# A Probabilistic Analysis on a Lattice Attack against DSA 

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#### Abstract

Analyzing the security of cryptosystems under attacks based on the malicious modification of memory registers is a research topic of high importance. This type of attacks may affect the randomness of the secret parameters by forcing a limited number of bits to a certain value which can be unknown to the attacker. In this context, we revisit the attack on DSA presented by Faugère, Goyet and Renault during the conference SAC 2012: we simplify their method and we provide a probabilistic approach in opposition to the heuristic proposed in the former to measure the limits of the attack. More precisely, the main problem is formulated as the search for a closest vector to a lattice, then we study the distribution of the vectors with bounded norms in a this family of lattices and we apply the result to predict the behavior of the attack. We validated this approach by computational experiments.


Keywords DSA • lattices • closest vector problem • exponential sums

## 1 Introduction

The security of the main public-key cryptosystems is based on the difficulty of solving certain mathematical problems. In this context, the most commonly used prob-

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lems come from Number Theory, most notably the integer factorization problem and the discrete logarithm on finite cyclic groups. DSA and RSA are two of the most used cryptosystems and their security relays in these problems. There are many software products like SSH, OpenPGP, S/MIME and SSL which use RSA for encrypting and signing and DSA for signing. The National Institute of Standards and Technology has also promoted the use of elliptic curve cryptography, whose security is based on the discrete logarithm problem in special groups. Although, the National Security Agency has recently advocated to start replacing these cryptosystems [2] because of the potential developments in quantum computing, the perspective is that these cryptosystems are going to be widely use in the short term.

In this paper, we study a cryptosystem based on the discrete logarithm problem. Apart from the advances in quantum computing and some recent results on quasipolynomial complexity algorithms for solving the discrete logarithm problem in multiplicative groups of small characteristic fields [3, 16, 17, 19] and for certain abelian groups [32], the discrete logarithm problem on finite fields of large characteristic remains solvable in subexponential time only and the best algorithm known is given by Adleman and DeMarrais [1].

The discrete logarithm problem in elliptic curves seems to be even harder. Although there are results for anomalous curves [4] and curves defined in extension fields [7], known approaches run in exponential time (see the survey by Galbraith and Gaudry [13]).

However, this fact does not mean that attacking secure cryptosystems is hopeless. Many practical attacks are possible because there is additional information available due to the knowledge of implementation. For example, Genkin, Shamir and Tromer [15] showed that it is possible to recover the private key of a 4096-RSA cryptosystem using the sound pattern generated during the decryption of some chosen data.

These advances point to a new research question: which information should be added in order to solve this problem in polynomial time. This question has been in the spotlight for a long time. Indeed, Rivest and Shamir [29] introduced the notion of oracle to formalize this approach in the context of factorization of RSA modules.

In this article, we focus on the Digital Signature Algorithm (DSA) [11] whose security is based on the difficulty of the DLP in multiplicative groups of finite fields (see Section 2 for more details). The first proposal of using an oracle on DSA comes from Howgrave-Graham and Smart [18] using the LLL lattice reduction algorithm [21] to take benefit from the knowledge of a small number of bits in many ephemeral keys. However, these results were only heuristics, even though confirmed by experimentation. Nguyen and Shparlinski [25,26] presented the first polynomial time algorithm that provably recovers the secret DSA key if about $\log ^{1 / 2}(q)$ LSB (or MSB) of each ephemeral key are known ( $q$ denoting the order of the chosen group, see Section 2) for a polynomially bounded number of corresponding signed messages. Other attacks take advantage using the bits in the ephemeral key and the Fast Fourier Transform [5], 6]. We remark that, although, these type of attacks normally need less bits, the computational cost is bigger. However, there is a common point between these attacks. They need explicit information about the bits used and they bypass the problem of computing discrete logarithms.

At SAC 2012, Faugère, Goyet and Renault [9] restricted the power of the oracle by introducing an implicit attack on DSA. More precisely, they do not assume that the oracle explicitly outputs bits of the ephemeral keys but rather provides only implicit information. In this implicit scenario, the oracle is stated in the following way: the attacker knows some signatures that were computed with ephemeral keys sharing some bits. Instead of an explicit information related to the value of these shared bits the implicit information provides only the positions of the shared bits. In an application point of view, this oracle can be instantiated by an invasive attack where some registers used by a pseudo random generator would be destroyed by a laser and keep always the same unknown value during the computation of many signatures. The introduction of implicit information given by an oracle where first presented by May and Ritzenhofen [22] in the context of the RSA cryptosystem and well studied since then (e.g. [10]30]). The attack proposed in [9] is heuristic based. The contribution of this article is to provide a rigorous proof and analyze the applicability of this attack. This article presents results for the DSA over a finite field, but we remark that these techniques can be adapted for the elliptic curve version (ECDSA) as well.

The paper is organized as follows. Section 2 gives an overview of DSA, recalls the attack proposed in [9] and presents the main contribution of this paper. Section 3 presents the background in uniform distribution theory necessary to understand the probabilistic approach. Section 4 presents the proofs of our main results and Section 5 shows the performance of the attack in experiments and discusses the relation with the theoretical results.

## 2 Implicit attack on DSA

We follow the same notation as in the article [9] and go through the technique proposed there. The next diagram represents the protocol to generate a public key and signing a message using DSA with finite fields. For readers not familiar with DSA, we provide the explicit details.


Let $M$ be a positive integer, $p$ be a a $L$-bit prime and $q$ be a prime divisor of $p-1$ satisfying $2^{M-1}<q<2^{M}$. The integers $p$ and $q$ are recommended to be chosen such that $(p, q) \in\{(1024,160),(2048,224),(2048,256),(3072,256)\}$ see [12].

The finite field of $q$ elements is denoted by $\mathbb{F}_{q}$ and each of its element is uniquely represented by an integer in the range $\{(1-q) / 2, \ldots,(q-1) / 2\}$. This also implies that in the sequel, any number modulo $q$ gives a number in the previous range. For the DSA signature scheme, the user selects a random element $a \in \mathbb{F}_{q}$, which must be
kept private, and then publishes $q, p$, an element $g \in \mathbb{F}_{p}$ of multiplicative order $q$ and $g^{a} \bmod p$.

For efficiency and security reasons, the bit-size of the messages signed with DSA has to be the same as the one of $q$ (e.g. 160 or 256 ). Thus for a general message it is necessary to consider its hash and only sign this hash. In the sequel, we denote by $\mathscr{H}$ this hash function and the hash of the message with $m$ (which its bit-size is assumed to be adapted to the chosen $q$ ). The hash function is not important in the results, if it has standard security requirements.

To sign $m$, the user generates a random number $k \in \mathbb{F}_{q}$ (called the ephemeral key) and calculates,

$$
\begin{equation*}
r:=g^{k} \quad \bmod p \quad \bmod q \quad \text { and } \quad s:=k^{-1}(m+a r) \bmod q . \tag{1}
\end{equation*}
$$

The user requires that $r$ and $s$ are not zero, and in this case, $(r, s)$ is a valid signature. Otherwise, the user generates another $k$ and calculates $(r, s)$ again.

### 2.1 Scenario of the attack

We suppose that the user wants to sign $n$ messages, whose hashes are $m_{1}, \ldots, m_{n}$, so he generates $k_{1}, \ldots, k_{n}$ and publishes the signatures $\left(s_{1}, r_{1}\right), \ldots,\left(s_{n}, r_{n}\right)$. We also suppose that, due to some malicious actions of the attacker, the corresponding ephemeral keys differs only in a block of bits of known length so the attacker knows that,

$$
\begin{gather*}
m_{1}+a r_{1}-s_{1} k_{1}=0 \bmod q, \\
m_{2}+a r_{2}-s_{2} k_{2}=0 \bmod q, \\
\vdots  \tag{2}\\
m_{n}+a r_{n}-s_{n} k_{n}=0 \bmod q,
\end{gather*}
$$

where $k_{i}$ have the following property,

$$
\begin{equation*}
k_{i}=k^{\prime}+2^{t} \tilde{k}_{i}+2^{t^{\prime}} k^{\prime \prime}, \quad\left|\tilde{k}_{i}\right| \leq 2^{M-\delta}, \quad \text { for } i=1, \ldots, n \tag{3}
\end{equation*}
$$

with $k^{\prime}, k^{\prime \prime}$ two unknown fixed $t$-bit and $\left(M-t^{\prime}\right)$-bit integers respectively. Thus, there is a total of $\delta=M-t^{\prime}+t$ shared bits. Notice that we can substitute $k_{i}$ using Equation (3) in Equation (2) and eliminate variables $k^{\prime}$ and $k^{\prime \prime}$ which results in the following set of equations,

$$
\begin{aligned}
a \beta_{2} & =\alpha_{2}+\tilde{k}_{1}-\tilde{k}_{2} \bmod q, \\
a \beta_{3} & =\alpha_{3}+\tilde{k}_{1}-\tilde{k}_{3} \bmod q, \\
\quad & \\
a \beta_{n} & =\alpha_{n}+\tilde{k}_{1}-\tilde{k}_{n} \bmod q,
\end{aligned}
$$

where $\tilde{k}_{i}$ come from (3) and $\alpha_{i}, \beta_{i} \in \mathbb{F}_{q}$ are public values defined as,

$$
\begin{align*}
\alpha_{i} & :=2^{-t}\left(s_{i}^{-1} m_{i}-s_{1}^{-1} m_{1}\right) \bmod q, \\
\beta_{i} & :=2^{-t}\left(s_{1}^{-1} r_{1}-s_{i}^{-1} r_{i}\right) \bmod q . \tag{4}
\end{align*}
$$

Next, we can build a lattice using the rows of the following matrix,

$$
\left(\begin{array}{cccccc}
2^{M-\delta} & 0 & \alpha_{2} & \alpha_{3} & \ldots & \alpha_{n}  \tag{5}\\
0 & 2^{-\delta} & \beta_{2} & \beta_{3} & \ldots & \beta_{n} \\
0 & 0 & q & 0 & \ldots & 0 \\
0 & 0 & 0 & q & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & q
\end{array}\right)
$$

and find a short vector in it using an appropriate algorithm, for example [31]. The attacker hopes to recover the following vector,

$$
\left(2^{M-\delta},-a 2^{-\delta}, \tilde{k}_{2}-\tilde{k}_{1}, \ldots, \tilde{k}_{n}-\tilde{k}_{1}\right)
$$

which has a rather short norm. This is the algorithm proposed in [9], with some discussion depending on the parameters.

### 2.2 Contributions

Our first contribution is a variant of this proposal, we still relate the recovering of the ephemeral keys in DSA with a lattice problem but we give rigorous results on the performance of the resulting algorithm.

To give this new attack, we follow the presentation in [24]. First, we define the lattice $\mathscr{L}$ by the rows of the following matrix,

$$
\left(\begin{array}{ccccc}
2^{-\delta} & \beta_{2} & \beta_{3} & \ldots & \beta_{n}  \tag{6}\\
0 & q & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & q & 0 \\
0 & 0 & \ldots & 0 & q
\end{array}\right)
$$

and two vectors $\mathbf{t}, \mathbf{u}$,

$$
\begin{align*}
\mathbf{t} & =\left(0, \alpha_{2}, \alpha_{3}, \ldots, \alpha_{n}\right) \\
\mathbf{u} & =r\left(a 2^{-\delta}, \gamma_{2}, \gamma_{3}, \ldots, \gamma_{n}\right) \tag{7}
\end{align*}
$$

where $\gamma_{i}:=a \beta_{i} \bmod q$ for $i=2, \ldots, n$.
Lattice $\mathscr{L}$ and $\mathbf{t}$ are known to the attacker and his goal is to recover $\mathbf{u}$ using this information. It is straightforward that $\mathbf{u} \in \mathscr{L}$ and $\|\mathbf{u}-\mathbf{t}\| \leq \sqrt{n} 2^{(M-\delta)}$. Thus $\mathbf{u}$ is a vector in this lattice, which is close to $\mathbf{t}$, and we hope that the solution of the closest vector problem is $\mathbf{u}$.

If we call $\mathbf{h} \in \mathscr{L}$ the solution to the closest vector problem then $\mathbf{v}=\mathbf{u}-\mathbf{h}$ verifies $\mathbf{v} \in \mathscr{L}$ and $\|\mathbf{v}\|=\|\mathbf{u}-\mathbf{h}\| \leq\|\mathbf{u}-\mathbf{t}\|+\|\mathbf{t}-\mathbf{h}\| \leq \sqrt{n} 2^{M-\delta+1}$.

The so-called Gaussian heuristic (see [27] page 27, Definition 8]) provides a way of analyzing this method's performance, describing those cases where $\mathbf{h}$ is expected to be $\mathbf{u}$. The shortest vector of lattice $\mathscr{L}$ is expected to have norm

$$
\sqrt{\frac{n+1}{e \pi}} \operatorname{vol}(\mathscr{L})^{1 / n} \approx \sqrt{\frac{n+1}{e \pi}} q^{1-1 / n} 2^{-\delta / n}
$$

so, as soon as

$$
q^{2-2 /(n+1)} 2^{-\delta / n} \geq(e \pi) 2^{2(M-\delta+1)},
$$

we hope to recover u.
Applying the Gaussian heuristic to the lattice defined in Equation (5) is equivalent to this situation because a short vector $\mathbf{u} \in \mathscr{L}$ defines the short vector $(0, \mathbf{u})$ which is in the lattice defined in (5). This argument is heuristic in nature, so an attacker who finds the closest vector in $\mathscr{L}$ to $\mathbf{t}$ has no theoretical guarantee to rediscover $a$. We extend this argument to a probabilistic-in-nature argument. This means, we can measure the success probability of this attack.

Assumption and Statement of the Main Result. In order to state our main result, we need that the hash function $\mathscr{H}$ used in DSA verifies a property (which is the case in practice):

Assumption 1 Let $\mathbb{M}$ and $\mathbb{M}^{\prime}$ be two different messages. The probability of a collision

$$
\mathscr{H}(\mathbb{M})=\mathscr{H}\left(\mathbb{M}^{\prime}\right)
$$

is supposed to be less than $q^{-d-1}$ where $d$ is some positive constant that will be defined later.

Under this assumption, we can now state our main result.
Theorem 1 Under the notations used above and Assumption 11 there exists $d>0$ such that the probability that $\mathbf{u}$ is the solution of the closest vector problem where $\mathbf{t}$ is the target vector in $\mathscr{L}$ is greater than $1-q\left(2^{-\delta+\log n+1}+q^{-d}\right)^{n-1}$.

This is equivalent to say that when $M<(n-1) \min \{\delta-\log n-1, d M\}$, then the attack has non negligible probability of being successful.

We remark that this assumption is not a big restriction because the expected probability of collision on a good hash function is of order $q^{-1}$.

Also, although $d$ is difficult to be evaluated exactly, if $q \geq p^{\varepsilon}$ for some positive $0<\varepsilon<1 / 2$, then $d \geq 2^{145-82 / \varepsilon}-\log \log p / \log q$ when $p$ is sufficiently big. This gives a lower bound for the success probability of the algorithm.

Moreover, we conjecture that the value of $d$ is close to 0.5 , so if $\delta g e \log ^{1 / 2}(M)$ and $n \geq M / \log ^{-1 / 2}(M)$, the probability of success is greater than $1-1 / n$.

This theorem can be generalized in the case where each ephemeral key is taken with $\ell$ blocks of bits fixed sharing a total of $\delta$ bits. This case was also considered in [9] with a heuristic approach. Again, in order to obtain a probability of success of the attack, the hash function has to verify Assumption 1 .

Theorem 2 Under the notation used above, and generalizing the attack above for $k_{1}, \ldots, k_{n}$ having $\ell$ blocks of bits sharing a total of $\delta$ bits, there exists $d>0$ and $a$ probabilistic algorithm to recover a in polynomial time in the size of the input such that the success probability is greater than $1-q\left(2^{-\delta+\log n+1}+q^{-d}(\log q)^{\ell}\right)^{n-1}$.

For practical purposes, the most interesting case is $\ell=1$, so we focus on this case, the proof of the general case follows the same ideas with more technicalities.

## 3 Short vectors and Discrepancy measures

Coming back at our original problem, we remark that we want to prove that the solution of the CVP is, in some way unique and this is related with the norm of the shortest vector in the lattice $\mathscr{L}$. This lattice has a vector of norm at most $2^{M-\delta}$ if and only if there exists $b \in \mathbb{F}_{q}$, such that

$$
b \beta_{i}=h_{i} \bmod q, \quad \text { where }\left|h_{i}\right| \leq 2^{M-\delta}, i=2, \ldots, n .
$$

If $\beta_{i}$ were taken randomly and independently in $\mathbb{F}_{q}$, then the probability of this event is approximately $q 2^{-\delta(n-1)}$. More precisely, we have the following result from [24].

Lemma 1 ([24]) Let $a \in \mathbb{F}_{q}$ be different from zero. Choose integers $\beta_{2}, \ldots, \beta_{n}$ uniformly and independently at random in $\mathbb{F}_{q}$. Then with probability $P \geq 1-q 2^{-\delta(n-1)}$ all vectors $\mathbf{v} \in \mathscr{L}$ such that $\|\mathbf{v}\|_{\infty} \leq 2^{M-\delta}$ are of the form

$$
\mathbf{v}=\left(b 2^{-\delta}, 0, \ldots, 0\right)
$$

where $b=0 \bmod q$ and $\|\mathbf{v}\|_{\infty}$ is the maximum of the absolutes values of vector $\mathbf{v}$.
Notice that this requires that $\beta_{i}$ are realizations of random independent variables in $\mathbb{F}_{q}$ and, unfortunately, as it is mentioned in [24], this is not necessary the case. However, if $\beta_{i}$ are sufficiently well-distributed, then the situation remains the same.

In order to keep the paper self-contained, we recall a way to measure welldistribution through the concept of discrepancy.

Definition 1 Let $\Gamma$ be a multiset of $N$ points contained in the real interval $[0,1)$, then the discrepancy of the set is defined as

$$
D_{N}(\Gamma)=\sup _{\mathscr{B} \subseteq[0,1)}\left|\frac{T_{\Gamma}(\mathscr{B})}{N}-|\mathscr{B}|\right|,
$$

where $T_{\Gamma}(\mathscr{B})$ is the number of points of $\Gamma$ inside the interval

$$
\mathscr{B}=\left[\mathscr{B}^{1}, \mathscr{B}^{2}\right) \subseteq[0,1)
$$

of volume $|\mathscr{B}|=\mathscr{B}^{2}-\mathscr{B}^{1}$ and the supremum is taken over all such boxes.
From the definition, it is easy to see that the discrepancy is a number between 0 and 1 . The closer the value is to 0 , more uniformly is distributed in the unit interval. For more information about discrepancy, see [8]. We also need to introduce the following definition.

Definition 2 A set $\mathscr{T}$ of integers is $\Delta$-homogeneously distributed modulo $q$ if for any integer $b$ coprime with $q$ the discrepancy of the set,

$$
\left\{\left.\frac{b t \bmod q}{q} \right\rvert\, t \in \mathscr{T}\right\}
$$

is at most $\Delta$.
We now state the following lemma. from [24].
Lemma 2 ([24]) Let $a \in \mathbb{F}_{q}$ be different from zero. Choose integers $\beta_{2}, \ldots, \beta_{n}$ uniformly and independently at random from $\mathscr{T}$, which is $\Delta$-homogeneously distributed modulo $q$. Then with probability $P \geq 1-q\left(2^{-\delta}+\Delta\right)^{n-1}$ all vectors $\mathbf{v} \in \mathscr{L}$ such that $\|\mathbf{v}\|_{\infty} \leq 2^{M-\delta}$ are of the form

$$
\mathbf{v}=\left(b 2^{-\delta}, 0, \ldots, 0\right)
$$

where $b=0 \bmod q$ and $\|\mathbf{v}\|_{\infty}$ is the maximum of the absolutes values of vector $\mathbf{v}$.
To show the limits of the attack, it is necessary to show that $\beta_{2}, \ldots, \beta_{n}$ defined in Equation (4) are taken from a set $\Delta$-homogeneously distributed. For this reason, we improve [24, lemma 10], which could be of independent interest and show that the following set,

$$
\Gamma=\left\{s^{-1} m \bmod q \mid \text { where } s, m \text { are defined in Equation (1) }\right\}
$$

is $q^{-d}$-homogeneously distributed.
Lemma 3 Fixed a real number $1 / 2>\varepsilon>0$, then for any sufficiently big $p$, there exists $d>0$ such that for any $g \in \mathbb{F}_{p}$ of multiplicative order $q \geq p^{\varepsilon}$, the set $\Gamma$ is $q^{-d} \mathbf{-}_{-}$ homogeneously distributed provided that the hash function verifies Assumption 1 .
The proof of this lemma will be given in Section 4 , By lemma2, a discrepancy bound for the set

$$
\bar{\Gamma}=\left\{\left.\frac{b s^{-1} m \bmod q}{q} \right\rvert\, \text { where } s, m \text { are defined in Equation (1) }\right\}
$$

for any $b$ coprime with $q$ gives a bound for the probability of $\mathscr{L}$ having a sufficiently short vector. Lemma 3 alone is not sufficient to measure the limits of the attack. Also, it is important to note that to find the closest vector in a lattice to a given target is an NP-complete problem if the dimension of the lattice is a parameter. The attacker relies on algorithms that provide only approximations for the closest vector in a lattice when the dimension is large. In particular, he can use a combination of Schnorr's modification [31] of the LLL algorithm with the result of Kannan to approximate the CVP [20]. We thus have the following result.

Lemma 4 There exists a polynomial time algorithm which, given an n-dimensional full rank lattice $\mathscr{L}$ and a vector $\mathbf{r}$, finds a a vector $\mathbf{v} \in \mathscr{L}$ satisfying the inequality,

$$
\|\mathbf{r}-\mathbf{v}\| \leq 2^{O\left(n \log ^{2} \log (n) / \log (n)\right)} \min \{\|\mathbf{r}-\mathbf{h}\| \mid \mathbf{h} \in \mathscr{L}\}
$$

where the implied constants are absolute.

This lemma shows that we must also consider the cases where the vector found is not so short, and this is the reason that proving results for $\delta$ small is difficult.

There is also an added difficulty, coming from $\delta$. Not all the bits of the ephemeral key $k_{i}$ are taken randomly and independently, indeed only $N-\delta$ are taken at random and the rest are fixed. The case of many blocks of shared bits is difficult because there are several blocks of bits which are fixed. However, in this case, one can prove a bound for the discrepancy of the set $\bar{\Gamma}$. ${ }^{1}$ We cite the following lemma without proof because its independent interest and mention that this follows the same lines as the previous result.

Lemma 5 Fixed a real number $1 / 2>\varepsilon>0$, then for any sufficiently big $p$, there exists $d>0$ such that for any $g \in \mathbb{F}_{p}$ of multiplicative order $q \geq p^{\varepsilon}, \Gamma$ is a $q^{-d}(\log q)^{\ell}$ homogeneously distributed when $k$ is taken with $\ell$ blocks of bits fixed provided that the hash function verifies Assumption [1.

As explained above, we focus on the case $\ell=1$ and thus we will prove this lemma in the next section for the case where $\ell=1$.

This result gives useful information whenever the discrepancy of the set $\bar{\Gamma}$ is smaller than $2^{-\sqrt{\log q}}$. We see that if $\ell$ is fixed and $q$ and $\delta$ are big enough, then the attack has a high probability of success.

## 4 Main results

### 4.1 Exponential Sums and Discrepancy

In this section, we study the discrepancy of the set $\bar{\Gamma}$ (see the definition on page 8 in the unit interval, Typically the bounds on the discrepancy of a sequence are derived from bounds of exponential sums with elements of this set. The relation is made explicit in the celebrated Koksma-Szüsz inequality which we present in the following form.

Lemma 6 (Corollary 3.11, [28]) Let $\Xi$ be a set of $N$ points in the range $[-q / 2, \ldots, q / 2]$ such that there exits a real number $B$ with the property

$$
\left|\sum_{x \in \Xi} \exp \left(2 \pi i \frac{u x}{q}\right)\right| \leq B
$$

for any integer $u$ with $u \neq 0$ and $-q / 2<u \leq q / 2$. Then, the discrepancy $D_{N}(\bar{\Xi})$ where,

$$
\bar{\Xi}=\left\{\left.\frac{x}{q} \right\rvert\, x \in \Xi\right\}
$$

satisfies

$$
D_{N}(\Xi) \ll \frac{B \log q}{N},
$$

where the implied constant is absolute.

[^1]For a positive integer $r$ we denote

$$
\mathbf{e}_{r}(z)=\exp (2 \pi i z / r), \text { where } \mathrm{z} \text { is an integer. }
$$

Notice that for a prime $r=q$, the function $\mathbf{e}_{q}(z)$ is an additive character of $\mathbb{F}_{q}$. Exponential sums are well studied and used extensively in number theory, uniform distribution theory and many other areas because of their applications. In the following lemmas, we outline several known properties.

Lemma 7 (Exercise 11.a,Chapter 3, [33]) Then, for any set $\mathscr{K} \subset \mathbb{F}_{q}$ and $k \in \mathbb{F}_{q}$, the formula

$$
\sum_{u \in \mathbb{F}_{q}} \sum_{k^{\prime} \in \mathscr{K}} \boldsymbol{e}_{q}\left(u\left(k-k^{\prime}\right)\right)= \begin{cases}0 & \text { if } k \notin \mathscr{K} \\ q & \text { otherwise },\end{cases}
$$

holds.
Lemma 8 (Exercise 11.c, Chapter 3, [33]) For any $1 \leq h \leq q$ and $u \in \mathbb{F}_{q}, u \neq 0$, the following inequality,

$$
\frac{1}{q} \sum_{x \in \mathbb{F}_{q}}\left|\sum_{y=1}^{h} \boldsymbol{e}_{q}(u x y)\right| \ll \log q
$$

holds, where the implicit constant is absolute.
We will need the following version of the Weil bound.
Lemma 9 ([23]) Let $F / G$ be a non-constant univariate rational function over $\mathbb{F}_{q}$ and let $v$ be the number of distinct roots of the polynomial $G$ in the algebraic closure of $\mathbb{F}_{q}$. Then

$$
\left|\sum_{x \in \mathbb{F}_{q}}^{*} \boldsymbol{e}_{q}\left(\frac{F(x)}{G(x)}\right)\right| \leq\left(\max (\operatorname{deg} F, \operatorname{deg} G)+v^{*}-2\right) q^{1 / 2}+\rho
$$

where $\Sigma^{*}$ indicates that the poles of $F / G$ are excluded from the summation, $v^{*}=v$ and $\rho=1$ if $\operatorname{deg} F \leq \operatorname{deg} G$, otherwise $\nu^{*}=v+1$ and $\rho=0$.

In order to prove the main result of this paper, we will need to study the number of solutions of the left part of Eq. (1) when $k$ has some fixed bits. Nguyen and Shparlinski [24, lemma 8] proved a similar result but we prove a stronger bound using the following result.

Lemma 10 (Theorem 4.1, [14]) Let $3 \leq m \leq 1.44 \log \log p$ be a positive integer, and $c>0$ an arbitrary fixed constant. Suppose that $X_{1}, \ldots, X_{m}$ are subsets of $\mathbb{F}_{p}$ not containing 0 and satisfying the condition

$$
\left|X_{1}\right| \cdot\left|X_{2}\right| \cdot\left(\left|X_{3}\right| \cdots\left|X_{m}\right|\right)^{1 / 81}>p^{1+c}
$$

Then,

$$
\left|\sum_{x_{1} \in X_{1}} \cdots \sum_{x_{m} \in X_{m}} \boldsymbol{e}_{p}\left(x_{1} \cdots x_{m}\right)\right| \leq\left|X_{1}\right| \cdots\left|X_{m}\right| p^{-0.45 c / 2^{m}}
$$

The following result is a particular case of the one given in [14, Corollary 4.1], but we prove here an explicit version for this case.

Lemma 11 Fixed a real number $1 / 2>\varepsilon>0$, then for any sufficiently big $p$ and $g \in \mathbb{F}_{p}$ of multiplicative order $q \geq p^{\varepsilon}$, the following bound,

$$
\max _{\operatorname{gcd}(c, p)=1}\left|\sum_{k=1}^{q} \boldsymbol{e}_{p}\left(c g^{k}\right)\right| \leq q^{1-2^{145-82 / \varepsilon}}
$$

holds.
Proof The proof is just the application of lemma 10 Fix the value of $c$ to $1 / 81$, and select an integer $m$ satisfying,

$$
q^{2+(m-2) / 81}>p^{82 / 81} \Longrightarrow \varepsilon(m+160)>82
$$

where the inequality on the right has been obtained by substituting $q=p^{\varepsilon}$ and taking logarithms in the equality on the right.

Now, considering $X_{1}=X_{2}=\cdots=X_{m-1}=\left\{g^{k} \bmod p \mid k=1, \ldots, q\right\}, X_{m}=\left\{c x_{1} \mid x_{1} \in\right.$ $\left.X_{1}\right\}$ and lemma 10 gives

$$
\left|\sum_{k=1}^{q} \mathbf{e}_{p}\left(c g^{k}\right)\right|^{m} \leq q^{m} p^{-2^{-m} / 180} \Longrightarrow\left|\sum_{k=1}^{q} \mathbf{e}_{p}\left(c g^{k}\right)\right| \leq q p^{-2^{-m} /(180 m)}
$$

Now, select $m$ the minimum integer satisfying $(m+160) \varepsilon>82$. If $p$ satisfies $m \leq$ $1.44 \log \log p$, i. e. it is sufficiently big, substituting the minimum value of $m$ and $q \geq p^{\varepsilon}$ give the result.

The next result is a generalization of a result by Shparlinski and Nguyen [24, lemma 8]. In this result, we prove an asymptotic bound for the discrepancy for the elements of the multiplicative group generated by $g$ for sufficiently big $p$.

Lemma 12 Fixed a real number $1 / 2>\varepsilon>0$, then for any sufficiently big $p$ and $g \in$ $\mathbb{F}_{p}$ of multiplicative order $q=p^{\varepsilon}$, the number of solutions of the following equation,

$$
g^{k} \quad \bmod p \quad \bmod q=\rho, \quad \rho \in \mathbb{F}_{q}
$$


Proof Defining $L=\lceil p / q\rceil$, it is only necessary to bound the number of solutions of

$$
\left(g^{k}-q x\right) \quad \bmod p=\rho, \quad 0 \leq x \leq L
$$

and where $\rho \in \mathbb{F}_{p}$ is fixed. By lemma 7 , the number of solutions is bounded by

$$
\left|\frac{1}{p} \sum_{c=1}^{p} \sum_{x=0}^{L} \sum_{k=1}^{q} \mathbf{e}_{p}\left(c\left(g^{k}-q x-\rho\right)\right)\right|=\left|\frac{1}{p} \sum_{c=1}^{p} \mathbf{e}_{p}(-c \rho) \sum_{k=1}^{q} \mathbf{e}_{p}\left(c g^{k}\right) \sum_{x=0}^{L} \mathbf{e}_{p}(-q c x)\right| .
$$

Now, we bound this sum using lemmas 8 and 11 so

$$
\begin{aligned}
&\left|\frac{1}{p} \sum_{c=1}^{p} \mathbf{e}_{p}(-c \rho) \sum_{k=1}^{q} \mathbf{e}_{p}\left(c g^{k}\right) \sum_{x=0}^{L} \mathbf{e}_{p}(-q c x)\right| \\
& \leq 2+\frac{1}{p} \sum_{c=1}^{p-1}\left|\sum_{k=1}^{q} \mathbf{e}_{p}\left(c g^{k}\right)\right|\left|\sum_{x=0}^{L} \mathbf{e}_{p}(-q c x)\right| \ll q^{1-2^{145-82 / \varepsilon} \log p}
\end{aligned}
$$

This finishes the proof.
Now, we will give an upper bound of the following exponential

$$
\sum_{m \in \mathscr{H}(\mathscr{M})} \sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}(c(\beta(k, m))+u k),
$$

where $\beta(k, m)$ is defined as,

$$
\beta(k, m):=2^{-t}\left(s^{-1} r\right) \bmod q,
$$

and $s, r$ are defined in (1). The symbol $\Sigma^{*}$ indicates that the poles are excluded from summation.

Lemma 13 Fixed a real number $1 / 2>\varepsilon>0$, then for any sufficiently big $p$ and any $g \in \mathbb{F}_{p}$ of multiplicative order $q=p^{\varepsilon}$, the bound

$$
\max _{\operatorname{gcd}(c, q)=1}\left|\sum_{m \in \mathscr{H}(\mathscr{M})} \sum_{k \in \mathbb{F}_{q}}^{*} \boldsymbol{e}_{q}(c(\beta(k, m))+u k)\right| \ll W^{1 / 2} q^{3 / 2-2^{145-82 / \varepsilon} \log ^{2} p,}
$$

holds, where the constant is absolute.
Proof Taking any integer $c$ coprime with $q$ and calling $\sigma$ the value of the exponential sum, we have

$$
\sigma=\left|\sum_{m \in \mathscr{H}(\mathscr{M})} \sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}(c(\beta(k, m))+u k)\right| \leq \sum_{m \in \mathscr{H}(. \mathscr{M})}\left|\sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}(c(\beta(k, m))+u k)\right| .
$$

For $\lambda \in \mathbb{F}_{q}$ we denote by $H(\lambda)$ the number of $m \in \mathscr{H}(\mathscr{M})$ with $m=\lambda$. We also define the integer $c_{0}:=2^{-t} c \bmod q$. Then,

$$
\sigma=\sum_{\lambda \in \mathbb{F}_{q}} H(\lambda)\left|\sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c_{0} \frac{k r(k)}{\lambda+\operatorname{ar}(k)}+u k\right)\right|,
$$

where $a$ is the private key, the symbol $\Sigma^{*}$ indicates that the poles are excluded from summation and $r(k):=g^{k} \bmod p \bmod q$

Now, we apply the Cauchy inequality,

$$
\begin{equation*}
\sigma^{2} \leq\left(\sum_{\lambda \in \mathbb{F}_{q}} H(\lambda)^{2}\right) \sum_{\lambda \in \mathbb{F}_{q}}\left|\sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c_{0} \frac{k r(k)}{\lambda+\operatorname{ar}(k)}+u k\right)\right|^{2} \tag{8}
\end{equation*}
$$

We remark that,

$$
\begin{equation*}
\sum_{\lambda \in \mathbb{F}_{q}} H(\lambda)^{2}=W \tag{9}
\end{equation*}
$$

We can operate with the other term in the right side of Equation (8).

$$
\begin{aligned}
& \sum_{\lambda \in \mathbb{F}_{q}}\left|\sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c_{0} \frac{k r(k)}{\lambda+\operatorname{ar}(k)}+u k\right)\right|^{2} \leq \\
& \sum_{\lambda \in \mathbb{F}_{q}} \sum_{k_{1}, k_{2} \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c_{0}\left(\frac{k_{1} r\left(k_{1}\right)}{\lambda+\operatorname{ar}\left(k_{1}\right)}-\frac{k_{2} r\left(k_{2}\right)}{\lambda+\operatorname{ar}\left(k_{2}\right)}\right)+u\left(k_{1}-k_{2}\right)\right)= \\
& \sum_{k_{1}, k_{2} \in \mathbb{F}_{q}} \sum_{\lambda \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c _ { 0 } \left(\frac{k_{1} r\left(k_{1}\right)}{\left.\left.\lambda+\operatorname{ar(k_{1})}-\frac{k_{2} r\left(k_{2}\right)}{\lambda+\operatorname{ar}\left(k_{2}\right)}\right)+u\left(k_{1}-k_{2}\right)\right) .}\right.\right.
\end{aligned}
$$

We write the inner sum in the following way:

$$
\sum_{\lambda \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c_{0}\left(\frac{k_{1} r\left(k_{1}\right)}{\lambda+\operatorname{ar}\left(k_{1}\right)}-\frac{k_{2} r\left(k_{2}\right)}{\lambda+\operatorname{ar}\left(k_{2}\right)}\right)+u\left(k_{1}-k_{2}\right)\right)=\sum_{\lambda \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(F_{k_{1}, k_{2}}(\lambda)\right)
$$

where

$$
F_{k_{1}, k_{2}}(X):=\left(c_{0}\left(\frac{k_{1} r\left(k_{1}\right)}{X+\operatorname{ar}\left(k_{1}\right)}-\frac{k_{2} r\left(k_{2}\right)}{X+\operatorname{ar}\left(k_{2}\right)}\right)+u\left(k_{1}-k_{2}\right)\right) .
$$

Notice that $F_{k_{1}, k_{2}}(X)$ is a rational function in $X$ when $k_{1}$ and $k_{2}$ are fixed. The function is not constant if $r\left(k_{1}\right) \neq r\left(k_{2}\right)$ because then $F_{k_{1}, k_{2}}$ has two different poles. If $r\left(k_{1}\right)=$ $r\left(k_{2}\right)$, the sum is constant only in two cases: either $k_{1}=k_{2}$ or $r\left(k_{1}\right)=r\left(k_{2}\right)=0$. By lemma 12, we see that the number of such pairs is $O\left(q^{2-2^{146-82 / \varepsilon}} \log ^{2} p+q\right)$. In other case, it is easy to see that $F_{k_{1}, k_{2}}(X)$ is not a constant function so it is possible to apply lemma 9 . This gives,

$$
\sum_{\lambda \in \mathbb{F}_{q}}\left|\sum_{k \in \mathbb{F}_{q}}^{*} \mathbf{e}_{q}\left(c_{0} \frac{k r(k)}{\lambda+\operatorname{ar}(k)}+u k\right)\right|^{2} \ll\left(q^{3-2^{146-82 / \varepsilon}} \log ^{2} p\right)
$$

Substituting this estimate in Equation (8) with Equation (9), we get the result.

### 4.2 Proof of lemma 5 for $\ell=1$

lemma 6 shows the relationship between bounds on exponential sums and bounds on the discrepancy. So, our goal is to find bounds of the following exponential sum:

$$
\left|\sum_{m \in \mathscr{H}(\mathscr{M})} \sum_{k \in \mathscr{K}} \mathbf{e}_{q}(c \beta(k, m))\right|=\frac{1}{q}\left|\sum_{k \in \mathbb{F}_{q}} \sum_{m \in \mathscr{H}(\mathscr{M})}{ }^{*} \mathbf{e}_{q}(c \beta(k, m)) \sum_{u \in \mathbb{F}_{q}} \sum_{k^{\prime} \in \mathscr{K}} \mathbf{e}_{q}\left(u\left(k-k^{\prime}\right)\right)\right|
$$

where $\mathscr{K}$ is the set of integers defined by Equation (3).

Notice that if $k$ does not meet the requisites, i. e. it does not has the correct bits fixed, the inner sum is equal to zero. Otherwise, the inner sum is equal to one. Doing the following transformations,

$$
\begin{aligned}
& \left|\sum_{k \in \mathbb{F}_{q}}\left(\frac{1}{q} \sum_{u \in \mathbb{F}_{q}} \sum_{k^{\prime} \in \mathscr{K}} \mathbf{e}_{q}\left(u\left(k^{\prime}-k\right)\right)\right) \sum_{m \in \mathscr{H}(\mathscr{M})} \mathbf{e}_{q}(c \beta(k, m))\right|= \\
& \left.\left\lvert\, \frac{1}{q} \sum_{u \in \mathbb{F}_{q}} \sum_{k \in \mathbb{F}_{q}}\left(\sum_{k^{\prime} \in \mathscr{K}} \mathbf{e}_{q}\left(u k^{\prime}\right)\right)\right.\right) \sum_{m \in \mathscr{H}(\mathscr{M})} \mathbf{e}_{q}(c \beta(k, m)-u k) \mid \leq \\
& \left.\left.\quad \frac{1}{q} \sum_{u \in \mathbb{F}_{q}}\left|\sum_{k \in \mathbb{F}_{q}} \sum_{m \in \mathscr{H}(\mathscr{M})} \mathbf{e}_{q}(c \beta(k, m)-u k)\right| \right\rvert\, \sum_{k^{\prime} \in \mathscr{K}} \mathbf{e}_{q}\left(u k^{\prime}\right)\right) \mid .
\end{aligned}
$$

By lemma 13 we have that

Recalling lemma 8

$$
\begin{aligned}
& \left|\sum_{k \in \mathscr{K}} \sum_{m \in \mathscr{H}(\mathscr{M})}{ }^{*} \mathbf{e}_{q}(c \beta(k, m))\right| \leq \\
& \left.\left.\quad \frac{1}{q} \sum_{u=1}^{q}\left|\sum_{k=1}^{q} \sum_{m \in \mathscr{H}(\mathscr{M})} \mathbf{e}_{q}(c \beta(k, m)-u k)\right| \right\rvert\, \sum_{k^{\prime} \in \mathscr{K}} \mathbf{e}_{q}\left(u k^{\prime}\right)\right) \mid \ll \\
& W^{1 / 2} q^{1 / 2-2^{145-82 / \varepsilon}} \log ^{2} p \sum_{u \in \mathbb{F}_{q}}\left|\sum_{k^{\prime} \in \mathscr{K}} \mathbf{e}_{q}\left(u k^{\prime}\right)\right| \ll W^{1 / 2} q^{3 / 2-2^{145-82 / \varepsilon} \log ^{3} p .}
\end{aligned}
$$

The above bound for the exponential sum and lemma 6 show that $\Gamma$ is a $2^{-\log ^{1 / 2} q_{-}}$ homogeneously distributed modulo $q$ provided that

$$
W \leq \frac{|\mathscr{M}|}{q^{1-2^{146-82 / \varepsilon} \log ^{6} p} .}
$$

### 4.3 Proof of theorem 1 and some comments

Now, we are ready to prove the main result.
Suppose that the attacker has obtained the following messages with their corresponding signatures,

$$
\left(m_{1}, s_{1}, r_{1}\right), \ldots,\left(m_{n}, s_{n}, r_{n}\right)
$$

Using this information, the attacker builds lattice $\mathscr{L}$ using the rows of the matrix defined in (6) and also vector $\mathbf{t}$ defined in (7). The attacker can find a closest vector
in $\mathscr{L}$ to $\mathbf{t}$ and suppose that the second component of this vector is $\gamma_{2}$. Let $\mathbf{h} \in \mathscr{L}$ be the solution found to the closest vector problem, so the norm of $\mathbf{h}-\mathbf{t}$ satisfies,

$$
\begin{equation*}
\|\mathbf{u}-\mathbf{h}\| \leq\|\mathbf{u}-\mathbf{t}\|+\|\mathbf{t}-\mathbf{h}\| \leq \sqrt{n} 2^{M-\delta+1}=2^{M-\delta+\log n+1} \tag{10}
\end{equation*}
$$

The attack success if any vector in the lattice with norm less than $2^{M-\delta+\log n+1}$ has a zero in the second coordinate. By lemma 2 , the probability of success is greater than $1-q\left(2^{-\delta+\log n+1}+\Delta\right)^{n-1}$, if $\Gamma$ is a $\Delta$-homogeneously distributed. Lemma 5 implies that it is possible to take $\Delta=q^{-d}$ and this finish the proof.

We want to mention that if the dimension is greater than 100, only approximation algorithms are practical. In those cases, it is necessary to multiply in the right hand side of equation (10) by the factor appearing in lemma4. This gives a lower bound in the probability of $1-q\left(2^{-\delta+O\left(n \log ^{2} \log n\right) / \log n}+\Delta\right)^{n-1}$.

## 5 Experimental results

We have empirically tested the performance of the attack.
In the first parameters set, the bit size of $p$ is 1024 and the bit size of $q$ is 160 . In the second set, the bit size of $p$ is 4096 and the bit size of $q$ is 250 . For the hash function, we have chosen SHA1, because it was widely used in DSA.

We note that the experimental results are better than what we expect from Theorem 2

The reason is that the lower bound, $d \geq 2^{145-82 / \varepsilon}-\log \log p / \log q$, is very pessimistic. Indeed, for $d \approx 0.5$, we have made the following calculations in Table 1 The calculation for the theoretical value of $n$ is finding the minimum value such that,

$$
1-q\left(2^{-\delta+\log n+1}+q^{-d}(\log q)^{\ell}\right)^{n-1}>0
$$

From the empirical results, we make the following conjecture.
Conjecture 1 Assuming that $\delta \geq \log ^{1 / 2} M$, then given $n$ messages with $n \geq M / \log ^{-1 / 2}(M)$, the probability of success is greater than $1-1 / n$.

Figure 1 show $M-\delta$ in the abscissa against the minimum number of signatures required to recover the ephemeral keys. For each experiment, we have selected randomly the value $k^{\prime}, k^{\prime \prime}$ and $\tilde{k}_{i}$ for $i=1, \ldots, n$ and repeated each experiment 10 times.

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| Known bits | Mean value of $n$ in simulations | Value of $n$ by Theorem 2 |
| :---: | :---: | :---: |
| 30 | 7.0 | 7 |
| 28 | 7.4 | 8 |
| 26 | 8.0 | 8 |
| 24 | 8.5 | 9 |
| 22 | 9.0 | 10 |
| 20 | 9.8 | 11 |
| 18 | 11.0 | 12 |
| 16 | 12.5 | 14 |
| 14 | 14.3 | 17 |
| 12 | 17.1 | 21 |
| 10 | 20.7 | 30 |
| 8 | 27.3 | 54 |
| 6 | 50 | - |

Table 1 Comparation of theoretical values and mean of computer simulations when employing $p$ of 1024 bits and $q$ of 160 bits. The mean value of messages needed has been simulated adding another message until the attack is successful.

## References

1. Leonard M. Adleman and Jonathan DeMarrais. A Subexponential Algorithm for Discrete Logarithms over All Finite Fields. In Advances in Cryptology - CRYPTO '93, 13th Annual International Cryptology Conference, Santa Barbara, California, USA, August 22-26, 1993, Proceedings, Lecture Notes in Computer Science, pages 147-158, 1993.
2. National Security Agency. Cryptography today.
3. Razvan Barbulescu, Pierrick Gaudry, Antoine Joux, and Emmanuel Thomé. A heuristic quasipolynomial algorithm for discrete logarithm in finite fields of small characteristic. In Advances in Cryptology-EUROCRYPT 2014-33rd Annual International Conference on the Theory and Applications of Cryptographic Techniques, Copenhagen, Denmark, May 11-15, 2014. Proceedings, pages 1-16, 2014.
4. Henri Cohen, Gerhard Frey, Roberto Avanzi, Christophe Doche, Tanja Lange, Kim Nguyen, and Frederik Vercauteren. Handbook of elliptic and hyperelliptic curve cryptography. CRC press, 2005.
5. Elke De Mulder, Michael Hutter, Mark E Marson, and Peter Pearson. Using Bleichenbacher s solution to the Hidden Number Problem to attack nonce leaks in 384-bit ECDSA. In Cryptographic Hardware and Embedded Systems-CHES 2013, pages 435-452. Springer, 2013.
6. Elke De Mulder, Michael Hutter, Mark E Marson, and Peter Pearson. Using Bleichenbachers solution to the hidden number problem to attack nonce leaks in 384-bit ecdsa: extended version. Journal of Cryptographic Engineering, 4(1):33-45, 2014.
7. Claus Diem. On the discrete logarithm problem in elliptic curves II. Algebra and Number Theory, 7(6):1281-1323, 2013.
8. Michael Drmota and Robert F. Tichy. Sequences, discrepancies, and applications. Lecture notes in mathematics. Springer, 1997.
9. Jean-Charles Faugère, Christopher Goyet, and Guénaël Renault. Attacking (EC)DSA given only an implicit hint. In Selected Areas in Cryptography, 19th International Conference, SAC 2012, Windsor, ON, Canada, August 15-16, 2012, Revised Selected Papers, pages 252-274, 2012.
10. Jean-Charles Faugère, Raphaël Marinier, and Guénaël Renault. Implicit Factoring with Shared Most Significant and Middle Bits. In Public Key Cryptography, volume 6056 of Lecture Notes in Computer Science, pages 70-87. Springer, 2010.
11. FIPS. Digital Signature Standard (DSS). National Institute of Standards and Technology (NIST), 1994.
12. FIPS. Digital Signature Standard (DSS). pub-NIST, pub-NIST:adr, 2013.
13. Steven D. Galbraith and Pierrick Gaudry. Recent progress on the elliptic curve discrete logarithm problem. Cryptology ePrint Archive, Report 2015/1022, 2015.
14. Mubaris Z. Garaev. Sums and products of sets and estimates of rational trigonometric sums in fields of prime order. Russian Mathematical Surveys, 65(4):599, 2010.
15. Daniel Genkin, Adi Shamir, and Eran Tromer. RSA key extraction via low-bandwidth acoustic cryptanalysis. In International Cryptology Conference, pages 444-461. Springer, 2014.
16. Faruk Göloglu, Robert Granger, Gary McGuire, and Jens Zumbrägel. On the function field sieve and the impact of higher splitting probabilities - application to discrete logarithms in and. In Advances in Cryptology - CRYPTO 2013-33rd Annual Cryptology Conference, Santa Barbara, CA, USA, August 18-22, 2013. Proceedings, Part II, pages 109-128, 2013.
17. Faruk Göloglu, Robert Granger, Gary McGuire, and Jens Zumbrägel. Solving a 6120 -bit DLP on a desktop computer. In Selected Areas in Cryptography - SAC 2013-20th International Conference, Burnaby, BC, Canada, August 14-16, 2013, Revised Selected Papers, pages 136-152, 2013.
18. Nick Howgrave-Graham and Nigel P. Smart. Lattice Attacks on Digital Signature Schemes. Des. Codes Cryptography, 23(3):283-290, 2001.
19. Antoine Joux. Faster index calculus for the medium prime case application to 1175 -bit and 1425 -bit finite fields. In Advances in Cryptology - EUROCRYPT 2013, 32nd Annual International Conference on the Theory and Applications of Cryptographic Techniques, Athens, Greece, May 26-30, 2013. Proceedings, pages 177-193, 2013.
20. Ravindran Kannan. Algorithmic geometry of numbers. Annual Review of Computer Science, 2(1):231-267, 1987.
21. Arjen Lenstra, Hendrik Lenstra, and László Lovász. Factoring polynomials with rational coefficients. In Mathematische Annalen, volume 261, no 4, pages 515-534, 1982.
22. Alexander May and Maike Ritzenhofen. Implicit factoring: On polynomial time factoring given only an implicit hint. In Public Key Cryptography, volume 5443 of Lecture Notes in Computer Science, pages 1-14. Springer, 2009.
23. Carlos Moreno and Oscar Moreno. Exponential sums and Goppa codes. Proceedings of the American Mathematical Monthly, 111:523-531, 1991.
24. Phong Q. Nguyen and Igor Shparlinski. The insecurity of the digital signature algorithm with partially known nonces. Journal of Cryptology, 15(3):151-176, 2002.
25. Phong Q. Nguyen and Igor Shparlinski. The Insecurity of the Digital Signature Algorithm with Partially Known Nonces. Journal of Cryptology, 15:151-176, 2002.
26. Phong Q. Nguyen and Igor Shparlinski. The Insecurity of the Elliptic Curve Digital Signature Algorithm with Partially Known Nonces. Designs, Codes and Cryptography, 30(2):201-217, 2003.
27. Phong Q. Nguyen and Brigitte Vallée. The LLL Algorithm. Springer, 2010.
28. Harald Niederreiter. Random Number Generation and Quasi-Monte Carlo Methods. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 1987.
29. Ronald Linn Rivest and Adi Shamir. Efficient factoring based on partial information. In Proc. of a workshop on the theory and application of cryptographic techniques on Advances in cryptologyEUROCRYPT '85, pages 31-34, New York, NY, USA, 1986. Springer-Verlag New York, Inc.
30. Santanu Sarkar and Subhamoy Maitra. Further Results on Implicit Factoring in Polynomial Time. Advances in Mathematics of Communications, 3(2):205-217, 2009.
31. Claus-Peter Schnorr. Efficient Identification and Signatures for Smart Cards. In Proceedings of the 9th Annual International Cryptology Conference on Advances in Cryptology, CRYPTO '89, pages 239-252. Springer-Verlag, 1990.
32. Andrew Sutherland. Structure computation and discrete logarithms in finite abelian p-groups. Mathematics of Computation, 80(273):477-500, 2011.
33. Ivan M. Vinogradov. Elements of Number Theory. Dover Phoenix Editions. Dover, 2003.


Fig. 1 Experimental results of the algorithm. The first plot employs the smaller parameter set, with $p$ of 1024 bits and $q$ of 160 bits while the second plot employs the parameter set with $p$ of 4096 bits and $q$ of 250 bits. The red dots represent the average of the number of message signatures needed to recover the ephemeral keys. The grey areas contain the 90 percentile of the number of message signatures needed


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[^1]:    ${ }^{1}$ We notice that $\beta_{i}$ are elements of the set $\bar{\Gamma}$ plus $s_{1}^{-1} m_{1}$ and then reduced modulo $q$. But this does not change the value of the discrepancy.

