# New Lower Bounds for Permutation Arrays Using Contraction 

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#### Abstract

A permutation array $A$ is a set of permutations on a finite set $\Omega$, say of size $n$. Given distinct permutations $\pi, \sigma \in \Omega$, we let $h d(\pi, \sigma)=|\{x \in \Omega: \pi(x) \neq \sigma(x)\}|$, called the Hamming distance between $\pi$ and $\sigma$. Now let $h d(A)=\min \{h d(\pi, \sigma): \pi, \sigma \in A\}$. For positive integers $n$ and $d$ with $d \leq n$, we let $M(n, d)$ be the maximum number of permutations in any array $A$ satisfying $h d(A) \geq d$. There is an extensive literature on the function $M(n, d)$, motivated in part by suggested applications to error correcting codes for message transmission over power lines.

A basic fact is that if a permutation group $G$ is sharply $k$-transitive on a set of size $n \geq k$, then $M(n, n-k+1)=|G|$. Motivated by this we consider the permutation groups $A G L(1, q)$ and $P G L(2, q)$ acting sharply 2-transitively on $G F(q)$ and sharply 3-transitively on $G F(q) \cup\{\infty\}$ respectively. Applying a contraction operation to these groups, we obtain the following new lower bounds for prime powers $q$ satisfying $q \equiv 1(\bmod 3)$.


1. $M(q-1, q-3) \geq\left(q^{2}-1\right) / 2$ for $q$ odd, $q \geq 7$,
2. $M(q-1, q-3) \geq(q-1)(q+2) / 3$ for $q$ even, $q \geq 8$,
3. $M(q, q-3) \geq K q^{2} \log q$ for some constant $K$ if $q$ is odd, $q \geq 13$.

These results resolve a case left open in a previous paper [2], where it was shown that $M(q-1, q-3) \geq q^{2}-q$ and $M(q, q-3) \geq q^{3}-q$ for all prime powers $q$ such that $q \not \equiv 1(\bmod$ $3)$. We also obtain lower bounds for $M(n, d)$ for a finite number of exceptional pairs $n, d$, by applying this contraction operation to the sharply 4 and 5 -transitive Mathieu groups.

## 1 Introduction

### 1.1 Notation and General Background

We consider permutations on a set $\Omega$ of size $n$. Given two such permutations $\pi$ and $\sigma$, we let $h d(\pi, \sigma)=|\{x \in \Omega: \pi(x) \neq \sigma(x)\}|$, so $h d(\pi, \sigma)$ is the number of elements of $\Omega$ at which $\pi$ and $\sigma$ disagree. When $h d(\pi, \sigma)=d$, we say that $\pi$ and $\sigma$ and are at Hamming distance $d$, or that the Hamming distance between $\pi$ and $\sigma$ is $d$. A permutation array $A$ is a set of permutations on $\Omega$. We say that $h d(A)=d$ if $d=\min \{h d(\pi, \sigma): \pi, \sigma \in A\}$. For positive integers $n$ and $d$

[^0]with $d \leq n$ we let $M(n, d)$ be the maximum number of permutations in any permutation array $A$ satisfying $h d(A) \geq d$.

Consider a fixed ordering $x_{1}, x_{2}, \cdots, x_{n}$ of the elements of $\Omega$. The image string of the permutation $\sigma \in A$ is the string $\sigma\left(x_{1}\right) \sigma\left(x_{2}\right) \cdots \sigma\left(x_{n}\right)$. Thus the permutation array $A$ can also be regarded as an $|A| \times n$ matrix whose rows are the image strings of the permutations in $A$. When $h d(A)=d$, any two rows of $A$ disagree in at least $d$ positions and some pair of rows disagree in exactly $d$ positions. In particular, if $G$ is a permutation group acting on $\Omega$, then we obtain a $|G| \times n$ permutation array whose rows consist of the mage strings of all the elements of $G$. We refer to this array by $G$, and we use $h d(G)$ to refer to the hamming distance of this array.

The study of permutation arrays began (to our knowledge) with the papers [9] and [13], where good bounds on $M(n, d)$ (together with other results) were developed based on combinatorial methods, motivated by the Gilbert-Varshamov bounds for binary codes. In recent years there has been renewed interest in permutation arrays, motivated by suggested applications in power line transmission [12], [20], [28], [15], block ciphers [27], and in multilevel flash memories [17] and 18].

We review here some of the known results and methods for estimating $M(n, d)$.
Some elementary exact values and bounds on $M(n, d)$ are the following (summarized with short proofs in [8] $; M(n, 2)=n!, M(n, 3)=\frac{n!}{2}, M(n, n)=n, M(n, d) \geq M(n-1, d)$, $M(n, d) \geq M(n, d+1), M(n, d) \leq n M(n-1, d)$, and $M(n, d) \leq \frac{n!}{(d-1)!}$. More sophisticated bounds were developed in the above cited papers [9] and [13], with a recent improvement in [29]. The smallest interesting case for $d$ is $d=4$. Here some interesting and non-elementary bounds for $M(n, 4)$ were developed in [11], using linear programming, characters on the symmetric group $S_{n}$, and Young diagrams. In [19] it is shown that if $K>0$ is a constant with $n>e^{30 / K^{2}}$ and $s<n^{1-K}$, then $M(n, n-s) \geq \theta(s!\sqrt{\log n})$. The lower bound is achieved by a polynomial time randomized construction, using the Lovasz Local Lemma in the analysis.

There are various construction methods for permutation arrays. First there is a connection with mutually orthogonal latin squares (MOLS). It was shown in [7] that if there are $m$ MOLS of order $n$, then $M(n, n-1) \geq m n$. From this it follows that if $q$ is a prime power, then $M(q, q-1)=q(q-1)$. Computational approaches for bounding $M(n, d)$ for small $n$ and $d$, including clique search, and the use of automorphisms are described in [8], [16], and [23]. There are also constructions of permutation arrays that arise from the use of permutation polynomials, also surveyed in [8, which we mention briefly below.

Additional construction methods are coset search [2] and partition and extension [3]. In the first of these, one starts with with a permutation group $G$ on $n$ letters with $h d(G)=d$, and which is a subgroup of some group $H$ (for example $H=S_{n}$ ). Now for disjoint permutation arrays $A, B$ on the same set of letters, let $h d(A, B)=\min \{h d(\sigma, \tau): \sigma \in A, \tau \in B\}$. For $x \notin G$ we observe that the coset $x G$ of $G$ in $H$ is a permutation array with $h d(x G)=h d(G)$. For cosets $x_{1} G, x_{1} G, \cdots, x_{k} G$ of $G$, the Hamming distance of the permutation array $\cup_{1 \leq i \leq k} x_{i} G$ is the minimum of $d$ and $m$, where $m=\min \left\{h d\left(x_{i} G, x_{j} G\right): 1 \leq i<j \leq k\right\}$. The method of coset search is to iteratively find coset representatives $x_{i}$ so that $m$, while in general less than $d$, is still reasonably large. The partition and extension method is a way of obtaining constructive lower bounds $M(n+1, d+1)$ from such bounds for $M(n, d)$.

Moving closer to the subject of this paper, we consider a class of optimal constructions which arise through sharply transitive groups. We say that a permutation array $A$ on a set $\Omega$ of size $n$ is sharply $k$-transitive on $\Omega$ if given any two $k$-tuples $x_{1}, x_{2}, \cdots, x_{k}$ and $y_{1}, y_{2}, \cdots, y_{k}$ of distinct elements of $\Omega$ there exists a unique $g \in A$ such that $g\left(x_{i}\right)=y_{i}$ for all $1 \leq i \leq k$. In our applications $A$ will be the set of image strings of a permutation group acting on $\Omega$. From the bound $M(n, d) \leq \frac{n!}{(d-1)!}$ we have for any positive integer $k$ that $M(n, n-k+1) \leq \frac{n!}{(n-k)!}$.

Now if $G$ is a sharply $k$-transitive group acting on $\Omega$, then $|G|=\frac{n!}{(n-k)!}$. Also, in such a $G$ any two distinct elements $g, h$ of $G$ can agree in at most $k-1$ positions, since otherwise $g h^{-1}$ is a nonidentity element of $G$ fixing at least $k$ elements of $\Omega$, contrary to sharp $k$-transitivity. Thus $h d(G) \geq n-k+1$. So a sharply $k$-transitive group $G$ implies the existence of an optimal array (the set of image strings of elements of $G$ ) realizing $M(n, n-k+1)=\frac{n!}{(n-k)!}$. The following theorem gives a strong converse to the above, including the generalization to arbitrary arrays that may not be groups.

Theorem 1 [4] Let $A$ be a permutation array on a set of $n$ letters satisfying $h d(A) \geq n-k+1$. Then $|A|=\frac{n!}{(n-k)!}=M(n, n-k+1) \Longleftrightarrow A$ is sharply $k$-transitive on this set.

The sharply $k$-transitive groups (for $k \geq 2$ ) are known, and these are as follows [6], [10], [22]; $k=2$ : the Affine General Linear Group $\operatorname{AGL}(1, q)$ acting on the finite field $G F(q)$, consisting of the transformations $\{x \rightarrow a x+b: x, a \neq 0, b \in G F(q)\}$,
$k=3$ : the Projective Linear Group $\operatorname{PGL}(2, q)$ acting on $G F(q) \cup\{\infty\}$, consisting of the transformations $\left\{x \rightarrow \frac{a x+b}{c x+d}: x, a, b, c, d \in G F(q), a d-b c \neq 0\right\}$,
$k=4$ : the Mathieu group $M_{11}$ acting on a set of size 11,
$k=5$ : the Mathieu group $M_{12}$ acting on a set of size 12,
arbitrary $k$ : the symmetric group $S_{k}$ acting on a set of size $k$ is sharply $k$ and $(k-1)$-transitive, as well as the alternating group $A_{k}$ acting on a set of size $k$ is sharply $(k-2)$-transitive ([22], Theorem 7.1.4).

In this paper we obtain new lower bounds on $M(n, d)$ for $n$ and $d$ near a prime power. Previous results of this kind are given in [8] where it is shown that for $n=2^{k}$ with $n \not \equiv 1(\bmod$ 3) we have $M(n, n-3) \geq(n+2) n(n-1)$ and $M(n, n-4) \geq \frac{1}{3} n(n-1)\left(n^{2}+3 n+8\right)$. It is also shown that for any prime power $n$ with $n \not \equiv 2(\bmod 3)$ we have $M(n, n-2) \geq n^{2}$. These results are based on permutation polynomials. Similar such results appearing in [2], are based on a contraction operation applied to permutation arrays defined in the next section. The latter results yield $M(n-1, n-3) \geq n^{2}-1$ when $n \not \equiv 1(\bmod 3)$, and $M(n-2, n-5) \geq n(n-1)$ when $n \not \equiv 2(\bmod 3)$ and $n \not \equiv 0,1(\bmod 5)$.

Our goal is to obtain new lower bounds when $q$ is prime power satisfying $q \equiv 1(\bmod 3)$; that is, to resolve the case left open in [2] where the methods of that paper do not apply. For such $q$, we accomplish this by applying the contraction operation to the permutation arrays $A G L(1, q)$ and $P G L(2, q)$ (acting on $G F(q)$ and on $G F(q) \cup\{\infty\}$ respectively), obtaining the following constructive lower bounds.

1. for $q \geq 7, M(q-1, q-3) \geq\left(q^{2}-1\right) / 2$ for $q$ odd and $M(q-1, q-3) \geq(q-1)(q+2) / 3$ for $q$ even, and
2. for $q \geq 13, M(q, q-3) \geq K q^{2} \log q$ for some constant $K$ if $q$ is odd, and
3. bounds for $M(n, d)$ for a finite number of exceptional pairs $n, d$, obtained from the Mathieu groups.

We will use standard graph theoretic notation. In particular for a graph $G$ and $S \subseteq V(G)$ we let $[S]_{G}$ be the graph with vertex set $S$ and edge set $E\left([S]_{G}\right)=\{x y: x, y \in S, x y \in E(G)\}$, and we call it the graph induced by $S$. When $G$ is understood by context, we abbreviate $[S]_{G}$ by $[S]$.

### 1.2 Contraction and the Contraction Graph

Consider a permutation array $A$ acting on a set $\Omega=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ of size $n$, where the elements of $\Omega$ are ordered by their subscripts. We distinguish some element, say $x_{n}$, by renaming it $F$.

Thus the image string of any element $\sigma \in A$ will be $\sigma\left(x_{1}\right) \sigma\left(x_{2}\right) \cdots \sigma(F)$, and we say that $\sigma\left(x_{i}\right)$ occurs in position or coordinate $x_{i}$ of the string. Now for any $\pi \in A$, define the permutation $\pi^{\triangle}$ on $\Omega$ by

$$
\pi^{\triangle}(x)= \begin{cases}\pi(F) & \text { if } x=\pi^{-1}(F) \\ F & \text { if } x=F \\ \pi(x) & \text { otherwise }\end{cases}
$$

Thus the image string of $\pi^{\triangle}$ is obtained from the image string of $\pi$ by interchanging the symbols $F$ and $\pi(F)$ if $\pi(F) \neq F$, while $\pi^{\triangle}=\pi$ if and only if $\pi(F)=F$. In either case, $\pi^{\triangle}$ has $F$ as its final symbol. We let $\pi_{-}^{\triangle}$ be the permutation on $n-1$ symbols obtained from $\pi^{\triangle}$ by dropping the last symbol $F$ from $\pi^{\triangle}$. As an example, if $\pi=a F b c d$, then $\pi^{\triangle}=a d b c F$, and $\pi_{-}^{\triangle}=a d b c$. We call the operation $\pi \rightarrow \pi_{-}^{\triangle}$ contraction, and we call $\pi_{-}^{\triangle}$ the contraction of the permutation $\pi$. Further, for any subset $R \subset A$, let $R^{\triangle}=\left\{\pi^{\triangle}: \pi \in R\right\}$, and $R_{-}^{\triangle}=\left\{\pi_{-}^{\triangle}: \pi \in R\right\}$. So $R_{-}^{\triangle}$ is a permutation array on the symbol set $\Omega-\{F\}$ of size $n-1$, and is called the contraction of $R$.

We note some basic properties related to the contraction operation.
Lemma 2 Let $G$ be a permutation group acting on the set $\Omega$ of size $n$, and let $\pi, \sigma \in G$.
a) The only coordinates in either $\pi$ or $\sigma$ whose values are affected by the $\triangle$ operation are $\pi^{-1}(F), \sigma^{-1}(F)$, and $F$. Thus $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right) \geq h d(\pi, \sigma)-3$.
b) Assume $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$. Then $\pi \sigma^{-1}$ contains a 3-cycle in its disjoint cycle factorization, and $|G|$ is divisible by 3.
c) Let $S \subseteq G$. Then $\left|S^{\triangle}\right|=\left|S_{-}^{\triangle}\right|$ and $h d\left(S^{\triangle}\right)=h d\left(S_{-}^{\triangle}\right)$. If also $h d(S)>3$, then $|S|=\left|S^{\triangle}\right|$.

Proof. Part a) follows immediately from the definition of the $\triangle$ operation.
For b), the assumption implies that there are positions $x_{i}, x_{j}, F$ at which the image strings of $\pi$ and $\sigma$ disagree and $\pi^{\triangle}$ and $\sigma^{\triangle}$ agree. So for some indices $s, t$ we must have $\pi\left(x_{i}\right)=$ $x_{s}, \pi\left(x_{j}\right)=F, \pi(F)=x_{t}$, while $\sigma\left(x_{i}\right)=F, \sigma\left(x_{j}\right)=x_{t}, \sigma(F)=x_{s}$. Then $\pi \sigma^{-1}$ (composing left to right) contains the 3 -cycle $\left(x_{i}, F, x_{j}\right)$ in its disjoint cycle factorization. Thus the subgroup of $G$ generated by $\pi \sigma^{-1}$ has order divisible by 3, and hence $|G|$ is divisible by 3 by Lagrange's theorem.

Consider c). The first two equalities follow from the fact that all image strings in $S^{\triangle}$ have $F$ as their last coordinate. To see $|S|=\left|S^{\Delta}\right|$ when $h d(S)>3$, suppose to the contrary that $\pi^{\triangle}=\sigma^{\triangle}$ for distinct $\pi, \sigma \in S$. As noted in the proof of part a), $\pi^{\triangle}$ and $\sigma^{\triangle}$ can agree in at most three positions where $\pi$ and $\sigma$ disagreed. Thus $\pi$ and $\sigma$ already agreed in at least $n-3$ positions. So $h d(\pi, \sigma) \leq 3$, a contradiction.

Consider a permutation array $H$ on $n$ symbols with $h d(H)=d$. The array $H_{-}^{\triangle}$ is on $n-1$ symbols and satisfes $h d\left(H_{-}^{\triangle}\right) \geq d-3$ by Lemma 2 a,c. For the arrays $H=A G L(1, q), P G L(2, q)$ and certain Mathieu groups, our goal is to find a subset $I \subset H$ with $h d\left(I_{-}^{\triangle}\right) \geq d-2$; that is, a subset $I$ whose contraction $I_{-}^{\triangle}$ has Hamming distance larger by 1 than the lower bound $d-3$ for $h d\left(H_{-}^{\triangle}\right)$ given by Lemma 2 a . The lower bound $M(n-1, d-2) \geq\left|I_{-}^{\triangle}\right|$ follows. Now our underlying arrays $H$ will satisfy $h d(H)>3$, so $h d(I) \geq h d(H)>3$. Thus by Lemma 2 c c we have $h d\left(I_{-}^{\triangle}\right)=h d\left(I^{\triangle}\right)$ and $\left|I_{-}^{\triangle}\right|=\left|I^{\triangle}\right|=|I|$. So we get $M(n-1, d-2) \geq|I|$, yielding the main results of this paper.

To find such a subset $I$ of $H$, we employ a graph $C_{H}$ defined as follows.

Definition 3 Let $H$ be a permutation array with $h d(H)=d$. Define the contraction graph for $H$, denoted $C_{H}$, by $V\left(C_{H}\right)=H$ and $E\left(C_{H}\right)=\left\{\pi \sigma: \pi, \sigma \in H, h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=d-3\right\}$.

For $\pi \in C_{H}$, notice that if $\pi(F)=F$, then $\pi$ is an isolated point in $C_{H}$. This is because then $\pi^{\triangle}=\pi$, so that for any other $\sigma \in C_{H}$ we have $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d\left(\pi, \sigma^{\triangle}\right) \geq h d(\pi, \sigma)-2$, implying no edge joining $\pi$ and $\sigma$ in $C_{H}$. We thus have the following characterization of edges in $C_{H}$ :

$$
\begin{equation*}
\pi \sigma \in E\left(C_{H}\right) \Longleftrightarrow\left\{\pi(F) \neq F, \sigma(F) \neq F, \sigma\left(\pi^{-1}(F)\right)=\pi(F), \pi\left(\sigma^{-1}(F)\right)=\sigma(F)\right\} \tag{1}
\end{equation*}
$$

This condition on edges is illustrated in Figure 1.


Figure 1: Neighbors $\pi, \sigma$ in $C_{H} ; \sigma\left(\pi^{-1}(F)\right)=\pi(F)$, and $\pi\left(\sigma^{-1}(F)\right)=\sigma(F)$
Since $h d\left(H^{\triangle}\right) \geq d-3$ by Lemma 2 a , it follows that any independent set $I$ of vertices in $C_{H}$ must satisfy $h d\left(I^{\triangle}\right) \geq d-2$. Now by using Lemma 2 a,c (together with $h d(H)>3$ for our arrays $H)$ get $M(n-1, d-2) \geq|I|$ as explained in the preceding paragraph. We are thus reduced to finding a large independent set in $C_{H}$ for each of the arrays $H=A G L(1, q), P G L(2, q)$, and Mathieu groups considered in this paper.

## 2 The contraction graph for $\operatorname{AGL}(1, q)$

Let $q$ be a prime power. Recall the Affine General Linear Group $A G L(1, q)$ acting as a permutation group on the finite field $G F(q)$ of size $q$, as the set of transformations $\{x \rightarrow a x+b: a \neq$ $0, x, b \in G F(q)\}$ under the binary operation of composition. For any $\pi \in A G L(1, q)$ the permutation $\pi^{\triangle}$ on $G F(q)$ is defined as in the previous section, based on some ordering $x_{1}, x_{2}, \cdots, x_{q}$ of the elements of $G F(q)$, where $F=x_{q}$ is a distinguished element. Clearly $|A G L(1, q)|=q(q-1)$. By standard facts $A G L(1, q)$ is sharply 2-transitive in this action, and it is straightforward to see that $h d(A G L(1, q))=q-1$ (see Lemma 4 for most of that short proof).

Our goal in this section is to obtain a lower bound on $M(q-1, q-3)$ for prime powers $q \geq 7$ satisfying $q \equiv 1(\bmod 3)$. Our method will involve the contraction graph $C_{A G L(1, q)}$ for $A G L(1, q)$, which we henceforth abbreviate by $C_{A}(q)$.

By definition we then have $V\left(C_{A}(q)\right)=A G L(1, q)$, and $E\left(C_{A}(q)\right)=\left\{\pi \sigma: h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=\right.$ $q-4\}$. Following the plan described in the previous section, we find an independent set $I$ in $C_{A}(q)$. Once we have such an $I$, then $I_{-}^{\triangle}$ is a permutation array on $q-1$ symbols, and by Lemma 2. satisfies $h d\left(I_{-}^{\triangle}\right)=h d\left(I^{\triangle}\right) \geq q-3$. This implies the lower bound $M(q-1, q-3) \geq$ $\left|I_{-}^{\triangle}\right|=\left|I^{\triangle}\right|=|I|$, the last equality by Lemma 2 , since $q \geq 7$ implies $h d(I) \geq q-1>3$. The actual size of $I$ will then yield our precise lower bound.

We are thus reduced to finding a large independent set $I$ in $C_{A}(q)$, and from this we get the bound $M(q-1, q-3) \geq|I|$. We begin on that in the following Lemma, which establishes relations in the the finite field $G F(q)$ that correspond to edges in the graph $C_{A}(q)$.

Lemma 4 Let $\pi$ and $\sigma$ be vertices of the graph $C_{A}(q), q \equiv 1(\bmod 3)$, say with $\sigma(x)=a x+r$ and $\pi(x)=b x+s$.
a) If $a \neq b$, then $h d(\pi, \sigma)=q-1$.
b) If $\pi(F)=F$, then $\pi$ is an isolated point in $C_{A}(q)$. There are $q-1$ points $\pi$ satisfying $\pi(F)=F$.
c) Suppose $\pi$ and $\sigma$ are neighbors in $C_{A}(q)$. Then
c1) $h d(\pi, \sigma)=q-1$, and $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$, and
c2) $\frac{a}{b}$ and $\frac{b}{a}$ are the distinct roots of the quadratic $t^{2}+t+1=0$ over $\operatorname{GF}(q)$.
Proof. For a), just observe that $\pi(x)=\sigma(x)$ has the unique solution $x=\frac{s-r}{a-b}$.
For the first claim in b) suppose not, and let $\sigma$ be a neighbor of $\pi$ in $C_{A}(q)$. Then we have $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=q-4$, implying also that $h d(\pi, \sigma)=q-1$ by Lemma 2 a. Let $i$ be the coordinate of agreement between $\pi$ and $\sigma$. Since $\pi(F)=F$, we have $\pi^{\triangle}=\pi$. Thus $h d\left(\pi, \sigma^{\triangle}\right)=q-4$. Now $\sigma^{\triangle}$ can have at most two coordinates, apart from $i$, in which it agrees with $\pi$, these being $F$ and $j=\sigma^{-1}(F)$. So altogether $\pi$ and $\sigma^{\triangle}$ agree in at most the 3 coordinates $i, j, F$. So $q-4=h d\left(\pi, \sigma^{\triangle}\right) \geq q-3$, a contradiction.

Now consider the second claim in b). For any fixed $i \in G F(q), i \neq F$
Since $\pi(F)=F$, we have $q-1$ choices for the value $\pi(i)$ for any fixed $i \in G F(q), i \neq F$. Hence there are $q-1$ choices for the ordered pair $(\pi(F)(=F), \pi(i))$, each such choice determining $\pi$ uniquely by the sharp 2-transitivity of $A G L(1, q)$ acting on $G F(q)$. The claim follows.

For c1), by the definition of edges in $C_{A}(q)$ we have $q-4=h d\left(\pi^{\triangle}, \sigma^{\triangle}\right) \geq h d(\pi, \sigma)-3$ using Lemma 2a. Since $h d(\pi, \sigma)=q$ or $q-1$, it follows that $h d(\pi, \sigma)=q-1$ and we have equality throughout, as required.

Consider c2). By part c1) we have $h d(\pi, \sigma)=q-1$ and $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$. So there are distinct $\alpha, \beta \in G F(q)$, with neither $\alpha$ nor $\beta$ being $F$, such that $\sigma(F)=i, \sigma(\alpha)=F$, and $\sigma(\beta)=j$, and $\pi(F)=j, \pi(\alpha)=i$, and $\pi(\beta)=F$ for distinct $i, j \in G F(q)$. This gives the following set of equations in $G F(q)$.

$$
\left\{\begin{array}{c}
\sigma(\alpha)-\sigma(\beta)=F-j=a(\alpha-\beta)  \tag{2}\\
\sigma(\alpha)-\sigma(F)=F-i=a(\alpha-F) \\
\pi(\alpha)-\pi(\beta)=i-F=b(\alpha-\beta) \\
\pi(\alpha)-\pi(F)=i-j=b(\alpha-F)
\end{array}\right.
$$

The second and third equations of (2) imply

$$
\begin{equation*}
a(\alpha-F)=-b(\alpha-\beta) . \tag{3}
\end{equation*}
$$

Now starting with the first equation of (2) we obtain

$$
\begin{aligned}
a(\alpha-\beta) & =F-j \\
& =(F-i)+(i-j) \\
& =(a+b)(\alpha-F) \quad(\text { by the second and fourth equations of (2) }) .
\end{aligned}
$$

Multiplying equation (3) by $a$ and the last equation by $b$, we obtain the equations

$$
\left\{\begin{align*}
a^{2}(\alpha-F) & =-a b(\alpha-\beta)  \tag{4}\\
a b(\alpha-\beta) & =b(a+b)(\alpha-F) .
\end{align*}\right.
$$

Thus $a^{2}(\alpha-F)=-b(a+b)(\alpha-F)$, and on dividing by $\alpha-F($ since $\alpha \neq F)$ we obtain

$$
\begin{equation*}
a^{2}+b(a+b)=0 \tag{5}
\end{equation*}
$$

Dividing equation (5) by $a^{2}$ or by $b^{2}$, we obtain that $a / b$ and $b / a$ are both roots of the equation $t^{2}+t+1=0$.

We show that $a / b$ and $b / a$ are distinct. Assuming otherwise, then $a / b=1$ or -1 . If $q$ is even then from $1+t+t^{2}=0$ we get the contradiction $1=0$ since the characteristic is 2 . Now assume $q$ is odd. If $a / b=1$, then we get $1+1+1=0$, forcing $q \equiv 0(\bmod 3)$, a contradiction. If $a / b=-1$, then we get $1=0$, again a contradiction.

Let $t_{1}$ and $\frac{1}{t_{1}}$ be the distinct roots of $t^{2}+t+1=0$ in $G F(q)$ for $q \equiv 1(\bmod 3)$ (by Corollary 23a). Let $\pi \in C_{A}(q)$ with $\pi(x)=a x+r$, and let $\sigma$ be a neighbor of $\pi$ in $C_{A}(q)$. Then by Lemma 4 2. 2 we have $\sigma(x)=a t_{1}+s_{1}$ or $\sigma(x)=a \frac{1}{t_{1}}+s_{2}$, so far with $s_{1}$ and $s_{2}$ undetermined. The next lemma shows that $s_{1}$ and $s_{2}$ are uniquely determined by $\pi$ and $t_{1}$.

Lemma 5 Let $q$ be a prime power with $q \equiv 1(\bmod 3)$. Suppose $\pi$ is not an isolated point of $C_{A}(q)$, say with $\pi(x)=a x+r$. Let $t_{1}$ be a root of $t^{2}+t+1=0$ in $G F(q)$. Then the neighbors of $\pi$ in $C_{A}(q)$ are $\sigma_{1}$ and $\sigma_{2}$, given by $\sigma_{1}(x)=a t_{1} x+\left(a-t_{1}\right) F+r\left(1+t_{1}\right)$ and $\sigma_{2}(x)=a \frac{1}{t_{1}} x+\left(a-\frac{1}{t_{1}}\right) F+r\left(1+\frac{1}{t_{1}}\right)$. In particular, each non-isolated point of $C_{A}(q)$ has degree 2 in $C_{A}(q)$.

Proof. : Let $N(\pi)$ be the set of neighbors of $\pi$ in $C_{A}(q)$. First we verify that $\sigma_{1}, \sigma_{2} \in N(\pi)$, giving details only for $\sigma_{1} \in N(\pi)$ as the containment $\sigma_{2} \in N(\pi)$ is proved similarly. To do this, we show that all conditions of (1) are satisfied with $\sigma_{1}$ playing the role of $\sigma$. Clearly $\pi(F) \neq F$ since $\pi$ is not isolated. To show $\sigma_{1}(F) \neq F$, assume not. Suppose first that $a \neq 1$. Then $\sigma_{1}(F)=F$ yields $F=\frac{r}{1-a}$. But now we get $\pi(F)=\frac{a r}{1-a}+r=\frac{r}{1-a}=F$, a contradiction. Next suppose $a=1$ so $\pi(x)=x+r$. Then $\sigma_{1}(F)=F$ together with $a=1$ yields $r\left(1+t_{1}\right)=0$. Combining this with $t_{1} \neq-1$ yields $r=0$. But then $\pi(F)=F$, a contradiction.

So it remains to show that $\sigma_{1}\left(\pi^{-1}(F)\right)=\pi(F)$ and $\pi\left(\sigma_{1}^{-1}(F)\right)=\sigma_{1}(F)$. For the first equality, solving $a x+r=F$ we obtain $\pi^{-1}(F)=\frac{F-r}{a}$. Thus $\sigma_{1}\left(\pi^{-1}(F)\right)=a t_{1}\left(\frac{F-r}{a}\right)+(a-$ $\left.t_{1}\right) F+r\left(1+t_{1}\right)=a F+r=\pi(F)$, as required. For the second equality, from the formula for $\sigma_{1}$ we obtain $\sigma_{1}^{-1}(F)=\frac{1}{a t_{1}}\left(F\left(1-a+t_{1}\right)-r\left(1+t_{1}\right)\right)$. Plugging this into $\pi$ and simplifying, we obtain $\pi\left(\sigma_{1}^{-1}(F)\right)=\frac{1}{t_{1}}\left(F\left(1-a+t_{1}\right)-r\left(1+t_{1}\right)\right)+r$. Working backwards from the equality $\pi\left(\sigma_{1}^{-1}(F)\right)=\sigma_{1}(F)$ we must show that $\frac{1}{t_{1}}\left(F\left(1-a+t_{1}\right)-r\left(1+t_{1}\right)\right)=F\left(a-t_{1}+a t_{1}\right)+r t_{1}$. This is equivalent to $F\left(1-a+t_{1}\right)=r\left(t_{1}^{2}+t_{1}+1\right)+F\left(a t_{1}-t_{1}^{2}+a t_{1}^{2}\right)=F\left(a t_{1}-t_{1}^{2}+a t_{1}^{2}\right)$. We are now reduced to showing $1-a+t_{1}=a t_{1}-t_{1}^{2}+a t_{1}^{2}$. This follows from $t_{1}^{2}+t_{1}+1=0$.

Now let $\sigma \in N(\pi)$, and we show that $\sigma=\sigma_{1}$ or $\sigma_{2}$. By Lemma 4, we know that $\sigma(x)=a t_{1}+s_{1}$ or $\sigma(x)=a \frac{1}{t_{1}}+s_{2}$ for suitable $s_{1}, s_{2} \in G F(q)$. Suppose first that $\sigma(x)=a t_{1}+s_{1}$. Applying the equality $\sigma\left(\pi^{-1}(F)\right)=\pi(F)$ together with $\pi^{-1}(F)=\frac{F-r}{a}$, we get $a t_{1}\left(\frac{F-r}{a}\right)+s_{1}=a F+r$. so $s_{1}=\left(a-t_{1}\right) F+r\left(1+t_{1}\right)$. Thus $\sigma=\sigma_{1}$. A very similar argument shows that if $\sigma(x)=a \frac{1}{t_{1}}+s_{2}$, then $s_{2}=\left(a-\frac{1}{t_{1}}\right) F+r\left(1+\frac{1}{t_{1}}\right)$, and thus $\sigma=\sigma_{2}$. So we have $N(\pi)=\left\{\sigma_{1}, \sigma_{2}\right\}$, completing the proof.

Consider the subgroup $Q=\{x+b: b \in G F(q)\}$ of $A G L(1, q)$. Clearly $|Q|=q$, and for each $h \in G F(q), h \neq 0, Q$ has the coset $h x Q=\{h x+b: b \in G F(q)\}$, which we abbreviate by $Q_{h}$.

Theorem 6 Let $q$ be a prime power with $q \equiv 1(\bmod 3)$. Then the connected components of $C_{A}(q)$ are as follows.
a) There are $q-1$ isolated points, these being the points $\pi$ satisfying $\pi(F)=F$.
b) If $q$ is odd, then each non-isolated point component is a cycle of length 6.
c) If $q$ is even, then each non-isolated point component is a cycle of length 3.

Proof. For part a), we show that $\pi \in C_{A}(q)$ is an isolated point if and only if $\pi(F)=F$. Then a) follows by Lemma 4 b .

If $\pi(F)=F$, then immediately $\pi$ is isolated in $C_{A}(q)$ by Lemma 4b. For the converse, suppose to the contrary that $\pi$ is isolated and that $\pi(F) \neq F$. Let $\sigma_{1}$ be given by $\sigma_{1}(x)=$ $a t_{1} x+\left(a-t_{1}\right) F+r\left(1+t_{1}\right)$ as in Lemma 5. Then the proof of Lemma 5, starting from the established claim $\pi(F) \neq F$ (this claim being an assumption here), shows that $\sigma_{1}$ is a neighbor of $\pi$ in $C_{A}(q)$. This contradicts $\pi$ being isolated.

Consider part b). By Lemma 5 each nontrivial component of $C_{A}(q)$ is a cycle. Let $\pi_{0}$ be a point on such a cycle $C$, say with $\pi_{0} \in Q_{a}$. Let $t_{1}$ be a fixed root of $t^{2}+t+1=0$. Consider a sequence of 4 vertices $\pi_{0} \pi_{1} \pi_{2} \pi_{3}$ on $C$ with $\pi_{j} \pi_{j+1} \in E\left(C_{A}(q)\right)$ for $0 \leq j \leq 2$. We may suppose that $\pi_{j} \in Q_{a t_{1}^{j}}$ by Lemma 5 and straightforward induction (otherwise replace $t_{1}$ by $\frac{1}{t_{1}}$ ). Thus $\pi_{0}, \pi_{1}, \pi_{2}$ are distinct since they belong distinct cosets of $Q$. Since $t_{1}^{3}=1$, we see also that $\pi_{0}$ and $\pi_{3}$ belong to the same coset $Q_{a}$ of $Q$. We now show that $\pi_{0} \neq \pi_{3}$. Writing $\pi_{1}(x)=b x+c$ (so $b=a t_{1}$ ), we apply the first and third equations of (2) with $\pi_{0}$ and $\pi_{1}$ playing the roles of $\sigma$ and $\pi$ respectively, to get $t_{1}=\frac{b}{a}=-\left(\frac{i-F}{j-F}\right)=-\left(\frac{\pi_{0}(F)-F}{\pi_{1}(F)-F}\right)$. Applying this equation two more times we get $1=t_{1}^{3}=-\left(\frac{\pi_{0}(F)-F}{\pi_{3}(F)-F}\right)$, so that $\left(\frac{\pi_{0}(F)-F}{\pi_{3}(F)-F}\right)=-1$. Thus $\pi_{0}(F) \neq \pi_{3}(F)$, so $\pi_{0} \neq \pi_{3}$. Thus each cycle component has length at least 4.

Consider now a sequence of 7 vertices $\pi_{0} \pi_{1} \cdots \pi_{6}$ on $C$ with $\pi_{j} \pi_{j+1} \in E\left(C_{A}(q)\right)$ for $0 \leq j \leq 5$. We claim the first 6 of these $\pi_{0}, \pi_{1}, \cdots, \pi_{5}$ must be distinct as follows. Clearly any two vertices $\pi_{j}, \pi_{j+3}$ are distinct, $0 \leq j \leq 2$, by the same argument that showed $\pi_{0} \neq \pi_{3}$. But any two vertices $\pi_{i}, \pi_{j}$ with $i \not \equiv j(\bmod 3)$ are distinct, since $t_{1} \neq 1$ and $t_{1}^{2} \neq 1$ imply that they belong to different cosets of $Q$, proving the claim. Finally note that $1=t_{1}^{6}=\left(\frac{\pi_{0}(F)-F}{\pi_{6}(F)-F}\right)$, so that $\pi_{0}(F)=\pi_{6}(F)$. Since also $\pi_{0}$ and $\pi_{6}$ also belong to the same coset $Q_{a}$ of $Q$, it follows that $\pi_{0}=\pi_{6}$. Thus the component $C$ containing $\pi_{0}$ is a cycle of length 6 , as required.

Now consider part c). Consider as above the sequence of 4 vertices $\pi_{0} \pi_{1} \pi_{2} \pi_{3}$ in a nontrivial component, with $\pi_{j} \pi_{j+1} \in E\left(C_{A}(q)\right)$ for $0 \leq j \leq 2$. We get $\left(\frac{\pi_{0}(F)-F}{\pi_{3}(F)-F}\right)=-1=1$ since $q$ is even. So since also $\pi_{0}$ and $\pi_{3}$ belong to the same coset $Q_{a}$ of $Q$, it follows that $\pi_{0}=\pi_{3}$. Thus the cycle containing $\pi_{0}$ has length 3 .

Corollary 7 Let $q$ be a prime power with $q \equiv 1(\bmod 3)$ and $q \geq 7$. Then
a) if $q$ is odd, then $M(q-1, q-3) \geq\left(q^{2}-1\right) / 2$, and
b) if $q$ is even, then $M(q-1, q-3) \geq(q-1)(q+2) / 3$.

Proof. For part a) we form an independent set $I$ in $C_{A}(q)$ by taking 3 independent points in each cycle component of of length 6 , together with the set $Y$ of isolated points. Then $M(q-1, q-3) \geq|Y|+\frac{1}{2}\left(\left|C_{A}(q)-Y\right|\right)=q-1+\frac{1}{2}(q(q-1)-(q-1))=\left(q^{2}-1\right) / 2$, as required.

For part b), we form an independent set $I$ in $C_{A}(q)$ by taking one point from each length 3 cycle component, together with the set $Y$ of isolated points. We then have $M(q-1, q-3) \geq$ $|Y|+\frac{1}{3}\left(\left|C_{A}(q)-Y\right|\right)=(q-1)(q+2) / 3$.

The lower bounds in this corollary should be compared to the lower bound $M(q, q-2) \geq q^{2}$ for prime powers $q \not \equiv 2(\bmod 3)$, derived by using permutation polynomials [8].

## 3 The contraction graph for $\operatorname{PGL}(2, q)$

Let $q$ be a power of a prime. The permutation group $P G L(2, q)$ is defined as the set of one to one functions $\sigma: G F(q) \cup\{\infty\} \rightarrow G F(q) \cup\{\infty\}$, under the binary operation of composition,
given by

$$
\begin{equation*}
\left\{\sigma(x)=\frac{a x+b}{c x+d}: a, b, c, d \in G F(q), a d \neq b c, x \in G F(q) \cup\{\infty\}\right\} \tag{6}
\end{equation*}
$$

Here $\sigma(x)$ is computed by the rules:

1. if $x \in G F(q)$ and $x \neq-(d / c)$, then $\sigma(x)=\frac{a x+b}{c x+d}$,
2. if $x \in G F(q)$ and $x=-(d / c)$, then $\sigma(x)=\infty$,
3. if $x=\infty$, and $c \neq 0$, then $\sigma(x)=a / c$, and
4. if $x=\infty$, and $c=0$, then $\sigma(x)=\infty$.

We regard $P G L(2, q)$ as a permutation group acting on the set $G F(q) \cup\{\infty\}$ of size $q+1$ via the one to one map $x \mapsto \sigma(x)$. One can show that $|P G L(2, q)|=(q+1) q(q-1)$, and it is well known that $P G L(2, q)$ is sharply 3-transitive in its action on $G F(q) \cup\{\infty\}$ (see [24] for a proof). It is straightforward to verify that $h d(P G L(2, q))=q-1$, and by Theorem 1 we have $M(q+1, q-1)=|P G L(2, q)|=(q+1) q(q-1)$.

Take a fixed ordering of $G F(q) \cup\{\infty\}$ with $\infty$ as final symbol, say $x_{1}, x_{2}, \cdots, x_{q}, \infty$ where the $x_{i}$ are the distinct elements of $G F(q)$. Then any element $\pi \in P G L(2, q)$ is identified with the length $q+1$ string $\pi\left(x_{1}\right) \pi\left(x_{2}\right) \pi\left(x_{3}\right) \cdots \pi\left(x_{q}\right) \pi(\infty)$, which again we call the image string of $\pi$. For any such $\pi \in P G L(2, q)$ the permutation $\pi^{\triangle}$ on $G F(q) \cup\{\infty\}$ is defined as in the section introducing contraction, where $F=\infty$ is the distinguished element of $G F(q) \cup\{\infty\}$ in that definition. As an example, if $\pi=a \infty b c d e$, then $\pi^{\Delta}=a e b c d \infty$, and $\pi_{-}^{\triangle}=a e b c d$. In the same way, for any subset $R \subset P G L(2, q)$, the sets $R^{\triangle}$, and $R_{-}^{\triangle}$ are defined as in that section, with $F=\infty$. Since $h d(P G L(2, q))=q-1=q+1-2$, the image strings of any two elements of $\operatorname{PGL}(2, q)$ agree in at most two positions. It follows from Lemma 2 a that for any $\pi, \sigma \in P G L(2, q)$ we have $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right) \geq h d(\pi, \sigma)-3 \geq q-4$. That is, $\pi^{\triangle}$ and $\sigma^{\triangle}$ can agree in at most 5 positions; up to 2 occurring from the original $\pi$ and $\sigma$, and up to 3 more occurring from the $\pi^{\triangle}$ and $\sigma^{\triangle}$ operation.

As noted earlier, lower bounds for $M(q, q-3)$ and $M(q, q-4)$ when $q \not \equiv 1(\bmod 3)$ based on permutation polynomials are known [8]. Thus in this section, we restrict ourselves to the case $q \equiv 1(\bmod 3), q$ an odd prime power, where such bounds are not known. For technical reasons we take $q \geq 13$.

The plan will be similar in some respects to the one we used in the previous section. That is, for a certain set $I \subset P G L(2, q)$ we will find a permutation array $I_{-}^{\triangle} \subset P G L(2, q)_{-}^{\triangle}$ on $q$ symbols with $h d\left(I_{-}^{\triangle}\right) \geq q-3$, thus obtaining the lower bound on $M(q, q-3) \geq\left|I_{-}^{\triangle}\right|$. This set $I$ will be an independent set in the contraction graph $C_{P G L(2, q)}$ for $P G L(2, q)$, which we abbreviate by $C_{P}(q)$.

Since $h d(P G L(2, q))=q-1, C_{P}(q)$ is given by $V\left(C_{P}(q)\right)=P G L(2, q)$, and $E\left(C_{P}(q)\right)=$ $\left\{\pi \sigma: h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=q-4\right\}$. So edges $\pi \sigma$ of $C_{P}(q)$ correspond to pairs $\pi, \sigma \in P G L(2, q)$ for which $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)$ achieves its least possible value of $q-4$, occurring when $\pi^{\triangle}$ and $\sigma^{\triangle}$ agree in 5 positions, so consequently $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$. Thus a set $I \subseteq V\left(C_{P}(q)\right)$ is independent in $C_{P}(q)$ if and only if it satisfies $h d\left(I^{\triangle}\right) \geq q-3$. By Lemma 2 , we get $h d\left(I_{-}^{\triangle}\right)=h d\left(I^{\triangle}\right) \geq q-3$, while $\left|I_{-}^{\triangle}\right|=\left|I^{\triangle}\right|=|I|$, with the last equality following from $h d\left(I^{\triangle}\right)=q-3>3$ since $q \geq 13$.

We are thus reduced to finding an independent set $I$ in $C_{P}(q)$, from which $M(q, q-3) \geq|I|$ follows.

To do this, it will be useful to represent functions in $\operatorname{PGL}(2, q)$ in a form different than the standard $\frac{a x+b}{c x+d}$ form.

Definition 8 Fix a prime power $q$.

1. Let $K, r, i \in G F(q)$ with $r \neq 0$. Define the function $f: G F(q) \cup\{\infty\} \rightarrow G F(q) \cup\{\infty\}$ by $f(x)=K+\frac{r}{x-i}$ for $x \notin\{i, \infty\}$, while $f(\infty)=K$ and $f(i)=\infty$.
2. Let $P=\left\{K+\frac{r}{x-i}: K, r, i \in G F(q), r \neq 0\right\}$ be the set of all functions defined in 1 .
3. Let $N \subset P G L(2, q)$ be given by $N=\left\{\pi \in P G L(2, q): \pi(x)=\frac{a x+b}{c x+d}, c \neq 0\right\}$.

We will now see that $P$ is the same set of functions as $N$.
Lemma 9 Let the map $\alpha: N \rightarrow P$ be defined as follows. For any $\pi \in N$ with $\pi(x)=\frac{a x+b}{c x+d}$, let $\alpha(\pi) \in P$ be given by $\alpha(\pi)(x)=\frac{a}{c}+\frac{\frac{b c-a d}{c^{2}}}{x+\frac{d}{c}}$. Then
a) $\pi$ and $\alpha(\pi)$ are the same function on $G F(q) \cup\{\infty\}$.
b) $|P|=|N|=q^{2}(q-1)$.
c) The map $\alpha$ is one to one and onto.

Proof. For a), straightforward manipulation shows that for $x \neq-\frac{d}{c}$ we have $\pi(x)=\frac{a}{c}+\frac{\frac{b c-a d}{c^{2}}}{x+\frac{d}{c}}=$ $\alpha(\pi)(x)$. Also by definition $\alpha(\pi)(\infty)=\frac{a}{c}=\pi(\infty)$ and $\alpha(\pi)\left(-\frac{d}{c}\right)=\infty=\pi\left(-\frac{d}{c}\right)$. so $\pi$ and $\alpha(\pi)$ are the same function.

Consider b). Clearly $|P|=q^{2}(q-1)$ since there are $q-1$ choices for $r$, and $q$ choices for each of $K$ and $i$, independent of each other. To show $|N|=q^{2}(q-1)$, observe first that for any $\pi \in P G L(2, q)$ with $\pi(x)=\frac{a x+b}{c x+d}$ we have $c=0 \Leftrightarrow \pi(\infty)=\infty$. The $\Rightarrow$ direction is immediate by definition. To see $\Leftarrow$, assume $c \neq 0$. Then $\pi(\infty)=\frac{a}{c} \neq \infty$, completing the proof of the observation. Next we have $\pi(\infty)=\infty \Leftrightarrow \pi(x)=A x+B \in A G L(1, q)$ for all $x$ for suitable $A \neq 0, B \in G F(q)$, by definition of computing in $P G L(2, q)$. Thus we have $|N|=|P G L(2, q)|-|A G L(2, q)|=(q+1) q(q-1)-q(q-1)=q^{2}(q-1)$.

Part c) is immediate from parts a) and b), since any two distinct elements of $N$ are distinct as functions. As an alternative (constructive) proof, let $f(x)=K+\frac{r}{x-i} \in P$ be given. Then for $\pi(x)=\frac{K x+r-i K}{x-i} \in N$ we have $\alpha(\pi)=f$. Thus $\alpha$ is onto, and since $|P|=|N|, \alpha$ is also one to one.

We now see how the above observations, together with results which come later, reduce the study of $C_{P}(q)$ to the set $P$ of permutations.. It was shown above that for $\pi \in P G L(2, q)$, we have $\pi(\infty)=\infty \Leftrightarrow c=0$. By condition (1) (with $\infty=F$ ) we see that $\pi(\infty)=\infty$ implies that $\pi$ is an isolated point in $C_{P}(q)$, and we will see later that for $C_{P}(q)$ the converse is also true. So to study the structure of $C_{P}(q)$ apart from its isolated points, we are reduced to studying its subgraph induced by the permutations in $N$. By the bijection $\alpha: N \leftrightarrow P$, under which $\pi \in N$ and $\alpha(\pi) \in P$ are the same permutation on $G F(q) \cup\{\infty\}$, we are then reduced to studying $P$.

Lemma 10 Let $\pi, \sigma \in P$ with $\pi(x)=a+\frac{r}{x-i}, \sigma(x)=b+\frac{s}{x-j}$ with $r, s \neq 0$. Then $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=$ $h d(\pi, \sigma)-3 \Longleftrightarrow(b-a)(j-i)=r$ and $r=s$.

Proof. $\Longrightarrow$ : By assumption we have $\pi(\infty)=a$ and $\pi(i)=\infty$, together with $\sigma(j)=\infty$ and $\sigma(\infty)=b$. By Lemma $2 a$ the only coordinates of either $\pi$ or $\sigma$ whose values are affected by the $\triangle$ operation are the 3 coordinates $\pi^{-1}(\infty)=i, \sigma^{-1}(\infty)=j$, and $\infty$. So the assumption $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$ implies that $\sigma(i)=a$ and $\pi(j)=b$. Thus we get $\pi(j)=a+\frac{r}{j-i}=b$, yielding $(b-a)(j-i)=r$ as required. Now interchanging the roles of $\pi$ and $\sigma$ in this argument, specifically, using $\sigma(i)=b+\frac{s}{i-j}=a$, we get $(a-b)(i-j)=s$, so also $r=s$.


Figure 2: The graph $P_{1}$, partitioned into levels $B_{0}$ and $B_{g^{i}}, 1 \leq i \leq q-1$.
$\Longleftarrow: ~ A g a i n ~ b y ~ a s s u m p t i o n ~ w e ~ h a v e ~ \pi(\infty)=a, \sigma(\infty)=b, \pi(i)=\infty, \sigma(j)=\infty$, and $(b-a)(j-i)=r$. To prove $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$, it remains only to show that $\pi(j)=b$ and $\sigma(i)=a$. For simplicity we let $r=s=1$, since the argument does not depend on $r=s$. Solving for $b$ in $(b-a)(j-i)=1$ we get $b=\frac{1}{j-i}+a=\pi(j)$. Solving for $a$ we get $a=b-\frac{1}{j-i}=b+\frac{1}{i-j}=\sigma(i)$, as required.

Lemma 11 Let $q=p^{m}$, where $p$ is an odd prime, with $q \equiv 1(\bmod 3), q \geq 13$. Let $\pi, \sigma \in P$, say with $\pi(x)=a+\frac{r}{x-i}, \sigma(x)=b+\frac{s}{x-j}$, with $r, s \neq 0$. Then $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3 \Longleftrightarrow$ $\pi \sigma \in E\left(C_{P}(q)\right)$.

Proof. $\Longleftarrow$ : By definition of edges in $C_{P}(q)$ and Lemma $2 a$ we have $q-4=h d\left(\pi^{\triangle}, \sigma^{\triangle}\right) \geq$ $h d(\pi, \sigma)-3$. Now since $q-1 \leq h d(\pi, \sigma) \leq q+1$, equality is forced together with $h d(\pi, \sigma)=q-1$. This yields $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$.
$\Longrightarrow$ : By the assumption $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$ and $h d(\pi, \sigma) \geq q-1$ we are reduced to showing that $h d(\pi, \sigma)=q-1$; that is, that $\pi$ and $\sigma$ already agree in two coordinates.

By assumption and Lemma 10 we have $r=s$, so write $\pi(x)=a+\frac{r}{x-i}$ and $\sigma(x)=b+\frac{r}{x-j}$, for $a, b, i, j, k \in G F(q)$ with $r \neq 0$. Note also $i \neq j$, since otherwise by Lemma 10 we get $r=0$, a contradiction.

We now derive a quadratic equation over $G F(q)$ whose distinct roots are the coordinates of agreement between $\pi$ and $\sigma$. Since $h d\left(\pi^{\triangle}, \sigma^{\triangle}\right)=h d(\pi, \sigma)-3$, by Lemma 10 we have $(b-a)(j-i)=r$. Thus $b=\frac{r}{j-i}+a$. Now we set $\pi(x)=\sigma(x)$ to find the possible coordinates $x$ at which $\pi$ and $\sigma$ agree, understanding that $x$ can be neither $i$ nor $j$ since $\pi$ and $\sigma$ can have no agreements in any of the coordinates $i=\pi^{-1}(\infty), j=\sigma^{-1}(\infty)$, or $\infty$ by Lemma 2 a . Substituting $\frac{r}{j-i}+a$ for $b$ and simplifying we obtain $\frac{1}{x-i}-\frac{1}{x-j}=\frac{1}{j-i}$. Hence $\frac{i-j}{(x-i)(x-j)}=\frac{1}{j-i}$, and we get the quadratic $x^{2}-(i+j) x+i j+(i-j)^{2}=0$. By Corollary 23b there are two distant roots to this equation, giving the two coordinates of agreement for $\pi$ and $\sigma$ as follows; $x_{1}=\frac{1}{2}[i(1+\sqrt{-3})+j(1-\sqrt{-3})]$, and $x_{2}=\frac{1}{2}[i(1-\sqrt{-3})+j(1+\sqrt{-3})]$.

Hence by our reduction at the beginning of the proof it follows that $\pi \sigma \in E\left(C_{P}(q)\right)$, as required.

The preceding two Lemmas yield the following.

Corollary 12 Let $q=p^{m}$, where $p$ is an odd prime, with $q \equiv 1(\bmod 3), q \geq 13$.
a) Let $\pi, \sigma \in P$, say with $\pi(x)=a+\frac{r}{x-i}, \sigma(x)=b+\frac{s}{x-j}, r, s \neq 0$. Then $\pi \sigma \in E\left(C_{P}(q)\right) \Longleftrightarrow$ $r=s$ and $(b-a)(j-i)=r$.
b) $\pi \in P G L(2, q)$ is an isolated point in $C_{P}(q) \Longleftrightarrow \pi(\infty)=\infty$.

Proof. Part a) follows immediately from Lemmas 10 and 11 .
For part b), suppose first that $\pi(\infty)=\infty$. Then immediately $\pi$ is isolated in $C_{P}(q)$ by the equivalence (1) (with $\infty=F$ ) applicable to any contraction graph.

Conversely, suppose to the contrary that $\pi$ is isolated in $C_{P}(q)$ and $\pi(\infty)=x \neq \infty$. Let $i=\pi^{-1}(\infty)$, and let $j$ be any coordinate with $j \notin\{i, \infty\}$, and let $\pi(j)=y$. Then by sharp 3 transitivity of $P G L(2, q)$ we can find an element $\sigma \in P G L(2, q)$ satisfying $\sigma(j)=\infty, \sigma(i)=x$, and $\sigma(\infty)=y$. Then we get $h d\left(\sigma^{\triangle}, \pi^{\triangle}\right)=h d(\sigma, \pi)-3$. So by Lemma 11 we have $\pi \sigma \in$ $E\left(C_{P}(q)\right)$, contradicting $\pi$ being isolated.

The next two theorems, which use the preceding Corollary, tell us more about $C_{P}(q)$. For $S \subset C_{P}(q)$, recall that $[S]$ is the subgraph of $C_{P}(q)$ induced by $S$. When $r$ is fixed by context, we denote a vertex $\pi \in C_{P}(q), \pi \in P$, with $\pi(x)=a+\frac{r}{x-i}$, by the abbreviation $(i, a)$.

Consider the partition of $P$ given by $P=\cup_{r \neq 0} P_{r}$, where for $r \in G F(q)$ with $r \neq 0, P_{r}=$ $\left\{a+\frac{r}{x-i}: a, i \in G F(q)\right\}$, so $\left|P_{r}\right|=q^{2}$. Further consider the partition of $P_{r}$ given by $P_{r}=$ $\cup_{i \in G F(q)} B_{i}(r)$, where $B_{i}(r)=\left\{a+\frac{r}{x-i}: a \in G F(q)\right\}$.

Theorem 13 Let $q=p^{m}$, where $p$ is an odd prime, with $q \equiv 1(\bmod 3), q \geq 13$. Then the following hold in the graph $C_{P}(q)$.
a) For any $r \neq s, r, s \neq 0$, we have $\left[P_{r}\right] \cong\left[P_{s}\right]$.
b) For any $r \neq 0$ and $i \neq j,\left[B_{i}(r) \cup B_{j}(r)\right]$ is a perfect matching, which matches $B_{i}(r)$ to $B_{j}(r)$.
c) For any $r \neq 0$, the subgraph $\left[P_{r}\right]$ is regular of degree $q-1$.
d) Let $v \in C_{P}(q)$ be a non isolated point, and $N(v)$ the set of neighbors of $v$ in $C_{P}(q)$. Then $[N(v)]$ is a disjoint union of cycles.

Proof. For a), consider for any $r \in G F(q), r \neq 0$, the map $\varphi: P_{1} \rightarrow P_{r}$ given by $\varphi\left(a+\frac{1}{x-i}\right)=$ $a+\frac{r}{x-r i}$. Let $v, w \in P_{1}$, say with $v(x)=a+\frac{1}{x-i}$ and $w(x)=b+\frac{1}{x-j}$. Then $v w \in E\left(\left[P_{1}\right]\right) \Leftrightarrow$ $(b-a)(j-i)=1 \Leftrightarrow(b-a)(r j-r i)=r \Leftrightarrow \varphi(v) \varphi(w) \in E\left(\left[P_{r}\right]\right)$. Thus $\varphi$ is a graph isomorphism, and since $r$ was arbitrary, it follows that for any $s \neq 0$ we have $\left[P_{r}\right] \cong\left[P_{1}\right] \cong\left[P_{s}\right]$.

Consider b). Fix $r$, and consider any two points $(i, a)$ and $(j, b)$ of $P_{r}$. By Corollary 12 we have $(i, a)(j, b) \in E\left(C_{P}(q)\right)$ if and only if $i \neq j$ and $(b-a)(j-i)=r$ in $G F(q)$. Let $H_{i j}=\left[B_{i}(r) \cup B_{j}(r)\right]$ for $i \neq j$. Note there can be no edge in $H_{i j}$ of the form $(i, a)(i, b)$ since $(b-a)(i-i)=0 \neq r$, and similarly no edge of the form $(j, a)(j, b)$. Now given $(i, a) \in B_{i}(r)$, a point $(j, b) \in B_{j}(r)$ is a neighbor of $(i, a)$ if and only if $(b-a)(j-i)=r$ by Corollary 12 .

Thus for this fixed $i$ and $j$ we can uniquely determine $b$ by the equation $b=r(j-i)^{-1}+a$, showing that $(j, b)$ is the only neighbor of $(i, a)$ in $B_{j}(r)$. A symmetric argument shows that each point in $B_{j}(r)$ has a unique neighbor in $B_{i}(r)$. Thus $E\left(H_{i j}\right)$ is a perfect matching, which matches $B_{i}(r)$ to $B_{j}(r)$.

For c), let $v \in C_{P}(q)$, say with $v \in B_{i}(r) \subset P_{r}$ for some $r \neq 0$. By Corollary 12, any neighbor of $v$ in $C_{P}(q)$ must also lie in $P_{r}$. By part b), the neighbors of $v$ are in one to one correspondence with the sets $B_{j}(r), j \neq i, j \in G F(q)$. Thus $v$ has exactly $|G F(q)|-1=q-1$ neighbors in $C_{P}(q)$.

For d), take $v \in C_{P}(q)$, and by the isomorphism of subgraphs [ $P_{r}$ ] from part a), we can take $v=(i, a) \in P_{1}$. By Corollary 12 we have $N(v) \subset P_{1}$. It suffices to show that $[N(v)]$ is regular of degree 2. Let $(j, b) \in N(v)$, so $j \neq i$ by part b). Now any neighbor $(k, c)$ of $(j, b)$ in $N(v)$ must


Figure 3: Perfect matching between any two levels of $P_{1}$.
lie in $N((i, a)) \cap N((j, b))$. So to show that $(j, b)$ has degree 2 in $[N(v)]$, it suffices to show that $(k, c) \in P_{1}$ satisfies $(k, c) \in N((i, a)) \cap N((j, b))$ if and only if $k$ is a root in $G F(q)$ of a quadratic equation over $G F(q)$ having two distinct roots in $G F(q)$.

Suppose first that $(k, c) \in N((i, a)) \cap N((j, b))$. By Corollary 12 we must have the equations

$$
(c-a)(k-i)=1,(b-c)(j-k)=1,(b-a)(j-i)=1
$$

Using the second and third equations we get $c=(j-i)^{-1}-(j-k)^{-1}+a$, and from the first equation $c=(k-i)^{-1}+a$. Setting these two expressions for $c$ equal we obtain $(k-i)^{-1}+(j-$ $k)^{-1}=(j-i)^{-1}$. Some simplification leads to the quadratic $k^{2}-k(i+j)+i j+(j-i)^{2}=0$ with coefficients over $G F(q)$ and unknown $k$. By Corollary 23b from the Appendix, we see that that there are two distinct solutions for $k$; namely $k_{1}=\frac{1}{2}[i(1+\sqrt{-3})+j(1-\sqrt{-3})]$, and $k_{2}=\frac{1}{2}[i(1-\sqrt{-3})+j(1+\sqrt{-3})]$.

Conversely suppose that $k$ is one of the two distinct solutions of $k^{2}-k(i+j)+i j+(j-i)^{2}=0$. Then $(k-i)(j-k)=-k^{2}+k(i+j)-i j=(j-i)^{2}$, and using $\frac{1}{(k-i)(j-k)}=\left(\frac{1}{j-i}\right)\left(\frac{1}{k-i}+\frac{1}{j-k}\right)$, one can derive $\frac{1}{k-i}+\frac{1}{j-k}=\frac{1}{j-i}$. Now set $c=\frac{1}{k-i}+a$, so immediately we get $(c-a)(k-i)=1$. Since $(i, a)$ and $(j, b)$ are neighbors we have $(b-a)(j-i)=1$, so $b=\frac{1}{j-i}+a$. It follows that $c=\frac{1}{k-i}+a=\frac{1}{j-i}-\frac{1}{j-k}+a=b-\frac{1}{j-k}$. Hence we get $(b-c)(j-k)=1$. Thus the three equations $(c-a)(k-i)=1,(b-c)(j-k)=1$, and $(b-a)(j-i)=1$ hold, showing that $(k, c) \in N((i, a)) \cap N((j, b))$ by Corollary 12 .

Note that once $k$ is determined (as one of the two distinct roots), then the point $(k, c)$ is uniquely determined by the perfect matching between $B_{k}(1)$ and $B_{i}(1)$ (or $\left.B_{j}(1)\right)$. Thus we obtain that an arbitrary point $(j, b) \in N(v)$ has exactly two neighbors in $N(v)$, completing d).

To round out the structure of $C_{P}(q)$ we consider the connected components of $C_{P}(q)$.
Theorem 14 Let $q=p^{m}$, where $p$ is an odd prime, with $q \equiv 1(\bmod 3), q \geq 13$. Then the connected components of $C_{P}(q)$ are as follows.

1) the isolated points - these are of the form $\pi(x)=a x+b, a \neq 0$, and there are $q(q-1)$ of them,
2) the $q-1$ many connected components $\left[P_{r}\right]$ induced by the sets $P_{r}$.

Proof. By Corollary 12b we have that $\pi \in P G L(2, q)$ is an isolated point in $C_{P}(q)$ if and only if $\pi(\infty)=\infty$. This is equivalent to $\pi(x)=a x+b, a \neq 0$ and there are $q(q-1)$ such points, completing part 1).

The remaining permutations are all of the form $\pi(x)=a+\frac{r}{x-i}$ for suitable $a, r, i \in G F(q)$ with $r \neq 0$ as shown earlier. Hence it suffices to analyze the connected component structure of $\left[\cup_{r \neq 0} P_{r}\right]$. By Corollary 12 and Theorem 13a, to prove part 2) it suffices to prove that any one of the $\left[P_{r}\right]$, say $\left[P_{1}\right]$, is connected.

Recall the partition $P_{1}=\cup_{i \in G F(q)} B_{i}(1)$ defined above, and from now on we abbreviate $B_{i}(1)$ by $B_{i}$. Let $g$ by a generator of the multiplicative cyclic subgroup of nonzero elements in $G F(q)$. Then we can write this partition as $P_{1}=B_{0} \cup\left(\cup_{1 \leq k \leq q-1} B_{g^{k}}\right)$. We regard the sets in this partition as "levels" of $C_{P}(q)$; where $B_{0}$ is level 0 and $B_{g^{k}}$ is level $k, 1 \leq k \leq q-1$. See Figure 2 for an illustration of $P_{1}$ from this viewpoint, where in that Figure we continue with the notation $(i, a)$ for $a+\frac{1}{x-i}$. In particular, $\left(g^{t}, a\right)$ refers to $a+\frac{1}{x-g^{t}}$. By Theorem 13 b the subgraph of $\left[P_{1}\right]$ induced by any two levels has edge set which is a perfect matching, as illustrated in Figure 3.

First we observe that to show that $\left[P_{1}\right]$ is connected it suffices to show that any two vertices in $B_{0}$ are joined by a path in $\left[P_{1}\right]$. For if that was true, then we can find a path in $\left[P_{1}\right]$ from $(0,0)$ to any vertex $w \in P_{1}$ (thus showing connectedness of $\left.\left[P_{1}\right]\right)$ as follows. If $w \in B_{0}$ we are done by assumption. So suppose $w \notin B_{0}$, say with $w \in B\left(g^{k}\right)$. Let $v$ be the unique neighbor in $B_{0}$ of $w$ under the perfect matching $E\left(\left[B_{0} \cup B\left(g^{k}\right)\right]\right)$. Let $P$ be the path from $(0,0)$ to $v$ in $\left[P_{1}\right]$ which exists by assumption. Then $P$ followed by the edge $v w$ is a walk joining $(0,0)$ to $w$, so $P$ contains a path from $(0,0)$ to $w$.

By Theorem 13b there is a (unique) path in $\left[P_{1}\right]$ starting at $(0,0)$ and passing through levels $1,2, \cdots, q-1$ in succession. Let $(0,0)-\left(g, \alpha_{1}\right)-\left(g^{2}, \alpha_{2}\right)-\ldots-\left(g^{q-1}, \alpha_{q-1}\right)$ be this path, illustrated in bold lines in Figure 4, for suitable $\alpha_{k} \in G F(q)$. For $k \geq 1$ let $\left(0, \beta_{k}\right) \in B_{0}$ be the unique neighbor in level 0 of the vertex $\left(g^{k}, \alpha_{k}\right)$ in level $k$. The edges $\left(g^{k}, \alpha_{k}\right)\left(0, \beta_{k}\right)$ are illustrated by the dotted lines in in Figure 4.

This path and the points $\left(0, \beta_{k}\right)$ are illustrated in Figure 4. Our first step is to obtain the values of $\alpha_{k}$ and $\beta_{k}$.
Claim 1: We have
a) $\alpha_{1}=\frac{1}{g}, \alpha_{2}=\frac{1}{g-1}$, and $\alpha_{k}=\frac{g^{k-1}+g^{k-3}+g^{k-4}+\cdots+g+1}{(g-1) g^{k-1}}$ for $k \geq 3$.
b) $\beta_{1}=0$, and $\beta_{k}=\frac{\left(g^{2}-g+1\right)\left(1+g+g^{2}+g^{3}+\cdots+g^{k-2}\right)}{g^{k}(g-1)}$ for $k \geq 2$.

Proof of Claim 1: We repeatedly use the fact, proved earlier, that if $(r, a)$ and $(s, b)$ are adjacent vertices in the contraction graph $C_{P}(q)$, then $(s-r)(b-a)=1$.

For part a), since $(0,0)-\left(g, \alpha_{1}\right)$ is an edge in $C_{P}(q)$ we have $\left(\alpha_{1}-0\right)(g-0)=1$, so $\alpha_{1}=\frac{1}{g}$. Since $\left(g, \alpha_{1}\right)-\left(g^{2}, \alpha_{2}\right)$ is an edge we have $\left(\alpha_{2}-\frac{1}{g}\right)\left(g^{2}-g\right)=1$, yielding $\alpha_{2}=\frac{1}{g-1}$, and similarly $\left(\alpha_{3}-\frac{1}{g-1}\right)\left(g^{3}-g^{2}\right)=1$, yielding $\alpha_{3}=\frac{g^{2}+1}{(g-1) g^{2}}$. Now for $k \geq 3$ we proceed by induction, having proved the base case $k=3$. Since $\left(g^{k}, \alpha_{k}\right)-\left(g^{k-1}, \alpha_{k-1}\right)$ is an edge, we have $\left(\alpha_{k}-\alpha_{k-1}\right)\left(g^{k}-g^{k-1}\right)=1$. Solving for $\alpha_{k}$ and applying the inductive hypothesis to $\alpha_{k-1}$, we obtain $\alpha_{k}=\frac{1}{g^{k}-g^{k-1}}+\frac{g^{k-2}+g^{k-4}+g^{k-5}+\cdots+g+1}{(g-1) g^{k-2}}$, which after simplification yields the claim.

For part b), we have $\beta_{1}=0$ since $(0,0)-\left(g, \alpha_{1}\right)$ is an edge by definition. Since $\left(g^{2}, \alpha_{2}\right)-\left(0, \beta_{2}\right)$ is an edge, we have $\left(\frac{1}{g-1}-\beta_{2}\right)\left(g^{2}-0\right)=1$, and solving for $\beta_{2}$ and simplifying we get the claim for $k=2$. Consider now $k \geq 2$. The existence of edge $\left(g^{k}, \alpha_{k}\right)-\left(0, \beta_{k}\right)$ gives $\left(\alpha_{k}-\beta_{k}\right) g^{k}=1$, so $\beta_{k}=\alpha_{k}-\frac{1}{g^{k}}$. Using the formula for $\alpha_{k}$ from part a), we have $\beta_{k}=\frac{g^{k-1}+g^{k-3}+g^{k-4}+\cdots+g+1}{(g-1) g^{k-1}}-\frac{1}{g^{k}}=$ $\frac{g^{k}+g^{k-2}+g^{k-3}+\cdots+g^{2}+1}{(g-1) g^{k}}=\frac{\left(g^{2}-g+1\right)\left(1+g+g^{2}+g^{3}+\cdots+g^{k-2}\right)}{g^{k}(g-1)}$. QED
 distinct.


Figure 4: The path $(0,0)-\left(g, \alpha_{1}\right)-\left(g^{2}, \alpha_{2}\right)-\cdots-\left(g^{q-1}, \alpha_{q-1}\right)$ in $P_{1}$, where $\left(0, \beta_{i}\right)$ is the level 0 neighbor of $\left(g^{i}, \alpha_{i}\right)$.

Proof of Claim 2: In applying Claim 1, we note first that $g$ could have been chosen so as not to be a root of $x^{2}-x+1=0$ as follows. The number of roots in $G F(q)$ to this quadratic is at most 2. Now the number of generators in the multiplicative cyclic group $G F(q)-\{0\}$ of order $q-1$ is the euler totient function $\phi(q-1)$, defined as the number of integers $1 \leq s \leq q-1$ which are relatively prime to $q-1$. Since $q$ is an odd prime power with $q \geq 13$, we know that $\phi(q-1)>2$, so such a $g$ exists.

We show that for for any pair $j, k$ with $1 \leq j<k \leq q-1$ we have $\beta_{k} \neq \beta_{j}$.
Consider first the case $j=1$. Since $\beta_{1}=0$, we need to show that $\beta_{k} \neq 0$ for $2 \leq k \leq$ $q-1$. Supposing the contrary and applying Claim 1 b we get $\frac{\left(g^{2}-g+1\right)\left(1+g+g^{2}+g^{3}+\cdots+g^{k-2}\right)}{g^{k}(g-1)}=\overline{0}$. Canceling the nonzero factor $\frac{g^{2}-g+1}{g^{k}(g-1)}$ (by the preceding paragraph) on the left side, we get $0=\left(1+g+g^{2}+g^{3}+\cdots+g^{k-2}\right)=\frac{g^{k-1}-1}{g-1}$. This implies that $g^{k-1}-1=0$, so $g$ has order $k-1$. This is impossible since $k-1 \leq q-2$ while $g$, being a generator of the cyclic group $G F(q)-\{0\}$, must have order $q-1$.

So now suppose that $j \geq 2$. Assuming the contrary that $\beta_{k}=\beta_{j}$ and applying Claim 1 b , we get after simplification that $1+g+g^{2}+g^{3}+\cdots+g^{k-2}=g^{k-j}\left(1+g+g^{2}+g^{3}+\cdots+g^{j-2}\right)=$ $g^{k-j}+g^{k-j+1}+\cdots+g^{k-2}$. Thus we have $0=1+g+g^{2}+\cdots+g^{k-j-1}=\frac{g^{k-j}-1}{g-1}$. So $g^{k-j}=1$, which is impossible since $k-j \leq q-3$, while $g$ has order $q-1$. QED

We introduce some notation in preparation for the rest of the argument. Let $Z=\left\{\left(0, \beta_{k}\right)\right.$ : $1 \leq k \leq q-1\} \subset B_{0}$. Since $\left|B_{0}\right|=q$, by Claim 2 we have $\left|B_{0}-Z\right|=1$, and we let $u$ be the unique vertex of $B_{0}-Z$. Further for any subset $T$ of vertices in $C_{P}(q)$, we let $N(T)=\left\{v \in C_{P}(q): v \notin T, v t \in E\left(C_{P}(q)\right)\right.$ for some $\left.t \in T\right\}$ be the neighbor set of $T$ in $C_{P}(q)$. Recall also that $[T]$ denotes the subgraph of $C_{P}(q)$ induced by $T$.
Claim 3: Let $H=[Z \cup N(Z)]_{C_{P}(q)}$, and $H^{\prime}=[\{u\} \cup N(u)]_{C_{P}(q)}$.
a) $H^{\prime}$ is connected.
b) $H$ is connected.
c) $V(H) \cap V\left(H^{\prime}\right)=\emptyset$
d) We have the partition $V\left(P_{1}\right)=V(H) \cup V\left(H^{\prime}\right)$.

Proof of Claim 3: For part a), we apply Theorem 13b to deduce that $H^{\prime}$ has the spanning star subgraph $K_{1, q-1}$, where the center is $u$ and the leaves, one in each level $B_{i}, i \neq 0$, form $N(u)$. Thus $H^{\prime}$ is connected.

Consider part b). Since $\beta_{1}=0$ we have $(0,0) \in Z \subset V(H)$. Thus it suffices to show that for any $w \in V(H)$ there is a path in $H$ joining $(0,0)$ to $w$.

Suppose first that $w \in Z$, so $w=\left(0, \beta_{k}\right)$ for some $k$. Observe that $\left(g^{i}, \alpha_{i}\right) \in N(Z)$ for all $i$ by definition. So the path $(0,0)-\left(g, \alpha_{1}\right)-\left(g^{2}, \alpha_{2}\right)-\ldots-\left(g^{k}, \alpha_{k}\right)$ followed by the edge $\left(g^{k}, \alpha_{k}\right)-\left(0, \beta_{k}\right)$ is path in $H$ joining $(0,0)$ to $w$.

Next suppose $w \in N(Z)$, say with $w$ adjacent to $\left(0, \beta_{k}\right) \in Z$. Then the path $(0,0)-\left(g, \alpha_{1}\right)-$ $\left(g^{2}, \alpha_{2}\right)-\ldots-\left(g^{k}, \alpha_{k}\right)$ followed by the length 2 path $\left(g^{k}, \alpha_{k}\right)-\left(0, \beta_{k}\right)-w$ is a walk in $H$ joining $(0,0)$ to $w$, and this walk contains the required path.

Next consider c). Suppose not, and let $z \in V(H) \cap V\left(H^{\prime}\right)$, say with $z \in B\left(g^{k}\right)$, noting that $k \geq 1$ since each level, in particular $B_{0}$, is an independent set in [ $P_{1}$ ]. Then $z$ has two distinct neighbors in $B_{0}$; namely $u$ and $\left(0, \beta_{j}\right)$, for some $1 \leq j \leq q-1$. This contradicts the fact that the edge set of $\left[B\left(g^{k}\right) \cup B_{0}\right.$ ] is a perfect matching between the levels $B\left(g^{k}\right)$ and $B_{0}$ by Theorem 13b. Thus $V(H) \cap V\left(H^{\prime}\right)=\emptyset$.

Consider now d). By part c), it suffices to show that $\left|V\left(P_{1}\right)\right|=|V(H)|+\left|V(H)^{\prime}\right|$. By Theorem 13b, it follows that $|V(H)|=|Z| q=(q-1) q$. For the same reason $\left|V\left(H^{\prime}\right)\right|=q$. Therefore $\left|V\left(P_{1}\right)\right|=q^{2}=|V(H)|+\left|V(H)^{\prime}\right|$ as required. QED

We can now complete the proof of the theorem by showing that $P_{1}$ is connected. In view of Claim 3, to do this we are reduced to showing that there is an edge $v w \in E\left(\left[P_{1}\right]\right)$ with $v \in H^{\prime}$ and $w \in H$. Suppose no such edge exists. Since $\left[P_{1}\right]$ is $(q-1)$-regular by Theorem 13 , it follows that $H^{\prime}$ is a simple $q-1$ regular graph on $q$ vertices. Thus $H^{\prime}=K_{q}$. Hence $[N(u)]=K_{q-1}$. But this is a contradiction for $q \geq 5$ since by Theorem 13d the neighborhood of any nonisolated point in $C_{P}(q)$ is regular of degree 2 , while $[N(u)$ ] is regular of degree $q-2>2$ since we have assumed $q \geq 13$.

We can now obtain our independent set in $C_{P}(q)$ as a consequence of our previous results and the following theorem of Alon [1].

Theorem 15 [1] Let $G=(V, E)$ be a graph on $N$ vertices with average degree $t \geq 1$ in which for every vertex $v \in V$ the induced subgraph on the set of all neighbors of $v$ is r-colorable. Then the maximum size $\alpha(G)$ of an independent set in $G$ satisfies $\alpha(G) \geq \frac{c}{\log (r+1)} \frac{N}{t} \log t$, for some absolute constant $c$.

Corollary 16 Let $q$ be a power of an odd prime $p$, with $q \equiv 1(\bmod 3), q \geq 13$.
a) $\alpha\left(C_{P}(q)\right) \geq K q^{2} \log q$ for some constant $K$.
b) $M(q, q-3) \geq K q^{2} \log q$ for some constant $K$.

Proof. Consider a). By Corollary 12 there is no edge between any two subgraphs $\left[P_{r}\right]$ and $\left[P_{s}\right]$ for $r \neq s$. Since there are $q$ such subgraphs, and by Theorem 13 a ) they are pairwise isomorphic, it suffices to show that $\alpha\left(P_{1}\right) \geq K q \log q$ for some constant $K$.

We now apply Alon's theorem to the subgraph $\left[P_{1}\right]$ of $C_{P}(q)$. Now $\left[P_{1}\right]$ is $(q-1)$-regular by Theorem 13k, and has $q^{2}$ points. Since the neighborhood of every point is a disjoint union of cycles by Theorem 13d, this neighborhood must be 3-colorable. It follows by Alon's theorem that $\left[P_{1}\right]$ contains an independent set of size $\frac{c}{\log 4} \frac{q^{2}}{q-1} \log (q-1) \sim K q \log q$, for some constant $K$.

For b), let $I$ be an independent set in $C_{P}(q)$ of size $K q^{2} \log q$ for suitable constant $K$, guaranteed to exist by by part a). Then by the reduction made in the discussion preceding Lemma 10 we have $M(q, q-3) \geq|I| \geq K q^{2} \log q$.

## 4 Special case lower bounds for $M(n, d)$ via the Mathieu groups

In this section we consider the Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$, discovered by E. Mathieu in 1861 and 1873. These permutation groups are the earliest known example of sporadic simple groups. See [10], [6], or [25] for a discussion of their construction. These groups act on $11,12,22,23,24$ letters respectively, with $M_{11}$ being a 1 point stabilizer of $M_{12}$, while $M_{23}$ and $M_{22}$ are 1 and 2 point stabilizers of $M_{24}$ respectively.

In this section we apply the contraction operation to these permutation groups to obtain new permutation arrays, with resulting lower bounds for $M(n, d)$ for suitable $n$ and $d$.

Since $M_{12}$ is sharply 5 -transitive we have by Theorem 1 that $h d\left(M_{12}\right)=8$ and $M(12,8)=$ $\left|M_{12}\right|=95040$. Similarly since $M_{11}$ is sharply 4-transitive we have $M(11,8)=\left|M_{11}\right|=7920$. For $M_{24}$ we do not have sharp transitivity. But observe that for any permutation group $G$ acting on some set, and three elements $\pi, \sigma, \tau \in G$, we have $h d(\pi, \sigma)=h d(\pi \tau, \sigma \tau)=h d(\tau \pi, \tau \sigma)$. Thus $h d(G)=\min \{h d(1, \sigma): \sigma \in G\}$. From the set of disjoint cycle structures of elements of $M_{24}$ (available at [30]) we find that the largest number of 1-cycles in the disjoint cycle structure of any nonidentity element of $M_{24}$ is 8 . Thus $h d\left(M_{24}\right)=24-8=16$, and from the stabilizer relation also $h d\left(M_{23}\right)=h d\left(M_{22}\right)=16$. We thus obtain $M(24,16) \geq\left|M_{24}\right|=24,423,040$, $M(23,16) \geq\left|M_{23}\right|=10,200,960$, and $M(22,16) \geq\left|M_{22}\right|=443,520$.

We now apply the contraction operation to these groups. Considering the action of $M_{12}$ on the 12 -letter set $\Omega=\left\{x_{1}, x_{2}, \cdots, x_{12}\right\}$, we designate some element, say $x_{12}$, of $\Omega$ as the distinguished element $F$ in the definition of $\pi^{\triangle}$. Then define for each $\pi \in M_{12}$ the permutation $\pi^{\triangle}$ on the set $\Omega$ exactly as in the introduction. Thus, using the natural ordering of elements of $\Omega$ by subscript, the image string of any $\sigma \in M_{12}$ can be written $\sigma\left(x_{1}\right) \sigma\left(x_{2}\right) \cdots \sigma\left(x_{11}\right) \sigma(F)$.

As before, we let $\pi_{-}^{\triangle}$ be the permutation on 11 symbols obtained from $\pi^{\triangle}$ by dropping the final symbol $F$, and for any subset $S \subset M_{12}$, we let $S_{-}^{\triangle}=\left\{\pi_{-}^{\triangle}: \pi \in S\right\}$, sometimes writing this as $(S)_{-}^{\triangle}$.

Proposition 17 a) $h d\left(\left(M_{12}\right)_{-}^{\triangle}\right) \geq 6$.
b) $M(11,6) \geq\left|M_{12}\right|=95040$.
c) $M(10,6) \geq 8640$.

Proof. We start with a). Suppose not. Since $h d\left(M_{12}\right)=8$, and for any $\alpha, \beta \in M_{12}$ we have $h d\left(\alpha^{\triangle}, \beta^{\triangle}\right) \geq h d(\alpha, \beta)-3$ by Lemma 2 a, the contrary assumption implies $h d\left(\left(M_{12}^{\triangle}\right)_{-}\right)=5$. Thus there is a pair $\sigma, \tau \in M_{12}$ such that $h d(\sigma, \tau)=8$ and $h d\left(\sigma^{\triangle}, \tau^{\triangle}\right)=5$; so $h d\left(\sigma^{\triangle}, \tau^{\triangle}\right)=$ $h d(\sigma, \tau)-3$. Thus by Lemma 2 b we know that $\pi \sigma^{-1}$ has a 3 -cycle in its disjoint cycle factorization so the order of $\pi \sigma^{-1}$ is divisible by 3 .

Since $h d(\sigma, \tau)=8$ and $\pi$ and $\sigma$ are permutations on 12 letters, it follows that there are four positions, call them $x_{i}, 1 \leq i \leq 4$, at which $\pi$ and $\sigma$ agree. Then $\pi \sigma^{-1}$ belongs to the subgroup $H$ of $M_{12}$ fixing these four positions; that is $H=\left\{\alpha \in M_{12}: \alpha\left(x_{i}\right)=x_{i}, 1 \leq i \leq 4\right\}$. This $H$, denoted $M_{8}$, is known to be isomorphic to $Q_{8}$, the quaternion group of order 8 ([5], section 3.2). We can also verify this directly by making use of GAP (Groups, Algorithms, Programming), a
system for computational discrete algebra. The following output employing GAP shows that $H \cong Q_{8}$, the quaternion group of order 8 ([14])
gap $>G:=$ MathieuGroup(12);;
gap $>H=\operatorname{Stabilizer}(G,[1,2,3,4]$, OnTuples);;
gap $>$ StructureDescription $(H)$;
" $Q_{8}^{\prime \prime}$
Now the order of $\pi \sigma^{-1}$ is divisible by 3 as noted above. But 3 does not divide $\left|Q_{8}\right|$, a contradiction to Lagrange's theorem.

Consider next b). Using Lemma 2 c and $h d\left(M_{12}\right)=8>3$, we have $\left|M_{12}\right|=\left|\left(M_{12}\right)_{-}^{\triangle}\right|$. Thus $\left(M_{12}\right)_{-}^{\triangle}$ is a permutation array on 11 letters of size $\left|M_{12}\right|$ with $h d\left(\left(M_{12}\right)_{-}^{\triangle}\right) \geq 6$. Part b) follows.

For part c), we recall from the introduction the elementary bound $M(n-1, d) \geq \frac{M(n, d)}{n}$. Using part a), we then obtain $M(10,6) \geq \frac{M(11,6)}{11} \geq 8640$.

We remark that using the same method as in part b) of the above proposition one can show $M(10,6) \geq\left|M_{11}\right|=7920$. But this is obviously weaker than the bound we give in part c).

We now consider the contraction of $M_{24}$ and resulting special case bounds for $M(n, d)$. Using similar notation as for $M_{12}$ above, we let $M_{24}$ act on the set of 24 letters $\Theta=\left\{x_{1}, x_{2}, \cdots, x_{24}\right\}$, and we designate $x_{24}$ as the distinguished symbol $F$ in the definition of $\pi^{\triangle}$ from the introduction. Now define $\pi^{\triangle}$ for any $\pi \in M_{24}$ as in the introduction, along with accompanying definitions $S^{\triangle}$ and $S_{-}^{\triangle}$ for $S \subseteq M_{24}$.

Proposition 18 a) $h d\left(\left(M_{24}\right)_{-}^{\triangle}\right) \geq 14$.
b) $M(23,14) \geq\left|M_{24}\right|=244,823,040$.
c) $M(22,14) \geq \frac{\left|M_{24}\right|}{23}=10,644,480$.
d) $M(21,14) \geq \frac{\left|M_{24}\right|}{23 \cdot 22}=483,840$.

Proof.
For a), suppose not. Since $h d\left(M_{24}\right)=16$, and for any $\alpha, \beta \in M_{24}$ we have $h d\left(\alpha^{\triangle}, \beta^{\triangle}\right) \geq$ $h d(\alpha, \beta)-3$, it follows that $h d\left(\left(M_{24}\right)_{-}^{\triangle}\right)=13$. Thus there is pair $\sigma, \tau \in M_{24}$ such that $h d(\sigma, \tau)=$ 16 and $h d\left(\sigma^{\triangle}, \tau^{\triangle}\right)=13$; so $h d\left(\sigma^{\triangle}, \tau^{\triangle}\right)=h d(\sigma, \tau)-3$. Hence by Lemma $2 \mathrm{~b}, \tau \sigma^{-1}$ has a 3 -cycle in its disjoint cycle structure factorization.

Since $h d(\sigma, \tau)=16$, and $\sigma$ and $\tau$ are permutations on 24 letters, it follows that $\sigma$ and $\tau$ must agree on 8 positions. Thus $\tau \sigma^{-1}$ belongs to the subgroup $H$ of $M_{24}$ fixing these 8 positions. From the structure theory of $M_{24}$, we know that if these 8 positions form an "octad" (among the 24 positions), then $H=M_{16} \cong Z_{2} \times Z_{2} \times Z_{2} \times Z_{2}$, the elementary Abelian group of order 16 ([25] Theorem 3.21, and [26] pp. 197-208). Again, this can also be verified directly using GAP from the following output ([14]).

```
gap \(>G:=\) MathieuGroup(24);;
gap \(>H:=\operatorname{Stabilizer}(G,[1,2,3,4,5]\), OnTuples);;
gap \(>S=\operatorname{SylowSubgroup}(H, 2) ;\);
gap \(>\) octad \(:=\) Filtered \(([1 . .24], x \rightarrow\) not \(x\) in MovedPoints \((S))\);
[1,2,3,4,5,8,11,13]
gap \(>H:=\) Stabilizer ( \(G\), octad, OnTuples);;
gap \(>\) StructureDescription \((H)\);
\(" C_{2} \times C_{2} \times C_{2} \times C_{2}^{\prime \prime}\).
```

If these 8 positions do not form an octad, then $H$ is the identity ([25), Lemma 3.1). Now the order of $\tau \sigma^{-1}$ is divisible by 3 , so 3 must divide $|H|$. By Lagrange's theorem, this contradicts that $|H|$ has order either 16 or 1 .

Consider next b). Using Lemma 2 c and $h d\left(M_{24}\right)=16>3$, we have $\left|M_{24}\right|=\left|\left(M_{24}\right)_{-}^{\triangle}\right|$. Thus $\left(M_{24}\right)_{-}^{\triangle}$ is a permutation array on 23 letters of size $\left|M_{24}\right|$, and by part a) we have $h d\left(\left(M_{24}\right)_{-}^{\triangle}\right) \geq$ 14. Part b) follows.

For part c), we again use the bound $M(n-1, d) \geq \frac{M(n, d)}{n}$. Using part b), we then obtain $M(22,14) \geq \frac{M(23,14)}{23} \geq \frac{\left|M_{24}\right|}{23}=10,644,480$.

For d), using $M(n-1, d) \geq \frac{M(n, d)}{n}$ again we get $M(21,14) \geq \frac{M(22,14)}{22} \geq \frac{\left|M_{24}\right|}{23 \cdot 22}=483,840$.

## 5 Concluding Remarks

We mention some problems left open from our work.

1. Recall that if $I$ is an independent set in $C_{P}(q)$, then $M(q, q-3) \geq|I|$. To find a large such $I$, one can focus on any nontrivial connected component, say $P_{1}$, of $C_{P}(q)$. If $P_{1}$ contains an independent set of size $k$, then by the isomorphism of the connected components $P_{i}, 1 \leq i \leq q-1$, we get an independent set of size $k(q-1)+q(q-1)=(q-1)(k+q)$ in $C_{P}(q)$, where $q(q-1)$ counts the number of isolated points in $C_{P}(q)$. Our lower bound $M(q, q-3) \geq K q^{2} \log q$ implies, again by the isomorphism of components, that $\alpha\left(P_{1}\right) \geq C q \log q$ (where $\alpha(G)$ is the maximum size of an independent set in a graph $G$ ), for some constant $C$. We therefore ask whether an improvement on this lower bound for $\alpha\left(P_{1}\right)$ can be found.

Now $V\left(P_{1}\right)$ can be viewed as a rectangular array $\{(i, a): i, a \in G F(q)\}$ as in Figure 2, where we let $i$ be the row index, and $a$ the column index. By Corollary 12 a an independent set in $P_{1}$ is just a subset $S$ of this array with the property that for any two points $(i, a),(j, b) \in S$ we have $(b-a)(j-i) \neq 1$ in $G F(q)$. Using the integer programming package GUROBI, we computed independent sets in $P_{1}$ of size $k$ for various $q$. This $k$, together with the resulting lower bound $(q-1)(k+q)$ for $M(q, q-3)$ are presented in Table 1. The primes $q=41,47,53,59,71,83,89$, for example, are not included in this table since $q \not \equiv 1(\bmod 3)$, and hence $M(q, q-3) \geq(q+1) q(q-1)$, an improvement over the lower bound obtained using GUROBI.
2. We also ask for good upper bounds on $\alpha\left(P_{1}\right)$.

## 6 Appendix - Some facts from Number Theory

In this section we review some facts from number theory that were used in this paper.
We start with some notation. For an odd prime $p$ and integer $r \not \equiv 0(\bmod p)$, define the Legendre symbol $\left(\frac{r}{p}\right)$ to be 1 (resp. -1 ) if $r$ is a quadratic residue (resp. nonresidue); that is a square (resp. nonsquare) $\bmod p$. If $r \equiv 0(\bmod p)$, then define $\left(\frac{r}{p}\right)=0$. A couple of simple facts about this symbol are these.

Lemma 19 For an odd prime $p$ and integers $r$ and $s$ we have the following.
a) $\left(\frac{-1}{p}\right)=1$ if $p \equiv 1(\bmod 4)$, and $\left(\frac{-1}{p}\right)=-1$ if $p \equiv 3(\bmod 4)$.
b) $\left(\frac{r s}{p}\right)=\left(\frac{r}{p}\right)\left(\frac{s}{p}\right)$.

Proof. For a), suppose $p \equiv 1(\bmod 4)$. So write $p=4 k+1$, and consider the multiplicative group of nonzero elements mod $p$, which has order $4 k$ and is cyclic. Let $x$ be a generator of this
group. Then note that in this group we have $1=x^{4 k}=\left(x^{2 k}\right)^{2}$, while also $(-1)^{2}=1$ in this group. Since the quadratic $z^{2}-1=0$ has exactly two solutions $z=1$ or -1 in $G F(q)$, and since $x^{2 k} \neq 1$ since $x$ is a generator, it follows that $x^{2 k}=-1$. Thus -1 is a square $\bmod p$.

If $p \equiv 3(\bmod 4)$, then this cyclic group has order $4 k+2$ for some integer $k$. This time we have $1=\left(x^{2 k+1}\right)^{2}$, so that by the same reasoning as above we have $x^{2 k+1}=-1$. This shows that -1 is not a square $\bmod p$, since it is on odd power of the generator.

Consider now b). Just observe that the product $r s$ is a square $\bmod p$ if and only if both $r$ and $s$ are squares $\bmod p$ or if both $r$ and $s$ are non-squares $\bmod p$. Part b) then follows immediately.

We now recall the quadratic reciprocity law.
Theorem 20 (Gauss Quadratic Reciprocity Law) For odd primes $p$ and $q$ we have

$$
\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)} .
$$

There are lots of proof of quadratic reciprocity in the literature, so we omit the proof here. Now let's apply these facts to determining $\left(\frac{-3}{p}\right)$ for odd primes $p$.

Theorem 21 Let $p>3$ be an odd prime. Then
a) If $p \equiv 1(\bmod 6)$, then -3 is a quadratic residue $\bmod p$.
b) If $p \equiv 5(\bmod 6)$, then -3 is a quadratic nonresidue $\bmod p$.

Proof. By the lemma above we have $\left(\frac{-3}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)$, while by quadratic reciprocity we have $\left(\frac{3}{p}\right)=\left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}}$. Thus

$$
\left(\frac{-3}{p}\right)=(-1)^{\frac{p-1}{2}}\left(\frac{-1}{p}\right)\left(\frac{p}{3}\right) .
$$

The factors on the right depend on the residue classes of $p \bmod 4$ and $p \bmod 3$. Thus we consider the four cases defined by the combinations of these two possibilities, obtaining results that initially depend on the residue class of $p \bmod 12$.
case 1: $p \equiv 1(\bmod 4)$ and $p \equiv 1(\bmod 3)$; equivalently $p \equiv 1(\bmod 12)$.
Now $p \equiv 1(\bmod 3)$ says that $\left(\frac{p}{3}\right)=1$. Also $p \equiv 1(\bmod 4)$ implies $(-1)^{\frac{p-1}{2}}=1$ and by Lemma 19 also implies $\left(\frac{-1}{p}\right)=1$. So by the formula above we have $\left(\frac{-3}{p}\right)=1$, showing that -3 is a quadratic residue when $p \equiv 1(\bmod 12)$.
case 2: $p \equiv 1(\bmod 4)$ and $p \equiv 2(\bmod 3)$; equivalently $p \equiv 5(\bmod 12)$.
Now $p \equiv 2(\bmod 3)$ says that $\left(\frac{p}{3}\right)=-1$. Also $p \equiv 1(\bmod 4)$ implies $(-1)^{\frac{p-1}{2}}=1$ and also Lemma 19 implies $\left(\frac{-1}{p}\right)=1$. So by the formula above we have $\left(\frac{-3}{p}\right)=-1$, showing that -3 is a quadratic nonresidue when $p \equiv 5(\bmod 12)$.
case 3: $p \equiv 3(\bmod 4)$ and $p \equiv 1(\bmod 3)$; equivalently $p \equiv 7(\bmod 12)$.
Since $p \equiv 1(\bmod 3)$ we have $\left(\frac{p}{3}\right)=1$. Also $p \equiv 3(\bmod 4)$ implies $(-1)^{\frac{p-1}{2}}=-1$, and also Lemma 19 implies $\left(\frac{-1}{p}\right)=-1$. So by the formula above we have $\left(\frac{-3}{p}\right)=1$, showing that -3 is a quadratic residue when $p \equiv 7(\bmod 12)$.
case 4: $p \equiv 3(\bmod 4)$ and $p \equiv 2(\bmod 3)$; equivalently $p \equiv 11(\bmod 12)$.
Since $p \equiv 2(\bmod 3)$ we have $\left(\frac{p}{3}\right)=-1$. Again $p \equiv 3(\bmod 4)$ implies that $(-1)^{\frac{p-1}{2}}=-1$, and also that $\left(\frac{-1}{p}\right)=-1$. So by the formula above we get $\left(\frac{-3}{p}\right)=-1$, showing that -3 is a quadratic nonresidue when $p \equiv 11(\bmod 12)$.

Putting together cases 1 and 3 , we see that -3 is a quadratic residue $\bmod p$ when $p \equiv 1(\bmod$ 6 ), while cases 2 and 4 show that -3 is a quadratic nonresidue $\bmod p$ when $p \equiv 5(\bmod 6)$, as required.

Corollary 22 Consider the prime power $q=p^{m}$, where $p>3$ is an odd prime. If $q \equiv 1$ (mod $3)$, then -3 is a square in the finite field $G F(q)$.

Proof. Since $p>3$ is an odd prime we have either $p \equiv 1(\bmod 6)$ or $p \equiv 5(\bmod 6)$. If $p \equiv 1(\bmod$ 6 ), then -3 is already a square in the prime subfield $G F(p) \subseteq G F(q)$ by Theorem 21, so -3 is a square in $G F(q)$, as required.

So suppose $p \equiv 5(\bmod 6)$. Consider the quadratic extension $G F(p)(\sqrt{-3})$ of $G F(p)$ obtained by adjoining to $G F(p)$ a root of the irreducible (by Theorem 21) polynomial $x^{2}+3$ over $G F(p)$. Then $G F(p)(\sqrt{-3}) \cong G F\left(p^{2}\right)$, and -3 is a square in $G F\left(p^{2}\right)$.

Since $q \equiv 1(\bmod 3)$, then since $p \equiv 5(\bmod 6)$ we have $p \equiv 2(\bmod 3)$, so it follows that $m$ must be even. We recall the basic fact from finite fields that $G F\left(p^{r}\right) \subseteq G F\left(p^{s}\right)$ if and only if $r \mid s$. It follows that $G F\left(p^{2}\right) \subseteq G F(q)$. Thus since -3 is a square in $G F\left(p^{2}\right)$, then -3 is a square in $G F(q)$.

Corollary 23 Let $q=p^{m}$ be a prime power, $q \equiv 1(\bmod 3)$.
a)The equation $x^{2}+x+1=0$ has two distinct solutions in $G F(q)$. If $x_{1}$ is such a root, then $\frac{1}{x_{1}}$ is the other distinct root.
b)For $q$ odd and distinct $i, j \in G F(q)$, the equation $x^{2}-(i+j) x+i j+(i-j)^{2}=0$ has two distinct roots in $G F(q)$.

Proof. Consider a), and suppose first that $p$ is odd. Since the characteristic of the field is odd, we may find the solutions by the standard quadratic formula. We obtain the solutions $x=\frac{1}{2}[-1+\sqrt{-3}], \frac{1}{2}[-1-\sqrt{-3}]$, where we have used the existence of $\sqrt{-3}$ in $G F(q)$ by Corollary 22. These solutions are distinct since $p$ is odd.

Now suppose $p=2$. Recall the trace function $\operatorname{Tr}_{G F(q) / G F(2)}(x)=\sum_{i=0}^{m-1} x^{2^{i}}$, defined for any $x \in G F(q)$, which we abbreviate by $\operatorname{Tr}(x)$. It can be shown (see [21]) that the quadratic equation $a x^{2}+b x+c=0$, with $a, b, c \in G F\left(2^{m}\right), a \neq 0$, has two distinct solutions in $G F\left(2^{m}\right)$ if and only if $b \neq 0$ and $\operatorname{Tr}\left(\frac{a c}{b^{2}}\right)=0$. In our case we have $a=b=c=1$, so $\frac{a c}{b^{2}}=1$. Since $p=2$ and $q \equiv 1(\bmod 3), m$ must be even. Thus there are an even number of terms in the sum defining $\operatorname{Tr}(x)$, each of them equal to 1 . So since the characteristic is 2 , we get $\operatorname{Tr}\left(\frac{a c}{b^{2}}\right)=0$ in our case. It follows that $x^{2}+x+1=0$ has two distinct solutions when $p=2$, as required.

Observe that if $x_{1}$ is a root of of $x^{2}+x+1=0$, then by direct substitution so is $\frac{1}{x_{1}}$. To show that $x_{1}$ and $\frac{1}{x_{1}}$ are distinct, assume not. Then $x_{1}=1$ or -1 . If $q$ is even, then $x_{1}^{2}+x_{1}+1=0$ implies that $1=0$ since the characteristic of the field is 2 , a contradiction. Assume $q$ is odd. Then if $x_{1}=1$ we get $1+1+1=0$, implying $q \equiv 0(\bmod 3)$, a contradiction. If $x_{1}=-1$, then we get $1=0$, contradiction. Thus $x_{1}$ and and $\frac{1}{x_{1}}$ are distinct.

Next consider b). Applying the quadratic formula in this field of odd characteristic, we get the two solutions $x=\frac{1}{2}\left[i+j \pm \sqrt{(i+j)^{2}-4\left(i j+(j-i)^{2}\right)}\right]=\frac{1}{2}\left[i+j \pm \sqrt{-3\left(i^{2}+j^{2}\right)+6 i j}\right]=$ $\frac{1}{2}\left[i+j \pm \sqrt{-3(i-j)^{2}}\right]=\frac{1}{2}[i+j \pm \sqrt{-3}(i-j)]$. Now since -3 is a square in $G F(q)$ for $q \equiv 1(\bmod 3)$ by Corollary 22 , it follows that the two solutions for $x$ can be written as $x_{1}=$ $\frac{1}{2}[i(1+\sqrt{-3})+j(1-\sqrt{-3})]$, and $x_{2}=\frac{1}{2}[i(1-\sqrt{-3})+j(1+\sqrt{-3})]$. Also these two solutions are distinct since $i \neq j$ and $q$ is odd.

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| $q$ | $k$ | $(q-1)(k+q) \leq M(q, q-3))$ |
| :---: | :---: | :---: |
| 7 | 13 | 120 |
| 13 | 33 | 552 |
| 19 | 81 | 1800 |
| 31 | 122 | 4590 |
| 37 | 191 | 8208 |
| 43 | 191 | 9828 |
| 49 | 226 | 13200 |
| 61 | 314 | 22500 |
| 67 | 340 | 26862 |
| 73 | 382 | 32760 |
| 79 | 415 | 38532 |
| 97 | 535 | 60672 |
| 103 | 598 | 71502 |
| 109 | 637 | 80568 |
| 121 | 2613 | 328080 |
| 127 | 768 | 112770 |
| 139 | 867 | 138828 |
| 151 | 945 | 164400 |
| 157 | 984 | 177996 |
| 163 | 1031 | 193428 |
| 169 | 1069 | 207984 |
| 181 | 1174 | 243900 |
| 193 | 1262 | 279360 |
| 199 | 1310 | 298782 |
| 211 | 1403 | 338940 |
| 223 | 1496 | 381618 |
| 229 | 1565 | 409032 |
| 241 | 1671 | 458880 |
| 277 | 1956 | 616308 |
| 283 | 2009 | 646344 |
| 289 | 2045 | 672192 |
| 307 | 2197 | 766224 |
| 313 | 2272 | 806528 |
| 331 | 2396 | 899910 |
| 337 | 2462 | 940464 |
| 343 | 2501 | 972648 |
|  |  |  |

Table 1: Independent set of size k in $P_{1}$ obtained by integer programming, and resulting lower bound $(q-1)(k+q)$ for $M(q, q-3)$, when $q \equiv 1(\bmod 3)$.


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