rooted tree with root x, vertex set $V = \operatorname{Pred}_x(f/X)$ and directed edges (y, f(y)) for $y \in V \setminus \{x\}$.

A directed graph is a pair G = (V, E) where V is the vertex set and $E \subseteq V \times V$ is the edge set. A homomorphism ϕ between two directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $\phi : G_1 \to G_2$, is a function $\phi : V_1 \to V_2$ such that if $(v, v') \in E_1$ then $(\phi(v), \phi(v')) \in E_2$. In the particular case of functional graphs, a homomorphism $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ is a function $\phi : X_1 \to X_2$ satisfying $\phi \circ f_1 = f_2 \circ \phi$, or equivalently such that the following diagram commutes

$$\begin{array}{ccc} X_1 & \stackrel{\phi}{\longrightarrow} & X_2 \\ f_1 & & & \uparrow f_2 \\ X_1 & \stackrel{\phi}{\longrightarrow} & X_2 \end{array}$$

It is easy to prove by induction that the relation $\phi \circ f_1 = f_2 \circ \phi$ implies $\phi \circ f_1^{(k)} = f_2^{(k)} \circ \phi$ for all $k \ge 1$, that is, $\phi : \mathcal{G}(f_1^{(k)}/X_1) \to \mathcal{G}(f_2^{(k)}/X_2)$ is also a homomorphism for all $k \ge 1$. If in addition ϕ is bijective (as function from X_1 to X_2) then $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ is an isomorphism of functional graphs. In this case the functional graphs are the same, up to the labelling of the vertices. The main result of this paper (Theorem 4) is an explicit description of $\mathcal{G}(T_n/\mathbb{F}_q)$, the functional graph of the Chebyshev polynomial T_n over a finite field \mathbb{F}_q .

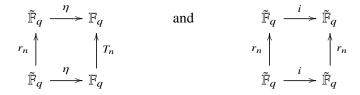
In the first part of this section we introduce the concept of θ -covering between two functional graphs and derive some properties. In the last part we apply these results to obtain some rooted tree isomorphism formulas which are used in the next sections.

A. θ -coverings

In our case of study (functional graph of Chebyshev polynomials) we consider the set $\tilde{\mathbb{F}}_q = \mathbb{F}_q^* \cup H$, where *H* is the multiplicative subgroup of $\mathbb{F}_{q^2}^*$ of order q + 1, and the following maps:

- The inversion map $i: \tilde{\mathbb{F}}_q \to \tilde{\mathbb{F}}_q$ given by $i(\alpha) = \alpha^{-1}$.
- The exponentiation map $r_n : \tilde{\mathbb{F}}_q \to \tilde{\mathbb{F}}_q$ given by $r_n(\alpha) = \alpha^n$.
- The map $\eta: \tilde{\mathbb{F}}_q \to \mathbb{F}_q$ given by $\eta(\alpha) = \alpha + \alpha^{-1}$.

A useful relationship between these maps and the Chebyshev map are $T_n \circ \eta = \eta \circ r_n$ and $r_n \circ i = i \circ r_n$. In other words we have the following commutative diagrams:



To describe the Chebyshev functional graph $\mathcal{G}(T_n/\mathbb{F}_q)$ it is helpful to consider the homomorphism $\eta : \mathcal{G}(r_n/\tilde{\mathbb{F}}_q) \to \mathcal{G}(T_n/\mathbb{F}_q)$ and to relate properties between these functional graphs. This homomorphism is not an isomorphism, but it has very nice properties that are captured in the next concept.

Definition 1. Let $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ be a homomorphism of functional graphs and $\theta : X_1 \to X_1$ be a permutation (bijection) which commutes with f_1 (that is, $f_1 \circ \theta = \theta \circ f_1$). Then ϕ is a θ -covering if for every $a \in X_2$ there is $\alpha \in X_1$ such that $\phi^{-1}(a) = \{\theta^{(i)}(\alpha) : i \in \mathbb{Z}\}$ (in other words, if the preimage of each point is a θ -orbit). The homomorphism ϕ is a covering if it is a θ -covering for some θ verifying the above properties.

We remark that a covering is necessarily onto and every isomorphism $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ is a covering (with respect to the identity map id : $X_1 \to X_1$, id(x) = x). We note that the condition of $\phi^{-1}(a)$ being a θ -orbit for all $a \in X_2$ implies that $\phi \circ \theta = \phi$.

In [7] it is proved several properties of the map η . Namely η is surjective, $\eta^{-1}(2) = \{1\}, \eta^{-1}(-2) = \{-1\},\$ and for $a \in \mathbb{F}_q$, $\eta^{-1}(a) = \{\alpha, \alpha^{-1}\}$ where α and α^{-1} are the roots (in $\mathbb{F}_{q^2}^*$) of $x^2 - ax + 1 = 0$ which are distinct if $a \neq \pm 2$. In particular, with our notation, we have that $\eta : \mathcal{G}(r_n/\tilde{\mathbb{F}}_q) \to \mathcal{G}(T_n/\mathbb{F}_q)$ is a *i*-covering between these functional graphs.

Next we prove some general properties for coverings of functional graphs that are used in the next section for the particular case of the covering $\eta : \mathcal{G}(r_n/\tilde{\mathbb{F}}_q) \to \mathcal{G}(T_n/\mathbb{F}_q)$. In the next propositions we denote by P_f and N_f the set of periodic and non-periodic points with respect to the map f, respectively. We note that if $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ is a homomorphism and $x \in P_{f_1}$ then there is a $k \ge 1$ such that $f_1^{(k)}(x) = x$. This implies $f_2^{(k)}(\phi(x)) = \phi(f_1^{(k)}(x)) = \phi(x)$, thus $x \in \phi^{-1}(P_{f_2})$ and we have $P_{f_1} \subseteq \phi^{-1}(P_{f_2})$. The next proposition shows that when ϕ is a covering this inclusion is in fact an equality.

Proposition 1. Let $\theta : X_1 \to X_1$ be a permutation satisfying $f_1 \circ \theta = \theta \circ f_1$. If $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ is a θ -covering then $\phi^{-1}(P_{f_2}) = P_{f_1}$.

Proof. Let ℓ be the order of θ (i.e. $\theta^{(\ell)} = id$). It suffices to prove $\phi^{-1}(P_{f_2}) \subseteq P_{f_1}$. If $\alpha \in \phi^{-1}(P_{f_2})$ then there is a $k \ge 1$ such that $f_2^{(k)}(\phi(\alpha)) = \phi(\alpha)$. Since $f_2^{(k)}(\phi(\alpha)) = \phi(f_1^{(k)}(\alpha))$ we conclude that $f_1^{(k)}(\alpha) = \theta^{(i)}(\alpha)$ for some $i \in \mathbb{Z}$. Applying $f_1^{(k)}$ on both sides we obtain $f_1^{(2k)}(\alpha) = f_1^{(k)}(\theta^{(i)}(\alpha)) = \theta^{(i)}(f_1^{(k)}(\alpha)) = \theta^{(2i)}(\alpha)$. In the same way, applying $f_1^{(k)}$ several times, we have by induction that $f_1^{(mk)}(\alpha) = \theta^{(mi)}(\alpha)$ for all $m \ge 1$. With $m = \ell$ we obtain $f_1^{(\ell k)}(\alpha) = \theta^{(\ell i)}(\alpha) = \alpha$, thus $\alpha \in P_{f_1}$.

Remark 1. The equation $\phi^{-1}(P_{f_2}) = P_{f_1}$ is equivalent to $\phi^{-1}(N_{f_2}) = N_{f_1}$ since $\phi^{-1}(X^c) = \phi^{-1}(X)^c$.

Proposition 2. Let $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ be a homomorphism satisfying $\phi^{-1}(P_{f_2}) = P_{f_1}$ and $\alpha \in X_1$. We have $Pred_{\alpha}(f_1/X_1) \subseteq \phi^{-1}(Pred_{\phi(\alpha)}(f_2/X_2))$.

Proof. Let $\beta \in \operatorname{Pred}_{\alpha}(f_1/X_1)$, $\beta \neq \alpha$ (in particular $\beta \in N_{f_1}$). By definition, there is an integer $k \geq 1$ such that $f_1^{(k)}(\beta) = \alpha$. This implies $f_2^{(k)}(\theta(\beta)) = \theta(f_1^{(k)}(\beta)) = \theta(\alpha)$. Since $\phi^{-1}(N_{f_2}) = N_{f_1}$ and $\beta \in N_{f_1}$ we have $\phi(\beta) \in N_{f_2}$, thus $\phi(\beta) \in \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$.

Remark 2. If $Pred_{\alpha}(f_1/X_1) \subseteq \phi^{-1}(Pred_{\phi(\alpha)}(f_2/X_2))$ then $\phi(Pred_{\alpha}(f_1/X_1)) \subseteq Pred_{\phi(\alpha)}(f_2/X_2)$ since ϕ is surjective.

Proposition 3. Let $\theta : X_1 \to X_1$ be a permutation satisfying $f_1 \circ \theta = \theta \circ f_1$, $\alpha \in X_1$ and $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ be a θ -covering. The equality $\phi(\operatorname{Pred}_{\alpha}(f_1/X_1)) = \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$ holds.

Proof. The inclusion $\phi(\operatorname{Pred}_{\alpha}(f_1/X_1)) \subseteq \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$ follows from Propositions 1 and 2 (see also Remark 2). To prove the other inclusion we consider $b \in \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$ with $b \neq \phi(\alpha)$ (in particular $b \in N_{f_2}$) and $\beta \in X_1$ such that $b = \phi(\beta)$ (this is possible because ϕ is surjective). We have to prove that there is a point $\beta' \in \operatorname{Pred}_{\alpha}(f_1/X_1)$ such that $\phi(\beta') = b$. By definition there is an integer $k \ge 1$ such that $f_2^{(k)}(b) = \phi(\alpha)$ and we have $\phi(f_1^{(k)}(\beta)) = f_2^{(k)}(\phi(\beta)) = \phi(\alpha)$. Since ϕ is a θ -covering, from $\phi(f_1^{(k)}(\beta)) = \phi(\alpha)$ we have that $\alpha = \theta^{(i)}(f_1^{(k)}(\beta))$ for some integer *i* and define $\beta' = \theta^{(i)}(\beta)$. Using that θ and f_1 commute we obtain $f_1^{(k)}(\beta') = \theta^{(i)}(f_1^{(k)}(\beta)) = \alpha$ and $\phi(\beta') = \phi(\theta^{(i)}(\beta)) = \phi(\beta) = b$ (because $\phi \circ \theta = \phi$). To conclude the proof we have to show that $\beta' \in \operatorname{Pred}_{\alpha}(f_1/X_1)$ and it suffices to prove that $\beta' \in N_{f_1}$. Since $\phi(\beta') = b \in N_{f_2}$ we have $\beta' \in \phi^{-1}(N_{f_2}) = N_{f_1}$ by Proposition 1 (see also Remark 1).

With the same notation and hypothesis of Proposition 3, if we denote by $P_1 = \operatorname{Pred}_{\alpha}(f_1/X_1)$ and $P_2 = \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$ we have that the restricted function $\phi|_{P_1} : P_1 \to P_2$ is onto. We want to find conditions to guarantee that $\phi|_{P_1} : P_1 \to P_2$ is a bijection. We recall that the order of a permutation $\theta : X_1 \to X_1$ is the smallest positive integer $\ell \ge 1$ such that $\theta^{(\ell)} = \operatorname{id}$. This implies that the cardinality of the θ -orbit of a point $\alpha \in X_1$, given by $\{\theta^{(i)}(\alpha) : 0 \le i < \ell\}$, is a divisor of ℓ .

Definition 2. Let $\theta : X_1 \to X_1$ be a permutation of order ℓ . A point $\alpha \in X_1$ is θ -maximal, if the sequence of iterates: $\alpha, \theta(\alpha), \theta^{(2)}(\alpha), \dots, \theta^{(\ell-1)}(\alpha)$ are pairwise distinct (that is, if the θ -orbit of α has exactly ℓ elements).

Remark 3. An important particular case is when $\theta : X_1 \to X_1$ is the identity map. In this case every point $\alpha \in X_1$ is θ -maximal.

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Proposition 4. Let $\theta : X_1 \to X_1$ be a permutation satisfying $f_1 \circ \theta = \theta \circ f_1$, α be a θ -maximal point of X_1 and $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ be a θ -covering. We denote by $P_1 = \operatorname{Pred}_{\alpha}(f_1/X_1)$ and $P_2 = \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$. Then the restricted map $\phi|_{P_1} : P_1 \to P_2$ is a bijection.

Proof. By Proposition 3 we have that $\phi|_{P_1} : P_1 \to P_2$ is onto. To prove that $\phi|_{P_1}$ is 1-to-1 we consider $\beta_1, \beta_2 \in P_1$ such that $\phi(\beta_1) = \phi(\beta_2)$. Then there is an integer $i \in \mathbb{Z}$ such that $\beta_2 = \theta^{(i)}(\beta_1)$. If the order of the permutation θ is ℓ , we can suppose that $0 \le i < \ell$ and we also have $\beta_1 = \theta^{(\ell-i)}(\beta_2)$. We consider the smallest integers $s_1, s_2 \ge 0$ such that $f_1^{(s_i)}(\beta_i) = \alpha$ for i = 1, 2 (they exist because $\beta_1, \beta_2 \in P_1$). We want to prove that $s_1 = s_2$. Consider the smallest integer $t \ge 0$ such that $f_1^{(t)}(\alpha) \in P_{f_1}$. We have that $\theta : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_1/X_1)$ is an isomorphism of a functional graph (since θ is bijective and $\theta \circ f_1 = f_1 \circ \theta$, thus, by Proposition 1, $\theta^{-1}(P_{f_1}) = P_{f_1}$. We have that $f_1^{(t+s_2)}(\beta_1) = \theta^{(\ell-i)}(f_1^{(t+s_2)}(\beta_2)) = \theta^{(\ell-i)}(f_1^{(t)}(\alpha)) \in \theta^{(\ell-i)}(P_{f_1}) = P_{f_1}$ (in particular $t + s_2 \ge s_1$ because $f_1^{(t+s_2)}(\beta_1) \in P_{f_1}$ and β_1 is a predecessor of α). We have that $f_1^{(t+s_2-s_1)}(\alpha) = f_1^{(t+s_2-s_1)}(f_1^{s_1}(\beta_1)) = f_1^{(t+s_2)}(\beta_1) \in P_{f_1}$ and by the minimality of t we conclude that $s_2 \ge s_1$. In a similar way we prove the other inequality $s_2 \le s_1$ obtaining $s_2 = s_1$; let us denote by $s = s_1 = s_2$. We have $\alpha = f_1^{(s)}(\beta_2) = f_i^{(s)}(\theta^{(i)}(\beta_1)) = \theta^{(i)}(f_1^{(s)}(\beta_1)) = \theta^{(i)}(\alpha)$ with $0 \le i < \ell$. Using that α is θ -maximal we conclude that i = 0 and $\beta_1 = \beta_2$ as desired.

B. Rooted tree isomorphism formulas

Let $\phi : \mathcal{G}(f_1/X_1) \to \mathcal{G}(f_2/X_2)$ be a homomorphism of functional graph. We consider a point $\alpha \in X_1$ and the sets $P_1 = \operatorname{Pred}_{\alpha}(f_1/X_1)$ and $P_2 = \operatorname{Pred}_{\phi(\alpha)}(f_2/X_2)$. When $\phi(P_1) \subseteq P_2$ and the restricted map $\phi|_{P_1} :$ $P_1 \to P_2$ is a bijection, this map determines an isomorphism between the rooted trees $T_1 = \operatorname{Tree}_{\alpha}(f_1/X_1)$ and $T_2 = \operatorname{Tree}_{\phi(\alpha)}(f_2/X_2)$ (i.e. a bijection between the vertices preserving directed edges). In this case we say that $\phi|_{P_1} : T_1 \to T_2$ is a rooted tree isomorphism and the trees T_1 and T_2 are isomorphic which is denoted by $T_1 \simeq T_2$. Sometimes, when the context is clear, we abuse notation and write $T_1 = T_2$ when these trees are isomorphic.

The first result is about the trees attached to the map $r_n(\alpha) = \alpha^n$. Since \mathbb{F}_q^* and H are closed under multiplication we have $r_n(\mathbb{F}_q^*) \subseteq \mathbb{F}_q^*$ and $r_n(H) \subseteq H$.

Proposition 5. Let $q - 1 = v_0 \omega_0$ and $q + 1 = v_1 \omega_1$ be the n-decomposition of q - 1 and q + 1, respectively. Let $\alpha \in \mathbb{F}_q^*$ and $\beta \in H$ be two r_n -periodic points. Then $Tree_\alpha(r_n/\mathbb{F}_q^*) = T_{\nu_0(n)}$ and $Tree_\beta(r_n/H) = T_{\nu_1(n)}$.

Proof. The sets \mathbb{F}_q^* and H are multiplicative cyclic groups of order q-1 and q+1, respectively. In general, if G is a multiplicative cyclic group of order $m = \nu \omega$ with $\operatorname{rad}(\nu) | \operatorname{rad}(n)$, $\operatorname{gcd}(n, \omega) = 1$, and $r_n : G \to G$ is the map given by $r_n(g) = g^n$ we prove that $\operatorname{Tree}_{g_0}(r_n/G) = T_{\nu(n)}$. Indeed, if ξ is a generator of G and

 $\phi : \mathbb{Z}_m \to G$ is the map given by $\phi(i) = \xi^i$, then $r_n \circ \phi(i) = (\xi^i)^n = \xi^{ni} = \phi \circ n(i)$ (where *n* denotes the multiplication-by-*n* map). This implies that $\phi : \mathcal{G}(n/\mathbb{Z}_m) \to \mathcal{G}(r_n/G)$ is an isomorphism of functional graphs. Since all the trees attached to periodic points in $\mathcal{G}(n/\mathbb{Z}_m)$ are isomorphic to $T_{\nu(n)}$ (Theorem 1) the same occurs for the trees attached to periodic points in $\mathcal{G}(r_n/G)$.

Proposition 6. If $n \ge 1$ is an odd integer and $a \in \mathbb{F}_q$, then $Tree_a(T_n/\mathbb{F}_q)$ and $Tree_{-a}(T_n/\mathbb{F}_q)$ are isomorphic.

Proof. Consider the map op : $\mathbb{F}_q \to \mathbb{F}_q$ given by op(x) = -x. Since *n* is an odd integer, the Chebyshev polynomial is an odd function and we have $op \circ T_n = T_n \circ op$. Thus $op : \mathcal{G}(T_n/\mathbb{F}_q) \to \mathcal{G}(T_n/\mathbb{F}_q)$ is an isomorphism of functional graphs and the results follows from Proposition 4.

Proposition 7. Let $\alpha \in \tilde{\mathbb{F}}_q$. Then, $Tree_{\alpha}(r_n/\tilde{\mathbb{F}}_q)$ and $Tree_{\alpha^{-1}}(r_n/\tilde{\mathbb{F}}_q)$ are isomorphic.

Proof. We consider the isomorphism of functional graphs $i : \mathcal{G}(r_n/\tilde{\mathbb{F}}_q) \to \mathcal{G}(r_n/\tilde{\mathbb{F}}_q)$ given by $i(x) = x^{-1}$ (it is an isomorphism because $i : \tilde{\mathbb{F}}_q \to \tilde{\mathbb{F}}_q$ is bijective and $i \circ r_n = r_n \circ i$). The results follows from Proposition 4.

Proposition 8. Let $\alpha \in \tilde{\mathbb{F}}_q$ with $\alpha \neq \pm 1$ and $a = \eta(\alpha)$. Then, $Tree_{\alpha}(r_n/\tilde{\mathbb{F}}_q)$ and $Tree_a(T_n/\mathbb{F}_q)$ are isomorphic.

Proof. We consider the homomorphism $\eta : \mathcal{G}(r_n/\tilde{\mathbb{F}}_q) \to \mathcal{G}(T_n/\mathbb{F}_q)$ (it is a homomorphism because $\eta \circ r_n = T_n \circ \eta$). This homomorphism is in fact a *i*-covering because $\eta^{-1}(a) = \{\alpha, i(\alpha) = \alpha^{-1}\}$ where $\alpha \in \tilde{\mathbb{F}}_q$ is a root of $x^2 - ax + 1 = 0$. We note that $\alpha \in \tilde{\mathbb{F}}_q$ is not *i*-maximal if and only if $\alpha = \alpha^{-1}$ since *i* is a permutation of order 2; this is equivalent to $\alpha = \pm 1$. If $\alpha \neq \pm 1$, then α is *i*-maximal and the result follows from Proposition 4.

IV. Splitting the functional graph $\mathcal{G}(T_n/\mathbb{F}_a)$ into uniform components

The most simple case of functional graph is when the trees attached to the periodic points are isomorphic. In this case describing the functional graph is equivalent to describing the cycle decomposition of the periodic points and the rooted tree attached to any periodic point. We start with a definition.

Definition 3. A functional graph $\mathcal{G}(f/X)$ is uniform if for every pair of periodic points $x, x' \in X$ the trees $Tree_x(f/X)$ and $Tree_{x'}(f/X)$ are isomorphic.

In this section we decompose the set \mathbb{F}_q in three T_n -invariant sets: R (the rational component), Q (the quadratic component) and S (the special component), obtaining a decomposition of the Chebyshev

functional graph

$$\mathcal{G}(T_n/\mathbb{F}_q) = \mathcal{G}(T_n/R) \oplus \mathcal{G}(T_n/Q) \oplus \mathcal{G}(T_n/S).$$
⁽²⁾

Moreover, we prove that the functional graphs of the right hand side are uniform (Proposition 10). We describe each component separately.

Lemma 1. We have $X \subseteq \mathbb{F}_q$ is T_n -invariant if and only if $\eta^{-1}(X)$ is r_n -invariant.

Proof. (\Rightarrow) Let $\alpha \in \eta^{-1}(X)$. We have $\eta(\alpha) \in X$ and $T_n(\eta(\alpha)) \in X$ (because X is forward T_n -invariant). Therefore $\eta(r_n(\alpha)) = T_n(\eta(\alpha)) \in X$ and then $r_n(\alpha) \in \eta^{-1}(X)$. This proves that $\eta^{-1}(X)$ is forward r_n -invariant. Now we consider $\beta \in \tilde{\mathbb{F}}_q$ such that $r_n(\beta) = \alpha \in \eta^{-1}(X)$. Then $T_n(\eta(\beta)) = \eta(r_n(\beta)) \in X$. Since X is backward T_n -invariant $\eta(\beta) \in X$, thus $\beta \in \eta^{-1}(X)$. This proves that $\eta^{-1}(X)$ is backward r_n -invariant. (\Leftarrow) Let $x \in X$. Since η is surjective we can write $x = \eta(\alpha)$ for some $\alpha \in \tilde{\mathbb{F}}_q$. We have $\alpha \in \eta^{-1}(X)$ and

using that $\eta^{-1}(X)$ is forward r_n -invariant we also have $r_n(\alpha) \in \eta^{-1}(X)$. Thus $T_n(x) = T_n(\eta(\alpha)) = \eta(r_n(\alpha)) \in X$. This proves that X is forward T_n -invariant. Now we consider $y \in \mathbb{F}_q$ such that $T_n(y) = x \in X$ and we can write $y = \eta(\beta)$ with $\beta \in \tilde{\mathbb{F}}_q$ since η is surjective. We have that $T_n(y) = T_n(\eta(\beta)) = \eta(r_n(\beta)) \in X$, thus $r_n(\beta) \in \eta^{-1}(X)$. Using that $\eta^{-1}(X)$ is backward r_n -invariant we conclude that $\beta \in \eta^{-1}(X)$. Therefore $y = \eta(\beta) \in X$ which proves that X is backward T_n -invariant.

Using the characterizations $\mathbb{F}_q^* = \{ \alpha \in \tilde{\mathbb{F}}_q : \operatorname{ord}(\alpha) \mid q-1 \}$ and $H = \{ \alpha \in \tilde{\mathbb{F}}_q : \operatorname{ord}(\alpha) \mid q+1 \}$, we obtain the following decomposition of $\tilde{\mathbb{F}}_q$ into r_n -invariant subsets.

Lemma 2. The subsets $\tilde{S} = \{ \alpha \in \tilde{\mathbb{F}}_q : \alpha^{n^k} = \pm 1 \text{ for some } k \ge 0 \}$, $\tilde{R} = \mathbb{F}_q^* \setminus \tilde{S} \text{ and } \tilde{Q} = H \setminus \tilde{S} \text{ form a partition of } \tilde{\mathbb{F}}_q \text{ in } r_n \text{-invariant subsets.}$

Proof. Since $(\pm 1)^n \subseteq {\pm 1}$, the set \tilde{S} is forward r_n -invariant. If $\alpha^n \in \tilde{S}$ there exists $k \ge 0$ such that $(\alpha^n)^{n^k} = \alpha^{n^{k+1}} = \pm 1$. Thus $\alpha \in \tilde{S}$ and \tilde{S} is backward r_n -invariant. This proves that \tilde{S} is r_n -invariant.

The proofs of the r_n -invariance of \tilde{R} and \tilde{Q} are similar. We only prove that \tilde{R} is r_n -invariant. It is easy to prove that the complement of an r_n -invariant is r_n -invariant and the intersection of two r_n invariant sets is also r_n -invariant. Since $R = \mathbb{F}_q^* \cap \tilde{S}^c$, it suffices to prove that \mathbb{F}_q^* is r_n -invariant. It is clear that \mathbb{F}_q^* is forward r_n -invariant. To prove that \mathbb{F}_q^* is backward r_n -invariant we use the characterization $\mathbb{F}_q^* = \{\alpha \in \tilde{\mathbb{F}}_q : \operatorname{ord}(\alpha) \mid q - 1\}$. We consider $\beta \in \tilde{\mathbb{F}}_q$ such that $r_n(\beta) = \beta^n \in \mathbb{F}_q^*$. The multiplicative order of β^n is given by $\operatorname{ord}(\beta^n) = \operatorname{ord}(\beta)/d$ with $d = \operatorname{gcd}(\operatorname{ord}(\beta), n)$. In particular $\operatorname{ord}(\beta) \mid q - 1$ (because $\operatorname{ord}(\beta) \mid \operatorname{ord}(\beta^n)$ and $\operatorname{ord}(\beta^n) \mid q - 1$), therefore $\beta \in \mathbb{F}_q^*$ by the above characterization of \mathbb{F}_q^* . \Box

Proposition 9. Let $R = \eta(\tilde{R}), Q = \eta(\tilde{Q})$ and $S = \eta(\tilde{S})$. The sets R, Q and S form a partition of \mathbb{F}_q in T_n -invariant sets. In particular the decomposition of $\mathcal{G}(T_n/\mathbb{F}_q)$ given by (2) holds.

Proof. It is straightforward to check that \tilde{R}, \tilde{Q} and \tilde{S} are *i*-invariant from which we obtain $\eta^{-1}(R) = \tilde{R}$, $\eta^{-1}(Q) = \tilde{Q}$ and $\eta^{-1}(S) = \tilde{S}$. By Lemma 2 these sets are r_n -invariant, and by Lemma 1 R, Q and S are T_n -invariant.

We finish this section proving that the functional graphs $\mathcal{G}(T_n/R)$, $\mathcal{G}(T_n/Q)$ and $\mathcal{G}(T_n/S)$ are uniform.

Proposition 10. The functional graphs $\mathcal{G}(T_n/R)$, $\mathcal{G}(T_n/Q)$ and $\mathcal{G}(T_n/S)$ are uniform. Moreover, every tree attached to a T_n -periodic point in $\mathcal{G}(T_n/R)$ is isomorphic to $T_{\nu_0(n)}$ and every tree attached to a T_n -periodic point in $\mathcal{G}(T_n/Q)$ is isomorphic to $T_{\nu_1(n)}$.

Proof. The easy case is to prove that $\mathcal{G}(T_n/S)$ is uniform, the other two cases are similar and we prove only that $\mathcal{G}(T_n/R)$ is uniform. If *n* or *q* is even, the only T_n -periodic point in *S* is 2 and there is nothing to prove. If *n* and *q* are odd there are two T_n -periodic points in *S*, 2 and -2, and the uniformity of $\mathcal{G}(T_n/S)$ follows from Proposition 6.

We denote by P_f the set of periodic points with respect to f and consider $a \in R \cap P_f$. We can write $a = \eta(\alpha)$ for some $\alpha \in \tilde{R}$ (in particular $a \in \mathbb{F}_q^*$ and $a \neq \pm 1$). By Proposition 8, Tree_a(T_n/\mathbb{F}_q) and Tree_a($r_n/\tilde{\mathbb{F}}_q$) are isomorphic. Using that \mathbb{F}_q^* is r_n -invariant and $a \in \mathbb{F}_q^*$ we have $\text{Tree}_{\alpha}(r_n/\tilde{\mathbb{F}}_q) = \text{Tree}_{\alpha}(r_n/\mathbb{F}_q^*)$ and by Proposition 1 (considering the *i*-covering $\eta : \mathcal{G}(r_n/\tilde{\mathbb{F}}_q) \to \mathcal{G}(T_n/\mathbb{F}_q)$) we have that α is an r_n -periodic point. By Proposition 5 we have that $\text{Tree}_{\alpha}(r_n/\tilde{\mathbb{F}}_q)$ is isomorphic to $T_{\nu_0(n)}$ and by transitivity $\text{Tree}_a(T_n/\mathbb{F}_q)$ is also isomorphic to $T_{\nu_0(n)}$.

V. THE RATIONAL AND QUADRATIC COMPONENTS

In this section we describe the functional graphs $\mathcal{G}(T_n/R)$ and $\mathcal{G}(T_n/Q)$.

The following proposition is a simple generalization of Proposition 2.1 of [7] for the general n case and is proved in a similar way.

Proposition 11. Let $a \in \mathbb{F}_q$, $\alpha \in \tilde{\mathbb{F}}_q$ such that $a = \alpha + \alpha^{-1}$ and $ord(\alpha) = ud$ the n-decomposition of the (multiplicative) order of α . Then $per(a) = \tilde{o}_d(n)$ and $pper(a) = \min\{k \ge 0 : u \mid n^k\}$.

Proof. Let $\pi = per(a)$ and $\rho = pper(a)$. Consider the following equivalences:

$$T_n^{\pi+\rho}(a) = T_n^{\rho}(a) \Leftrightarrow T_{n^{\pi+\rho}}(a) = T_{n^{\rho}}(a)$$

$$\Leftrightarrow \alpha^{n^{\pi+\rho}} + \alpha^{-n^{\pi+\rho}} = \alpha^{n^{\rho}} + \alpha^{-n^{\rho}}$$

$$\Leftrightarrow (\alpha^{n^{\pi+\rho}} - \alpha^{n^{\rho}})(\alpha^{n^{\pi+\rho}} - \alpha^{-n^{\rho}}) = 0$$

$$\Leftrightarrow \alpha^{n^{\pi+\rho}} = \alpha^{n^{\rho}} \text{ or } \alpha^{n^{\pi+\rho}} = \alpha^{-n^{\rho}}$$

$$\Leftrightarrow n^{\pi+\rho} \equiv \pm n^{\rho} \pmod{ud}$$

$$\Leftrightarrow n^{\pi} \equiv \pm 1 \pmod{d} \text{ and } u \mid n^{\rho}.$$
de that $\pi = \tilde{o}_d(n)$ and $\rho = \min\{k \ge 0 : u \mid n^k\}.$

By minimality, we conclude that $\pi = \tilde{o}_d(n)$ and $\rho = \min\{k \ge 0 : u \mid n^k\}$.

Corollary 1. Let $\alpha \in \tilde{\mathbb{F}}_q$. The point $a = \alpha + \alpha^{-1} \in \mathbb{F}_q$ is T_n -periodic point if and only if the multiplicative order of α (as element of $\mathbb{F}_{a^2}^*$) is coprime with n.

Proof. Let $a = \alpha + \alpha^{-1} \in \mathbb{F}_q$ and $\operatorname{ord}(\alpha) = ud$ be the *n*-decomposition of the (multiplicative) order of α . We have that a is T_n -periodic point if and only if pper(a) = 0 and by Proposition 11 this happens if and only if $u \mid 1$, that is, if and only if u = 1 and $gcd(ord(\alpha), n) = 1$.

Corollary 2. Let P_{T_n} be the set of T_n -periodic points, $\alpha \in \tilde{\mathbb{F}}_q$ and $a = \alpha + \alpha^{-1}$.

- 1. $a \in R \cap P_{T_n}$ if and only if $ord(\alpha) > 2$ and $ord(\alpha) \mid \omega_0$;
- 2. $a \in Q \cap P_{T_n}$ if and only if $ord(\alpha) > 2$ and $ord(\alpha) \mid \omega_1$;
- 3. $a \in S \cap P_{T_n}$ if and only if $ord(\alpha) \le 2$ and $gcd(ord(\alpha), n) = 1$.

Proof. Since η is surjective, $\eta(\eta^{-1}(X)) = X$ for all $X \subseteq \mathbb{F}_q$ (in particular $a \in X$ if and only if $\alpha \in \eta^{-1}(X)$). Denote $\tilde{P}_{T_n} := \eta^{-1}(P_{T_n})$. By Corollary 1, $\tilde{P} = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = 1\}$. First we prove that $\tilde{P}_{T_n} \cap \tilde{S} = \tilde{P}_{T_n} \cap \{\pm 1\}$. Indeed, if $\alpha \in \tilde{P}_{T_n} \cap \tilde{S}$, then $gcd(ord(\alpha), n) = 1$ and $\alpha^{n^k} = \pm 1$ for some $k \ge 0$. Thus $\operatorname{ord}(\alpha) = \frac{\operatorname{ord}(\alpha)}{\gcd(\operatorname{ord}(\alpha), n^k)} = \operatorname{ord}(\alpha^{n^k}) = \operatorname{ord}(\pm 1)|2$ which implies $\alpha = \pm 1$. This proves that $\tilde{P}_{T_n} \cap \tilde{S} \subseteq \mathbb{C}$ $\tilde{P}_{T_n} \cap \{+1\}$ and the other inclusion is clear. We note that this is equivalent to $\tilde{P}_{T_n} \cap \tilde{S}^c = \tilde{P}_{T_n} \cap \{+1\}^c$. Now we prove the statements.

1. $a \in R \cap P_{T_n}$ if and only if $\alpha \in \tilde{R} \cap \tilde{P}_{T_n} = \mathbb{F}_q^* \cap \tilde{S}^c \cap \tilde{P}_{T_n} = \tilde{P}_{T_n} \cap \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) \in \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) \in \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) \in \mathbb{F}_q^* \cap \{\pm 1\}^c = \{\alpha \in \mathbb{F}_q : \gcd(\operatorname{ord}(\alpha), n) \in \mathbb{F}_q^* \cap \{\pm 1\}^c \in \mathbb{F}_q^*$ 1, $\operatorname{ord}(\alpha)|q-1, \alpha \neq \pm 1\} = \{\alpha \in \tilde{\mathbb{F}}_q : \operatorname{ord}(\alpha)|\omega_0, \operatorname{ord}(\alpha) > 2\}.$

2. This part is similar to 1.; here we use $\alpha \in H$ if and only if $\operatorname{ord}(\alpha) \mid q+1$.

3. $a \in S \cap P_{T_n}$ if and only if $\alpha \in \tilde{S} \cap \tilde{P}_{T_n} = \tilde{P}_{T_n} \cap \{\pm 1\} = \{\alpha \in \tilde{\mathbb{F}}_q : \gcd(\operatorname{ord}(\alpha), n) = 1, \operatorname{ord}(\alpha) \le 2\}.$ Next we obtain an isomorphism formula for the rational component and the quadratic component of $\mathcal{G}(T_n/\mathbb{F}_q)$.

Theorem 2. Let $q - 1 = v_0 \omega_0$ and $q + 1 = v_1 \omega_1$ be their n-decompositions. The rational component of the Chebyshev's graph $\mathcal{G}(T_n/\mathbb{F}_q)$ is given by:

$$\mathcal{G}(T_n/R) = \bigoplus_{\substack{d \mid \omega_0 \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} \times Cyc\left(\tilde{o}_d(n), T_{\nu_0(n)}\right);$$

the quadratic component is given by

$$\mathcal{G}(T_n/Q) = \bigoplus_{\substack{d \mid \omega_1 \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} \times Cyc\left(\tilde{o}_d(n), T_{\nu_1(n)}\right).$$

Proof. We only prove the statement for the rational component since the proof for the quadratic component is similar. Let P_{T_n} be the set of T_n -periodic points and $R_d = \{\alpha + \alpha^{-1} : \alpha \in \tilde{\mathbb{F}}_q, \operatorname{ord}(\alpha) = d\}$. By Corollary 2, $R \cap P_{T_n}$ is the disjoint union of R_d with $d \mid \omega_0, d > 2$. If $\operatorname{ord}(\alpha) = d \mid \omega_0$ we have that $\operatorname{gcd}(d, n) = 1$ and $\operatorname{ord}(\alpha^n) = \operatorname{ord}(\alpha)/\operatorname{gcd}(\operatorname{ord}(\alpha), n) = \operatorname{ord}(\alpha)$. Then we have the following decomposition $\mathcal{G}(T_n/R \cap P_{T_n}) = \bigoplus_{\substack{d \mid \omega_0 \\ d > 2}} \mathcal{G}(T_n/R_d)$. By Proposition 11, every point in $\mathcal{G}(T_n/R_d)$ belongs to a cycle of length $\tilde{o}_d(n)$. Thus,

$$\mathcal{G}(T_n/R \cap P_{T_n}) = \bigoplus_{\substack{d \mid \omega_0 \\ d > 2}} \frac{\#R_d}{\tilde{o}_d(n)} \times \operatorname{Cyc}\left(\tilde{o}_d(n), \bullet\right).$$
(3)

For each $d \mid \omega_0, d > 2$, we consider the set $\tilde{R}_d = \{\alpha \in \tilde{\mathbb{F}}_q : \operatorname{ord}(\alpha) = d\}$. By a standard counting argument $\#\tilde{R}_d = \varphi(d)$ and using that the restriction of η to \tilde{R} is a 2-to-1 map from \tilde{R} onto R we obtain $\#R = \#\tilde{R}/2 = \varphi(d)/2$. Substituting this expression into Equation (3) and using the uniformity of $\mathcal{G}(T_n/R)$ (Proposition 10) we obtain $\mathcal{G}(T_n/R) = \bigoplus_{\substack{d \mid \omega_0 \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} \times \operatorname{Cyc}(\tilde{o}_d(n), T_{\nu_0(n)})$.

VI. THE SPECIAL COMPONENT OF $\mathcal{G}(T_n/\mathbb{F}_q)$

In this section we describe the special component of the Chebyshev functional graph $\mathcal{G}(T_n/S)$ where $S = \{a \in \mathbb{F}_q : T_n(a)^{(k)} = \pm 2, \text{ for some } k \ge 0\}$. If *n* and *q* are odd, $T_n(-2) = -2$ and $T_n(2) = 2$ then the only periodic points of T_n in *S* are 2 and -2. In this case the trees attached to the fixed points 2 and -2 are isomorphic (Proposition 10). If either *n* is even or *q* is even, $T_n(-2) = 2 = T_n(2)$ and the only periodic point of T_n in *S* is 2 (if *q* is even this is true because 2 = -2). The next proposition summarizes the above discussion.

Proposition 12. Let $\mathcal{T} = Tree_2(T_n/\mathbb{F}_q)$ be the rooted tree attached to the fixed point 2 for the Chebyshev polynomial T_n restricted to the set $S = \{a \in \mathbb{F}_q : T_n(a)^{(k)} = \pm 2, \text{ for some } k \ge 0\}$. Then

$$\mathcal{G}(T_n/S) = \begin{cases} 2 \times Cyc(1,T) & \text{if } n \text{ is odd and } q \text{ is odd} \\ Cyc(1,T) & \text{otherwise.} \end{cases}$$

We remark that $\text{Tree}_2(T_n/S) = \text{Tree}_2(T_n/\mathbb{F}_q)$, which is a consequence of S being T_n -invariant (Proposition 9). By Proposition 12, to describe the special component it suffices to describe the tree $\mathcal{T} = \text{Tree}_2(T_n/\mathbb{F}_q)$. If $q - 1 = v_0\omega_0$ and $q + 1 = v_1\omega_1$ is the *n*-decomposition of q - 1 and q + 1, respectively, the rooted trees attached to the periodic points are isomorphic to $T_{v_0(n)}$ in the rational component and isomorphic to $T_{v_1(n)}$ in the quadratic component (Proposition 10). In the case of the special component the situation is different, the tree $\mathcal{T} = \text{Tree}_2(T_n/\mathbb{F}_q)$ is not isomorphic to a tree associated to a *v*-series (that is, the trees associated to the multiplication by *n* map over \mathbb{Z}_m for some $m \in \mathbb{Z}^+$). However we show in this section that the tree \mathcal{T} can be expressed as a "mean" of the trees $T_{v_0(n)}$ and $T_{v_10(n)}$. In the first part of this section we define the bisection of trees together some of their main properties. In the second part we deduce an isomorphism formula for the special component of the Chebyshev graph.

A. Bisection of rooted trees

We start by defining the sum of rooted trees.

Definition 4. Let $T = \langle T_1 \oplus T_2 \oplus \cdots \oplus T_r \rangle$ and $T' = \langle T'_1 \oplus T'_2 \oplus \cdots \oplus T'_s \rangle$ be two rooted trees. We define their sum as $T + T' = \langle T_1 \oplus T_2 \oplus \cdots \oplus T_r \oplus T'_1 \oplus T'_2 \oplus \cdots \oplus T'_s \rangle$.

We remark that the tree consisting of a unique node $T = \bullet = \langle 0 \rangle$ is the neutral element of the sum. The tree T - T' denotes a tree such that T = T' + (T - T') in case this tree exists (if it exists, it is unique up to isomorphism). We note that $(T_1 + T_2) - T'$ is defined if and only if $T_i - T'$ is defined for some i = 1, 2. If $T_1 - T'$ is defined then $(T_1 + T_2) - T' = (T_1 - T') + T_2$ and if $T_2 - T'$ is defined then $(T_1 + T_2) - T' = T_1 + (T_2 - T')$. Therefore when $(T_1 + T_2) - T'$ is defined we can write this tree as $T_1 + T_2 - T'$ without ambiguity.

A forest is a graph that can be expressed as a disjoint union of rooted trees. A tree *T* is *even* if it can be expressed as $T = \langle 2 \times F \rangle$ for some forest *F* and it is *quasi-even* if it can be expressed as $T = \langle 2 \times F \oplus T' \rangle$ for some forest *F* and some even tree *T'* (i.e. $T' = \langle 2 \times F' \rangle$ for some forest *F'*). In particular the tree $T = \bullet$ is even because $T = \langle 2 \times \emptyset \rangle$. For these classes of trees we define the bisection as follows.

Definition 5. If $T = \langle 2 \times F \rangle$ is an even tree, its bisection is the tree $\frac{1}{2}T = \langle F \rangle$. If $T = \langle 2 \times F \oplus \langle 2 \times F' \rangle \rangle$ is a quasi-even tree its bisection is defined as the tree $\frac{1}{2}T = \langle F \oplus \langle F' \rangle \rangle$.

Example 2. The tree associated with the v-series 18(30) = (6,3) is given by $T_{(6,3)} = \langle 2 \times T \oplus 3 \times T' \rangle$ where $T = \langle 6 \times \bullet \rangle$ and $T' = \bullet$. Thus $T_{(6,3)}$ is quasi-even since it can be written as $T_{(6,3)} = \langle 2 \times F \oplus T' \rangle$ with $F = T \oplus T'$ and $T' = \langle 2 \times \emptyset \rangle$ is even. The bisection of this tree is given by $\frac{1}{2}T_{(6,3)} = \langle F \oplus \langle \emptyset \rangle \rangle = \langle T \oplus 2 \times T' \rangle$.

Even and quasi-even trees are very restricted classes of trees, however they contain all trees associated with ν -series as stated in the following proposition.

Proposition 13. If $T_{\nu(n)}$ is the tree associated with $\nu(n) = (\nu_1, \ldots, \nu_D)$, then $T_{\nu(n)}$ is even when ν is odd and quasi-even when ν is even.

Proof. By Equation (1) we have $T_{\nu(n)} = \langle (\nu_D - 1) \times T^{D-1} \oplus \bigoplus_{i=1}^{D-1} (\nu_i - \nu_{i+1}) \times T^{i-1} \rangle$, where the T_i are pairwise non-isomorphic rooted trees. When ν is odd, ν_i is odd for $1 \le i \le D$. Then, $\nu_D - 1$ and $\nu_i - \nu_{i+1}$ are even for $1 \le i \le D - 1$ and the tree $T_{\nu(n)}$ is even. When ν is even, we have that ν_1, \ldots, ν_k are even and ν_{k+1}, \ldots, ν_D are odd for some $k, 1 \le k \le D$. If k = D, then $\nu_D - 1$ is odd and $\nu_i - \nu_{i+1}$ are even for $1 \le i \le D - 1$ and the tree $T_{\nu(n)}$ is quasi-even. If k < D, then $\nu_D - 1$ and $\nu_i - \nu_{i+1}$ are even for $1 \le i \le D - 1$ and the tree $T_{\nu(n)}$ is quasi-even. If k < D, then $\nu_D - 1$ and $\nu_i - \nu_{i+1}$ are even for $1 \le i \le L - 1$ and $k + 1 \le i \le D$, and $\nu_k - \nu_{k+1}$ is odd. Thus, $T_{\nu(n)}$ is also quasi-even.

We note that the if T_1 and T_2 are rooted trees, then $|T_1 + T_2| = |T_1| + |T_2| - 1$ where, as usual, |T| denotes the number of nodes of T. The next proposition establishes a relation between |T| and $|\frac{1}{2}T|$.

Proposition 14. Let T be a rooted tree with |T| = N nodes. We have

$$|1/2 \cdot T| = \begin{cases} \frac{N+1}{2} & \text{if } T \text{ is even;} \\ \frac{N+2}{2} & \text{if } T \text{ is quasi-even} \end{cases}$$

Proof. If *T* is even, there is a forest *S* with *s* nodes such that $T = \langle 2 \times S \rangle$. We have N = |T| = 1 + 2s from which we obtain $s = \frac{N-1}{2}$. Since $\frac{1}{2}T = \langle S \rangle$, $|\frac{1}{2}T| = s + 1 = \frac{N-1}{2} + 1 = \frac{N+1}{2}$. If *T* is quasi-even, there is a pair of forests *S* and *R* with *s* and *r* nodes, respectively, such that $T = \langle 2 \times S \oplus \langle 2 \times R \rangle \rangle$. We have N = |T| = 1 + 2s + 1 + 2r = 2(r + s + 1) from which we obtain $r + s + 1 = \frac{N}{2}$. Since $\frac{1}{2}T = \langle S \oplus \langle R \rangle \rangle$, $|\frac{1}{2}T| = 1 + s + 1 + r = 1 + \frac{N}{2} = \frac{N+2}{2}$.

B. The tree $Tree_2(T_n/\mathbb{F}_q)$

The next theorem describe the rooted tree attached to the fixed point 2 for the Chebyshev polynomial $T_n : \mathbb{F}_q \to \mathbb{F}_q$. We require the following lemma.

Lemma 3. Let n > 1 be an even integer, \mathbb{F}_q be an odd characteristic finite field and H be the multiplicative subgroup of $\mathbb{F}_{q^2}^*$ with order q + 1.

(i) If $q \equiv 3 \pmod{4}$, the equation $x^n = -1$ has no solution in \mathbb{F}_q^* .

Proof. Let $\alpha \in \tilde{\mathbb{F}}_q$ be a solution of $x^n = -1$. From the relations $\operatorname{ord}(\alpha^n) = \operatorname{ord}(\alpha)/\operatorname{gcd}(\operatorname{ord}(\alpha), n)$ and $\operatorname{ord}(-1) = 2$, we conclude that if *n* is even, then 4 | $\operatorname{ord}(\alpha)$. By Lagrange theorem, $\alpha \in \mathbb{F}_q^*$ implies 4 | q-1 and $q \neq 3 \pmod{4}$; and $\alpha \in H$ implies 4 | q-3 and $q \neq 1 \pmod{4}$.

Theorem 3. Let $q - 1 = v_0\omega_0$ and $q + 1 = v_1\omega_1$ be their *n*-decompositions. The rooted tree associated with the fixed point 2 is described as follows:

$$Tree_2(T_n/\mathbb{F}_q) = \begin{cases} 1/2 \cdot T_{\nu_0(n)} + 1/2 \cdot T_{\nu_1(n)} & \text{if } n \text{ is odd or } q \text{ is even;} \\ 1/2 \cdot T_{\nu_0(n)} + 1/2 \cdot T_{\nu_1(n)} - \langle \bullet \rangle & \text{if } n \text{ is even and } q \text{ is odd.} \end{cases}$$

Proof. The isomorphism formula is obtained after relating $\text{Tree}_2(T_n/\mathbb{F}_q)$ and $\text{Tree}_1(r_n/\tilde{\mathbb{F}}_q)$. First we consider the case when *n* is odd or *q* is even. In this case $r_n(-1) = -1$ or -1 = 1, in both cases we have that the predecessors of 1 in $\text{Tree}_1(r_n/\tilde{\mathbb{F}}_q)$ are in \mathbb{F}_q^* or in *H* (but not in both). Since the sets \mathbb{F}_q^* and *H* are backward r_n -invariant (Lemma 2), we have

$$\operatorname{Tree}_{1}(r_{n}/\tilde{\mathbb{F}}_{q}) = \operatorname{Tree}_{1}(r_{n}/\mathbb{F}_{q}^{*}) + \operatorname{Tree}_{1}(r_{n}/H) = T_{\nu_{0}(n)} + T_{\nu_{1}(n)},$$

where in the last equality we use Proposition 5. Now, we write $r_n^{-1}(1) \cap \mathbb{F}_q^* = \{\alpha_1, \ldots, \alpha_{2s}, 1\}$ with $\alpha_{s+i} = \alpha_i^{-1}, \alpha_i \neq \pm 1$, for all $i : 1 \le i \le s$ and $r_n^{-1}(1) \cap H = \{\beta_1, \ldots, \beta_{2t}, 1\}$ with $\beta_{t+j} = \beta_j^{-1}, \beta_j \neq \pm 1$, for all $j : 1 \le j \le t$. Denote by $\tilde{T}(\alpha_i) := \operatorname{Tree}_{\alpha_i}(r_n/\mathbb{F}_q^*)$ for $1 \le i \le 2s$ and $\tilde{T}(\beta_j) := \operatorname{Tree}_{\beta_j}(r_n/H)$ for $1 \le j \le 2t$. Using Proposition 7 we have that $T_{\nu_0(n)} = \operatorname{Tree}_1(r_n/\mathbb{F}_q^*) = \langle \tilde{T}(\alpha_1) \oplus \cdots \oplus \tilde{T}(\alpha_{2s}) \rangle = \langle 2 \times (\tilde{T}(\alpha_1) \oplus \cdots \oplus \tilde{T}(\alpha_s)) \rangle$, from which we obtain

$$1/2 \cdot T_{\nu_0(n)} = \langle \tilde{T}(\alpha_1) \oplus \cdots \oplus \tilde{T}(\alpha_s) \rangle.$$

In the same way we obtain

$$1/2 \cdot T_{\nu_1(n)} = \langle \tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_t) \rangle$$

Let $a_i = \eta(\alpha_i), T(a_i) = \text{Tree}_{a_i}(T_n/\mathbb{F}_q), b_j = \eta(\alpha_j) \text{ and } T(b_j) = \text{Tree}_{b_j}(T_n/\mathbb{F}_q) \text{ for } 1 \le i \le s, 1 \le j \le t$. We have $T_n^{-1}(2) = \{a_1, \dots, a_s, b_1, \dots, b_t, 2\}$ and

$$Tree_{2}(T_{n}/\mathbb{F}_{q}) = \langle T(a_{1}) \oplus \cdots \oplus T(a_{s}) \oplus T(b_{1}) \oplus \cdots \oplus T(b_{t}) \rangle$$
$$= \langle \tilde{T}(\alpha_{1}) \oplus \cdots \oplus \tilde{T}(\alpha_{s}) \oplus \tilde{T}(\beta_{1}) \oplus \cdots \oplus \tilde{T}(\beta_{t}) \rangle \quad \text{(by Prop. 8)}$$
$$= \langle \tilde{T}(\alpha_{1}) \oplus \cdots \oplus \tilde{T}(\alpha_{s}) \rangle + \langle \tilde{T}(\beta_{1}) \oplus \cdots \oplus \tilde{T}(\beta_{t}) \rangle$$
$$= 1/2 \cdot T_{\nu_{0}(n)} + 1/2 \cdot T_{\nu_{1}(n)}.$$

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Now we consider the case when *n* is even and *q* is odd. Here we can write $r_n^{-1}(1) \cap \mathbb{F}_q^* = \{\alpha_1, \dots, \alpha_{2s}, -1, 1\}$ with $\alpha_{s+i} = \alpha_i^{-1}$, $\alpha_i \neq \pm 1$, for all $i : 1 \le i \le s$, $r_n^{-1}(1) \cap H = \{\beta_1, \dots, \beta_{2t}, -1, 1\}$ with $\beta_{t+j} = \beta_j^{-1}$, $\beta_j \neq \pm 1$, for all $j : 1 \le j \le t$ and $r_n^{-1}(-1) = \{\gamma_1, \dots, \gamma_{2r}\}$ with $\gamma_{r+k} = \gamma_k^{-1}$, $\gamma_k \neq \pm 1$, for all $k : 1 \le k \le r$.

Denote by $\tilde{T}(\alpha_i) := \operatorname{Tree}_{\alpha_i}(r_n/\mathbb{F}_q^*)$ for $1 \le i \le 2s$, $\tilde{T}(\beta_j) := \operatorname{Tree}_{\beta_j}(r_n/H)$ for $1 \le j \le 2t$, $\tilde{T}(\gamma_k) := \operatorname{Tree}_{\gamma_k}(r_n/\tilde{\mathbb{F}}_q)$ for $1 \le k \le 2r$ and $\tilde{T}(-1) := \operatorname{Tree}_{-1}(r_n/\tilde{\mathbb{F}}_q)$. In this case we have, by Proposition 7, $\tilde{T}(-1) = \langle \tilde{T}(\gamma_1) \oplus \cdots \oplus \tilde{T}(\gamma_{2r}) \rangle = \langle 2 \times (\tilde{T}(\gamma_1) \oplus \cdots \oplus \tilde{T}(\gamma_r)) \rangle$, thus

$$1/2 \cdot \tilde{T}(-1) = \langle \tilde{T}(\gamma_1) \oplus \dots \oplus \tilde{T}(\gamma_r) \rangle.$$
(4)

Let $a_i = \eta(\alpha_i), T(a_i) = \text{Tree}_{a_i}(T_n/\mathbb{F}_q), b_j = \eta(\alpha_j), T(b_j) = \text{Tree}_{b_j}(T_n/\mathbb{F}_q), c_k = \eta(\gamma_k), T(c_k) = \text{Tree}_{c_k}(T_n/\mathbb{F}_q)$ for $1 \le i \le s, 1 \le j \le t, 1 \le k \le r$ and $T(-2) = \text{Tree}_{-2}(T_n/\mathbb{F}_q)$. We have $T_n^{-1}(2) = \{a_1, \ldots, a_s, b_1, \ldots, b_t, -2, 2\},$ $T_n^{-1}(-2) = \{c_1, \ldots, c_r\}$. By Proposition 7 and Equation (4) we have $T(-2) = \langle T(c_1) \oplus \cdots \oplus T(c_r) \rangle = \langle \tilde{T}(\gamma_1) \oplus \cdots \oplus \tilde{T}(\gamma_r) \rangle = 1/2 \cdot \tilde{T}(-1)$, thus

$$\operatorname{Tree}_{2}(T_{n}/\mathbb{F}_{q}) = \langle T(a_{1}) \oplus \cdots \oplus T(a_{s}) \oplus T(b_{1}) \oplus \cdots \oplus T(b_{t}) \oplus T(-2) \rangle$$
$$= \langle \tilde{T}(\alpha_{1}) \oplus \cdots \oplus \tilde{T}(\alpha_{s}) \oplus \tilde{T}(\beta_{1}) \oplus \cdots \oplus \tilde{T}(\beta_{t}) \oplus 1/2 \cdot \tilde{T}(-1) \rangle.$$
(5)

Now we consider two subcases: $q \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{4}$. First we consider the subcase $q \equiv 1 \pmod{4}$. (mod 4). By Lemma 3 we have $r_n^{-1}(-1) \cap H = \emptyset$ and $r_n^{-1}(-1) \subseteq \mathbb{F}_q^*$. Thus $\tilde{T}(-1) = \text{Tree}_{-1}(T_n/\mathbb{F}_q^*)$ and we have, by Propositions 5 and 7, $T_{\nu_0(n)} = \text{Tree}_1(r_n/\mathbb{F}_q^*) = \langle \tilde{T}(\alpha_1) \oplus \cdots \oplus \tilde{T}(\alpha_{2s}) \oplus \tilde{T}(-1) \rangle = \langle 2 \times (\tilde{T}(\alpha_1) \oplus \cdots \oplus \tilde{T}(\alpha_s)) \oplus \tilde{T}(-1) \rangle$. Therefore

$$1/2 \cdot T_{\nu_0(n)} = \langle \tilde{T}(\alpha_1) \oplus \dots \oplus \tilde{T}(\alpha_s) \oplus 1/2 \cdot \tilde{T}(-1) \rangle.$$
(6)

Since $r_n^{-1}(-1) \cap H = \emptyset$, we have $T_{\nu_1(n)} = \text{Tree}_1(r_n/H) = \langle \tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_{2t}) \oplus \bullet \rangle = \langle 2 \times (\tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_t)) \oplus \bullet \rangle$ $\tilde{T}(\beta_t)) \oplus \bullet \rangle$ and $1/2 \cdot T_{\nu_1(n)} = \langle \tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_t) \oplus \bullet \rangle = \langle \tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_t) \rangle + \langle \bullet \rangle$; from which we obtain

 $1/2 \cdot T_{\nu_1(n)} - \langle \bullet \rangle = \langle \tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_t) \rangle.$ ⁽⁷⁾

Substituting Equations (6) and (7) in Equation (5) we have

$$\operatorname{Tree}_{2}(T_{n}/\mathbb{F}_{q}) = \langle \tilde{T}(\alpha_{1}) \oplus \cdots \oplus \tilde{T}(\alpha_{s}) \oplus \tilde{T}(\beta_{1}) \oplus \cdots \oplus \tilde{T}(\beta_{t}) \oplus 1/2 \cdot \tilde{T}(-1) \rangle$$
$$= \langle \tilde{T}(\alpha_{1}) \oplus \cdots \oplus \tilde{T}(\alpha_{s}) \oplus 1/2 \cdot \tilde{T}(-1) \rangle + \langle \tilde{T}(\beta_{1}) \oplus \cdots \oplus \tilde{T}(\beta_{t}) \rangle$$
$$= 1/2 \cdot T_{\nu_{0}(n)} + 1/2 \cdot T_{\nu_{1}(n)} - \langle \bullet \rangle.$$

The proof of the subcase $q \equiv 3 \pmod{4}$ is similar. In this case applying Lemma 3 we obtain $\tilde{T}(-1) =$ Tree₋₁(T_n/H) and using the same arguments used for the subcase $q \equiv 1 \pmod{4}$ we obtain

$$1/2 \cdot T_{\nu_1(n)} = 1/2 \cdot \text{Tree}_1(r_n/H) = \langle \tilde{T}(\beta_1) \oplus \cdots \oplus \tilde{T}(\beta_t) \oplus 1/2 \cdot \tilde{T}(-1) \rangle$$
(8)

and

$$1/2 \cdot T_{\nu_0(n)} - \langle \bullet \rangle = \langle \tilde{T}(\alpha_1) \oplus \cdots \oplus \tilde{T}(\alpha_s) \rangle.$$
(9)

Using Equations (5), (8) and (9) we have $\text{Tree}_2(T_n/\mathbb{F}_q) = 1/2 \cdot T_{\nu_0(n)} + 1/2 \cdot T_{\nu_1(n)} - \langle \bullet \rangle$. \Box

VII. STRUCTURE THEOREM FOR CHEBYSHEV POLYNOMIAL AND CONSEQUENCES

A. Isomorphism formula for $\mathcal{G}(T_n/\mathbb{F}_q)$

We summarize all the information in the following main theorem of this paper, which follows from Theorems 2 and 3 and Proposition 12.

Theorem 4. Let $q - 1 = v_0 \omega_0$ and $q + 1 = v_1 \omega_1$ be the n-decomposition of q - 1 and q + 1, respectively. The Chebyshev graph admits a decomposition of the form $\mathcal{G}(T_n/\mathbb{F}_q) = \mathcal{G}^R \oplus \mathcal{G}^Q \oplus \mathcal{G}^S$ where the rational component \mathcal{G}^R is given by

$$\mathcal{G}^{R} = \bigoplus_{\substack{d \mid \omega_{0} \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_{d}(n)} \times Cyc\left(\tilde{o}_{d}(n), T_{\nu_{0}(n)}\right);$$

the quadratic component \mathcal{G}^Q is given by

$$\mathcal{G}^{Q} = \bigoplus_{\substack{d \mid \omega_{1} \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_{d}(n)} \times Cyc\left(\tilde{o}_{d}(n), T_{\nu_{1}(n)}\right);$$

and the special component \mathcal{G}^S is given by

$$\mathcal{G}^{S} = \begin{cases} Cyc(1, 1/2 \cdot T_{\nu_{0}(n)} + 1/2 \cdot T_{\nu_{1}(n)} - \langle \bullet \rangle) & \text{if } n \text{ is even and } q \text{ is odd,} \\ 2 \times Cyc(1, 1/2 \cdot T_{\nu_{0}(n)} + 1/2 \cdot T_{\nu_{1}(n)}) & \text{if } n \text{ is odd and } q \text{ is odd;} \\ Cyc(1, 1/2 \cdot T_{\nu_{0}(n)} + 1/2 \cdot T_{\nu_{1}(n)}) & \text{if } q \text{ is even.} \end{cases}$$

B. Examples

We provide a series of examples showing our main result.

Example 3. We consider the Chebyshev polynomial T_{30} over \mathbb{F}_{19} (see Figure 1a). We have $19-1 = 18 = v_0\omega_0$ with $v_0 = 18$, $\omega_0 = 1$ and $19+1 = 20 = v_1\omega_1$ with $v_1 = 20$, $\omega_1 = 1$. Since $\omega_0, \omega_1 \le 2$ both the rational and the quadratic components of $\mathcal{G}(T_{30}/\mathbb{F}_{19})$ are empty. We calculate the v-series 18(30) = (6, 3) and 20(30) = (10, 2) obtaining

$$\mathcal{G}(T_{30}/\mathbb{F}_{19}) = Cyc\left(1, \frac{1}{2}T_{(6,3)} + \frac{1}{2}T_{(10,2)} - \langle \bullet \rangle\right).$$

Thus, the graph $\mathcal{G}(T_{30}/\mathbb{F}_{19})$ consist of a loop corresponding to the fix point 2 and a tree $T = \frac{1}{2}T_{(6,3)} + \frac{1}{2}T_{(10,2)} - \langle \bullet \rangle$ attached to this point; see Figure 3.

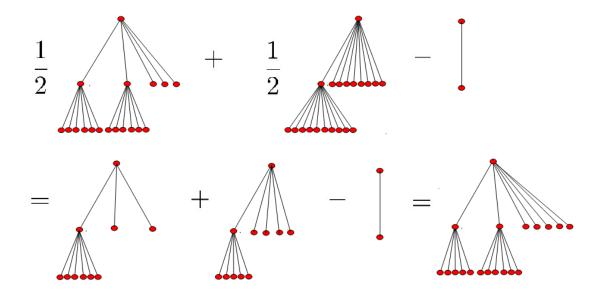


Fig. 3. Construction of the tree $T = \frac{1}{2}T_{(6,3)} + \frac{1}{2}T_{(10,2)} - \langle \bullet \rangle$.

Example 4. Now we consider again the Chebyshev polynomial T_{30} but this time over \mathbb{F}_{23} (see Figure 1b). We have $23 - 1 = 22 = v_0\omega_0$ with $v_0 = 2, \omega_0 = 11$ and $23 + 1 = 24 = v_1\omega_1$ with $v_1 = 24, \omega_1 = 1$. In this case the quadratic component of $\mathcal{G}(T_{30}/\mathbb{F}_{23})$ is empty and the rational component is $\frac{\varphi(11)}{2\tilde{o}_{11}(30)} \times Cyc(\tilde{o}_{11}(30), T_{2(30)})$. Since $\varphi(11) = 10$, $\tilde{o}_{11}(30) = 5$ and $T_{2(30)} = T_{(2)} = \langle \bullet \rangle$, it is given by $Cyc(5, \langle \bullet \rangle)$. We calculate the v-series 24(30) = (6, 2, 2). Then, the Chebyshev's graph of T_{30} over \mathbb{F}_{23} is given by:

$$\mathcal{G}(T_{30}/\mathbb{F}_{23}) = Cyc\left(5, \langle\bullet\rangle\right) \oplus Cyc\left(1, \frac{1}{2}T_{(2)} + \frac{1}{2}T_{(6,2,2)} - \langle\bullet\rangle\right).$$

We have $\frac{1}{2}T_{(2)} = \frac{1}{2}\langle\langle 2 \times \emptyset \rangle\rangle = \langle\langle \emptyset \rangle\rangle = \langle \bullet \rangle$ (i.e. $T_{(2)}$ is invariant under bisection), and after simplifying we obtain $\mathcal{G}(T_{30}/\mathbb{F}_{23}) = Cyc(5, \langle \bullet \rangle) \oplus Cyc(1, \frac{1}{2}T_{(6,2,2)})$. To obtain a more explicit formula we calculate the bisection of $T_{(6,2,2)}$. Using the recursive formula (1), we obtain $T_{(6,2,2)} = \langle 4 \times \bullet \oplus T \rangle$ where $T = \langle 4 \times \bullet \oplus 2 \times \langle 6 \times \bullet \rangle \rangle$, then $T_{(6,2,2)}$ is quasi-even and T is even. Since $\frac{1}{2}T = \langle 2 \times \bullet \oplus \langle 6 \times \bullet \rangle \rangle$, we have $\frac{1}{2}T_{(6,2,2)} = \langle 2 \times \bullet \oplus \frac{1}{2}T \rangle = \langle 2 \times \bullet \oplus \langle 6 \times \bullet \rangle \rangle$.

Example 5. We consider again the Chebyshev polynomial T_{30} , this time over the reasonably large finite field \mathbb{F}_{739} where the symmetries can be better appreciated; see Figure 4. We calculate the 30-decomposition of 738 = 18 · 41 (v_0 = 18, ω_0 = 41) and 740 = 120 · 37 (v_1 = 20, ω_1 = 37). Since $\varphi(41) = 40$, $\tilde{o}_{41}(30) = 20$, $\varphi(37) = 36$, $\tilde{o}_{37}(30) = 9$, the rational component \mathcal{G}^R and the quadratic component \mathcal{G}^Q are given by $\mathcal{G}^R = Cyc(20, T_{18(30)})$ and $\mathcal{G}^Q = 2 \times Cyc(9, T_{20(30)})$. We have 18(30) = (6, 3) and 20(30) = (10, 2). Thus the special component is $\mathcal{G}^S = Cyc\left(1, \frac{1}{2}T_{(6,3)} + \frac{1}{2}T_{(10,2)} - \langle \bullet \rangle\right)$ and the structure of the whole graph is given by $\mathcal{G}(T_{30}/\mathbb{F}_{739}) = Cyc(20, T_{(6,3)}) \oplus 2 \times Cyc(9, T_{(10,2)}) \oplus Cyc\left(1, \frac{1}{2}T_{(6,3)} + \frac{1}{2}T_{(10,2)} - \langle \bullet \rangle\right)$.

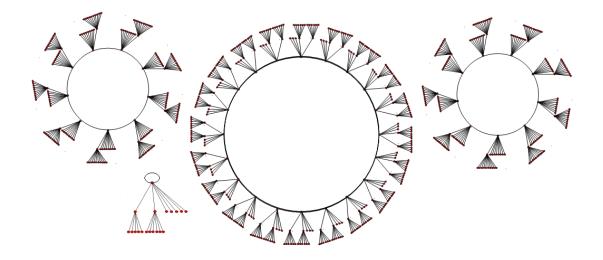


Fig. 4. Structure of the functional graph $\mathcal{G}(T_{30}/\mathbb{F}_{739})$.

п	\mathcal{G}^{R}	\mathcal{G}^Q	\mathcal{G}^{S}
2	$\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$	$2 \times \operatorname{Cyc}(4, \bullet)$	Cyc(1, ●)
3	$\operatorname{Cyc}(2, \langle 2 \times \bullet \rangle)$	Cyc(8, ●)	$\operatorname{Cyc}(1, \langle \bullet \rangle)$
4	$3 \times \operatorname{Cyc}(1, \bullet) \oplus 2 \times \operatorname{Cyc}(2, \bullet)$	$4 \times \operatorname{Cyc}(2, \bullet)$	Cyc(1, ●)
5	$\operatorname{Cyc}(1, \langle 4 \times \bullet \rangle)$	Cyc(8, ●)	$\operatorname{Cyc}(1, \langle 2 \times \bullet \rangle)$
6	$2 \times \operatorname{Cyc}(1, \langle 2 \times \bullet \rangle)$	Cyc(8, ●)	$\operatorname{Cyc}(1, \langle \bullet \rangle)$
7	$\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$	Cyc(8, ●)	Cyc(1, ●)
8	$\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$	$2 \times \operatorname{Cyc}(4, \bullet)$	Cyc(1, ●)
9	$2 \times \operatorname{Cyc}(1, \langle 2 \times \bullet \rangle)$	$2 \times \operatorname{Cyc}(4, \bullet)$	$\operatorname{Cyc}(1, \langle \bullet \rangle)$
10	$\operatorname{Cyc}(1, \langle 4 \times \bullet \rangle)$	Cyc(8, ●)	$\operatorname{Cyc}(1, \langle 2 \times \bullet \rangle)$
15	Ø	$2 \times \operatorname{Cyc}(4, \bullet)$	$\operatorname{Cyc}(1, \langle 7 \times \bullet \rangle)$
17	$\operatorname{Cyc}(1, \bullet) \oplus \operatorname{Cyc}(2, \bullet) \oplus \operatorname{Cyc}(4, \bullet)$	Ø	$\operatorname{Cyc}(1, \langle 8 \times \bullet \rangle)$
34	$3 \times \operatorname{Cyc}(1, \bullet) \oplus 2 \times \operatorname{Cyc}(2, \bullet)$	Ø	$\operatorname{Cyc}(1, \langle 8 \times \bullet \rangle)$
255	Ø	Ø	$\operatorname{Cyc}(1, \langle 15 \times \bullet \rangle)$
TABLE II			

GRAPH STRUCTURE FOR CHEBYSHEV POLYNOMIALS T_n over the binary field \mathbb{F}_{16} . We recall that $T = \langle m \times \bullet \rangle$ Denotes a tree consisting of a root with *m* predecessors.

Example 6. We consider the action of Chebyshev polynomials over the binary field \mathbb{F}_{16} . Using Theorem 4 we obtain the structure of the rational component \mathcal{G}^R , the quadratic component \mathcal{G}^Q and the special component \mathcal{G}^S of the Chebyshev graph $\mathcal{G}(T_n/\mathbb{F}_{16})$ for $2 \le n \le 10$ and n = 15, 17, 34 and 255; see Table II.

C. Chebyshev involutions and permutations

It is well known that the Chebyshev polynomial T_n is a permutation polynomial over \mathbb{F}_q if and only if $gcd(q^2 - 1, n) = 1$. Using that $T_{\nu(n)} = \bullet$ if and only if $\nu = 1$, this condition can be obtained as a direct corollary of Theorem 4 together with the decomposition into disjoint cycles.

Corollary 3. The Chebyshev polynomial T_n is a permutation polynomial over \mathbb{F}_q if and only if $gcd(q^2 - 1, n) = 1$. In this case, if $q - 1 = v_0\omega_0$ and $q + 1 = v_1\omega_1$ are their n-decompositions, we have the following decomposition of $\mathcal{G}(T_n/\mathbb{F}_q)$ into disjoint cycles:

$$\bigoplus_{\substack{d \mid \omega_0 \\ d>2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} \times Cyc\left(\tilde{o}_d(n), \bullet\right) \oplus \bigoplus_{\substack{d \mid \omega_1 \\ d>2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} \times Cyc\left(\tilde{o}_d(n), \bullet\right) \oplus k \times Cyc(1, \bullet),$$

where k = 2 if nq is odd, and k = 1 otherwise.

A particular case of cryptographic interest is permutation polynomials that are involutions [2], [3], that is, when the composition with itself is the identity map. For Chebyshev polynomials we obtain the following characterization.

Corollary 4. Let $q - 1 = v_0 \omega_0$ and $q + 1 = v_1 \omega_1$ be the n-decomposition of q - 1 and q + 1, respectively. The Chebyshev polynomial T_n is an involution over \mathbb{F}_q if and only if $v_0 = v_1 = 1$, $n^2 \equiv \pm 1 \pmod{\omega_1}$ and $n^2 \equiv \pm 1 \pmod{\omega_2}$.

Proof. The condition $v_0 = v_1 = 1$ is equivalent to $gcd(q^2 - 1, n) = 1$ which is equivalent to T_n being a permutation by Corollary 3. If this condition is satisfied, T_n is an involution if and only if $\tilde{o}_d(n) \in \{1, 2\}$ for all d such that $d \mid \omega_0$ or $d \mid \omega_1$, if and only if $n^2 \equiv \pm 1$ for all d with $d \mid \omega_0$ or $d \mid \omega_1$, if and only if $n^2 \equiv \pm 1 \pmod{\omega_1}$ and $n^2 \equiv \pm 1 \pmod{\omega_2}$.

Example 7. Consider the Chebyshev polynomial T_{31} over \mathbb{F}_{25} . Here $n = 31, q = 25, v_0 = v_1 = 1, \omega_1 = 24, \omega_1 = 26$. Since $31^2 \equiv 1 \pmod{24}$ and $31^2 \equiv -1 \pmod{26}$, the polynomial T_{31} is an involution over \mathbb{F}_{25} .

D. Explicit formulas for N, T_0, C, T and R

Let $\mathcal{G} = \mathcal{G}(f/X)$ be a functional graph where X is a finite set. Given $x_0 \in X$ there are integers $c \ge 1$ and $t \ge 0$ such that $x_0^{c+t} = x_0^t$. The smallest integers with this property are denoted by $per(x_0) := c$ (the *period* of x_0) and $pper(x_0) := t$ (the *preperiod* of x_0). The *rho length* of x_0 is $rho(x_0) := per(x_0) + pper(x_0)$. We also consider the parameters N, T_0, C, T and R where

• $N(\mathcal{G})$ is the number of connected component of \mathcal{G} ;

- $T_0(\mathcal{G})$ is the number of periodic points;
- $C(\mathcal{G}) = \frac{1}{|X|} \sum_{x \in X} \operatorname{per}(x)$ is the expected value of the period;
- $T(\mathcal{G}) = \frac{1}{|X|} \sum_{x \in X} \operatorname{pper}(x)$ is the expected value of the preperiod and
- $R(\mathcal{G}) = \frac{1}{|X|} \sum_{x \in X} \operatorname{rho}(x)$ is the expected value of the rho length.

We apply our structural theorem to deduce explicit formulas for the parameters N, T_0 , C and T for Chebyshev polynomials over \mathbb{F}_q (the average rho length can be obtained from R = C + T). These parameters were studied in [4] for the exponentiation map and in [15] for Rédei functions.

We remark that the above parameters are invariant under isomorphism (i.e. isomorphic functional graphs have the same value). Related to *C* and *T* we consider the parameters \widehat{C} and \widehat{T} defined as the sum of the values of the periods and preperiods, respectively, from which we can easily obtain *C* and *T*. The advantage of working with these parameters instead of *C* and *T* is that they are additive (i.e. $\widehat{C}(\mathcal{G}_1 \oplus \mathcal{G}_2) = \widehat{C}(\mathcal{G}_1) + \widehat{C}(\mathcal{G}_2)$ and $\widehat{T}(\mathcal{G}_1 \oplus \mathcal{G}_2) = \widehat{T}(\mathcal{G}_1) + \widehat{T}(\mathcal{G}_2)$) as well as the parameters *N* and *T*₀. For additive parameters it suffices to know their values on each connected component. In the case of Chebyshev polynomials over finite fields, each connected component of its functional graph is uniform. It is immediate to check that if $\mathcal{G} = \text{Cyc}(m, T)$ where *T* is a rooted tree with depth *D*, then $N(\mathcal{G}) = 1$; $T_0(\mathcal{G}) = m$; $\widehat{C}(\mathcal{G}) = m^2 |T|$ and $\widehat{T}(\mathcal{G}) = m \sum_{j=1}^D jh(j)$ where h(j) denotes the number of nodes in *T* at depth¹ *j*. When the rooted tree *T* is the tree attached to a *v*-series $T = T_{\nu(n)}$ we have the following formulas, whose proof is the same as the given one in [15] for Rédei functions.

Lemma 4 ([15], Proposition 2.2.). Let n, v, m be positive integers with rad(v) | rad(n). Consider $v(n) = (v_1, v_2, ..., v_D)$ and $\mathcal{G} = Cyc(m, T_{v(n)})$. Then $N(\mathcal{G}) = 1, T_0(\mathcal{G}) = m, \widehat{C}(\mathcal{G}) = m^2 v$ and $\widehat{T}(\mathcal{G}) = m \sum_{j=1}^{D-1} v_1 \cdots v_j$.

The next lemma shows how the parameter \hat{T} behaves regarding to addition and bisection of trees.

Lemma 5. The following statements hold.

1) If $G_1 = Cyc(1, T_1)$, $G_2 = Cyc(1, T_2)$ and $G = Cyc(1, T_1 + T_2)$, then $\widehat{T}(G) = \widehat{T}(G_1) + \widehat{T}(G_2)$.

2) If $\mathcal{G} = Cyc(1,T)$ where T is an even or quasi-even rooted tree and $\mathcal{G}' = Cyc(1,\frac{1}{2}T)$, then $\widehat{T}(\mathcal{G}') = \begin{cases} \frac{\widehat{T}(\mathcal{G})}{2} & \text{if } T \text{ is even;} \\ \frac{\widehat{T}(\mathcal{G})+1}{2} & \text{if } T \text{ is quasi-even.} \end{cases}$

Proof. 1. Denote by $h_1(j)$, $h_2(j)$ and h(j) the number of nodes at depth j in T_1 , T_2 and T_1+T_2 , respectively. Clearly we have h(0) = 1 and $h(j) = h_1(j) + h_2(j)$ for $j \ge 1$, from which we obtain $\widehat{T}(\mathcal{G}) = \sum jh(j) = \sum jh_1(j) + \sum jh_2(j) = \widehat{T}(\mathcal{G}_1) + \widehat{T}(\mathcal{G}_2)$.

¹The depth of a node x in a rooted tree T with root r is the length of the smallest path connecting x to r. If T is a rooted tree attached to a cyclic node in a functional graph, the depth of a node is the same as its preperiod.

2. First we consider the case when T is even. We can write $T = \langle 2 \times S \rangle$ for some forest S. We denote by $h_S(j)$ the number of nodes at depth j in $\frac{1}{2}T = \langle S \rangle$. We have that $\widehat{T}(\mathcal{G}) = \sum j \cdot 2h_S(j) = 2\sum jh_S(j) = 2\widehat{T}(\mathcal{G}')$. Now we consider the case when T is quasi-even. We can write $T = \langle 2 \times S \oplus \langle 2 \times R \rangle \rangle$. We denote by $h_S(j)$ and $h_R(j)$ the number of nodes at depth j in $\langle S \rangle$ and $\langle R \rangle$, respectively. We have that $\widehat{T}(\mathcal{G}) = (\sum j \cdot 2h_R(j)) + 1 + \sum (j+1) \cdot 2h_S(j)$ and $\widehat{T}(\mathcal{G}') = (\sum jh_R(j)) + 1 + \sum (j+1)h_S(j)$. Thus $2\widehat{T}(\mathcal{G}') = \widehat{T}(\mathcal{G}) + 1$.

Next we calculate formulas for \widehat{C} and \widehat{T} for the special component \mathcal{G}^S of the Chebyshev functional graph $\mathcal{G}(T_n/\mathbb{F}_q)$.

Lemma 6. Let *n* be a positive integer, $q - 1 = v_0 \omega_0$ and $q + 1 = v_1 \omega_1$ be the *n*-decompositions of q - 1 and q + 1, respectively. Let $v_0(n) = (a_1, \ldots, a_D)$, $v_1(n) = (b_0, \ldots, b_{D'})$, $A = \sum_{i=1}^{D-1} a_1 \cdots a_i$ and $B = \sum_{i=1}^{D'-1} b_1 \cdots b_i$. Denote by \mathcal{G}^S the special component of the Chebyshev graph $\mathcal{G}(T_n/\mathbb{F}_q)$. The following formulas for \widehat{C} and \widehat{T} hold.

$$\widehat{C}(\mathcal{G}^S) = \begin{cases} v_0 + v_1, & \text{if } nq \text{ is } odd; \\ \frac{v_0 + v_1}{2}, & \text{otherwise.} \end{cases} \quad and \quad \widehat{T}(\mathcal{G}^S) = \begin{cases} A + B, & \text{if } nq \text{ is } odd; \\ \frac{A + B}{2}, & \text{otherwise.} \end{cases}$$

Proof. First we consider the case when qn is odd. In this case both v_0 and v_1 are odd and, by Proposition 13, both rooted trees $T_{v_0(n)}$ and $T_{v_1(n)}$ are even. From Proposition 14, Theorem 4 and the fact that $|T_{v(n)}| = v$ (see Equation (1) and the following paragraph), we have $\widehat{C}(\mathcal{G}^S) = 2|\frac{1}{2}T_{v_0(n)} + \frac{1}{2}T_{v_1(n)}| = 2\left(\frac{v_0+1}{2} + \frac{v_1+1}{2} - 1\right) = v_0 + v_1$. Applying Lemmas 4 and 5 we obtain $\widehat{T}(\mathcal{G}^S) = 2 \cdot (\frac{A}{2} + \frac{B}{2}) = A + B$.

Now we consider the case when q is even. In this case again both v_0 and v_1 are odd and consequently both rooted trees $T_{v_0(n)}$ and $T_{v_1(n)}$ are even. By Proposition 14 and Theorem 4, we have $\widehat{C}(\mathcal{G}^S) = |\frac{1}{2}T_{v_0(n)} + \frac{1}{2}T_{v_0(n)}| = \frac{v_0+1}{2} + \frac{v_1+1}{2} - 1 = \frac{v_0+v_1}{2}$. Applying Lemmas 4 and 5 we obtain $\widehat{T}(\mathcal{G}^S) = \frac{A}{2} + \frac{B}{2} = \frac{A+B}{2}$.

The remainder case is when *n* is even and *q* is odd. In this case both v_0 and v_1 are even. By Proposition 13 both $T_{v_0(n)}$ and $T_{v_1(n)}$ are quasi-even. By Proposition 14 and Theorem 4 we have $\widehat{C}(\mathcal{G}^S) = |\frac{1}{2}T_{v_0(n)} + \frac{1}{2}T_{v_0(n)} - \langle \bullet \rangle| = \frac{v_0+2}{2} + \frac{v_1+2}{2} - 1 - 1 = \frac{v_0+v_1}{2}$. Applying Lemmas 4 and 5 we obtain $\widehat{T}(\mathcal{G}^S) = \frac{A+1}{2} + \frac{B+1}{2} - 1 = \frac{A+B}{2}$.

Theorem 5. Let *n* be a positive integer. Let $q - 1 = v_0\omega_0$ and $q + 1 = v_1\omega_1$ be the *n*-decompositions of q - 1 and q + 1, respectively. Let $v_0(n) = (a_1, \ldots, a_D)$ and $v_1(n) = (b_0, \ldots, b_{D'})$. Then, the following holds for $\mathcal{G} = \mathcal{G}(T_n/\mathbb{F}_q)$:

- the number of cycles in $\mathcal{G}(T_n/\mathbb{F}_q)$ is $N(\mathcal{G}) = \frac{1}{2} (\sum_{d|\omega_0} \frac{\varphi(d)}{\tilde{o}_d(n)} + \sum_{d|\omega_1} \frac{\varphi(d)}{\tilde{o}_d(n)});$
- the number of periodic points is given by $T_0(\mathcal{G}) = \frac{\omega_0 + \omega_1}{2}$;

- the expected value of per(a) where a runs over the elements of \mathbb{F}_q is $C(\mathcal{G}) = \frac{q-1}{2q} (\frac{1}{\omega_0} \sum_{d|\omega_0} \varphi(d) \tilde{o}_d(n)) + \frac{q+1}{2q} (\frac{1}{\omega_1} \sum_{d|\omega_1} \varphi(d) \tilde{o}_d(n));$
- the expected value of pper(a) where a runs over the elements of \mathbb{F}_q is $T(\mathcal{G}) = \frac{q-1}{2q} (\frac{1}{\nu_0} \sum_{i=1}^{D-1} a_1 \dots a_i) + \frac{q+1}{2q} (\frac{1}{\nu_1} \sum_{i=1}^{D'-1} b_1 \dots b_i).$

Proof. Applying Theorem 4 we have

$$N(\mathcal{G}) = N(\mathcal{G}^{R}) + N(\mathcal{G}^{Q}) + N(\mathcal{G}^{S})$$

$$= \sum_{\substack{d \mid \omega_{0} \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_{d}(n)} + \sum_{\substack{d \mid \omega_{1} \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_{d}(n)} + \begin{cases} 2, & \text{if } nq \text{ is odd;} \\ 1, & \text{if } nq \text{ is even.} \end{cases}$$
(10)

Since both ω_0 and ω_1 are even when nq is odd and both ω_0 and ω_1 are odd when nq is even, we have

$$\sum_{\substack{d \mid \omega_0 \\ d \le 2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} + \sum_{\substack{d \mid \omega_1 \\ d \le 2}} \frac{\varphi(d)}{2\tilde{o}_d(n)} = \begin{cases} 1+1=2, & \text{if } nq \text{ is odd;} \\ \frac{1}{2} + \frac{1}{2} = 1, & \text{if } nq \text{ is even.} \end{cases}$$
(11)

By Equations (10) and (11) we have $N(\mathcal{G}) = \sum_{d|\omega_0} \frac{\varphi(d)}{2\tilde{o}_d(n)} + \sum_{d|\omega_1} \frac{\varphi(d)}{2\tilde{o}_d(n)}$.

Applying Theorem 4 we have

$$T_{0}(\mathcal{G}) = T_{0}(\mathcal{G}^{R}) + T_{0}(\mathcal{G}^{Q}) + T_{0}(\mathcal{G}^{S})$$

$$= \sum_{\substack{d \mid \omega_{0} \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n) + \sum_{\substack{d \mid \omega_{1} \\ d > 2}} \frac{\varphi(d)}{2\tilde{o}_{d}(n)} \cdot \tilde{o}_{d}(n) + T_{0}(\mathcal{G}^{S})$$

$$= \sum_{\substack{d \mid \omega_{0} \\ d > 2}} \frac{\varphi(d)}{2} + \sum_{\substack{d \mid \omega_{1} \\ d > 2}} \frac{\varphi(d)}{2} + \begin{cases} 2, & \text{if } nq \text{ is odd;} \\ 1, & \text{if } nq \text{ is even.} \end{cases}$$

$$(12)$$

Since ω_0 and ω_1 are even when nq is odd and ω_0 and ω_1 are odd otherwise, we have

$$\sum_{\substack{d \mid \omega_0 \\ d \le 2}} \frac{\varphi(d)}{2} + \sum_{\substack{d \mid \omega_1 \\ d \le 2}} \frac{\varphi(d)}{2} = \begin{cases} 1+1=2, & \text{if } nq \text{ is odd;} \\ \frac{1}{2} + \frac{1}{2} = 1, & \text{if } nq \text{ is even.} \end{cases}$$
(13)

By Equations (12) and (13) we have

$$T_0(\mathcal{G}) = \sum_{d \mid \omega_0} \frac{\varphi(d)}{2} + \sum_{d \mid \omega_1} \frac{\varphi(d)}{2} = \frac{\omega_0}{2} + \frac{\omega_1}{2} = \frac{\omega_0 + \omega_1}{2}.$$

Using that $Cyc(m, T_{\nu(n)})$ has exactly $m\nu$ nodes (see Equation (1) and the following paragraph) and applying Theorem 4 and Lemma 6 we have

$$\widehat{C}(\mathcal{G}) = \widehat{C}(\mathcal{G}^R) + \widehat{C}(\mathcal{G}^Q) + \widehat{C}(\mathcal{G}^S)$$

$$= \sum_{\substack{d \mid \omega_0 \\ d > 2}} \frac{\varphi(d)}{2\widetilde{o}_d(n)} \cdot \widetilde{o}_d(n) \cdot \widetilde{o}_d(n) v_0 + \sum_{\substack{d \mid \omega_1 \\ d > 2}} \frac{\varphi(d)}{2\widetilde{o}_d(n)} \cdot \widetilde{o}_d(n) \cdot \widetilde{o}_d(n) v_1 + \widehat{C}(\mathcal{G}^S)$$

$$= \frac{v_0}{2} \sum_{\substack{d \mid \omega_0 \\ d > 2}} \varphi(d) \widetilde{o}_d(n) + \frac{v_1}{2} \sum_{\substack{d \mid \omega_1 \\ d > 2}} \varphi(d) \widetilde{o}_d(n) + \begin{cases} v_0 + v_1, & \text{if } nq \text{ is odd;} \\ \frac{v_0 + v_1}{2}, & \text{otherwise.} \end{cases}$$
(14)

Since ω_0 and ω_1 are even when nq is odd and ω_0 and ω_1 are odd otherwise, we have

$$\frac{\nu_0}{2} \sum_{\substack{d \mid \omega_0 \\ d \le 2}} \varphi(d) \tilde{o}_d(n) + \frac{\nu_1}{2} \sum_{\substack{d \mid \omega_1 \\ d \le 2}} \varphi(d) \tilde{o}_d(n) = \begin{cases} \nu_0 + \nu_1, & \text{if } nq \text{ is odd;} \\ \frac{\nu_0 + \nu_1}{2}, & \text{otherwise.} \end{cases}$$
(15)

By Equations (14) and (15) we have

$$\widehat{C}(\mathcal{G}) = \frac{\nu_0}{2} \sum_{d|\omega_0} \varphi(d) \widetilde{o}_d(n) + \frac{\nu_1}{2} \sum_{d|\omega_1} \varphi(d) \widetilde{o}_d(n)$$
$$= \frac{q-1}{2} \left(\frac{1}{\omega_0} \sum_{d|\omega_0} \varphi(d) \widetilde{o}_d(n) \right) + \frac{q+1}{2} \left(\frac{1}{\omega_1} \sum_{d|\omega_1} \varphi(d) \widetilde{o}_d(n) \right).$$

Dividing both sides by q we obtain the formula for $C(\mathcal{G})$.

Now we deduce the formula for *T*. Denote $A = \sum_{i=1}^{D-1} a_1 \cdots a_i$ and $B = \sum_{i=1}^{D'-1} B_1 \cdots B_i$. Using Lemmas 4 and 6 and Theorem 4 we obtain

$$\begin{aligned} \widehat{T}(\mathcal{G}) &= \widehat{T}(\mathcal{G}^R) + \widehat{T}(\mathcal{G}^Q) + \widehat{T}(\mathcal{G}^S) \\ &= \sum_{\substack{d \mid \omega_0 \\ d > 2}} \frac{\varphi(d)}{2\widetilde{o}_d(n)} \cdot \widetilde{o}_d(n)A + \sum_{\substack{d \mid \omega_1 \\ d > 2}} \frac{\varphi(d)}{2\widetilde{o}_d(n)} \cdot \widetilde{o}_d(n)B + \widehat{T}(\mathcal{G}^S) \\ &= \frac{A}{2} \sum_{\substack{d \mid \omega_0 \\ d > 2}} \varphi(d) + \frac{B}{2} \sum_{\substack{d \mid \omega_1 \\ d > 2}} \varphi(d) + \begin{cases} A + B, & \text{if } nq \text{ is odd;} \\ \frac{A + B}{2}, & \text{otherwise.} \end{cases} \end{aligned}$$
(16)

Since ω_0 and ω_1 are even when nq is odd and ω_0 and ω_1 are odd otherwise, we have

$$\frac{A}{2} \sum_{\substack{d \mid \omega_0 \\ d \le 2}} \varphi(d) + \frac{B}{2} \sum_{\substack{d \mid \omega_1 \\ d \le 2}} \varphi(d) = \begin{cases} A+B, & \text{if } nq \text{ is odd;} \\ \frac{A+B}{2}, & \text{otherwise.} \end{cases}$$
(17)

By Equations (16) and (17) we have

$$\widehat{T}(\mathcal{G}) = \frac{A}{2} \sum_{d \mid \omega_0} \varphi(d) + \frac{B}{2} \sum_{d \mid \omega_1} \varphi(d) = \frac{A\omega_0}{2} + \frac{B\omega_1}{2} = \frac{q-1}{2} \cdot \frac{A}{v_0} + \frac{q+1}{2} \cdot \frac{B}{v_1}.$$

Dividing both sides by q we obtain the formula for $T(\mathcal{G})$.

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