# Optimal optical orthogonal signature pattern codes with weight three and cross-correlation constraint one <sup>1</sup>

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Abstract: Optical orthogonal signature pattern codes (OOSPCs) have attracted wide attention as signature patterns of spatial optical code division multiple access networks. In this paper, an improved upper bound on the size of an  $(m, n, 3, \lambda_a, 1)$ -OOSPC with  $\lambda_a = 2, 3$  is established. The exact number of codewords of an optimal  $(m, n, 3, \lambda_a, 1)$ -OOSPC is determined for any positive integers  $m, n \equiv 2 \pmod{4}$  and  $\lambda_a \in \{2, 3\}$ .

**Keywords**: optical orthogonal signature pattern code; optical orthogonal code; OCDMA **Mathematics Subject Classification:** 05B40; 94B25

# 1 Introduction

An optical orthogonal signature pattern code is a family of (0, 1)-matrices with good auto- and cross-correlation. Its study has been motivated by an application in an optical code division multiple access (OCDMA) network for image transmission, called a spatial OCDMA network. Compared with the traditional OCDMA, the spatial OCDMA provides higher throughput (cf. [18-20, 28]).

Denote by  $\mathbb{Z}_v$  the additive group of integers modulo v. Let  $m, n, k, \lambda_a$  and  $\lambda_c$  be positive integers. An  $(m, n, k, \lambda_a, \lambda_c)$  optical orthogonal signature pattern code (briefly,  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC) is a family C of  $m \times n$  (0, 1)-matrices of Hamming weight k satisfying the following properties:

- (1) the auto-correlation property:  $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x_{i,j} x_{i \oplus s, j \oplus t} \leq \lambda_a \text{ for any } (x_{ij}) \in \mathcal{C} \text{ and any } (s,t) \in \mathbb{Z}_m \times \mathbb{Z}_n \setminus \{(0,0)\};$
- (2) the cross-correlation property:  $\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} x_{i,j} y_{i \oplus s, j \oplus t} \leq \lambda_c \text{ for any distinct } (x_{ij}), (y_{ij}) \in \mathcal{C}$ and any  $(s,t) \in \mathbb{Z}_m \times \mathbb{Z}_n$ ,

where the additions  $\oplus$  and  $\widehat{\oplus}$  are, respectively, reduced modulo m and n. When  $\lambda_a = \lambda_c = \lambda$ , the notation  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC is briefly written as  $(m, n, k, \lambda)$ -OOSPC.

The number of codewords in an OOSPC is called the *size* of the OOSPC. For given positive integers m, n, k,  $\lambda_a$  and  $\lambda_c$ , denote by  $\Theta(m, n, k, \lambda_a, \lambda_c)$  the largest possible size among all  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPCs. An  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC with size  $\Theta(m, n, k, \lambda_a, \lambda_c)$  is said to be optimal (or maximum).

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When  $\lambda_a = \lambda_c = \lambda$ ,  $\Theta(m, n, k, \lambda_a, \lambda_c)$  is simply written as  $\Theta(m, n, k, \lambda)$ . Based on the Johnson bound [17] for constant weight codes, an upper bound on  $\Theta(m, n, k, \lambda)$  was given below

$$\Theta(m, n, k, \lambda) \le J(mn, k, \lambda),$$

where

$$J(mn,k,\lambda) = \left\lfloor \frac{1}{k} \left\lfloor \frac{mn-1}{k-1} \left\lfloor \frac{mn-2}{k-2} \left\lfloor \cdots \left\lfloor \frac{mn-\lambda}{k-\lambda} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor \right\rfloor.$$

When  $\lambda_a > \lambda_c$ ,  $\Theta(m, n, k, \lambda_a, \lambda_c)$  is upper bounded in [28] by

$$\left\lfloor \frac{\lambda_a(mn-1)(mn-2)\cdots(mn-\lambda_c)}{k(k-1)(k-2)\cdots(k-\lambda_c)} \right\rfloor.$$
(1.1)

When *m* and *n* are coprime, it has been shown in [28] that an  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC is equivalent to a 1-dimensional  $(mn, k, \lambda_a, \lambda_c)$ -optical orthogonal code (OOC). See [1–4, 7, 10, 13, 14, 29] and the references therein for more details on OOCs.

When m and n are not coprime, various OOSPCs have been constructed via algebraic and combinatorial methods for the case of  $\lambda_a = \lambda_c$  (see [5, 6, 15, 23–26, 28]). We only quote the following result for later use.

**Theorem 1.1** [24]

$$\Theta(m,n,3,1) = \begin{cases} J(mn,3,1) - 1, & \text{if } mn \equiv 14,20 \pmod{24}, \\ & \text{or } \text{if } mn \equiv 8,16 \pmod{24} \text{ and } \gcd(m,n,4) = 2, \\ & \text{or } \text{if } mn \equiv 2 \pmod{6} \text{ and } \gcd(m,n,4) = 4; \\ J(mn,3,1), & \text{otherwise.} \end{cases}$$

On the other hand, for the case of  $\lambda_a \neq \lambda_c$ , very little has been done on  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPCs with maximum size. Compared with (1.1), an improved upper bound on  $\Theta(m, n, 3, 2, 1)$  was given by Sawa and Kageyama [26]. That is

$$\Theta(m, n, 3, 2, 1) \le \begin{cases} \frac{mn}{4}, & \text{if } mn \equiv 0 \pmod{4}, \\ \left\lfloor \frac{mn-1}{4} \right\rfloor, & \text{otherwise.} \end{cases}$$
(1.2)

And they proved the following theorem.

### **Theorem 1.2** [26]

 $\Theta(m,n,3,2,1) = \begin{cases} \frac{mn-1}{4}, & \text{if } m = n \equiv 1 \pmod{4} \text{ is a prime and } 2 \text{ is a primitive root in } \mathbb{Z}_m; \\ \frac{mn-2}{4}, & \text{if } mn \equiv 2 \pmod{4}. \end{cases}$ 

In Section 2, we shall give an equivalent combinatorial description of  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPCs by using set-theoretic notation. Section 3 is devoted to improving Sawa and Kageyama's bound (1.2), especially for the case of  $mn \equiv 0 \pmod{4}$ . Throughout this paper, let  $\xi$  denote the number of subgroups of order 3 in  $\mathbb{Z}_m \times \mathbb{Z}_n$ , i.e.,

$$\xi = \begin{cases} 0, & \text{if } 3 \nmid mn; \\ 1, & \text{if } 3 \mid mn \text{ and } \gcd(m, n, 3) = 1; \\ 4, & \text{if } \gcd(m, n, 3) = 3. \end{cases}$$

Let

$$\omega = \begin{cases} 0, & \text{if } \lambda_a = 2; \\ \xi, & \text{if } \lambda_a = 3. \end{cases}$$

We are to prove the following theorem.

**Theorem 1.3** Let  $\lambda_a \in \{2,3\}$ . Then  $\Theta(m, n, 3, \lambda_a, 1) \leq$ 

$$\begin{bmatrix} \frac{mn+2\omega}{4} \end{bmatrix}, \qquad mn \equiv 1, 2, 3 \pmod{4}; \\ \begin{bmatrix} \frac{7mn+16\omega}{32} \end{bmatrix}, \qquad mn \equiv 0 \pmod{8} \text{ and } \gcd(m, n, 2) = 1; \\ \begin{bmatrix} \frac{7mn+4+16\omega}{32} \end{bmatrix}, \qquad mn \equiv 4 \pmod{8}, \gcd(m, n, 2) = 1, \text{ and } (m, n) \notin \{(12, 3), (3, 12)\}; \\ 7 + \frac{\omega}{2}, \qquad (m, n) \in \{(12, 3), (3, 12)\}; \\ \begin{bmatrix} \frac{5mn+4+8\omega}{24} \end{bmatrix}, \qquad mn \equiv 4 \pmod{8} \text{ and } \gcd(m, n, 2) = 2; \\ \begin{bmatrix} \frac{13mn+40+32\omega}{64} \end{bmatrix}, \qquad mn \equiv 8 \pmod{16}, \gcd(m, n, 2) = 2, (m, n) \notin \{(2, 4), (4, 2)\}, \text{ and } (m, n) \notin \{(2, 12), (4, 6), (6, 4), (6, 12), (12, 2), (12, 6)\} \text{ when } \lambda_a = 3; \\ 1, \qquad (m, n) \notin \{(2, 12), (4, 6), (6, 4), (6, 12), (12, 2), (12, 6)\} \text{ when } \lambda_a = 3; \\ \begin{bmatrix} \frac{13mn+32+32\omega}{64} \end{bmatrix}, \qquad mn \equiv 0 \pmod{32} \text{ and } \gcd(m, n, 2) = 2, \text{ or } mn \equiv 16 \pmod{32} \text{ and } \gcd(m, n, 4) = 2, \\ \exp t \text{ for } mn \equiv 32 \pmod{64} \text{ and } \gcd(m, n, 8) = 4 \text{ when } \lambda_a = 2; \\ \\ \begin{bmatrix} \frac{13mn-32}{64}, \qquad mn \equiv 16 \pmod{32} \text{ and } \gcd(m, n, 4) = 4, \\ \exp t \text{ for } mn \equiv 144 \pmod{192} \text{ and } \gcd(m, n, 4) = 4 \text{ when } \lambda_a = 2; \\ \\ \end{bmatrix}$$

In Section 4, we shall establish three recursive constructions for  $(m, n, 3, \lambda_a, 1)$ -OOSPCs. Especially, a very efficient doubling construction is presented in Construction 4.6 to facilitate determining the exact value of  $\Theta(m, n, 3, \lambda_a, 1)$  for  $m, n \equiv 2 \pmod{4}$  and  $\lambda_a = 2, 3$ . We are to prove the following theorem in Section 5.

**Theorem 1.4** Let  $\lambda_a \in \{2,3\}$ . For any  $m, n \equiv 2 \pmod{4}$ ,

$$\Theta(m,n,3,\lambda_a,1) = \left\lfloor \frac{5mn+4+8\omega}{24} \right\rfloor.$$

Finally, in Section 6, it is conjectured that when  $mn \equiv 0 \pmod{4}$ , our bound for  $\Theta(m, n, 3, \lambda_a, 1)$  with  $\lambda_a \in \{2, 3\}$  shown in Theorem 1.3 is tight.

# 2 Preliminaries

### 2.1 Set-theoretic descriptions

A convenient way of viewing optical orthogonal signature pattern codes is from a set-theoretic perspective.

Let  $\mathcal{C}$  be an  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC. For each (0, 1)-matrix  $M = (a_{ij}) \in \mathcal{C}$ , whose rows are indexed by  $\mathbb{Z}_m$  and columns are indexed by  $\mathbb{Z}_n$ , define  $X_M = \{(i, j) \in \mathbb{Z}_m \times \mathbb{Z}_n : a_{ij} = 1\}$ . Then,  $\mathcal{F} = \{X_M : M \in \mathcal{C}\}$  is a set-theoretic representation of  $\mathcal{C}$ . Conversely, let  $\mathcal{F}$  be a set of k-subsets of  $\mathbb{Z}_m \times \mathbb{Z}_n$ . Then  $\mathcal{F}$  is an  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC if the following properties are satisfied:

- (1') the auto-correlation property:  $|X \cap (X + (s,t))| \leq \lambda_a$  for any  $X \in \mathcal{F}$  and any  $(s,t) \in \mathbb{Z}_m \times \mathbb{Z}_n \setminus \{(0,0)\};$
- (2') the cross-correlation property:  $|X \cap (Y + (s, t))| \leq \lambda_c$  for any distinct  $X, Y \in \mathcal{F}$  and any  $(s,t) \in \mathbb{Z}_m \times \mathbb{Z}_n$ ,

where the addition "+" performs in  $\mathbb{Z}_m \times \mathbb{Z}_n$ . Throughout this paper, we shall use the settheoretic notation to list codewords of a given OOSPC.

For a given set  $\mathcal{F}$  of k-subsets of  $\mathbb{Z}_m \times \mathbb{Z}_n$ , it is not convenient to check whether it satisfies the auto- and cross-correlation property according to Conditions (1') and (2'). However, when  $\lambda_c = 1$ , a more efficient description can be given by using the difference method. Let  $X \in \mathcal{F}$ . Define the list of *differences* of X by

$$\Delta X = \{ (x_i, y_i) - (x_j, y_j) : 0 \le i, j \le k - 1, i \ne j \}$$

as a multiset, and define the support of  $\Delta X$ , denoted by  $\operatorname{supp}(\Delta X)$ , as the set of underlying elements in  $\Delta X$ . Let  $\lambda(X)$  denote the maximum multiplicity of elements in the multiset  $\Delta X$ . Then  $\mathcal{F}$  constitutes an  $(m, n, k, \lambda_a, 1)$ -OOSPC if the following properties are satisfied:

(1") the auto-correlation property:  $\lambda(X) \leq \lambda_a$  for any  $X \in \mathcal{F}$ ;

(2") the cross-correlation property:  $\operatorname{supp}(\Delta X) \cap \operatorname{supp}(\Delta Y) = \emptyset$  for any distinct  $X, Y \in \mathcal{F}$ .

**Example 2.1** We here give an example of a  $(6, 6, 3, \lambda_a, 1)$ -OOSPC with  $\lambda_a \in \{2, 3\}$  defined on  $\mathbb{Z}_6 \times \mathbb{Z}_6$  as follows:

$$\begin{split} \lambda_a &= 2 \colon \begin{array}{ll} \{(0,0),(0,3),(3,0)\}, & \{(0,0),(0,1),(0,2)\}, & \{(0,0),(1,0),(2,0)\}, \\ & \{(0,0),(1,1),(2,2)\}, & \{(0,0),(1,2),(2,1)\}, & \{(0,0),(1,3),(3,2)\}, \\ & \{(0,0),(1,4),(3,1)\}; \end{array} \\ \lambda_a &= 3 \colon \begin{array}{ll} \{(0,0),(0,2),(0,4)\}, & \{(0,0),(2,0),(4,0)\}, & \{(0,0),(2,2),(4,4)\}, \\ & \{(0,0),(2,4),(4,2)\}, & \{(0,0),(0,1),(1,0)\}, & \{(0,0),(1,1),(2,3)\}, \\ & \{(0,0),(1,3),(3,2)\}, & \{(0,0),(1,4),(3,5)\}, & \{(0,0),(0,3),(3,0)\}. \end{split}$$

### 2.2 Notation and basic propositions

Throughout this paper, let  $A = \mathbb{Z}_m \times \mathbb{Z}_n$ . For each  $(x, y) \in A \setminus \{(0, 0)\}$ , denote by  $\pm(x, y)$  the two elements (x, y) and (-x, -y) in A.

**Proposition 2.2** All possible subgroups of order 3 in A are

 $\{ (0,0), (0,\frac{n}{3}), (0,\frac{2n}{3}) \}, \qquad \{ (0,0), (\frac{m}{3},0), (\frac{2m}{3},0) \}, \\ \{ (0,0), (\frac{m}{3},\frac{n}{3}), (\frac{2m}{3},\frac{2n}{3}) \}, \qquad \{ (0,0), (\frac{m}{3},\frac{2n}{3}), (\frac{2m}{3},\frac{n}{3}) \}.$ 

**Proposition 2.3** All possible cyclic subgroups of order 4 in A are

 $\begin{array}{ll} \{(0,0),\pm(\frac{m}{4},0),(\frac{m}{2},0)\}, & \{(0,0),\pm(\frac{m}{4},\frac{n}{4}),(\frac{m}{2},\frac{n}{2})\}, & \{(0,0),\pm(\frac{m}{4},\frac{n}{2}),(\frac{m}{2},0)\}, \\ \{(0,0),\pm(\frac{m}{4},\frac{3n}{4}),(\frac{m}{2},\frac{n}{2})\}, & \{(0,0),\pm(0,\frac{n}{4}),(0,\frac{n}{2})\}, & \{(0,0),\pm(\frac{m}{2},\frac{n}{4}),(0,\frac{n}{2})\}. \end{array}$ 

**Proposition 2.4** The unique possible subgroup of order 4 isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  in A is  $\{(0,0), (\frac{m}{2}, 0), (0, \frac{n}{2}), (\frac{m}{2}, \frac{n}{2})\}.$ 

Let (G, +) be an abelian group with the identity 0. Let  $X \subseteq G$ . The *G*-orbit of X is the set  $\operatorname{Orb}_G(X) = \{X + g : g \in G\}$ , where  $X + g = \{x + g : x \in X\}$ . For any positive integer *i*, let

$$\Omega_G(i) = \{ \alpha \in G : i\alpha = 0 \}$$

**Proposition 2.5**  $\Omega_A(2) = \{(0,0), (0,\frac{n}{2}), (\frac{m}{2},0), (\frac{m}{2},\frac{n}{2})\}.$ 

**Remark 2.6** In Proposition 2.5, the notation  $\Omega_A(2)$  should be understood as follows

$$\Omega_A(2) = \begin{cases} \{(0,0)\}, & \text{if } m, n \equiv 1 \pmod{2}; \\ \{(0,0), \left(\frac{m}{2}, 0\right)\}, & \text{if } m \equiv 0 \pmod{2} \text{ and } n \equiv 1 \pmod{2}; \\ \{(0,0), \left(0, \frac{n}{2}\right)\}, & \text{if } m \equiv 1 \pmod{2} \text{ and } n \equiv 1 \pmod{2}; \\ \{(0,0), \left(0, \frac{n}{2}\right), \left(\frac{m}{2}, 0\right), \left(\frac{m}{2}, \frac{n}{2}\right)\}, & \text{if } m, n \equiv 0 \pmod{2}. \end{cases}$$

In what follows, we always use a similar method to denote sets. The reader can judge it according to the context.

Proposition 2.7 (1) 
$$\Omega_A(3) = \{(0,0), \pm (0,\frac{n}{3}), \pm (\frac{m}{3},0), \pm (\frac{m}{3},\frac{n}{3}), \pm (\frac{m}{3},\frac{2n}{3})\}.$$
  
(2)  $\Omega_A(4) = \Omega_A(2) \cup \{\pm (0,\frac{n}{4}), \pm (\frac{m}{4},0), \pm (\frac{m}{4},\frac{n}{4}), \pm (\frac{m}{4},\frac{n}{2}), \pm (\frac{m}{4},\frac{3n}{4}), \pm (\frac{m}{2},\frac{n}{4})\}$ 

# **3** Upper bound on the size of $(m, n, 3, \lambda_a, 1)$ -OOSPCs

In this section, we shall estimate the upper bound of  $\Theta(m, n, 3, \lambda_a, 1)$  with  $\lambda_a \in \{2, 3\}$  for any positive integers m and n. Without loss of generality assume that each codeword in an OOSPC contains the element (0, 0). Let

$$\mathcal{T}_1 = \{ \operatorname{Orb}_A(\{(0,0), \alpha, 2\alpha\}) : \alpha \in A \setminus (\Omega_A(3) \cup \Omega_A(4)) \}, \text{ and}$$
$$\mathcal{T}_2 = \{ \operatorname{Orb}_A(\{(0,0), \alpha, \beta\}) : \alpha \in A \setminus \Omega_A(4) \text{ and } \beta \in \Omega_A(2) \setminus \{(0,0)\} \}$$

The following two lemmas play key roles to derive our bound.

**Lemma 3.1** [26, Lemma 4.1] Let  $X = \{(0,0), \alpha, \beta\}$  be a 3-subset of A. Then

$$|\operatorname{supp}(\Delta X)| = \begin{cases} 2, & \text{if } < \alpha, \beta > \cong \mathbb{Z}_3; \\ 3, & \text{if } < \alpha, \beta > \cong \mathbb{Z}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2; \\ 4, & \text{if } \operatorname{Orb}_A(X) \in \mathcal{T}_1; \\ 5, & \text{if } \operatorname{Orb}_A(X) \in \mathcal{T}_2; \\ 6, & \text{otherwise}, \end{cases}$$

where  $\langle \alpha, \beta \rangle$  denotes the additive subgroup of A generated by  $\alpha$  and  $\beta$ .

**Lemma 3.2** [26, Lemma 4.2] Let X be a 3-subset of A. Then

$$\lambda(X) = \max_{(s,t)\in A\setminus\{(0,0)\}} |X \cap (X + (s,t))| = \begin{cases} 3, \text{ if } |\operatorname{supp}(\Delta X)| = 2; \\ 2, \text{ if } |\operatorname{supp}(\Delta X)| = 3, 4, 5; \\ 1, \text{ if } |\operatorname{supp}(\Delta X)| = 6. \end{cases}$$

### 3.1 General upper bound

Let  $\lambda_a \in \{2,3\}$ . For any codeword X of an  $(m, n, 3, \lambda_a, 1)$ -OOSPC  $\mathcal{F}$ , if  $|\operatorname{supp}(\Delta X)| = i$ , then X is said to be of *Type i*. By Lemma 3.1,  $i \in \{2, 3, 4, 5, 6\}$ . Let  $N_i$  denote the number of codewords in  $\mathcal{F}$  of Type *i*. The cross-correlation property (2'') implies that  $\Delta \mathcal{F} = \bigcup_{X \in \mathcal{F}} \operatorname{supp}(\Delta X)$  covers each nonzero element of A at most once. Thus we have

$$2N_2 + 3N_3 + 4N_4 + 5N_5 + 6N_6 \le mn - 1. \tag{3.3}$$

Lemma 3.3

$$N_2 \le \omega. \tag{3.4}$$

**Proof** Let  $X = \{(0,0), \alpha, \beta\}$  be a codeword satisfying  $|\operatorname{supp}(\Delta X)| = 2$ . By Lemma 3.2,  $\lambda(X) = 3$ . Note that the auto-correlation property (1'') requires  $\lambda(X) \leq \lambda_a$  for any  $X \in \mathcal{F}$ . So if  $\lambda_a = 2$ , then  $N_2 = 0$ . If  $\lambda_a = 3$ , by Lemma 3.1,  $\langle \alpha, \beta \rangle$  forms an additive subgroup of order 3 in A. Since all possible subgroups of order 3 in A are  $\{(0,0), (0, \frac{n}{3}), (0, \frac{2n}{3})\}, \{(0,0), (\frac{m}{3}, 0), (\frac{2m}{3}, 0)\}, \{(0,0), (\frac{m}{3}, \frac{n}{3}), (\frac{2m}{3}, \frac{2n}{3})\}$  and  $\{(0,0), (\frac{m}{3}, \frac{2n}{3}), (\frac{2m}{3}, \frac{n}{3})\}$ , the value of  $N_2$  depends on whether m and n could be divided by 3.

For each codeword  $X = \{(0,0), \alpha, \beta\}$  of Type 3, if  $\langle \alpha, \beta \rangle \cong \mathbb{Z}_4$ , then by Proposition 2.3, w.l.o.g., X is one of the following forms:

$$\{ (0,0), (\frac{m}{4}, 0), (\frac{m}{2}, 0) \}, \quad \{ (0,0), (\frac{m}{4}, \frac{n}{4}), (\frac{m}{2}, \frac{n}{2}) \}, \quad \{ (0,0), (\frac{m}{4}, \frac{n}{2}), (\frac{m}{2}, 0) \}; \\ \{ (0,0), (\frac{m}{4}, \frac{3n}{4}), (\frac{m}{2}, \frac{n}{2}) \}, \quad \{ (0,0), (0, \frac{n}{4}), (0, \frac{n}{2}) \}, \quad \{ (0,0), (\frac{m}{4}, \frac{n}{2}), (0, \frac{n}{2}) \}.$$

$$(3.5)$$

If  $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , then by Proposition 2.4, w.l.o.g., X is of the form

$$\{(0,0), (\frac{m}{2}, 0), (0, \frac{n}{2})\}.$$
(3.6)

Therefore, all codewords of Type 3, which are in the form of  $\{(0,0), \alpha = (a,b), \beta\}$ , can be divided into the following three types:

**Type 3.1**:  $\alpha, \beta \in \Omega_A(2) \setminus \{(0,0)\};$ 

**Type 3.2**:  $\alpha \in \Omega_A(4) \setminus \{(0,0)\}$  and  $a \in \Omega_{\mathbb{Z}_m}(4) \setminus \Omega_{\mathbb{Z}_m}(2);$ 

**Type 3.3**:  $\alpha \in \Omega_A(4) \setminus \{(0,0)\}, a \in \Omega_{\mathbb{Z}_m}(2) \text{ and } b \in \Omega_{\mathbb{Z}_n}(4) \setminus \Omega_{\mathbb{Z}_n}(2).$ 

Let  $N_3^{(1)}$ ,  $N_3^{(2)}$  and  $N_3^{(3)}$  denote the number of codewords in  $\mathcal{F}$  of Types 3.1, 3.2 and 3.3, respectively. Then,

$$N_3 = N_3^{(1)} + N_3^{(2)} + N_3^{(3)}.$$

**Remark 3.4** By (3.5) and (3.6), one can check the following facts.

- (1) For any codeword X of Type 3.1,  $supp(\Delta X)$  is of the form  $\{(\frac{m}{2}, 0), (0, \frac{n}{2}), (\frac{m}{2}, \frac{n}{2})\}$ .
- (2) For any codeword X of Type 3.2,  $supp(\Delta X)$  is one of the forms:  $\{\pm(\frac{m}{4},0),(\frac{m}{2},0)\}, \{\pm(\frac{m}{4},\frac{m}{4}),(\frac{m}{2},\frac{n}{2})\}, \{\pm(\frac{m}{4},\frac{n}{2}),(\frac{m}{2},0)\}$  and  $\{\pm(\frac{m}{4},\frac{3n}{4}),(\frac{m}{2},\frac{n}{2})\}.$
- (3) For any codeword X of Type 3.3,  $supp(\Delta X)$  is one of the forms:  $\{\pm(0, \frac{n}{4}), (0, \frac{n}{2})\}$  and  $\{\pm(\frac{m}{2}, \frac{n}{4}), (0, \frac{n}{2})\}$ .

Lemma 3.5

$$N_3^{(1)} \le \begin{cases} 1, & \text{if } \gcd(m, n, 2) = 2; \\ 0, & \text{if } \gcd(m, n, 2) = 1. \end{cases}$$
(3.7)

**Proof** For each codeword X of Type 3.1,  $|\operatorname{supp}(\Delta X) \cap \Omega_A(2)| = 3$  by Remark 3.4(1). By Remark 2.6, we have

$$3N_3^{(1)} \le |\Omega_A(2) \setminus \{(0,0)\}| = \begin{cases} 3, & \text{if } \gcd(m,n,2) = 2; \\ 0, & \text{if } \gcd(m,n,2) = 1. \end{cases}$$

Lemma 3.6 Let

$$\varepsilon = \begin{cases} 2, & \text{if } m, n \equiv 0 \pmod{4}; \\ 1, & \text{if } m \equiv 0 \pmod{4} \text{ and } n \not\equiv 0 \pmod{4}; \\ 0, & \text{if } m \not\equiv 0 \pmod{4}. \end{cases}$$

Then

(1) 
$$N_3^{(2)} \le \varepsilon;$$
 (3.8)  
(2)  $N_3^{(3)} \le \begin{cases} 1, & \text{if } n \equiv 0 \pmod{4}; \\ 0, & \text{if } n \not\equiv 0 \pmod{4}. \end{cases}$  (3.9)

**Proof** Each codeword of Type 3 is one of the forms shown in (3.5) and (3.6).

(1) When  $m \not\equiv 0 \pmod{4}$ , there is no codeword of Type 3.2, and so  $N_3^{(2)} = 0$ . When  $m \equiv 0 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , there is at most one codeword of Type 3.2, which is  $\{(0,0), (\frac{m}{4}, 0), (\frac{m}{2}, 0)\}$ , and so  $N_3^{(2)} \leq 1$ . When  $m \equiv 0 \pmod{4}$  and  $n \equiv 2 \pmod{4}$ , there are at most two codewords of Type 3.2, which are  $\{(0,0), (\frac{m}{4}, 0), (\frac{m}{2}, 0)\}$  and  $\{(0,0), (\frac{m}{4}, \frac{n}{2}), (\frac{m}{2}, 0)\}$ . Here  $(\frac{m}{2}, 0)$  is shared as a difference. Hence,  $N_3^{(2)} \leq 1$ . When  $m, n \equiv 0 \pmod{4}$ , there are at most four codewords of Type 3.2, which are  $\{(0,0), (\frac{m}{4}, 0), (\frac{m}{2}, 0)\}$ ,  $\{(0,0), (\frac{m}{4}, \frac{n}{4}), (\frac{m}{2}, \frac{n}{2})\}$ ,  $\{(0,0), (\frac{m}{4}, \frac{n}{2}), (\frac{m}{2}, 0)\}$  and  $\{(0,0), (\frac{m}{4}, \frac{3n}{4}), (\frac{m}{2}, \frac{n}{2})\}$ . Since  $\lambda_c = 1$ , it is readily checked that  $N_3^{(2)} \leq 2$ .

(2) When  $n \neq 0 \pmod{4}$ , there is no codeword of Type 3.3, and so  $N_3^{(3)} = 0$ . When  $n \equiv 0 \pmod{4}$ , there are at most two codewords of Type 3.3, which are  $\{(0,0), (0, \frac{n}{4}), (0, \frac{n}{2})\}$  and  $\{(0,0), (\frac{m}{2}, \frac{n}{4}), (0, \frac{n}{2})\}$ . Here  $(0, \frac{n}{2})$  is shared as a difference. Hence,  $N_3^{(3)} \leq 1$ .

**Lemma 3.7**  $\Theta(m, n, 3, \lambda_a, 1) \leq \lfloor \frac{mn+2\omega}{4} \rfloor$  for any  $m \equiv 1 \pmod{2}$  and  $\lambda_a \in \{2, 3\}$ .

**Proof** For  $m \equiv 1 \pmod{2}$ , by (3.4), (3.7), (3.8) and (3.9), we have  $N_2 \leq \omega$ ,  $N_3^{(1)} = N_3^{(2)} = 0$  and  $N_3^{(3)} \leq 1$ . Then by (3.3),  $4(N_2 + N_3 + N_4 + N_5 + N_6) \leq mn - 1 + 2N_2 + N_3 - N_5 - 2N_6 \leq mn + 2\omega$ . Thus  $\Theta(m, n, 3, \lambda_a, 1) \leq \lfloor \frac{mn + 2\omega}{4} \rfloor$ .

Lemma 3.8 Let

$$\rho = \begin{cases} 3, & \text{if } \gcd(m, n, 2) = 2; \\ 1, & \text{if } 2 \mid mn \text{ and } \gcd(m, n, 2) = 1; \\ 0, & \text{if } 2 \nmid mn. \end{cases}$$

Then

$$3N_3^{(1)} + N_3^{(2)} + N_3^{(3)} + N_5 \le \rho.$$
(3.10)

**Proof** By Proposition 2.5,  $\Omega_A(2) = \{(0,0), (0, \frac{n}{2}), (\frac{m}{2}, 0), (\frac{m}{2}, \frac{n}{2})\}$ . If X is a codeword of Type 3.1, then  $|\operatorname{supp}(\Delta X) \cap \Omega_A(2)| = 3$  by Remark 3.4(1). If X is a codeword of Types 3.2 or 3.3 or Type 5, then  $|\operatorname{supp}(\Delta X) \cap \Omega_A(2)| = 1$  by Remark 3.4(2)(3) and Lemma 3.1. Hence,  $3N_3^{(1)} + N_3^{(2)} + N_3^{(3)} + N_5 \leq |\Omega_A(2) \setminus \{(0,0)\}| = \rho$ .

Without loss of generality, each codeword of Type 4 is of the form  $\{(0,0), (a,b), (2a,2b)\}$ , where  $(a,b) \in A \setminus (\Omega_A(3) \cup \Omega_A(4))$ . We divide the codewords of Type 4 into the following two types according to the parity of a:

**Type 4.1**:  $a \equiv 1 \pmod{2}$ ;

Type 4.2:  $a \equiv 0 \pmod{2}$ .

Let  $N_4^{(1)}$  and  $N_4^{(2)}$  denote the number of codewords in  $\mathcal{F}$  of Types 4.1 and 4.2, respectively. Then,

$$N_4 = N_4^{(1)} + N_4^{(2)}.$$

Throughout this paper, we always set

 $\begin{array}{ll} A_{e\cdot} = \left\{ (x,y) \in A: \ x \equiv 0 \pmod{2} \right\}, & A_{eo} = \left\{ (x,y) \in A: \ x \equiv 0 \pmod{2}, \ y \equiv 1 \pmod{2} \right\}, \\ A_{s\cdot} = \left\{ (x,y) \in A: \ x \equiv 2 \pmod{4} \right\}, & A_{oe} = \left\{ (x,y) \in A: \ x \equiv 1 \pmod{2}, \ y \equiv 0 \pmod{2} \right\}, \\ A_{d\cdot} = \left\{ (x,y) \in A: \ x \equiv 0 \pmod{4} \right\}, & A_{se} = \left\{ (x,y) \in A: \ x \equiv 2 \pmod{4}, \ y \equiv 0 \pmod{2} \right\}, \\ A_{\cdot o} = \left\{ (x,y) \in A: \ y \equiv 1 \pmod{2} \right\}, & A_{de} = \left\{ (x,y) \in A: \ x \equiv 0 \pmod{4}, \ y \equiv 0 \pmod{2} \right\}, \\ A_{ee} = \left\{ (x,y) \in A: \ x,y \equiv 0 \pmod{2} \right\}, & A_{ds} = \left\{ (x,y) \in A: \ x \equiv 0 \pmod{4}, \ y \equiv 2 \pmod{2} \right\}, \\ A_{oo} = \left\{ (x,y) \in A: \ x,y \equiv 1 \pmod{2} \right\}, & A_{ds} = \left\{ (x,y) \in A: \ x \equiv 0 \pmod{4}, \ y \equiv 2 \pmod{4} \right\}, \\ A_{oo} = \left\{ (x,y) \in A: \ x,y \equiv 1 \pmod{2} \right\}. \end{array}$ 

### Lemma 3.9

$$2N_2 + 2N_4 \leq \begin{cases} \frac{mn}{4} - 1, & \text{if } m, n \equiv 2 \pmod{4}; \\ \frac{mn}{4} - 2, & \text{if } m \equiv 2 \pmod{4} \text{ and } n \equiv 0 \pmod{4}, \\ & \text{or } m \equiv 0 \pmod{4} \text{ and } n \equiv 2 \pmod{4}; \\ \frac{mn}{4} - 4, & \text{if } m, n \equiv 0 \pmod{4}. \end{cases}$$
(3.11)

**Proof** Due to  $|A_{ee}| = \frac{m}{2} \times \frac{n}{2} = \frac{mn}{4}$  and

$$A_{ee} \cap \Omega_A(2) = \begin{cases} \{(0,0)\}, & \text{if } m, n \equiv 2 \pmod{4}; \\ \{(0,0), (0, \frac{n}{2})\}, & \text{if } m \equiv 2 \pmod{4} \text{ and } n \equiv 0 \pmod{4}; \\ \{(0,0), (\frac{m}{2}, 0)\}, & \text{if } m \equiv 0 \pmod{4} \text{ and } n \equiv 2 \pmod{4}; \\ \{(0,0), (0, \frac{n}{2}), (\frac{m}{2}, 0), (\frac{m}{2}, \frac{n}{2})\}, & \text{if } m, n \equiv 0 \pmod{4}, \end{cases}$$

we have

$$|A_{ee} \setminus \Omega_A(2)| = |A_{ee}| - |A_{ee} \cap \Omega_A(2)| = \begin{cases} \frac{mn}{4} - 1, & \text{if } m, n \equiv 2 \pmod{4}; \\ \frac{mn}{4} - 2, & \text{if } m \equiv 2 \pmod{4} \text{ and } n \equiv 0 \pmod{4}, \\ & \text{or } m \equiv 0 \pmod{4} \text{ and } n \equiv 2 \pmod{4}; \\ \frac{mn}{4} - 4, & \text{if } m, n \equiv 0 \pmod{4}. \end{cases}$$

Since *m* and *n* are even, it is readily checked that for any codeword *X* of Type 2, supp $(\Delta X) \subseteq A_{ee} \setminus \Omega_A(2)$ . For each codeword  $X = \{(0,0), (a,b), (2a,2b)\}$  of Type 4, supp $(\Delta X)$  contributes at least two differences,  $\pm (2a,2b)$ , in  $A_{ee} \setminus \Omega_A(2)$ . Thus  $2N_2 + 2N_4 \leq |A_{ee} \setminus \Omega_A(2)|$ .

**Lemma 3.10** For any  $m \equiv 2 \pmod{4}$  and  $\lambda_a \in \{2, 3\}$ ,

$$\Theta(m, n, 3, \lambda_a, 1) \leq \begin{cases} \left\lfloor \frac{5mn + 4 + 8\omega}{24} \right\rfloor, & \text{if } n \equiv 2 \pmod{4}; \\ \left\lfloor \frac{5mn + 8 + 8\omega}{24} \right\rfloor, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

**Proof** For  $m \equiv 2 \pmod{4}$ , by (3.8),  $N_3^{(2)} = 0$  and hence  $N_3 = N_3^{(1)} + N_3^{(3)}$ . We rewrite (3.3) and (3.10) as follows:

$$2N_2 + 3N_3^{(1)} + 3N_3^{(3)} + 4N_4 + 5N_5 + 6N_6 \le mn - 1, \tag{3.12}$$

$$3N_3^{(1)} + N_3^{(3)} + N_5 \le 3. (3.13)$$

By  $2 \times (3.4) + 2 \times (3.9) + (3.11) + (3.12) + (3.13)$ , we have

$$6(N_2 + N_3 + N_4 + N_5 + N_6) \le \begin{cases} \frac{5mn}{4} + 2\omega + 1, & \text{if } n \equiv 2 \pmod{4};\\ \frac{5mn}{4} + 2\omega + 2, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

Since  $\Theta(m, n, 3, \lambda_a, 1) = N_2 + N_3 + N_4 + N_5 + N_6$ , the conclusion follows.

Remark 3.11 Examining the proof of Lemma 3.10, we have that if

$$\Theta(m, n, 3, \lambda_a, 1) = \begin{cases} \left\lfloor \frac{5mn+4+8\omega}{24} \right\rfloor, & \text{if } n \equiv 2 \pmod{4}; \\ \left\lfloor \frac{5mn+8+8\omega}{24} \right\rfloor, & \text{if } n \equiv 0 \pmod{4}, \end{cases}$$

where  $m \equiv 2 \pmod{4}$  and  $\lambda_a \in \{2,3\}$ , then the equalities must hold in (3.4), (3.9), (3.11), (3.12) and (3.13).

Lemma 3.12 Let

$$\eta = \begin{cases} \frac{mn}{8}, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 0 \pmod{2}; \\ \frac{mn}{4}, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 1 \pmod{2}; \\ \frac{mn}{8} - 2, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 0 \pmod{4}; \\ \frac{mn}{8} - 1, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 2 \pmod{4}; \\ \frac{mn}{4} - 1, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 1 \pmod{2}. \end{cases}$$

Then

$$2N_4^{(1)} \le \eta.$$
 (3.14)

**Proof Case 1**:  $m \equiv 0 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ . Due to  $|A_{s}| = \frac{m}{4} \times n = \frac{mn}{4}$  and

$$A_{s} \cap \Omega_A(2) = \begin{cases} \emptyset, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 1 \pmod{2}; \\ \{(\frac{m}{2}, 0)\}, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 1 \pmod{2}, \end{cases}$$

we have

$$|A_{s} \setminus \Omega_A(2)| = |A_{s}| - |A_{s} \cap \Omega_A(2)| = \begin{cases} \frac{mn}{4}, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 1 \pmod{2}; \\ \frac{mn}{4} - 1, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 1 \pmod{2}. \end{cases}$$

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Each codeword  $\{(0,0), (a,b), (2a,2b)\}$  of Type 4.1 contributes exactly two differences,  $\pm(2a,2b)$ , in  $A_s$ .  $\setminus \Omega_A(2)$ . Hence,  $2N_4^{(1)} \leq |A_s \setminus \Omega_A(2)|$ . **Case 2**:  $m \equiv 0 \pmod{4}$  and  $n \equiv 0 \pmod{2}$ . Due to  $|A_{se}| = \frac{m}{4} \times \frac{n}{2} = \frac{mn}{8}$  and

$$A_{se} \cap \Omega_A(2) = \begin{cases} \emptyset, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 0 \pmod{2}; \\ \{(\frac{m}{2}, 0), (\frac{m}{2}, \frac{n}{2})\}, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 0 \pmod{4}; \\ \{(\frac{m}{2}, 0)\}, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 2 \pmod{4}, \end{cases}$$

we have

$$|A_{se} \setminus \Omega_A(2)| = |A_{se}| - |A_{se} \cap \Omega_A(2)| = \begin{cases} \frac{mn}{8}, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 0 \pmod{2}; \\ \frac{mn}{8} - 2, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 0 \pmod{4}; \\ \frac{mn}{8} - 1, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 2 \pmod{4}. \end{cases}$$

Each codeword  $\{(0,0), (a,b), (2a,2b)\}$  of Type 4.1 contributes exactly two differences,  $\pm(2a,2b)$ , in  $A_{se} \setminus \Omega_A(2)$ . Hence,  $2N_4^{(1)} \le |A_{se} \setminus \Omega_A(2)|$ . 

Lemma 3.13 Let

$$\gamma = \begin{cases} 1, & \text{if } m \equiv 4 \pmod{8}; \\ 3, & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$

Then

$$2N_2 + 3N_3^{(1)} + \gamma N_3^{(2)} + 3N_3^{(3)} + 2N_4^{(1)} + 4N_4^{(2)} + N_5 + 2N_6 \le \frac{mn}{2} - 1.$$
(3.15)

**Proof** For  $m \equiv 0 \pmod{4}$ ,  $|A_{e\cdot}| = \frac{m}{2} \times n = \frac{mn}{2}$ . Take any codeword X of an  $(m, n, 3, \lambda_a, 1)$ -OOSPC,  $\mathcal{F}$ , with  $\lambda_a \in \{2, 3\}$ . It is readily checked that

$$|\text{supp}(\Delta X) \cap A_{e.}| \ge \begin{cases} 1, & \text{if } X \text{ is of Type 5;} \\ 2, & \text{if } X \text{ is of Type 2 or 4.1 or 6;} \\ 3, & \text{if } X \text{ is of Type 3.1 or 3.3;} \\ 4, & \text{if } X \text{ is of Type 4.2;} \\ \gamma, & \text{if } X \text{ is of Type 3.2.} \end{cases}$$

Hence,

$$N_5 + 2(N_2 + N_4^{(1)} + N_6) + 3(N_3^{(1)} + N_3^{(3)}) + 4N_4^{(2)} + \gamma N_3^{(2)}$$
  
$$\leq \sum_{X \in \mathcal{F}} |\operatorname{supp}(\Delta X) \cap A_{e \cdot}| \leq |A_{e \cdot} \setminus \{(0,0)\}| = |A_{e \cdot}| - 1 = \frac{mn}{2} - 1$$

Note that the second inequality comes from the cross-correlation parameter  $\lambda_c = 1$ .

**Lemma 3.14** For any  $m \equiv 0 \pmod{4}$  and  $\lambda_a \in \{2, 3\}$ ,

$$\Theta(m, n, 3, \lambda_a, 1) \le \left\lfloor \frac{1}{8} \left( \frac{3mn}{2} + \eta + 2\rho + 4\omega - 2 + (3 - \gamma)\varepsilon \right) \right\rfloor.$$

**Proof** By  $(3.3)+4\times(3.4)+2\times(3.10)+(3.14)+(3.15)$ , we have

$$8N_2 + 12N_3^{(1)} + (5+\gamma)N_3^{(2)} + 8N_3^{(3)} + 8N_4^{(1)} + 8N_4^{(2)} + 8N_5 + 8N_6 \le \frac{3mn}{2} + \eta + 2\rho + 4\omega - 2.$$

It follows that

$$8(N_2 + N_3^{(1)} + N_3^{(2)} + N_3^{(3)} + N_4^{(1)} + N_4^{(2)} + N_5 + N_6) \le \frac{3mn}{2} + \eta + 2\rho + 4\omega - 2 + (3-\gamma)N_3^{(2)} - 4N_3^{(1)}.$$

By (3.8),  $N_3^{(2)} \leq \varepsilon$ . Due to  $N_3^{(1)} \geq 0$ , we have

$$8\Theta(m, n, 3, \lambda_a, 1) \le \frac{3mn}{2} + \eta + 2\rho + 4\omega - 2 + (3 - \gamma)\varepsilon$$

Then the conclusion follows immediately.

**Remark 3.15** Examining the proof of Lemma 3.14, we have that if

$$\Theta(m, n, 3, \lambda_a, 1) = \frac{1}{8} \left( \frac{3mn}{2} + \eta + 2\rho + 4\omega - 2 + (3 - \gamma)\varepsilon \right),$$

where  $m \equiv 0 \pmod{4}$  and  $\lambda_a \in \{2,3\}$ , then  $N_3^{(1)} = 0$ , and the equalities must hold in (3.3), (3.4), (3.10), (3.14) and (3.15). Especially, when  $3 - \gamma \neq 0$ , i.e.,  $m \equiv 4 \pmod{8}$ ,  $N_3^{(2)} = \varepsilon$ .

Input the exact values of  $\omega$ ,  $\varepsilon$ ,  $\rho$ ,  $\eta$ ,  $\gamma$  to Lemma 3.14, and combine with Lemmas 3.7 and 3.10. We get an explicit upper bound of  $\Theta(m, n, 3, \lambda_a, 1)$  for any positive integers m and n.

**Theorem 3.16** Let  $\lambda_a \in \{2,3\}$ . Then  $\Theta(m, n, 3, \lambda_a, 1) \leq$ 

$$\begin{bmatrix} \frac{mn+2\omega}{4} \end{bmatrix}, \qquad mn \equiv 1, 2, 3 \pmod{4}; \\ \begin{bmatrix} \frac{7mn+16\omega}{32} \end{bmatrix}, \qquad mn \equiv 0 \pmod{8} \text{ and } \gcd(m, n, 2) = 1; \\ \begin{bmatrix} \frac{7mn+4+16\omega}{32} \end{bmatrix}, \qquad mn \equiv 4 \pmod{8} \text{ and } \gcd(m, n, 2) = 1; \\ \begin{bmatrix} \frac{5mn+4+8\omega}{24} \end{bmatrix}, \qquad mn \equiv 4 \pmod{8} \text{ and } \gcd(m, n, 2) = 2; \\ \begin{bmatrix} \frac{13mn+40+32\omega}{64} \end{bmatrix}, \qquad mn \equiv 8 \pmod{16}, \gcd(m, n, 2) = 2, \text{ and} \\ (m, n) \notin \{(2, 12), (4, 6), (6, 4), (6, 12), (12, 2), (12, 6)\} \text{ when } \lambda_a = 3; \\ \begin{bmatrix} \frac{5mn+8+8\omega}{24} \end{bmatrix}, \qquad (m, n) \in \{(2, 12), (4, 6), (6, 4), (6, 12), (12, 2), (12, 6)\} \text{ and } \lambda_a = 3; \\ \begin{bmatrix} \frac{13mn+32+32\omega}{64} \end{bmatrix}, \qquad mn \equiv 0 \pmod{32} \text{ and } \gcd(m, n, 2) = 2, \text{ or} \\ mn \equiv 16 \pmod{32} \text{ and } \gcd(m, n, 4) = 2; \\ \end{bmatrix}, \qquad mn \equiv 16 \pmod{32} \text{ and } \gcd(m, n, 4) = 4.$$

**Proof** Note that  $\Theta(m, n, 3, \lambda_a, 1) = \Theta(n, m, 3, \lambda_a, 1)$ .

For  $mn \equiv 1, 2, 3 \pmod{4}$ , at least one of m and n is odd. Then the conclusion follows from Lemma 3.7.

For  $mn \equiv 0 \pmod{4}$  and gcd(m, n, 2) = 1, w.l.o.g., assume that  $n \equiv 1 \pmod{2}$  and  $m \equiv 0 \pmod{4}$ . Apply Lemma 3.14 with  $\varepsilon = \rho = 1$ ,  $(\eta, \gamma) = (\frac{mn}{4}, 3)$  and  $(\frac{mn}{4} - 1, 1)$  when  $m \equiv 0 \pmod{8}$  and  $m \equiv 4 \pmod{8}$ , respectively. Then we have

$$\Theta(m,n,3,\lambda_a,1) \le \begin{cases} \left\lfloor \frac{7mn+16\omega}{32} \right\rfloor, & \text{if } mn \equiv 0 \pmod{8} \text{ and } \gcd(m,n,2) = 1; \\ \left\lfloor \frac{7mn+4+16\omega}{32} \right\rfloor, & \text{if } mn \equiv 4 \pmod{8} \text{ and } \gcd(m,n,2) = 1. \end{cases}$$

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For  $mn \equiv 4 \pmod{8}$  and gcd(m, n, 2) = 2, we have  $m, n \equiv 2 \pmod{4}$ . Then the conclusion follows from Lemma 3.10.

For  $mn \equiv 8 \pmod{16}$  and gcd(m, n, 2) = 2, w.l.o.g., assume that  $m \equiv 2 \pmod{4}$  and  $n \equiv 4 \pmod{8}$ . By Lemma 3.10, we have

$$\Theta(m, n, 3, \lambda_a, 1) \le \left\lfloor \frac{5mn + 8 + 8\omega}{24} \right\rfloor := U_1.$$

W.l.o.g., we can also assume that  $m \equiv 4 \pmod{8}$  and  $n \equiv 2 \pmod{4}$ . Applying Lemma 3.14 with  $\varepsilon = \gamma = 1$ ,  $\rho = 3$  and  $\eta = \frac{mn}{8} - 1$ , we have

$$\Theta(m, n, 3, \lambda_a, 1) \le \left\lfloor \frac{13mn + 40 + 32\omega}{64} \right\rfloor := U_2.$$

It follows that  $\Theta(m, n, 3, \lambda_a, 1) \leq \min\{U_1, U_2\}$ . Comparing the values of  $U_1$  and  $U_2$ , we have

$$\min\{U_1, U_2\} = \begin{cases} U_2, & \text{if } m \equiv 4 \pmod{8}, n \equiv 2 \pmod{4}, \text{ and} \\ & (m, n) \notin \{(4, 6), (12, 2), (12, 6)\} \text{ when } \lambda_a = 3; \\ U_1, & \text{if } (m, n) \in \{(4, 6), (12, 2), (12, 6)\} \text{ and } \lambda_a = 3. \end{cases}$$

For  $mn \equiv 0 \pmod{32}$  and gcd(m, n, 2) = 2, w.l.o.g., assume that  $m \equiv 0 \pmod{8}$  and  $n \equiv 0 \pmod{2}$ . By Lemma 3.14 with  $\rho = \gamma = 3$  and  $\eta = \frac{mn}{8}$ , we have

$$\Theta(m, n, 3, \lambda_a, 1) \le \left\lfloor \frac{13mn + 32 + 32\omega}{64} \right\rfloor := U_3.$$

W.l.o.g., we can also assume that  $m \equiv 0 \pmod{2}$  and  $n \equiv 0 \pmod{8}$ . Applying Lemma 3.10 with  $m \equiv 2 \pmod{4}$ , and Lemma 3.14 with  $\varepsilon = 2$ ,  $\rho = 3$ ,  $(\eta, \gamma) = (\frac{mn}{8} - 2, 1)$  and  $(\frac{mn}{8}, 3)$  for  $m \equiv 4 \pmod{8}$  and  $m \equiv 0 \pmod{8}$ , respectively, we have

$$\Theta(m,n,3,\lambda_a,1) \le \begin{cases} U_1, & \text{if } m \equiv 2 \pmod{4}; \\ \lfloor \frac{13mn+48+32\omega}{64} \rfloor := U_4, & \text{if } m \equiv 4 \pmod{8}; \\ U_3, & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$

Therefore,

$$\Theta(m, n, 3, \lambda_a, 1) \leq \begin{cases} \min\{U_3, U_1\} = U_3, & \text{if } m \equiv 2 \pmod{4}; \\ \min\{U_3, U_4\} = U_3, & \text{if } m \equiv 4 \pmod{8}; \\ U_3, & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$

For  $mn \equiv 16 \pmod{32}$  and  $\gcd(m, n, 2) = 2$ , we consider two subcases. If  $mn \equiv 16 \pmod{32}$ and  $\gcd(m, n, 4) = 2$ , then assume that  $m \equiv 2 \pmod{4}$  and  $n \equiv 8 \pmod{16}$ . By Lemma 3.10, we have  $\Theta(m, n, 3, \lambda_a, 1) \leq U_1$ . We can also assume that  $m \equiv 8 \pmod{16}$  and  $n \equiv 2 \pmod{4}$ . Then applying Lemma 3.14 with  $\rho = \gamma = 3$  and  $\eta = \frac{mn}{8}$ , we have  $\Theta(m, n, 3, \lambda_a, 1) \leq U_3$ . Therefore, we get  $\Theta(m, n, 3, \lambda_a, 1) \leq \min\{U_1, U_3\} = U_3$ . If  $mn \equiv 16 \pmod{32}$  and  $\gcd(m, n, 4) = 4$ , which implies  $m, n \equiv 4 \pmod{8}$ , then by Lemma 3.14 with  $\varepsilon = 2$ ,  $\rho = 3$ ,  $\eta = \frac{mn}{8} - 2$  and  $\gamma = 1$ , we have  $\Theta(m, n, 3, \lambda_a, 1) \leq U_4$ .

#### 3.2Improved upper bound for two subclasses when $\lambda_a = 2$

A codeword of Type 4.2 is of the form  $\{(0,0), (a,b), (2a,2b)\}$ , where  $a \equiv 0 \pmod{2}$ . All codewords of Type 4.2 can be divided into the following three types:

**Type 4.2.1**:  $a \equiv 2 \pmod{4}$  and  $b \equiv 0 \pmod{2}$ ;

**Type 4.2.2**:  $a \equiv 0 \pmod{4}$  and  $b \equiv 0 \pmod{2}$ ;

**Type 4.2.3**:  $a \equiv 0 \pmod{2}$  and  $b \equiv 1 \pmod{2}$ .

Let  $N_4^{(2,1)}$ ,  $N_4^{(2,2)}$  and  $N_4^{(2,3)}$  denote the number of codewords in  $\mathcal{F}$  of Types 4.2.1, 4.2.2 and 4.2.3, respectively. Then,

$$N_4^{(2)} = N_4^{(2,1)} + N_4^{(2,2)} + N_4^{(2,3)}$$

**Lemma 3.17** If  $\lambda_a = 2$  and  $2N_4^{(1)} = \eta$ , then

$$4N_4^{(2)} \leq \begin{cases} \frac{mn}{4} - 2\xi - 1, & \text{if } m \equiv 4 \pmod{8} \text{ and } n \equiv 1 \pmod{2}; \\ \frac{3mn}{16} - 3, & \text{if } m \equiv 4, 20 \pmod{24} \text{ and } n \equiv 4 \pmod{8}; \\ \frac{3mn}{16} - 5, & \text{if } m \equiv 12 \pmod{24} \text{ and } n \equiv 4, 20 \pmod{24}; \\ \frac{3mn}{16} - 7, & \text{if } m, n \equiv 12 \pmod{24}; \\ \frac{3mn}{16} - 6, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 4 \pmod{8}. \end{cases}$$

Furthermore, when  $m \equiv 0 \pmod{8}$  and  $n \equiv 4 \pmod{8}$ , if  $4N_4^{(2)} = \frac{3mn}{16} - 6$ , then  $2N_4^{(2,3)} = \frac{3mn}{16} - 6$ .  $\frac{mn}{16} - 2.$ 

**Proof** Case 1:  $m \equiv 4 \pmod{8}$  and  $n \equiv 1 \pmod{2}$ .

Due to  $A_{s} \setminus \Omega_A(2) = A_{s} \setminus \{(\frac{m}{2}, 0)\}$ , we have  $|A_s \setminus \Omega_A(2)| = \frac{m}{4} \times n - 1 = \eta$ . Since each codeword of Type 4.1 contributes exactly two differences in  $A_s$ ,  $\langle \Omega_A(2),$  the condition  $2N_4^{(1)} = \eta$ implies that every element in  $A_s$ .  $\langle \Omega_A(2) \rangle$  is used as a difference of some codeword of Type 4.1. Hence for each codeword  $\{(0,0), (a,b), (2a,2b)\}$  of Type 4.2, we have  $a \equiv 0 \pmod{4}$ . It follows that each codeword of Type 4.2 contributes four differences in  $A_{d}$ . Since  $m \equiv 4 \pmod{8}$ and  $n \equiv 1 \pmod{2}$ , it is readily checked that for any codeword X of Type 4.2, we have  $\operatorname{supp}(\Delta X) \cap \Omega_A(3) = \emptyset$  (otherwise, either  $\lambda_a = 3$  or  $4 \mid \frac{m}{6}$ ). Due to

$$|A_{d\cdot} \cap \Omega_A(3)| = \begin{cases} 1, & \text{if } 3 \nmid mn; \\ 3, & \text{if } 3 \mid mn \text{ and } \gcd(m, n, 3) = 1; \\ 9, & \text{if } \gcd(m, n, 3) = 3, \end{cases}$$

we have  $|A_d \cap \Omega_A(3)| = 2\xi + 1$ . Therefore,

$$4N_4^{(2)} \le |A_d \setminus \Omega_A(3)| = |A_d | - |A_d \cap \Omega_A(3)| = \frac{m}{4} \times n - (2\xi + 1) = \frac{mn}{4} - 2\xi - 1.$$

Case 2:  $m \equiv 0 \pmod{4}$  and  $n \equiv 4 \pmod{8}$ .

For  $m \equiv 0 \pmod{8}$  and  $n \equiv 4 \pmod{8}$ ,  $A_{se} \setminus \Omega_A(2) = A_{se}$ . Hence  $|A_{se} \setminus \Omega_A(2)| = \frac{m}{4} \times \frac{n}{2} = \frac{mn}{8} = \eta$ . For  $m \equiv 4 \pmod{8}$  and  $n \equiv 4 \pmod{8}$ ,  $A_{se} \setminus \Omega_A(2) = A_{se} \setminus \{(\frac{m}{2}, 0), (\frac{m}{2}, \frac{n}{2})\}$ . Hence  $|A_{se} \setminus \Omega_A(2)| = \frac{m}{4} \times \frac{n}{2} - 2 = \frac{mn}{8} - 2 = \eta$ . Let  $\{(0, 0), (a, b), (2a, 2b)\}$  be a codeword of Type 4.1, where  $a \equiv 1 \pmod{2}$ . It contributes

two differences (2a, 2b) and (-2a, -2b) in  $A_{se} \setminus \Omega_A(2)$ . Due to  $|A_{se} \setminus \Omega_A(2)| = \eta$ , the condition

 $2N_4^{(1)} = \eta$  implies that every element in  $A_{se} \setminus \Omega_A(2)$  is used as a difference of some codeword of Type 4.1. It follows that

$$N_4^{(2,1)} = 0. (3.16)$$

Due to

$$A_{ds} \cap \Omega_A(2) = \begin{cases} \{(0, \frac{n}{2})\}, & \text{if } m \equiv 4 \pmod{8}; \\ \{(0, \frac{n}{2}), (\frac{m}{2}, \frac{n}{2})\}, & \text{if } m \equiv 0 \pmod{8}, \end{cases}$$

we have

$$|A_{ds} \cap \Omega_A(2)| = \begin{cases} 1, & \text{if } m \equiv 4 \pmod{8}; \\ 2, & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$

For any codeword X of Type 4.2.3,  $|\operatorname{supp}(\Delta X) \cap A_{ds}| = |\operatorname{supp}(\Delta X) \cap A_{de}| = 2$ . By the definition of  $\mathcal{T}_1$ , each codeword X of Type 4.2 satisfies  $\operatorname{supp}(\Delta X) \cap \Omega_A(2) = \emptyset$ . Therefore,

$$2N_4^{(2,3)} \le |A_{ds} \setminus \Omega_A(2)| = |A_{ds}| - |A_{ds} \cap \Omega_A(2)| = \begin{cases} \frac{mn}{16} - 1, & \text{if } m \equiv 4 \pmod{8}; \\ \frac{mn}{16} - 2, & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$
(3.17)

Due to

$$A_{de} \cap \Omega_A(2) = \begin{cases} \{(0,0), (0,\frac{n}{2})\}, & \text{if } m \equiv 4 \pmod{8}; \\ \{(0,0), (0,\frac{n}{2}), (\frac{m}{2},0), (\frac{m}{2},\frac{n}{2})\}, & \text{if } m \equiv 0 \pmod{8}, \end{cases}$$

we have

$$|A_{de} \cap \Omega_A(2)| = \begin{cases} 2, & \text{if } m \equiv 4 \pmod{8}; \\ 4, & \text{if } m \equiv 0 \pmod{8}. \end{cases}$$

For any codeword X of Type 4.2.2, we have  $|\operatorname{supp}(\Delta X) \cap A_{de}| = 4$ . Therefore, when  $m \equiv 0 \pmod{8}$ ,

$$4N_4^{(2,2)} + 2N_4^{(2,3)} \le |A_{de} \setminus \Omega_A(2)| = |A_{de}| - |A_{de} \cap \Omega_A(2)| = \frac{mn}{8} - 4.$$
(3.18)

Let  $W = \left\{ \pm (0, \frac{n}{3}), \pm (\frac{2m}{3}, \frac{n}{3}) \right\}$ . Then  $W \subset \Omega_A(3) \subset A_{de}$ . Hence,

$$|A_{de} \cap (\Omega_A(3) \setminus W)| = |\Omega_A(3) \setminus W| = \begin{cases} 1, & \text{if } m \equiv 4, 20 \pmod{24} \text{ and } n \equiv 4 \pmod{8}; \\ 3, & \text{if } m \equiv 12 \pmod{24} \text{ and } n \equiv 4, 20 \pmod{24}; \\ 5, & \text{if } m, n \equiv 12 \pmod{24}. \end{cases}$$

When  $m, n \equiv 4 \pmod{8}$ , it is readily checked that for any codeword X of Type 4.2.2 or Type 4.2.3, we have  $\operatorname{supp}(\Delta X) \cap \Omega_A(3) \subseteq W$  (otherwise, either  $\lambda_a = 3$  or  $4 \mid \frac{m}{6}$ ). Therefore,

$$4N_{4}^{(2,2)} + 2N_{4}^{(2,3)} \leq |A_{de} \setminus (\Omega_{A}(2) \cup (\Omega_{A}(3) \setminus W))| \\ = |A_{de}| - |A_{de} \cap \Omega_{A}(2)| - |\Omega_{A}(3) \setminus W| + 1 \\ = \begin{cases} \frac{mn}{8} - 2, & \text{if } m \equiv 4, 20 \pmod{24} \text{ and } n \equiv 4 \pmod{8}; \\ \frac{mn}{8} - 4, & \text{if } m \equiv 12 \pmod{24} \text{ and } n \equiv 4, 20 \pmod{24}; \\ \frac{mn}{8} - 6, & \text{if } m, n \equiv 12 \pmod{24}. \end{cases}$$

By  $4 \times (3.16) + (3.17) + (3.18)$  and  $4 \times (3.16) + (3.17) + (3.19)$ , we obtain

$$4N_4^{(2)} = 4N_4^{(2,2)} + 4N_4^{(2,3)} \le \begin{cases} \frac{3mn}{16} - 3, & \text{if } m \equiv 4,20 \pmod{24} \text{ and } n \equiv 4 \pmod{8};\\ \frac{3mn}{16} - 5, & \text{if } m \equiv 12 \pmod{24} \text{ and } n \equiv 4,20 \pmod{24};\\ \frac{3mn}{16} - 7, & \text{if } m, n \equiv 12 \pmod{24};\\ \frac{3mn}{16} - 6, & \text{if } m \equiv 0 \pmod{8} \text{ and } n \equiv 4 \pmod{8}. \end{cases}$$

Furthermore, when  $m \equiv 0 \pmod{8}$  and  $n \equiv 4 \pmod{8}$ , if  $4N_4^{(2)} = \frac{3mn}{16} - 6$ , then the equality holds in (3.17), i.e.,  $2N_4^{(2,3)} = \frac{mn}{16} - 2$ .

**Lemma 3.18** Let  $m, n \equiv 4 \pmod{16}$  or  $m, n \equiv 12 \pmod{16}$  be positive integers such that  $3 \mid m$ . Then

$$\Theta(m, n, 3, 2, 1) \le \frac{13mn - 16}{64}$$

**Proof** For  $m, n \equiv 4 \pmod{16}$ , or  $m, n \equiv 12 \pmod{16}$ , by Theorem 3.16 with  $\omega = 0$ ,  $\Theta(m, n, 3, 2, 1) \leq \frac{13mn+48}{64}$ . Assume that  $\Theta(m, n, 3, 2, 1) = \frac{13mn+48}{64}$ . By Remark 3.15,  $N_3^{(1)} = 0$ ,  $N_3^{(2)} = \varepsilon = 2$  and the equalities hold in (3.3), (3.10), (3.14) and (3.15). Then by (3.10),  $N_3^{(3)} + N_5 = 1$ . By (3.14),  $N_4^{(1)} = \frac{mn}{16} - 1$ . By (3.3) - (3.15), we have

$$2N_3^{(2)} + 2N_4^{(1)} + 4N_5 + 4N_6 = \frac{mn}{2},$$

which yields

$$N_6 = \frac{3mn - 16}{32} - N_5 \le \frac{3mn - 16}{32}$$

Note that  $N_2 = 0$ . Therefore,

$$4N_4^{(2)} = 4\left(\Theta(m, n, 3, 2, 1) - N_4^{(1)} - N_3^{(2)} - (N_3^{(3)} + N_5) - N_6\right) = \frac{9mn}{16} - 5 - 4N_6 \ge \frac{3mn}{16} - 3.$$

However, the condition  $3 \mid m$  implies that  $m \equiv 12 \pmod{24}$ . Hence by Lemma 3.17,  $4N_4^{(2)} \leq \frac{3mn}{16} - 7$  or  $\frac{3mn}{16} - 5$  according to whether *n* is divided by 3 or not, a contradiction.

A codeword of Type 4.1 is of the form  $\{(0,0), (a,b), (2a,2b)\}$ , where  $a \equiv 1 \pmod{2}$ . All the codewords of Type 4.1 can be divided into the following two types according to the parity of b:

**Type 4.1.1**:  $a, b \equiv 1 \pmod{2}$ ;

**Type 4.1.2**:  $a \equiv 1 \pmod{2}$  and  $b \equiv 0 \pmod{2}$ ;

Let  $N_4^{(1,1)}$  and  $N_4^{(1,2)}$  denote the number of codewords in  $\mathcal{F}$  of Types 4.1.1 and 4.1.2, respectively. Then,

$$N_4^{(1)} = N_4^{(1,1)} + N_4^{(1,2)}.$$

**Lemma 3.19** Let  $m \equiv 0 \pmod{8}$  and  $n \equiv 0 \pmod{4}$ . Assume that  $2N_4^{(1)} = \frac{mn}{8}$ . Then  $N_4^{(1,1)} = N_4^{(1,2)} = \frac{mn}{32}$ . Furthermore, if  $m \equiv 8 \pmod{16}$ , then  $N_3^{(2)} \leq 1$ , and for any codeword X of Type 3.2,  $\operatorname{supp}(\Delta X) = \left\{ \pm \left(\frac{m}{4}, \frac{n}{4}\right), \left(\frac{m}{2}, \frac{n}{2}\right) \right\}$  or  $\left\{ \pm \left(\frac{m}{4}, \frac{3n}{4}\right), \left(\frac{m}{2}, \frac{n}{2}\right) \right\}$ .

**Proof** Let  $\{(0,0), (a,b), (2a,2b)\}$  with  $a \equiv 1 \pmod{2}$  be a codeword of Type 4.1. It contributes two differences (2a,2b) and (-2a,-2b) in  $A_{se}$ . Due to  $|A_{se}| = \frac{m}{4} \times \frac{n}{2} = \frac{mn}{8}$ , the condition  $2N_4^{(1)} = \frac{mn}{8}$  implies that every element in  $A_{se}$  is used as a difference of some codeword of Type 4.1. Therefore,  $n \equiv 0 \pmod{4}$  yields  $N_4^{(1,1)} = N_4^{(1,2)} = \frac{1}{2}N_4^{(1)} = \frac{mn}{32}$ . Furthermore, when  $m \equiv 8 \pmod{16}$ ,  $\{\pm \left(\frac{m}{4}, 0\right), \pm \left(\frac{m}{4}, \frac{n}{2}\right)\} \subset A_{se}$ . So  $\pm \left(\frac{m}{4}, 0\right)$  and  $\pm \left(\frac{m}{4}, \frac{n}{2}\right)$  are used as differences of some codewords of Type 4.1, and cannot be produced by other types of codewords. Therefore, for any codeword X of Type 3.2, by Remark 3.4(2),  $\sup(\Delta X) = \{\pm \left(\frac{m}{4}, \frac{n}{4}\right), \left(\frac{m}{2}, \frac{n}{2}\right)\}$ or  $\{\pm \left(\frac{m}{4}, \frac{3n}{4}\right), \left(\frac{m}{2}, \frac{n}{2}\right)\}$ . The two sets share a common element  $\left(\frac{m}{2}, \frac{n}{2}\right)$ , so  $N_3^{(2)} \leq 1$ .

**Lemma 3.20** For any  $m \equiv 8 \pmod{16}$  and  $n \equiv 4 \pmod{8}$ ,

$$\Theta(m, n, 3, 2, 1) \le \frac{13mn - 32}{64}.$$

**Proof** For  $m \equiv 8 \pmod{16}$  and  $n \equiv 4 \pmod{8}$ , applying Theorem 3.16 with  $\omega = 0$ , we have  $\Theta(m, n, 3, 2, 1) \leq \frac{13mn+32}{64}$ . Assume that  $\Theta(m, n, 3, 2, 1) = \frac{13mn+32}{64}$ . By Remark 3.15,  $N_3^{(1)} = 0$  and the equalities hold in (3.3), (3.10), (3.14) and (3.15). By (3.14),  $2N_4^{(1)} = \frac{mn}{8}$ . It follows that  $N_3^{(2)} \leq 1$  by Lemma 3.19. Note that  $N_3^{(3)} \leq 1$  by (3.9). Thus, by (3.10),  $N_3^{(2)} + N_3^{(3)} + N_5 = 3$  yields  $N_5 \geq 1$ . By (3.3)–(3.15), we have  $2N_4^{(1)} + 4N_5 + 4N_6 = \frac{mn}{2}$ , which yields

$$N_6 = \frac{3mn}{32} - N_5 \le \frac{3mn}{32} - 1.$$

Note that  $N_2 = 0$  by Lemma 3.3. Therefore,

$$N_4^{(2)} = \Theta(m, n, 3, 2, 1) - N_4^{(1)} - (N_3^{(2)} + N_3^{(3)} + N_5) - N_6 = \frac{9mn - 160}{64} - N_6 \ge \frac{3mn - 96}{64}.$$

By Lemma 3.17,  $N_4^{(2)} \leq \frac{3mn-96}{64}$ . Thus  $N_4^{(2)} = \frac{3mn-96}{64}$ . It follows that  $N_3^{(2)} = N_3^{(3)} = N_5 = 1$  and  $N_6 = \frac{3mn}{32} - 1$ .

Recall that the equality holds in (3.3). It follows that the elements in  $A_{.o}$  are used up as the differences of codewords of Types 3.2, 3.3, 4.1.1, 4.2.3, 5 and 6. Since  $N_3^{(2)} = 1$ , by Lemma 3.19, the unique codeword X of Type 3.2 satisfies  $|\operatorname{supp}(\Delta X) \cap A_{.o}| = 2$ . Since  $N_3^{(3)} = 1$ , the unique codeword X of Type 3.3 also satisfies  $|\operatorname{supp}(\Delta X) \cap A_{.o}| = 2$  by Remark 3.4(3). Due to  $2N_4^{(1)} = \frac{mn}{8}$ , by Lemma 3.19, there are  $N_4^{(1,1)} = \frac{mn}{32}$  codewords of Type 4.1.1, and by Lemma 3.17, there are  $N_4^{(2,3)} = \frac{mn}{32} - 1$  codewords of Type 4.2.3. Each codeword of Type 4.1.1 (resp. of Type 4.2.3) contributes two distinct differences in  $A_{.o}$ . Let Q denote the number of distinct differences in  $A_{.o}$  from all codewords of Type 5 and of Type 6. Then it is readily checked that  $Q \equiv 0 \pmod{4}$ . However, since  $|A_{.o}| = m \times \frac{n}{2} = \frac{mn}{2}$ , we get

$$2 + 2 + \frac{mn}{32} \times 2 + (\frac{mn}{32} - 1) \times 2 + Q = \frac{mn}{2},$$

which yields  $Q = \frac{3mn}{8} - 2 \equiv 2 \pmod{4}$ , a contradiction.

### 3.3 Sporadic values

**Lemma 3.21**  $\Theta(4, 2, 3, \lambda_a, 1) \leq 1$  for any  $\lambda_a \in \{2, 3\}$ .

**Proof** By (3.7), (3.8), (3.9) and (3.11),  $N_2 = N_3^{(3)} = N_4 = 0$ ,  $N_3^{(1)} \le 1$  and  $N_3^{(2)} \le 1$ . By (3.3), we have

$$3N_3^{(1)} + 3N_3^{(2)} + 5N_5 + 6N_6 \le 7.$$
(3.20)

By Theorem 3.16,  $\Theta(4, 2, 3, \lambda_a, 1) \leq 2$ . Assume that  $\Theta(4, 2, 3, \lambda_a, 1) = 2$ , that is,  $N_3^{(1)} + N_3^{(2)} + N_5 + N_6 = 2$ . It follows from (3.20) that  $N_3^{(1)} = N_3^{(2)} = 1$  and  $N_5 = N_6 = 0$ . Thus  $3N_3^{(1)} + N_3^{(2)} + N_3^{(3)} + N_5 = 4$ . It contradicts with (3.10).

**Lemma 3.22**  $\Theta(12, 3, 3, 3, 1) \le 9.$ 

**Proof** By Theorem 3.16,  $\Theta(12, 3, 3, 3, 1) \leq 10$ . Assume that  $\Theta(12, 3, 3, 3, 1) = 10$ . By Remark 3.15,  $N_3^{(1)} = 0$ ,  $N_3^{(2)} = 1$  and the equalities hold in (3.3), (3.4), (3.10) and (3.14). By (3.10),  $N_3^{(3)} = N_5 = 0$ . By (3.4) and (3.14),  $N_2 = 4$  and  $N_4^{(1)} = 4$ . By (3.3),  $2N_2 + 3N_3 + 4N_4 + 5N_5 + 6N_6 = 35$ , which yields  $2N_4^{(2)} + 3N_6 = 4$ . It follows that  $N_4^{(2)} = 2$  and  $N_6 = 0$ . Therefore,  $N_2 + N_3 + N_4 + N_5 + N_6 = 11$ . It contradicts with  $\Theta(12, 3, 3, 3, 1) = 10$ .

**Lemma 3.23**  $\Theta(12, 3, 3, 2, 1) \leq 7.$ 

**Proof** The proof is subtly different from that of Lemma 3.22. By Theorem 3.16,  $\Theta(12, 3, 3, 2, 1) \le 8$ . Assume that  $\Theta(12, 3, 3, 2, 1) = 8$ . By Remark 3.15,  $N_3^{(1)} = 0$ ,  $N_3^{(2)} = 1$  and the equalities hold in (3.3), (3.10), (3.14) and (3.15). By (3.10),  $N_3^{(3)} = N_5 = 0$ . By (3.4) and (3.14),  $N_2 = 0$  and  $N_4^{(1)} = 4$ . Therefore, by (3.3),  $2N_4^{(2)} + 3N_6 = 8$ , and by (3.15),  $2N_4^{(2)} + N_6 = 4$ . It follows that  $N_6 = 2$  and  $N_4^{(2)} = 1$ . However, by Lemma 3.17,  $N_4^{(2)} = 0$ , a contradiction.

### 3.4 Proof of Theorem 1.3

For  $(m,n) \in \{(2,4), (4,2), (3,12), (12,3)\}$ , the conclusion follows from Lemmas 3.21, 3.22 and 3.23. For  $mn \equiv 32 \pmod{64}$ , gcd(m,n,8) = 4 and  $\lambda_a = 2$ , exactly one of m and n is divided by 4 but not by 8. W.l.o.g., assume that  $n \equiv 4 \pmod{8}$  and  $m \equiv 8 \pmod{16}$ . Then the conclusion follows from Lemma 3.20. For  $mn \equiv 144 \pmod{192}$ , gcd(m,n,4) = 4 and  $\lambda_a = 2$ , we have  $3 \mid mn$  and  $mn \equiv 16 \pmod{64}$ . W.l.o.g., assume that  $3 \mid m$ , and  $m, n \equiv 4 \pmod{16}$  or  $m, n \equiv 12 \pmod{16}$ . Then the conclusion follows by Lemma 3.18. All the other cases follow from Theorem 3.16.

## 4 **Recursive constructions**

Let  $\mathcal{F}$  be an  $(m, n, k, \lambda_a, 1)$ -OOSPC. Define the (difference) leave of  $\mathcal{F}$ , briefly  $DL(\mathcal{F})$ , as the set of all nonzero elements in  $\mathbb{Z}_m \times \mathbb{Z}_n$  which are not covered by  $\Delta \mathcal{F} = \bigcup_{X \in \mathcal{F}} \operatorname{supp}(\Delta X)$ .  $\mathcal{F}$ is said to be (s, t)-regular if  $DL(\mathcal{F}) \cup \{(0, 0)\}$  forms an additive subgroup  $S \times T$  of  $\mathbb{Z}_m \times \mathbb{Z}_n$ , where S and T are, respectively, the additive subgroups of order s in  $\mathbb{Z}_m$  and order t in  $\mathbb{Z}_n$ .

Construction 4.1 (Filling Construction) Suppose that there exist

(1) an (s,t)-regular  $(m, n, k, \lambda_a, 1)$ -OOSPC  $\mathcal{F}_1$  with  $b_1$  codewords;

(2) an  $(s, t, k, \lambda_a, 1)$ -OOSPC  $\mathcal{F}_2$  with  $b_2$  codewords.

Then there exists an  $(m, n, k, \lambda_a, 1)$ -OOSPC with  $b_1 + b_2$  codewords. Furthermore, if the given  $(s, t, k, \lambda_a, 1)$ -OOSPC is (g, h)-regular, then the resulting  $(m, n, k, \lambda_a, 1)$ -OOSPC is (g, h)-regular.

**Proof** Let us interpret all codewords of  $\mathcal{F}_2$  as codewords in  $(\frac{m}{s}\mathbb{Z}_m) \times (\frac{n}{t}\mathbb{Z}_n)$  and add them to the codewords of  $\mathcal{F}_1$ . We then get the desired  $(m, n, k, \lambda_a, 1)$ -OOSPC with  $b_1 + b_2$  codewords, whose leave is exactly  $DL(\mathcal{F}_2)$ .

Let G be an abelian group of order v. A  $(G, k, \lambda)$  difference matrix (briefly,  $(G, k, \lambda)$ -DM) is a  $k \times \lambda v$  matrix  $D = (d_{ij})$  with entries from G such that for any distinct rows x and y, the multiset  $\{d_{xi} - d_{yi} : 1 \le i \le \lambda v\}$  contains each element of G exactly  $\lambda$  times. If  $G = \mathbb{Z}_v$ , the difference matrix is called *cyclic* and denoted by a  $(v, k, \lambda)$ -CDM.

When  $\lambda_a = 1$ , the notation (s, t)-regular (m, n, k, 1, 1)-OOSPC is simply written as (s, t)-regular (m, n, k, 1)-OOSPC.

**Construction 4.2** [23, Construction 3.3] (Inflation Construction) Let m, n and v be positive integers. Suppose that there exist

- (1) an (s,t)-regular (m, n, k, 1)-OOSPC;
- (2) a(v, k, 1)-CDM.

Then there exist an (sv,t)-regular (mv,n,k,1)-OOSPC and an (s,tv)-regular (m,nv,k,1)-OOSPC.

**Lemma 4.3** [8] Let v and k be positive integers such that gcd(v, (k-1)!) = 1. Then there exists a (v, k, 1)-CDM.

Let  $m, n \equiv 2 \pmod{4}$ , and H be the normal subgroup  $\{(0,0), (0,\frac{n}{2}), (\frac{m}{2},0), (\frac{m}{2},\frac{n}{2})\}$  of  $\mathbb{Z}_m \times \mathbb{Z}_n$ . Then the quotient group  $(\mathbb{Z}_m \times \mathbb{Z}_n)/H$  is isomorphic to  $\mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ . For  $(x, y) \in \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ , let D(x, y) = (x, y) + H be a coset of H in  $\mathbb{Z}_m \times \mathbb{Z}_n$ , namely,

$$D(x,y) = \left\{ (x,y), (x,y+\frac{n}{2}), (x+\frac{m}{2},y), (x+\frac{m}{2},y+\frac{n}{2}) \right\}.$$

The following proposition is straightforward from group theory.

**Proposition 4.4** (1) For any distinct (x, y) and (x', y') from  $\mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ ,  $D(x, y) \cap D(x', y') = \emptyset$ .

(2)  $\bigcup_{(x,y)\in\mathbb{Z}_{\frac{m}{2}}\times\mathbb{Z}_{\frac{n}{2}}} D(x,y) = \mathbb{Z}_m\times\mathbb{Z}_n.$ 

Recall that  $A = \mathbb{Z}_m \times \mathbb{Z}_n$  and

$$\begin{aligned} A_{ee} &= \{(x,y) \in A : \ x,y \equiv 0 \pmod{2}\}, \quad A_{eo} = \{(x,y) \in A : \ x \equiv 0 \pmod{2}, y \equiv 1 \pmod{2}\}, \\ A_{oo} &= \{(x,y) \in A : \ x,y \equiv 1 \pmod{2}\}, \quad A_{oe} = \{(x,y) \in A : \ x \equiv 1 \pmod{2}, y \equiv 0 \pmod{2}\}. \end{aligned}$$

**Proposition 4.5** Let  $m, n \equiv 2 \pmod{4}$ . For any  $(x, y) \in \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ ,

$$|D(x,y) \cap A_{oo}| = |D(x,y) \cap A_{oe}| = |D(x,y) \cap A_{eo}| = |D(x,y) \cap A_{ee}| = 1.$$

**Construction 4.6** (Doubling Construction) Let  $m, n \equiv 2 \pmod{4}$ . Suppose there exists an  $(\frac{m}{2}, \frac{n}{2}, 3, 1)$ -OOSPC  $\mathcal{F}$  whose leave is L. Then there exists an (m, n, 3, 2, 1)-OOSPC with  $5 |\mathcal{F}|$  codewords whose leave is

$$L' = \left(\bigcup_{(x,y)\in L\cup\{(0,0)\}} (D(x,y)\setminus A_{ee})\right) \cup \left(\bigcup_{(x,y)\in L} \{(2x,2y)\}\right).$$
(4.21)

Especially, if the given  $(\frac{m}{2}, \frac{n}{2}, 3, 1)$ -OOSPC is (s, t)-regular, then the resulting (m, n, 3, 2, 1)-OOSPC is (2s, 2t)-regular.

**Proof** For each codeword  $F = \{(0,0), (x_1, y_1), (x_2, y_2)\} \in \mathcal{F}$ , construct a set  $\mathcal{B}_F$  which consists of the following five 3-subsets in A:

$$\{ (0,0), \alpha_1, 2\alpha_1 \}, \quad \{ (0,0), \alpha_2, 2\alpha_2 \}, \quad \{ (0,0), \alpha_3, 2\alpha_3 \}, \\ \{ (0,0), \beta_1, \beta_2 \}, \quad \{ (0,0), \beta_3, \beta_4 \},$$

satisfying that  $\{\alpha_1, \beta_1, \beta_3\} = D(x_1, y_1) \setminus A_{ee}, \{\alpha_2, \beta_2, \beta_4\} = D(x_2, y_2) \setminus A_{ee}$  and  $\{\alpha_3, \beta_2 - \beta_1, \beta_4 - \beta_3\} = D(x_2 - x_1, y_2 - y_1) \setminus A_{ee}$ . This can be done because for any  $(x, y) \in \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ , by Proposition 4.5,  $|D(x, y) \cap A_{oo}| = |D(x, y) \cap A_{oe}| = |D(x, y) \cap A_{eo}| = |D(x, y) \cap A_{ee}| = 1$ . So we can take  $\alpha_1 \in A_{oo}, \beta_1 \in A_{eo}, \beta_3 \in A_{oe}, \alpha_2 \in A_{eo}, \beta_2 \in A_{oe}, \beta_4 \in A_{oo}$  and  $\alpha_3 \in A_{oe}$ .

Note that  $\pm 2\alpha_1 = \pm (2x_1, 2y_1), \pm 2\alpha_2 = \pm (2x_2, 2y_2)$  and  $\pm 2\alpha_3 = \pm (2(x_2 - x_1), 2(y_2 - y_1)).$ They are distinct elements in  $A_{ee}$ . It follows that

$$\begin{aligned} \Delta \mathcal{B}_F &= (\pm \{\alpha_1, \beta_1, \beta_3\}) \cup (\pm \{\alpha_2, \beta_2, \beta_4\}) \cup (\pm \{\alpha_3, \beta_2 - \beta_1, \beta_4 - \beta_3\}) \cup (\pm \{2\alpha_1, 2\alpha_2, 2\alpha_3\}) \\ &= \left( \bigcup_{(x,y) \in \Delta F} (D(x,y) \setminus A_{ee}) \right) \cup \left( \bigcup_{(x,y) \in \Delta F} \{(2x, 2y)\} \right). \end{aligned}$$

Let  $\mathcal{B} = \bigcup_{F \in \mathcal{F}} \mathcal{B}_F$  and  $V = \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}} \setminus (L \cup \{(0,0)\})$ . Then

$$\Delta \mathcal{B} = \left(\bigcup_{(x,y)\in V} (D(x,y)\setminus A_{ee})\right) \cup \left(\bigcup_{(x,y)\in V} \{(2x,2y)\}\right).$$

Thus  $\mathcal{B}$  forms an (m, n, 3, 2, 1)-OOSPC with  $5|\mathcal{F}|$  codewords whose leave is of the form (4.21).

Especially, if the given  $(\frac{m}{2}, \frac{n}{2}, 3, 1)$ -OOSPC is (s, t)-regular, then its leave L along with  $\{(0,0)\}$  forms an additive subgroup  $S \times T$  of  $\mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ , where S and T are, respectively, the additive subgroups of order s in  $\mathbb{Z}_{\frac{m}{2}}$  and order t in  $\mathbb{Z}_{\frac{n}{2}}$ . It is readily checked that the leave L' of the resulting (m, n, 3, 2, 1)-OOSPC satisfies

$$L' \cup \{(0,0)\} = \left(\bigcup_{(x,y)\in S\times T} (D(x,y)\setminus A_{ee})\right) \cup \left(\bigcup_{(x,y)\in S\times T} \{(2x,2y)\}\right)$$
$$= \left(\bigcup_{(x,y)\in S\times T} D(x,y)\right) = S'\times T',$$

where S' and T' are, respectively, the additive subgroups of order 2s in  $\mathbb{Z}_m$  and order 2t in  $\mathbb{Z}_n$ . Therefore, the resulting OOSPC is (2s, 2t)-regular.

# 5 Determination of $\Theta(m, n, 3, \lambda_a, 1)$ with $m, n \equiv 2 \pmod{4}$

This section is devoted to constructing optimal  $(m, n, 3, \lambda_a, 1)$ -OOSPCs with  $\lambda_a = 2, 3$  for  $m, n \equiv 2 \pmod{4}$ . In this case,  $mn \equiv 4 \pmod{8}$  and gcd(m, n, 2) = 2. By Theorem 1.3, we have the following corollary.

**Corollary 5.1** For any  $m, n \equiv 2 \pmod{4}$  and  $\lambda_a \in \{2, 3\}$ ,

$$\Theta(m, n, 3, \lambda_a, 1) \le \left\lfloor \frac{5mn + 4 + 8\omega}{24} \right\rfloor.$$

**Proposition 5.2** For any m and n such that gcd(mn,3) = 1, an (m,n,3,2,1)-OOSPC is equivalent to an (m,n,3,3,1)-OOSPC.

**Proof** Let gcd(mn, 3) = 1 and X be a 3-subset of A. By Lemma 3.1,  $|supp(\Delta X)| \ge 3$ . Then by Lemma 3.2,  $\lambda(X) \le 2$ . Hence, by the auto-correlation property (1''), for gcd(mn, 3) = 1, an (m, n, 3, 2, 1)-OOSPC is equivalent to an (m, n, 3, 3, 1)-OOSPC.

## 5.1 (s,t)-regular $(m, n, 3, \lambda_a, 1)$ -OOSPCs

**Lemma 5.3** [24, Theorem 4.8] For any m and n such that  $mn \equiv 1 \pmod{6}$ , there exists a (1,1)-regular (m, n, 3, 1)-OOSPC.

**Lemma 5.4** There exists a (1,3)-regular (m, n, 3, 1)-OOSPC for any  $m \equiv 1, 5 \pmod{6}$  and  $n \equiv 3 \pmod{6}$  except for (m, n) = (1, 9).

**Proof** Let  $m \equiv 1, 5 \pmod{6}$ . For  $n \in \{3, 9\}$ , due to gcd(m, 3) = gcd(m, 9) = 1, a (1, 3)-regular (m, n, 3, 1)-OOSPC is equivalent to a cyclic Steiner triple system (CSTS) of order mn. It is known that a CSTS(mn) exists if and only if  $mn \equiv 1, 3 \pmod{6}$  and  $mn \neq 9$  (see [9, Theorem 2.25]).

For  $n \ge 15$ , start from a (1,3)-regular (1, n, 3, 1)-OOSPC, which is equivalent to a CSTS(n). Apply Construction 4.2 with an (m, 3, 1)-CDM to obtain an (m, 3)-regular (m, n, 3, 1)-OOSPC. Then apply Construction 4.1 to obtain a (1,3)-regular (m, n, 3, 1)-OOSPC.

**Lemma 5.5** There exists a (1,3)-regular (m,n,3,2,1)-OOSPC with  $\frac{5mn-12}{24}$  codewords for any  $m \equiv 2, 10 \pmod{12}$  and  $n \equiv 6 \pmod{12}$ .

**Proof** For  $(m, n) \in \{(2, 6), (2, 18)\}$ , all  $\frac{5mn-12}{24}$  codewords of a (1, 3)-regular (m, n, 3, 2, 1)-OOSPC are listed below:

 $\begin{array}{ll} (m,n) = (2,6) : & \{(0,0),(0,1),(1,2)\}, & \{(0,0),(0,3),(1,0)\}; \\ (m,n) = (2,18) : & \{(0,0),(0,1),(0,2)\}, & \{(0,0),(1,4),(0,8)\}, & \{(0,0),(1,7),(0,14)\}, \\ & \{(0,0),(0,3),(1,1)\}, & \{(0,0),(0,5),(1,8)\}, & \{(0,0),(0,7),(1,12)\}, \\ & \{(0,0),(0,9),(1,0)\}. \end{array}$ 

For  $m \equiv 2, 10 \pmod{12}$ ,  $n \equiv 6 \pmod{12}$  and  $(m, n) \neq (2, 18)$ , by Lemma 5.4, there exists a (1,3)-regular  $(\frac{m}{2}, \frac{n}{2}, 3, 1)$ -OOSPC with  $\frac{mn-12}{24}$  codewords. Apply Construction 4.6 to obtain a (2, 6)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-60}{24}$  codewords. Then apply Construction 4.1 with a (1,3)-regular (2, 6, 3, 2, 1)-OOSPC, which has 2 codewords, to obtain a (1,3)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-12}{24}$  codewords. **Lemma 5.6** There exists a (3,3)-regular (m, n, 3, 1)-OOSPC for any  $m, n \equiv 3 \pmod{6}$ ,  $m \neq 9$  and  $n \neq 9$ .

**Proof** Without loss of generality, assume that  $n \ge m$ .

**Case 1:** m = 3. When n = 3, a (3, 3)-regular (3, 3, 3, 1)-OOSPC is trivial. When n > 9, start from a (1,3)-regular (1, n, 3, 1)-OOSPC, which is equivalent to a CSTS(n). Then apply Construction 4.2 with a (3,3,1)-CDM to obtain a (3,3)-regular (3, n, 3, 1)-OOSPC.

**Case 2:** m > 9. Then n > 9. Start from a (3,3)-regular (3, n, 3, 1)-OOSPC, which exists by Case 1, and apply Construction 4.2 with an  $(\frac{m}{3}, 3, 1)$ -CDM to obtain an (m, 3)-regular (m, n, 3, 1)-OOSPC. Then apply Construction 4.1 with a (3,3)-regular (m, 3, 3, 1)-OOSPC to obtain a (3,3)-regular (m, n, 3, 1)-OOSPC.

**Lemma 5.7** There exists a (3,3)-regular (9, n, 3, 1)-OOSPC for any  $n \equiv 3 \pmod{6}$ .

**Proof** For n = 3, a (3, 3)-regular (9, 3, 3, 1)-OOSPC has three codewords:  $\{(0, 0), (1, 0), (2, 1)\}$ ,  $\{(0, 0), (1, 2), (5, 0)\}$ ,  $\{(0, 0), (2, 0), (4, 2)\}$ . For n = 9, there exists a (3, 3)-regular (9, 9, 3, 1)-OOSPC by [24, Lemma 4.5]. For  $n \ge 15$ , start from the (3, 3)-regular (9, 3, 3, 1)-OOSPC, and apply Construction 4.2 with an  $(\frac{n}{3}, 3, 1)$ -CDM to obtain a (3, n)-regular (9, n, 3, 1)-OOSPC. Then apply Construction 4.1 with a (3, 3)-regular (3, n, 3, 1)-OOSPC, which exists by Lemma 5.6, to obtain a (3, 3)-regular (9, n, 3, 1)-OOSPC.

Denote by [a, b] the set of integers v such that  $a \le v \le b$ .

**Lemma 5.8** For any  $m \equiv 2 \pmod{12}$  and  $n \equiv 2 \pmod{4}$ , there exists a (2, n)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5n(m-2)}{24}$  codewords.

**Proof** For m = 2, the conclusion is trivial. For  $m \ge 14$ , all  $\frac{5n(m-2)}{24}$  codewords of a (2, n)-regular (m, n, 3, 2, 1)-OOSPC are listed as follows:

where  $i \in [0, \frac{n}{2} - 1], j \in [0, \frac{m-14}{12}], s \in [0, \lfloor \frac{m-26}{24} \rfloor]$  and  $t \in [0, \lfloor \frac{m-14}{24} \rfloor]$ .

### 5.2 Proof of Theorem 1.4

**Lemma 5.9**  $\Theta(m, n, 3, 2, 1) = \frac{5mn+4}{24}$  for any  $m, n \equiv 2 \pmod{12}$  and  $m, n \equiv 10 \pmod{12}$ .

**Proof**  $m, n \equiv 2 \pmod{12}$  and  $m, n \equiv 10 \pmod{12}$  both imply  $mn \equiv 4 \pmod{24}$ . By Corollary 5.1,  $\Theta(m, n, 3, 2, 1) \leq \frac{5mn+4}{24}$ . For (m, n) = (2, 2), an optimal (2, 2, 3, 2, 1)-OOSPC has only one codeword  $\{(0, 0), (1, 0), (0, 1)\}$ . For  $(m, n) \neq (2, 2)$ , start from a (1, 1)-regular  $(\frac{m}{2}, \frac{n}{2}, 3, 1)$ -OOSPC with  $\frac{mn-4}{24}$  codewords, which exists by Lemma 5.3, and apply Construction 4.6 to obtain a (2, 2)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-20}{24}$  codewords. Then apply Construction 4.1 with an optimal (2, 2, 3, 2, 1)-OOSPC with 1 codeword to obtain an optimal (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn+4}{24}$  codewords.

**Lemma 5.10**  $\Theta(m, n, 3, 2, 1) = \frac{5mn-4}{24}$  for any  $m \equiv 2 \pmod{12}$  and  $n \equiv 10 \pmod{12}$ .

**Proof** By Corollary 5.1,  $\Theta(m, n, 3, 2, 1) \leq \frac{5mn-4}{24}$ . For m = 2, set n = 2t, where  $t \equiv 5 \pmod{6}$ . We here give an explicit construction for an optimal (2, n, 3, 2, 1)-OOSPC on  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_t$  with  $\frac{5n-2}{12} = \frac{5t-1}{6}$  codewords:

$$\begin{array}{lll} \text{Type 3:} & \{(0,0,0),(0,1,0),(1,0,0)\},\\ \text{Type 4:} & \{(0,0,0),(0,1,2i-1),(0,0,4i-2)\}, & i\in[1,\frac{t+1}{6}],\\ & \{(0,0,0),(1,0,2i),(0,0,4i)\}, & i\in[1,\frac{t+1}{6}],\\ & \{(0,0,0),(1,1,\frac{t+1}{2}-i),(0,0,t-2i+1)\}, & i\in[1,\frac{t-5}{6}],\\ \text{Type 6:} & \{(0,0,0),(1,1,2i-1),(0,1,4i-2)\}, & i\in[1,\frac{t+1}{6}],\\ & \{(0,0,0),(0,1,4i),(1,0,\frac{t+1}{3}+2i)\}, & i\in[1,\frac{t-5}{6}]. \end{array}$$

For  $m \geq 14$ , begin with a (2, n)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5n(m-2)}{24}$  codewords, which comes from Lemma 5.8. Apply Construction 4.1 with a (2, n, 3, 2, 1)-OOSPC with  $\frac{5n-2}{12}$  codewords to obtain an optimal (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-4}{24}$  codewords.

**Lemma 5.11** For any  $m \equiv 2, 10 \pmod{12}$  and  $n \equiv 6 \pmod{12}$ ,  $\Theta(m, n, 3, 2, 1) = \frac{5mn-12}{24}$  and  $\Theta(m, n, 3, 3, 1) = \frac{5mn+12}{24}$ .

**Proof** The condition  $m \equiv 2, 10 \pmod{12}$  and  $n \equiv 6 \pmod{12}$  implies  $mn \equiv 12 \pmod{24}$ . By Corollary 5.1,  $\Theta(m, n, 3, 2, 1) \leq \frac{5mn-12}{24}$  and  $\Theta(m, n, 3, 3, 1) \leq \frac{5mn+12}{24}$ . By Lemma 5.5, there exists a (1, 3)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-12}{24}$  codewords for any  $m \equiv 2, 10 \pmod{12}$  and  $n \equiv 6 \pmod{12}$ . Thus  $\Theta(m, n, 3, 2, 1) = \frac{5mn-12}{24}$ .

Start from the resulting (1,3)-regular OOSPC, and apply Construction 4.1 with an optimal (1,3,3,3,1)-OOSPC, which consists of the unique codeword  $\{(0,0), (0,1), (0,2)\}$ , to obtain an optimal (m, n, 3, 3, 1)-OOSPC with  $\frac{5mn+12}{24}$  codewords. Thus  $\Theta(m, n, 3, 3, 1) = \frac{5mn+12}{24}$ .

**Lemma 5.12** For any  $m, n \equiv 6 \pmod{12}$ ,  $\Theta(m, n, 3, 2, 1) = \frac{5mn-12}{24}$  and  $\Theta(m, n, 3, 3, 1) = \frac{5mn+36}{24}$ .

**Proof** The condition  $m, n \equiv 6 \pmod{12}$  implies  $mn \equiv 12 \pmod{24}$ . By Corollary 5.1,  $\Theta(m, n, 3, 2, 1) \leq \frac{5mn-12}{24}$  and  $\Theta(m, n, 3, 3, 1) \leq \frac{5mn+36}{24}$ . When (m, n) = (6, 6), the conclusion follows from Example 2.1. When  $(m, n) \neq (6, 6)$ , there is a (3, 3)-regular  $(\frac{m}{2}, \frac{n}{2}, 3, 1)$ -OOSPC with  $\frac{mn-36}{24}$  codewords by Lemmas 5.6 and 5.7. Apply Construction 4.6 to obtain a (6, 6)-regular (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-180}{24}$  codewords. Then apply Construction 4.1 with a (6, 6, 3, 2, 1)-OOSPC with 7 codewords to obtain a (m, n, 3, 2, 1)-OOSPC with  $\frac{5mn-12}{24}$  codewords. Thus  $\Theta(m, n, 3, 2, 1) = \frac{5mn-12}{24}$ . Apply Construction 4.1 with a (6, 6, 3, 3, 1)-OOSPC with 9 codewords to obtain a (m, n, 3, 3, 1)-OOSPC with  $\frac{5mn+36}{24}$  codewords.  $\Box$ 

By Proposition 5.2, for any m and n such that gcd(mn, 3) = 1, an (m, n, 3, 2, 1)-OOSPC is equivalent to an (m, n, 3, 3, 1)-OOSPC. Note that  $\Theta(m, n, 3, \lambda_a, 1) = \Theta(n, m, 3, \lambda_a, 1)$ . Now combining the results from Lemmas 5.9-5.12, one can complete the proof of Theorem 1.4.

# 6 Concluding remarks

Compared with (1.2), Theorem 1.3 provides a much more complicated upper bound on the size of an  $(m, n, 3, \lambda_a, 1)$ -OOSPC with  $\lambda_a \in \{2, 3\}$ . It seems that this bound is good for  $mn \equiv$ 

m	n	$\Theta(m, n, 3, \lambda_a, 1)$		m	m	$\Theta(m, n, 3, \lambda_a, 1)$		m	n	$\Theta(m, n, 3, \lambda_a, 1)$	
		$\lambda_a = 3$	$\lambda_a = 2$	111	11	$\lambda_a = 3$	$\lambda_a = 2$	$\Pi \iota$	n	$\lambda_a = 3$	$\lambda_a = 2$
2	4	1	1	3	12	9	7	4	32	26	26
2	8	3	3	3	24	17	15	4	34	28	28
2	12	5	5	3	36	25	23	4	36	30	29
2	16	7	7	3	48	33	31	5	20	22	22
2	20	8	8	4	4	4	4	6	8	10	10
2	24	10	10	4	6	5	5	6	12	16	15
2	28	12	12	4	8	6	6	6	16	20	20
2	32	13	13	4	10	8	8	6	20	25	25
2	36	15	15	4	12	11	10	6	24	31	29
2	40	16	16	4	14	12	12	8	8	13	13
2	44	18	18	4	16	13	13	8	10	16	16
2	48	20	20	4	18	15	15	8	12	20	19
2	52	21	21	4	20	17	17	8	14	23	23
2	56	23	23	4	22	18	18	8	16	25	25
2	60	25	25	4	24	20	19	8	18	30	29
2	64	26	26	4	26	21	21	9	12	25	23
2	68	28	28	4	28	23	23	10	12	25	25
2	72	30	29	4	30	25	25	12	12	32	29

Table 1:  $\Theta(m, n, 3, \lambda_a, 1)$  not covered by Theorem 1.4 for  $mn \equiv 0 \pmod{4}$ ,  $mn \leq 150$  and  $gcd(m, n) \neq 1$ 

0 (mod 4). On one hand, when gcd(m, n) = 1, an  $(m, n, k, \lambda_a, \lambda_c)$ -OOSPC is equivalent to a 1-D  $(mn, k, \lambda_a, \lambda_c)$ -OOC [28]. Let  $\Phi(mn, k, \lambda_a, \lambda_c)$  denote the largest possible size among all 1-D  $(mn, k, \lambda_a, \lambda_c)$ -OOCs. Then  $\Theta(m, n, k, \lambda_a, \lambda_c) = \Phi(mn, k, \lambda_a, \lambda_c)$  for gcd(m, n) = 1. When  $v \equiv 0 \pmod{4}$ , the exact value of  $\Phi(v, 3, \lambda_a, 1)$  has been determined in the literature. Note that a 1-D (v, k, k, 1)-OOC is often referred to as a *conflict-avoiding code*, which finds its application on a multiple-access collision channel without feedback.

**Theorem 6.1** [11, 27]

$$\Phi(v,3,2,1) = \begin{cases} \left\lfloor \frac{7v}{32} \right\rfloor, & \text{if } v \equiv 0 \pmod{8} \text{ and } v \neq 64; \\ 13, & \text{if } v = 64; \\ \left\lfloor \frac{7v+4}{32} \right\rfloor, & \text{if } v \equiv 4 \pmod{8}. \end{cases}$$

**Theorem 6.2** [12, 16, 21, 22]

$$\Phi(v,3,3,1) = \begin{cases} \left\lfloor \frac{7v+16}{32} \right\rfloor, & \text{if } v \equiv 0 \pmod{24} \text{ and } v \neq 48; \\ 10, & \text{if } v = 48; \\ \left\lfloor \frac{7v+4}{32} \right\rfloor, & \text{if } v \equiv 4, 20 \pmod{24}; \\ \left\lfloor \frac{7v}{32} \right\rfloor, & \text{if } v \equiv 8, 16 \pmod{24} \text{ and } v \neq 64; \\ 13, & \text{if } v = 64; \\ \left\lfloor \frac{7v+20}{32} \right\rfloor, & \text{if } v \equiv 12 \pmod{24}. \end{cases}$$

It is easy to check that Theorems 6.1 and 6.2 satisfy the bound for  $\Theta(m, n, 3, \lambda_a, 1)$  in Theorem 1.3 when gcd(m, n) = 1 except for  $mn \in \{48, 64\}$ .

On the other hand, when  $gcd(m, n) \neq 1$ , Theorem 1.4 determines the values of  $\Theta(m, n, 3, \lambda_a, 1)$ with  $\lambda_a = 2, 3$  for  $m, n \equiv 2 \pmod{4}$ , which coincides with the bound in Theorem 1.3. By computer search, it is shown that for any m and n such that  $mn \equiv 0 \pmod{4}$  and  $mn \leq 150$ , there exists an  $(m, n, 3, \lambda_a, 1)$ -OOSPC attaining the bound in Theorem 1.3 (see Table 1). The interested reader may get a copy of these data from the authors. We conjecture that when  $mn \equiv 0 \pmod{4}$ , our bound for  $\Theta(m, n, 3, \lambda_a, 1)$  with  $\lambda_a \in \{2, 3\}$  shown in Theorem 1.3 is tight.

Theorem 1.4 determines the value of  $\Theta(m, n, 3, \lambda_a, 1)$  with  $\lambda_a \in \{2, 3\}$  for  $m, n \equiv 2 \pmod{4}$ . To prove Theorem 1.4, the doubling construction (Construction 4.6) plays an important role. It seems that to solve other cases of m and n such that  $mn \equiv 0 \pmod{4}$ , one must explore a quadrupling construction.

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