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THE WEIGHT SPECTRUM OF CERTAIN AFFINE GRASSMANN CODE

FERNANDO L. PIÑERO AND PRASANT SINGH

ABSTRACT. We consider the linear code $C^{\mathcal{A}}(2,m)$ associated to a special affine part of the Grassmannian $G_{2,m}$. This affine part is the complement of the Schubert divisor of $G_{2,m}$. In view of this, we show that there is a projection of Grassmann code onto the affine Grassmann code and the projection is a linear isomorphism. Using this isomorphism, we give a skew-symmetric matrix in some standard block form corresponding to every codeword of $C^{\mathcal{A}}(2,m)$. The weight of a codeword is given in terms of the rank of some blocks of this form and it is shown that the weight of every codeword is divisible by some power of q. We also count the number of skew-symmetric matrices in the block form to compute the weight spectrum of the affine Grassmann code $C^{\mathcal{A}}(2,m)$.

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, ℓ and m are positive integers satisfying $\ell \leq m$. Let $G_{\ell,m}$ be the Grassmannian of all ℓ-planes in $\mathbb{F}_q^m$, i.e.

$$G_{\ell,m} = \{W \subseteq \mathbb{F}_q^m : W \text{ is a subspace and } \dim W = \ell\}.$$

The Grassmannian $G_{\ell,m}$ can be embedded into the projective space $\mathbb{P}^{\binom{m}{\ell}-1}$ via the Plücker map and via this embedding, it is a closed algebraic subset of the projective space $\mathbb{P}^{\binom{m}{\ell}-1}$. Every subset of a projective space naturally corresponds to a linear code [18], therefore it is natural to study the code associated with the Grassmannian $G_{\ell,m}$. The linear code associated to the Grassmannian $G_{\ell,m}$ is known as the Grassmann code and is denoted by $C(\ell,m)$. The study of the Grassmann code goes back to C.T. Ryan [[16],[17]] over $\mathbb{F}_2$, and to Nogin [11] over any finite field. Nogin [11] prove that the Grassmann code $C(\ell,m)$ is an $[n,k,d]_q$ linear code with parameters n, k and d are given by

$$n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q, \quad k = \binom{m}{\ell} \quad \text{and} \quad d = q^{\ell(m-\ell)} \tag{1}$$

where $\begin{bmatrix} m \\ \ell \end{bmatrix}_q$ is the Gaussian binomial coefficient. In the case when $\ell = 2$, codewords of the Grassmann code $C(2,m)$ are in one-to-one correspondence with the space of skew-symmetric matrices of dimension $m \times m$. Nogin [11] used the classification of skew-symmetric matrices to give the weight spectrum of the Grassmann

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code $C(2, m)$. In later work, Nogin [12] and Kaipa-Pillai [9] computed the weight spectrum of the Grassmann codes $C(3, 6)$ and $C(3, 7)$ respectively. In general, the weight spectrum of Grassmann codes is not known.

Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$ be an ℓ-tuple of positive integers satisfying $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell \leq m$ and let A_i be the subspace of $\mathbb{F}_q^m$ spanned by $\{e_1, \ldots, e_{\alpha_i}\}$, where e_j is the jth canonical basis element of $\mathbb{F}_q^m$. The Schubert variety corresponding to the sequence α is defined by

$$\Omega_\alpha(\ell, m) = \{W \in G_{\ell,m} : \dim(W \cap A_i) \geq i \text{ for } 1 \leq i \leq \ell\}.$$

Since every $W \in \Omega_\alpha(\ell, m)$ is contained in A_ℓ, we may assume $\alpha_\ell = m$. Schubert varieties are subvarieties of Graasmannian therefore, one may think them as a subset of the projective space $\mathbb{P}^{\binom{m}{\ell}-1}$. Linear codes corresponding to Schubert varieties are known as Schubert codes and are denoted by $C_\alpha(\ell, m)$. Ghorpade-Lachaud [4] initiated the study of Schubert codes and gave an upper bound for the minimum distance of these codes. They conjectured that the minimum distance of the Schubert code $C_\alpha(\ell, m)$ is $q^{\delta(\alpha)}$, where $\delta(\alpha) = \sum_{i=1}^{\ell}(\alpha_i - i)$. This conjecture is known as the Minimum Distance Conjecture or the MDC. The MDC was proved, first by H. Chen [3] and Guerra-Vincenti [8] when $\ell = 2$, then by Ghorpade-Tsfasman [7] for the Schubert divisor and finally by Xiang [19] and [6] for general Schubert codes. It is now well known that the Schubert code $C_\alpha(\ell, m)$ is an $[n_\alpha, k_\alpha, d_\alpha]$ linear code where

$$n_\alpha = \sum_{\beta \leq \alpha} q^{\delta(\beta)}, \quad k_\alpha = \det_{1 \leq i,j \leq \ell}\left(\binom{\alpha_j - j + 1}{i - j + 1}\right) \quad \text{and} \quad d_\alpha = q^{\delta(\alpha)} \tag{2}$$

where by $\beta = (\beta_1, \ldots, \beta_\ell) \leq \alpha = (\alpha_1, \ldots, \alpha_\ell)$ we mean $\beta_i \leq \alpha_i$ for each $1 \leq i \leq \ell$. The weight spectrum of Schubert code is in general not known. Though an attempt to understand the weight spectrum of Schubert code $C_\alpha(2, m)$ was made by the authors [15] and a formula for the weight spectrum has been given. It was shown that unlike Grassmann code $C(2, m)$, the weight spectrum of the Schubert code $C_\alpha(2, m)$ is given in terms of 2 parameters.

Let ℓ, ℓ' be two positive integers satisfying $\ell \leq \ell'$ and $m = \ell + \ell'$. Let $\mathbb{M}_{\ell \times \ell'}$ be the collection of all $\ell \times \ell'$ matrices over $\mathbb{F}_q$. Note that $\mathbb{M}_{\ell \times \ell'}$ can be thought of as the affine subset of $G_{\ell,m}$ given by setting the Plücker coordinate corresponding to the columns $\ell + 1, \ldots, m$ to be non zero. Let $\mathbf{X} = (X_{ij})$ be the $\ell \times \ell'$ matrix of indeterminates X_{ij} over $\mathbb{F}_q$. Let $\mathcal{F}^{\mathcal{A}}(\ell, m)$ denote the $\mathbb{F}_q$ span of all $i \times i$ minors of $\mathbf{X}$ for $0 \leq i \leq \ell$, where by 0 minor we mean 1. As the set $\mathbb{M}_{\ell \times \ell'}$ is an affine part of the Grassmannian $G_{\ell,m}$, the evaluation code obtained by evaluating functions $f \in \mathcal{F}^{\mathcal{A}}(\ell, m)$ at the points of $\mathbb{M}_{\ell \times \ell'}$ is known as the *affine Grassmann code* and is denoted by $C^{\mathcal{A}}(\ell, m)$. Affine Grassmann codes were introduced in [1] and it was shown that the affine Grassmann code $C^{\mathcal{A}}(\ell, m)$ is an $[n_{\mathcal{A}}, k_{\mathcal{A}}, d_{\mathcal{A}}]$ code where

$$n_{\mathcal{A}} = q^{\ell(m-\ell)}, \qquad k_{\mathcal{A}} = \binom{m}{\ell} \qquad \text{and} \qquad d_{\mathcal{A}} = q^{\ell(m-2\ell)} \prod_{i=0}^{\ell-1} (q^\ell - q^i). \tag{3}$$

The dual code $C^{\mathcal{A}}(\ell, m)^{\perp}$ of the affine Grassmann code $C^{\mathcal{A}}(\ell, m)$ was studied in [2] and classification of the minimum weight codewords of the affine Grassmann code is also known. But the weight spectrum of this code is known in any but the trivial cases.

In this article, we study the weight spectrum of the affine Grassmann code $C^{\mathcal{A}}(2, m)$. First, we discuss how the Grassmann code $C(2, m)$, the Schubert code $C_\alpha(2, m)$ associated to the Schubert divisor $\Omega_\alpha(2, m)$ and the affine Grassmann code $C^{\mathcal{A}}(2, m)$ are related. We use this to associate a skew-symmetric matrix in some "block form" corresponding to each codeword of $C^{\mathcal{A}}(2, m)$. This allows us to determine the weight distribution. As a corollary, we obtain that the weight of each codeword is divisible by q^{m-3} or q^{m-2}.

2. The Grassmann Code $C(2, m)$, The Schubert code $C_\alpha(2, m)$ and The Affine Grassmann Code $C^{\mathcal{A}}(2, m)$

In this section, we consider the Grassmann code $C(2, m)$ and show that every codeword in $C(2, m)$ can be written as the extension of a codeword in the affine Grassmann code via a codeword in the Schubert code $C_\alpha(2, m)$ corresponding to the Schubert divisor, i.e., the Schubert code corresponding to the sequence $\alpha = (m-2, m)$. In this way, we establish a projection of the Grassmann code $C(2, m)$ onto the affine Grassmann code $C^{\mathcal{A}}(2, m)$. We will see that the projection is a linear isomorphism of vector spaces. Throughout this article, by a skew-symmetric matrix of size m (or dimension $m \times m$) we mean, an $m \times m$ matrix $\mathbf{A}$ with diagonal entries zero and $\mathbf{A} = -\mathbf{A}^{\mathbf{T}}$.

Let m be a positive integer, $m \geq 4$ and $G_{2,m}$ be the corresponding Grassmannian. For any $W \in G_{2,m}$, let M_W be a $2 \times m$ matrix whose rows forms a basis of W. Let

$$G_{2,m}(\mathbb{A}) = \{\mathbf{M}_1, \ldots, \mathbf{M}_{\left[{m \atop 2}\right]_q}\}$$

be the set of $2 \times m$ matrices in some order, corresponding to distinct points of the Grassmannian $G_{2,m}$. Let $\mathbf{X} = (X_{ij})$ be the $2 \times m$ matrix of indeterminates X_{ij} over $\mathbb{F}_q$ and $[m] = \{1, 2, \ldots, m\}$. For $I \subset [m]$ with $I = \{i, j\}$, let $\det_I(\mathbf{X}) = \mathbf{X}_{\{i,j\}}$ be the 2×2 minor of $\mathbf{X}$ with respect to columns of $\mathbf{X}$ whose first column is labeled by i and second column is labeled by j. Let $\mathcal{F}(2, m)$ be the vector space over $\mathbb{F}_q$ spanned by all possible minors $\det_I(\mathbf{X})$. Consider the evaluation map

$$\mathrm{Ev} : \mathcal{F}(2, m) \to \mathbb{F}_q^{\left[{m \atop 2}\right]_q}, \qquad f \mapsto c_f = (f(\mathbf{M}_1), \ldots, f(\mathbf{M}_{\left[{m \atop 2}\right]_q})). \tag{4}$$

where $f(\mathbf{M}_i)$ is the evaluation of the function f on the matrix M_i. This is a linear, injective map and the image of this map is the Grassmann code $C(2, m)$. Note that a different set of choices of matrices $\mathbf{M}_i$ gives an equivalent code. As functions on matrices, $\det_I(\mathbf{X})$ is a linear and alternating function on I, every codeword of $C(2,m)$ corresponds to a unique skew-symmetric $m \times m$ matrix. More precisely, for any codeword $c \in C(2,m)$, there exists a unique $f \in \mathcal{F}(2,m)$ such that $c = c_f$. Let $f = \sum_{1 \leq i,j \leq m} a_{ij} \det_{\{i,j\}}(\mathbf{X})$ be the corresponding function. Clearly, we have $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for all $1 \leq i,j \leq m$. If $\mathbf{F} = (a_{ij})$ be the coefficient matrix then $\mathbf{F}$ is a skew-symmetric matrix. This is the associated matrix. Also, for $\mathbf{M} \in G_{2,m}(\mathbb{A})$ we have $f(\mathbf{M}) = \mathbf{x}\mathbf{F}\mathbf{y}^T$, where $\mathbf{x}$ and $\mathbf{y}$ are the first and the second rows of $\mathbf{M}$ respectively. The matrix $\mathbf{F}$ is called as the *standard form* corresponding to the codeword c. For any matrix $\mathbf{A}$, we denote by $r(\mathbf{A})$ the rank of the matrix $\mathbf{A}$. We know that the rank of a skew-symmetric matrix $\mathbf{F}$ is always even, therefore we set $2r_{\mathbf{F}} = r(\mathbf{F})$. In the following theorem, we state the result of Nogin [11] to calculate the weight spectrum of the Grassmann code $C(2,m)$.

Theorem 2.1. *Let $c \in C(2,m)$ be a codeword and $\mathbf{F}$ be the corresponding standard form. The weight of the codeword c is given by*

$$wt(c) = q^{2(m-r_{\mathbf{F}}-1)} \frac{q^{r(\mathbf{F})} - 1}{q^2 - 1}. \tag{5}$$

Furthermore, for any positive integers r, the number of codewords in $C(2,m)$ of weight $q^{2(m-r-1)} \frac{q^{2r}-1}{q^2-1}$ is given by $N(m,2r)$, where

$$N(m,2r) = q^{r(r-1)} \frac{\prod_{i=0}^{2r-1} (q^{m-i} - 1)}{\prod_{i=0}^{r-1} (q^{2(r-i)} - 1)}. \tag{6}$$

The number $N(m,2r)$ is the number of skew-symmetric matrices of size m and rank $2r$.

Fix $\alpha = (m-2,m)$ and let A_1 be the $m-2$ dimensional subspace of $\mathbb{F}_q^m$ spanned by the first $m-2$ canonical (standard) basis of $\mathbb{F}_q^m$. Let $\Omega_\alpha(2,m)$ be the corresponding Schubert variety. For any $W \in \Omega_\alpha(2,m)$ choose a matrix M_W whose rows forms a basis of W and last two entries of the first row of M_W are zero. Let $\Omega_\alpha(2,m)(\mathbb{A})$ be the collection of such matrices corresponding to each point of $\Omega_\alpha(2,m)$. Therefore, there is a choice of matrices in $G_{2,m}(\mathbb{A})$, such that we may write

$$G_{2,m}(\mathbb{A}) = \Omega_\alpha(2,m)(\mathbb{A}) \coprod \mathcal{A}(2,m). \tag{7}$$

Also, the Schubert code $C_\alpha(2,m)$ is the image of the restriction of the evaluation map defined in equation (4) to the set $\Omega_\alpha(2,m)(\mathbb{A})$. Thus, we may assume a codeword $c = c_f \in C(2,m)$ can be written as a vector $(c_S|c_A)$ where c_S is the evaluation of f on $\Omega_\alpha(2,m)(\mathbb{A})$ and c_A is the evaluation of f on $\mathcal{A}(2,m)$. But the evaluation of f on $\Omega_\alpha(2,m)\mathring{A}$ is a codeword in $C_\alpha(2,m)$.

Finally, let $\ell' = m-2$ and $C^{\mathcal{A}}(2,m)$ be the corresponding affine Grassmann code. We have the following embedding

$$\mathbb{M}_{2,m-2} \to G_{2,m}(\mathbb{A}), \qquad \mathbf{M} \mapsto (\mathbf{M}|\mathbf{I}_2)$$

where $\mathbf{I}_2$ is the identity matrix of size 2. We may use this embedding to identify $\mathbb{M}_{2,m-2}$ as a subset of $G_{2,m}(\mathbb{A})$. Moreover, equation (7) may be written as

$$G_{2,m}(\mathbb{A}) = \Omega_\alpha(2,m)(\mathbb{A}) \coprod \mathbb{M}_{2,m-2}. \tag{8}$$

Furthermore, for any codeword $c = c_f \in C(2,m)$, the evaluation of f on $\Omega_\alpha(2,m)(\mathbb{A})$ corresponds to a codeword in $C_\alpha(2,m)$ and the evaluation of f on $\mathbb{M}_{2,m-2}$ corresponds to a codeword in $C^{\mathcal{A}}(2,m)$. In particular, every codeword c_G of the Grassmann code $C(2,m)$ can be written uniquely as

$$c_G = (c_S|c_A)\,. \tag{9}$$

where c_S is a codeword in the Schubert code $C_\alpha(2,m)$ and c_A is a codeword in the affine Grassmann code $C^{\mathcal{A}}(2,m)$. This gives a projection of the Grassmann code $C(2,m)$ onto the affine Grassmann code $C^{\mathcal{A}}(2,m)$. and the map is linear as well as injective. On the other hand, both codes are of dimension $\binom{m}{2}$ (see equations (1) and (3)). This means, for every $c_A \in C^{\mathcal{A}}(2,m)$, there exist codewords $c_G \in C(2,m)$ and $c_S \in C_\alpha(2,m)$ such that $c_G = (c_S|c_A)$. This is the key idea of this article. In the next lemma, we use equation (9) to get a skew symmetric matrix corresponding to every codeword of $C^{\mathcal{A}}(2,m)$.

Lemma 2.2. *For every codeword $c_A \in C^{\mathcal{A}}(2,m)$ there exist a unique skew symmetric matrix $\mathbf{F}$ of size m such that*

$$\mathrm{wt}(c_A) = \mathrm{wt}(\mathbf{F}|C(2,m)) - \mathrm{wt}(\mathbf{F}|C_\alpha(2,m)), \tag{10}$$

where $\mathrm{wt}(\mathbf{F}|C(2,m))$ and $\mathrm{wt}(\mathbf{F}|C_\alpha(2,m))$ denotes the weight of the codewords in $C(2,m)$ and $C_\alpha(2,m)$, associated to the restriction of $\mathbf{F}$ to the Grassmannian $G_{2,m}$ and the Schubert divisor $\Omega_\alpha(2,m)$ respectively.

Proof. The proof of this lemma is trivial but this lemma is the heart of the article. As we discussed, for a given codeword $c_A \in C^{\mathcal{A}}(2,m)$, there exist $c_G \in C(2,m)$ and $c_S \in C_\alpha(2,m)$ such that $c_G = (c_S|c_A)$. The codeword c_G is unique. Consequently, $\mathrm{wt}(c_A) = \mathrm{wt}(c_G) - \mathrm{wt}(c_S)$. Let $\mathbf{F}$ be the skew-symmetric matrix corresponding to the codeword c_G and hence $\mathrm{wt}(c_G) = \mathrm{wt}(\mathbf{F}|C(2,m))$. In (9) we have seen that the codeword c_S is the restriction of the form $\mathbf{F}$ to the Schubert variety $\Omega_\alpha(2,m)$. This completes the proof. $\square$

If the skew-symmetric matrix $\mathbf{F}$ is associated to a codeword $c_A \in C^{\mathcal{A}}(2,m)$, we may write $\mathbf{F}$ as

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^{\mathbf{T}} & \mathbf{D} \end{pmatrix}, \tag{11}$$

where $\mathbf{A}$ and $\mathbf{D}$ are skew-symmetric matrices of size $m-2$ and 2 respectively and $\mathbf{B}$ is an $(m-2) \times 2$ matrix.

Definition 2.3. Let $c_A \in C^{\mathcal{A}}(2,m)$ be a codeword and $\mathbf{F}$ be the corresponding skew-symmetric matrix in the form written as in equation (11). The matrix $\mathbf{F}$ is called the *standard block form* corresponding to the codeword c_A.

For a skew-symmetric matrix $\mathbf{F}$, we know that the evaluation of $\mathbf{F}$ at some point $W \in G_{2,m}$ is given by $\mathbf{x}\mathbf{F}\mathbf{y}^T$, where $\mathbf{x}$ and $\mathbf{y}$ is a basis of W. From equations (8) and (9), it is clear that if $\mathbf{F}$ is a skew-symmetric matrix corresponding to a codeword $c_A \in C^{\mathcal{A}}(2,m)$ then

$$c_A = (\mathbf{x}\mathbf{F}\mathbf{y}^{\mathbf{T}}) \tag{12}$$

where $\mathbf{x},\ \mathbf{y} \in \mathbb{F}_q^m$ runs over vectors of $\mathbb{F}_q^m$ such that the last two columns of $\mathbf{x},\ \mathbf{y}$ forms the 2×2 identity matrix. For the rest of the article, if $\mathbf{F}$ is a skew-symmetric matrix in the standard block form as in equation (11), then the corresponding codeword of affine Grassmann code is given by equation (12). Therefore, if we write $\mathbf{x}\mathbf{F}\mathbf{y}^{\mathbf{T}}$, the evaluation of $\mathbf{F}$ as a codeword in $C^{\mathcal{A}}(2,m)$ we always mean $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{F}_q^m$ such that $\mathbf{x} = (x_1,\ldots,x_{m-2},1,0)$ and $\mathbf{y} = (y_1,\ldots,y_{m-2},0,1)$. Also, if $\mathbf{F}$ is a skew-symmetric matrix in the standard block form, we fix the following notation

$$r(\mathbf{A}|\mathbf{B}) = rank((\mathbf{A}|\mathbf{B})).$$

Since each skew symmetric matrix corresponds to a codeword in the Grassmann code $C(2,m)$ as well as in the affine Grassmann code $C^{\mathcal{A}}(2,m)$, therefore, if $\mathbf{F}$ is a skew-symmetric matrix we use the notations $c_G(\mathbf{F})$ and $c_A(\mathbf{F})$ to denote the corresponding codeword in the Grassmann code and in the affine Grassmann code respectively.

3. Weight Spectrum of Affine Grassmann Code $C^{\mathcal{A}}(2,m)$

In the last section, we saw how we can get a skew-symmetric matrix to each codeword of $C^{\mathcal{A}}(2,m)$ written in some standard block form. In this part of the article, we will see that the weight of a codeword can be given in terms of the rank $r(\mathbf{A})$ where $\mathbf{A}$ is the upper $(m-2)\times(m-2)$ block of the standard block form. We use the rank of these block matrices to give a formula for the weight spectrum of this code. But first, recall that the *weight spectrum* of an $[n,k]_q$ linear code C is a sequence $(A_0,\ldots,A_n)$ of positive integers where A_i denotes the number of codewords in C of weight i. The weight enumerator polynomial of a code is defined and denoted by

$$W_C(X) = \sum_{i=0}^{n} A_i X^i.$$

In the next table, the weight enumerator polynomial of the affine Grassmann code $C^{\mathcal{A}}(2,m)$ for some small values of m and q are listed. We used SAGEMATH to compute it.

3.1. **Weight Enumerator Polynomial of Some affine Grassmann codes Codes.**

q	m	weight enumerator
2	4	$X^{16} + 16X^{10} + 30X^8 + 16X^6 + 1$
2	5	$X^{64} + 112X^{40} + 798X^{32} + 112X^{24} + 1$
2	6	$X^{256} + 560X^{160} + 7168X^{136} + 17310X^{128} + 7168X^{120} + 560X^{96} + 1$
3	4	$2X^{81} + 324X^{57} + 240X^{54} + 162X^{48} + 1$
3	5	$2X^{729} + 4212X^{513} + 52728X^{486} + 2106X^{432} + 1$
4	4	$3X^{256} + 2304X^{196} + 1020X^{192} + 768X^{180} + 1$
4	5	$3X^{4096} + 48384X^{3136} + 984060X^{3072} + 16128X^{2880} + 1$
5	4	$4X^{625} + 10000X^{505} + 3120X^{500} + 2500X^{480} + 1$
5	5	$4X^{15625} + 310000X^{12625} + 9378120X^{12500} + 77500X^{12000} + 1$

After finding the formula for the weight spectrum of $C^{\mathcal{A}}(2,m)$ we will compare some of the values given in this table. We would like to get some canonical form corresponding to each standard block form and to do so we recall the following well known result from [10] on the classification of skew-symmetric matrices.

Proposition 3.1. *Let* $\mathbf{A}$ *be a skew-symmetric matrix of size* m *and* $r(\mathbf{A}) = 2r_{\mathbf{A}}$. *Then there exists a nonsingular matrix* $\mathbf{C}$ *such that*

$$\mathbf{CAC^T} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} \\ -\mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{CAC^T}$ *is an* $m \times m$ *matrix in the block from and the matrices in the first or second row have* $r_\mathbf{A}$ *rows, the matrices in the third row have* $m - 2r_\mathbf{A}$ *rows. Likewise, the matrices in the first or second column have* $r_\mathbf{A}$ *columns, the matrices in the third column have* $m - 2r_\mathbf{A}$ *columns. The matrix* $\mathbf{I}_{r_\mathbf{A}}$ *is the* $r_\mathbf{A} \times r_\mathbf{A}$ *identity matrix.*

Let $\mathbf{F}$ a skew symmetric matrix written in the standard block form as in (11) and $c_G(\mathbf{F})$ and $c_A(\mathbf{F})$ be the corresponding codewords in the Grassmann code $C(2,m)$ and the affine Grassmann code $C^{\mathcal{A}}(2,m)$. We know that the weight of the $c_G(\mathbf{F})$ depends on the rank of $\mathbf{F}$. Therefore, we would like to understand the weight of

$c_A(\mathbf{F})$ in terms of the rank of matrices $\mathbf{F}$, $\mathbf{A}$ and $\mathbf{B}$. To do so we would first give another standard form corresponding to a skew-symmetric matrix $\mathbf{F}$.

Lemma 3.2. *Let* $\mathbf{F}$ *be a skew-symmetric block form of size* m *given by*

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B^T} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{A}$ *and* $\mathbf{D}$ *are skew-symmetric matrices of size* $m-2$ *and* 2 *respectively and* $\mathbf{B}$ *is an* $m-2\times 2$ *matrix. Then there exists a nonsingular matrix* $\mathbf{C_m}$ *such that*

$$\mathbf{C_m F C_m^T} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f} \\ -\mathbf{0} & -\mathbf{0} & -\mathbf{f}^T & \mathbf{D} \end{pmatrix} \tag{13}$$

where $\mathbf{C_m A C_m^T}$ *is an* $m\times m$ *block matrix such that the matrices in first or second row have* $r_\mathbf{A}$ *rows, the matrices in the third row have* $m-2r_\mathbf{A}-2$ *rows and the matrices in the fourth row have* 2 *rows. Likewise, the matrices in the first or second column have* $r_\mathbf{A}$ *columns, the matrices in the third column have* $m-2r_\mathbf{A}-2$ *columns and the matrices in the fourth column have* 2 *columns. The matrix* $\mathbf{I}_{r_\mathbf{A}}$ *is the* $r_\mathbf{A}\times r_\mathbf{A}$ *identity matrix. The matrices* $\mathbf{0}$ *are zero matrices of the appropriate size. The matrix* $\mathbf{f}$ *is a generic matrix of the corresponding dimensions.*

Proof. The proof of this lemma is easy. Using Proposition 3.1 we get a nonsingular matrix $\mathbf{C}$ of size $m-2$ such that $\mathbf{CAC^T}$ has the following block form

$$\mathbf{CAC^T} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{r_A} & \mathbf{0} \\ -\mathbf{I}_{r_A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

The matrix $\mathbf{CAC^T}$ has the required form for the upper $m-2\times m-2$ principal minor of the matrix in equation (13) of the lemma. Define a new matrix $\mathbf{P_m} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix}$ where $\mathbf{I}_2$ is the 2×2 identity matrix. Then $\mathbf{P_m}$ is a nonsingular matrix of size m and we have

$$\mathbf{P_m F P_m^T} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} \mathbf{F} \begin{pmatrix} \mathbf{C^T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{CAC^T} & \mathbf{CB} \\ -\mathbf{BC^T} & \mathbf{D} \end{pmatrix}$$

and we may rewrite $\mathbf{CB}$ such that

$$\begin{pmatrix} \mathbf{CAC^T} & \mathbf{CB} \\ -\mathbf{BC^T} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{d} \\ -\mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} & \mathbf{e} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f} \\ -\mathbf{d^T} & -\mathbf{e^T} & -\mathbf{f}^T & \mathbf{D} \end{pmatrix}.$$

Define $\mathbf{Q_m}$ to be the following nonsingular matrix

$$\mathbf{Q_m} = \begin{pmatrix} \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{m-2-2r_\mathbf{A}} & \mathbf{0} \\ -\mathbf{e^T} & -\mathbf{d^T} & \mathbf{0} & \mathbf{I}_2 \end{pmatrix}.$$

If we take $\mathbf{C_m} = \mathbf{Q_m P_m}$, then the matrix $\mathbf{C_m F C_m^T}$ has the required form as in equation (13). □

For any $f \in \mathcal{F}^{\mathcal{A}}(2,m)$, let $c_f \in C^{\mathcal{A}}(2,m)$ denote the corresponding codeword. Recall that $\mathbf{X}$ is the $2 \times m$ matrix of indeterminates X_{ij} and $\mathbf{X}_{\{i,j\}}$ is the minor of $\mathbf{X}$ corresponding to the columns i and j. Also, if $\mathbf{F}$ is a skew-symmetric matrix of size m, then $f = \mathbf{X_1 F X_2}^T \in \mathcal{F}^{\mathcal{A}}(2,m)$ is the function such that $c_A(\mathbf{F}) = c_f$, where $\mathbf{X_1} = (X_{11}, \ldots, X_{1m-2}, 1, 0)$ and $\mathbf{X_2} = (X_{21}, \ldots, X_{2m-2}, 0, 1)$. In the next lemma we give a canonical function $f \in \mathcal{F}^{\mathcal{A}}(2,m)$ corresponding to each codeword $c_A \in C^{\mathcal{A}}(2,m)$ such that $\mathrm{wt}(c_f) = \mathrm{wt}(c_A)$. More precisely,

Lemma 3.3. *For every codeword $c_A \in C^{\mathcal{A}}(2,m)$ there exist some $f \in \mathcal{F}^{\mathcal{A}}(2,m)$ such that $\mathrm{wt}(c_A) = \mathrm{wt}(c_f)$ and*

$$f = \sum_{i=1}^{r} \mathbf{X}_{\{i,r+i\}} + \sum_{i,j>2r} f_{ij} \mathbf{X}_{\{i,j\}} + c,$$

where $0 \le r \le \frac{m-2}{2}$ is an integer and c is a constant.

Proof. Let $\mathbf{F}$ be the standard block form corresponding to the codeword c_A. Let $\mathbf{C_m}$ be the nonsingular matrix such that $\mathbf{C_m F C_m^T}$ takes the form as in equation (13). Let $f = \mathbf{X_1 F X_2}^T \in \mathcal{F}^{\mathcal{A}}(2,m)$ be the function corresponding to the skew-symmetric matrix $\mathbf{C_m F C_m^T}$. Clearly, f is in the required form with $2r = r(\mathbf{A})$ and the weight of these two codewords c_A and c_f are same.

□

The benefit of Lemma 3.3 is that corresponding to every codeword $c_A \in C^{\mathcal{A}}(2,m)$ we can get a function $f \in \mathcal{F}^{\mathcal{A}}(2,m)$ that can be written as the sum of r disjoint 2×2 minors, many disjoint 1×1 minors and a constant.

Lemma 3.4. *Let $0 \le 2r \le m-2$ be an even number. There are $(q^{2r}-1)(q^{2r}-q^{2r-1})$ vector pairs $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^{2r}$ such that* $\mathbf{x} \begin{pmatrix} \mathbf{0} & \mathbf{I}_r \\ -\mathbf{I}_r & \mathbf{0} \end{pmatrix} \mathbf{y^T} \neq 0$

Proof. The proof is a simple consequence of equation (5) in Theorem 2.1. □

Lemma 3.5. *Let $f = \sum_{i=1}^{r} \mathbf{X}_{\{i,r+i\}} \in \mathcal{F}^{\mathcal{A}}(2,m)$ and $c_f \in C^{\mathcal{A}}(2,m)$ be the corresponding codeword. Then* $\mathrm{wt}(c_f) = (q^{2r}-1)(q^{2r}-q^{2r-1})q^{2(m-2-2r)}$

Proof. Note that $\mathrm{wt}(c_f)$ is the number of $2 \times m$ matrices with rows $\mathbf{x},\ \mathbf{y} \in \mathbb{F}_q^m$ such that the last two entries of $\mathbf{x}$ and $\mathbf{y}$ gives the identity matrix and $\mathbf{xFy^T} \neq 0$ where

$$\mathbf{F} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_r \\ -\mathbf{I}_r & \mathbf{0} \end{pmatrix}.$$

Now the result follows from the above lemma. $\square$

Next, we determine the weight of a codeword $c_g \in C^{\mathcal{A}}(2,m)$ that is the evaluation of a function $g \in \mathcal{F}^{\mathcal{A}}(2,m)$ which is the sum of some disjoint 2×2 minors and 1×1 minors.

Lemma 3.6. *Let* $f = \sum_{i=1}^{r} \mathbf{X}_{\{i,r+i\}} \in \mathcal{F}^{\mathcal{A}}(2,m)$ *and* $h = \sum_{j>2r} h_{1,j}X_{1,j} + h_{2,j}X_{2,j}$ *be a nonzero function. Then* $f + h \in \mathcal{F}^{\mathcal{A}}(2,m)$ *and* $\mathrm{wt}(c_{f+h}) = q^{2(m-2)-1}(q-1)$.

Proof. Note that h is a nonzero linear map $\mathbb{F}_q^{2(m-2-2r)} \longrightarrow \mathbb{F}_q$. Therefore it assumes each value exactly $q^{2(m-2-2r)-1}$ times. Also, any $\mathbf{M} \in \mathbb{M}_{2,m-2}$ can be written as $(\mathbf{M_1}|\mathbf{M_2})$ such that $\mathbf{M_1}$ is a $2 \times 2r$ matrix, $\mathbf{M_2}$ is a $2 \times m-2-2r$ matrix and $f + h(\mathbf{M}) = f(\mathbf{M_1}) + h(\mathbf{M_2})$. Therefore $f + h(\mathbf{M}) \neq 0$ iff $f(\mathbf{M_1}) \neq -h(\mathbf{M_2})$. For each $\mathbf{M_1}$ there are exactly $q^{2(m-2-2r)-1}(q-1)$ matrices $\mathbf{M_2}$ such that $f(\mathbf{M_1}) \neq -h(\mathbf{M_2})$, or in other words $f + h(\mathbf{M}) \neq 0$. But there are q^{4r} matrices $\mathbf{M_1}$. Therefore, we get $q^{2(m-2)-1}(q-1)$ matrices $\mathbf{M} \in \mathbb{M}_{2,m-2}$ such that $f + h(\mathbf{M}) \neq 0$. Hence $\mathrm{wt}\, c_{f+h} = q^{2(m-2)-1}(q-1)$.

$\square$

Lemma 3.7. *Let* $f = \sum_{i=1}^{r} \mathbf{X}_{\{i,r+i\}} \in \mathcal{F}^{\mathcal{A}}(2,m)$ *and* $\lambda \neq 0$ *be an element of* $\mathbb{F}_q$. *Then* $\mathrm{wt}(c_{f+\lambda}) = q^{2(m-2-2r)}(q^{4r} - q^{4r-1} + q^{2r-1})$.

Proof. First, we shall prove that the evaluation of f assumes each nonzero value for exactly $(q^{2r}-1)q^{2r-1}$ many pairs $\mathbf{x},\ \mathbf{y} \in \mathbb{F}_q^{2r}$. Let $c \in \mathbb{F}_q$ be non zero and $\mathbf{F}$ be the skew-symmetric matrix associated to f. Then the evaluation of f on $\mathbf{M} \in \mathbb{M}_{2,m-2}$ is given by $\mathbf{xFy^T} \neq 0$ where

$$\mathbf{F} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_r \\ -\mathbf{I}_r & \mathbf{0} \end{pmatrix}$$

with $\mathbf{x}$ and $\mathbf{y}$ are rows of $\mathbf{M}$. For any nonzero $\mathbf{x} \in \mathbb{F}_q^{2r}$, the partial evaluation $\mathbf{x}$ is a nonzero linear map $\mathbf{xF} : \mathbb{F}_q^{2r} \longrightarrow \mathbb{F}_q$. Clearly, for any such $\mathbf{x}$, there are q^{2r-1} $\mathbf{y} \in \mathbb{F}_q^{2r}$ such that $\mathbf{xFy^T} = c$.

Now, the number of pairs $\mathbf{x},\ \mathbf{y} \in \mathbb{F}_q^{m-2}$ such that the evaluation of $f+\lambda$ onto matrices $\mathbf{M}$ having rows $\mathbf{x},\ \mathbf{y}$ is zero, can be given by $(q^{2r}-1)q^{2r-1}q^{2(m-2r-2)}$. Consequently, the weight of the codeword $c_{f+\lambda}$ is given by $q^{2(m-2)} - (q^{2r}-1)q^{2r-1}q^{2(m-2r-2)}$ which is the desired number.

$\square$

This is a good time to determine the weight distribution of the affine Grassmann code $C^{\mathcal{A}}(2,m)$. To do so, recall that corresponding to every codeword $c_A \in C^{\mathcal{A}}(2,m)$ we can get a skew-symmetric form $\mathbf{F}$, where

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B^T} & \mathbf{D} \end{pmatrix}$$

and the codeword c_A is given by equation (12). In the next theorem, we determine the weight of a given codeword $c_A \in C^{\mathcal{A}}(2,m)$ depending on the rank of the matrices $\mathbf{A}$, $(\mathbf{A}|\mathbf{B})$ and $\mathbf{D}$.

Theorem 3.8. *Let $c_A \in C^{\mathcal{A}}(2,m)$ be a codeword and $\mathbf{F}$ written as above, be the skew-symmetric matrix in the block form corresponding to this codeword. We have the following.*

(a) *If $r(\mathbf{A}) = r((\mathbf{A}|\mathbf{B})) = r(\mathbf{F})$, then $\mathrm{wt}(c_A) = (q^{2r_\mathbf{A}} - 1)(q-1)q^{2(m-2-r_\mathbf{A})-1}$.*

(b) *If $r(\mathbf{A}) = r((\mathbf{A}|\mathbf{B}))$ and $r(\mathbf{F}) = r(\mathbf{A}) + 2$, then*

$$\mathrm{wt}(c_A) = (q^{2r_\mathbf{A}+1} - q^{2r_\mathbf{A}} + 1)q^{2(m-2-r_\mathbf{A})-1}.$$

(c) *If $r(\mathbf{A}) \neq r((\mathbf{A}|\mathbf{B}))$ then $\mathrm{wt}(c_A) = q^{2(m-2)-1}(q-1)$*

Proof. By Lemma 3.3 we know that, corresponding to the codeword c_A we can associate a codeword $c_f \in C^{\mathcal{A}}(2,m)$ such that $\mathrm{wt}(c_A) = \mathrm{wt}(c_f)$ and the skew-symmetric matrix corresponding to c_f is given by

$$\mathbf{C_m F C_m^T} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f} \\ -\mathbf{0} & -\mathbf{0} & -\mathbf{f}^T & \mathbf{D} \end{pmatrix}.$$

(a) In this case, we must have $\mathbf{f} = \mathbf{0}$ and $\mathbf{D} = \mathbf{0}$. Therefore the corresponding function g in this case look like $f = \sum_{i=1}^{r_\mathbf{A}} \mathbf{X}_{\{i, r_\mathbf{A}+i\}} \in \mathcal{F}^{\mathcal{A}}(2,m)$. The result now follows by Lemma 3.5.

(b) In this case, we must have $\mathbf{f} = \mathbf{0}$ and $\mathbf{D} \neq \mathbf{0}$. Since $\mathbf{D}$ is a nonzero skew-symmetric matrix, it must be given by $\mathbf{D} = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ where $\lambda \neq 0$. The corresponding function f, is given by $f = \sum_{i=1}^{r_\mathbf{A}} \mathbf{X}_{\{i, r_\mathbf{A}+i\}} + \lambda^2 \in \mathcal{F}^{\mathcal{A}}(2,m)$. The result now follows by Lemma 3.7.

(c) Finally, in this case, we get $\mathbf{f} \neq \mathbf{0}$. After applying the rows and the columns operations that affects only the last two block rows and columns of the matrix $\mathbf{C_m F C_m^T}$ we can get another skew-symmetric matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_{r_\mathbf{A}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f} \\ -\mathbf{0} & -\mathbf{0} & -\mathbf{f}^T & \mathbf{0} \end{pmatrix}$$

such that the corresponding codeword c_g has the same weight as the codeword c_f and hence the same weight as the codeword c_A. In this case the function $g \in \mathcal{F}^{\mathcal{A}}(2,m)$ is $g = \sum_{i=1}^{r_\mathbf{A}} \mathbf{X}_{\{i, r_\mathbf{A}+i\}} + \sum_{j>2r_\mathbf{A}} h_{1,j}X_{1,j} + h_{2,j}X_{2,j}$. Now the weight of c_g can be given by Lemma 3.6. This completes the proof of the theorem.

□

The following corollary says that the weight of every codeword in $C^{\mathcal{A}}(2,m)$ is divisible by some power of q.

Corollary 3.9. *Let $c_A \in C^{\mathcal{A}}(2,m)$ be a codeword.*
(a) If m is even, then $q^{m-3} | \operatorname{wt}(c_A)$.
(b) If m is odd, then $q^{m-2} | \operatorname{wt}(c_A)$.

Proof. Note that the $\operatorname{wt}(c_A)$ is either $(q^{2r}-1)(q-1)q^{2(m-2-r)-1}$ or $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$ or $q^{2(m-2)-1}(q-1)$ for some even number $0 \le 2r \le m-2$. In either case, the weight of c_A is divisible by $q^{2(m-2-r)-1}$ and the power is minimal when r is maximal. Now if m is even then $2r = m-2$ gives the minimal power and if m is odd then $2r = m-3$ gives the minimal power.

□

Remark 3.10. For every even number $0 \le 2r \le m-2$, there are codewords of weights $(q^{2r}-1)(q-1)q^{2(m-2-r)-1}$, $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$ or $q^{2(m-2)-1}(q-1)$. It is not hard to construct a codeword of these weights. All we have to do is to choose a skew-symmetric matrix $\mathbf{A}$ of size $m-2$ and rank $2r$. Now choose any $m-2\times 2$ matrix $\mathbf{B}$ such that the columns of $\mathbf{B}$ are contained in the column space of $\mathbf{A}$. Choose any skew symmetric matrix $\mathbf{D}$ of size 2. Then the codeword $c_A(\mathbf{F}) \in C^{\mathcal{A}}(2,m)$ associated with the form

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B^T} & \mathbf{D} \end{pmatrix}$$

has weight either $(q^{2r}-1)(q-1)q^{2(m-2-r)-1}$ or $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$ depending on whether $\mathbf{D}$ is the zero or a non zero matrix. On the other hand if we choose $\mathbf{B}$ such that the columns of $\mathbf{B}$ are not in the column space of $\mathbf{A}$, then for any choice of $\mathbf{D}$, the codeword associated to the form $\mathbf{F}$ has weight $q^{2(m-2)-1}(q-1)$ where

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B^T} & \mathbf{D} \end{pmatrix}$$

Finally, we are ready to give the weight spectrum of the affine Grassmann code $C^{\mathcal{A}}(2,m)$. Recall that, for any non-negative, even integer $2r \leq m$, $N(m,2r)$ denotes the number of skew-symmetric matrices of size m and rank $2r$, where

$$N(m,2r) = q^{r(r-1)} \frac{\prod_{i=0}^{2r-1} (q^{m-i}-1)}{\prod_{i=0}^{r-1} (q^{2(r-i)}-1)}.$$

For the next theorem, we fix $0 \leq 2r \leq m-2$.

Theorem 3.11. *There are A_i codewords in $C^{\mathcal{A}}(2,m)$ of weight i, where for every $0 \leq 2r \leq m-2$,*

(a) $A_i = N(m-2,2r)q^{4r}$ *when* $i = (q^{2r}-1)(q-1)q^{2(m-2-r)-1}$
(b) $A_i = N(m-2,2r)q^{4r}(q-1)$ *when* $i = (q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$.
(c) $A_i = \sum_{0\leq 2r\leq m-2} N(m-2,2r)q(q^{2(m-2)}-q^{4r})$ *when* $i = q^{2(m-2)-1}(q-1)$.
(d) $A_i = 0$ *in all other cases.*

Proof. Every codeword $c_A \in C^{\mathcal{A}}(2,m)$ correspond to a skew-symmetric matrix $\mathbf{F}$, where

$$\mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B}^{\mathbf{T}} & \mathbf{D} \end{pmatrix}.$$

The weight of the codeword depends on the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$. We have seen that if $r(\mathbf{A}) = r(\mathbf{A}|\mathbf{B})$, i.e., the columns of $\mathbf{B}$ is contained in the column space of $\mathbf{A}$ then the weight of the corresponding codeword is $(q^{2r}-1)(q-1)q^{2(m-2-r)-1}$ or $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$ depending on whether $\mathbf{D}$ is zero or nonzero. In all other cases i.e. when $r(\mathbf{A}) \neq r(\mathbf{A}|\mathbf{B})$, the weight of the corresponding codeword is $q^{2(m-2)-1}(q-1)$. Therefore, to complete the proof we need only to count how many matrices $\mathbf{F}$ satisfy these conditions. The weight of the codeword c_A is $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$ iff the standard representation block form of corresponding skew symmetric matrix $\mathbf{F}$ has $\mathbf{D} = \mathbf{0}$ and $2r = r(\mathbf{A}) = r(\mathbf{A}|\mathbf{B})$. The number of choices for $\mathbf{A}$, in this case, is $N(m-2,2r)$ and for every such choice of $\mathbf{A}$, we have exactly q^{4r} choices of $\mathbf{B}$ satisfying $r(\mathbf{A}) = r(\mathbf{A}|\mathbf{B})$ as $r(\mathbf{A}) = 2r$. Therefore there are $N(m-2,2r)q^{4r}$ many codewords in $C^{\mathcal{A}}(2,m)$ of weight $(q^{2r}-1)(q-1)q^{2(m-2-r)-1}$.

However, the weight of the codeword c_A is $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$ if and only if the standard representation of the corresponding skew symmetric matrix $\mathbf{F}$ satisfies $\mathbf{D} \neq \mathbf{0}$ and $2r = r(\mathbf{A}) = r(\mathbf{A}|\mathbf{B})$. Since $\mathbf{D}$ is a nonzero skew-symmetric matrix of size 2, there are exacly $(q-1)$ choices of $\mathbf{D}$. Hence we get $N(m-2,2r)q^{4r}(q-1)$ codewords in $C^{\mathcal{A}}(2,m)$ of weight $(q^{2r+1}-q^{2r}+1)q^{2(m-2-r)-1}$.

Finally, the weight of a codeword is $q^{2(m-2)-1}(q-1)$ iff $r(\mathbf{A}|\mathbf{B}) \neq r(\mathbf{A})$. In this case, for every $0 \leq 2r \leq m-2$ we get $N(m-2,2r)$ choices of $\mathbf{A}$. For any such choice of a skew-symmetric matrix $\mathbf{A}$ of rank $2r$ and size $m-2$, there are exactly $(q^{2(m-2)}-q^{4r})$ choices of $\mathbf{B}$ satisfying $r(\mathbf{A}|\mathbf{B}) \neq r(\mathbf{A})$ and q choices of $\mathbf{D}$.

Therefore, we get $\sum\limits_{0\leq 2r\leq m-2} N(m-2,2r)q(q^{2(m-2)}-q^{4r})$ many codewords of weight $q^{2(m-2)-1}(q-1)$. This completes the proof of the theorem.

□

Remark 3.12. A formula for the weight spectrum of the affine Grassmann code $C^{\mathcal{A}}(2,m)$ has been given. Now we show that the sum of all values of A_i given in Theorem 3.11 for different values of r is $q^{\frac{m(m-1)}{2}} = |C^{\mathcal{A}}(2,m)|$ which is expected.

First, assume that m is even and $m = 2k+2$ for some $k \in \mathbb{N}$ Then by equation (3), we get $|C^{\mathcal{A}}(2,m)| = q^{\frac{(2k+2)(2k+1)}{2}}$. Also, in this case, $m-2 = 2k$ and possible values of r are $0,1,\ldots,k$. Therefore, If we add all values of A_i given in all three cases of Theorem 3.11, we get

$$\begin{aligned}
&\sum_{r=0}^{k} N(2k,2r)q^{4r} + \sum_{r=0}^{k} N(2k,2r)q^{4r}(q-1) + \sum_{r=0}^{k} N(2k,2r)q(q^{4k}-q^{4r})\\
&= N(2k,0)q^{4k+1} + N(2k,2)q^{4k+1}\ldots + N(2k,2k)q^{4k+1}\\
&= \left(N(2k,0) + N(2k,2) + \ldots + N(2k,2k)\right)q^{4k+1},
\end{aligned}$$

where $N(2k,2r)$ denotes the number of skew-symmetric matrix of size $2k$ of rank $2r$. Since any skew-symmetric matrix of size $2k$ can have rank any even number between 0 and $2k$, therefore the sum of the numbers in the bracket is the number of skew-symmetric matrices of size $2k$. Consequently, we get

$$\begin{aligned}
&\sum_{r=0}^{k} N(2k,2r)q^{4r} + \sum_{r=0}^{k} N(2k,2r)q^{4r}(q-1) + \sum_{r=0}^{k} N(2k,2r)q(q^{4k}-q^{4r})\\
&= q^{k(2k-1)}q^{4k+1}\\
&= q^{\frac{(2k+1)(k+1)}{2}}
\end{aligned}$$

and this is the number of codewords in $C^{\mathcal{A}}(2,m)$.

Computation of the case when m is odd is quite similar. For example, if m is odd, we may assume $m = 2k+3$ fr some $k \in \mathbb{N}$. In this case $m-2 = 2k+1$ and all possible values of r in this case are also $0,2,\ldots,2k$. Rest of the computation is exactly same as in the previous case, where m is even.

In the next example we compute the weight enumerator polynomial of $C^{\mathcal{A}}(2,m)$ for some small values of m over any field and compare it with the Table 3.1

Example 3.13. We compute the weight spectrum of the affine Grassmann code $C^{\mathcal{A}}(2,m)$ when $m = 4,5$ In both these cases we have only two possibilities of $2r$ namely, $2r = 0,2$.

If $m = 4$, then using Theorem 3.11 we find there are 1 codeword of weight 0, $(q-1)$ codewords of weight q^4, $(q-1)q^4$ codewords of weight $(q^2-1)(q-1)q$, $(q-1)q^4(q-1)$ codewords of weight $(q^3-q^2+1)q$ and $q(q^4-1)$codewords of weight

$q^3(q-1)$. The weight enumerator polynomial of this code is given by

$$W_{C^{\mathcal{A}}(2,4)}(X) = q(q^4-1)X^{q^3(q-1)} + (q-1)^2q^4X^{q(q^3-q^2+1)} + 1 \\ +(q-1)X^{q^4} + (q-1)q^4X^{q(q^2-1)(q-1)}.$$

It is easy to see this polynomial matches with the weight enumerator polynomials given in Table 3.1.

Similarly, in the case when $m = 5$, we get 1 codeword of weight 0, $(q-1)$ codewords of weight q^6, $(q^3-1)q^4$ codewords of weight $(q^2-1)(q-1)q^3$, $(q^3-1)q^4(q-1)$ codewords of weight $(q^3-q^2+1)q^3$ and $q(q^6-1)+(q^3-1)q(q^6-q^4)$ codewords of weight $q^5(q-1)$. In this case the weight enumerator polynomial of the code $C^{\mathcal{A}}(2,5)$ is given by

$$W_{C^{\mathcal{A}}(2,5)}(X) = (q(q^6-1) + q^5(q^2-1)(q^3-1))X^{q^5(q-1)} + 1 + (q-1)X^{q^6} \\ +q^4(q^3-1)(q-1)X^{q^3(q^3-q^2+1)} + (q^3-1)q^4X^{q^3(q^2-1)(q-1)}.$$

One may compare this too with the weight enumerator polynomial given in Table 3.1.

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