Random Construction of Partial MDS Codes

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Abstract

This work deals with partial MDS (PMDS) codes, a special class of locally repairable codes, used for distributed storage system. We first show that a known construction of these codes, using Gabidulin codes, can be extended to use any maximum rank distance code. Then we define a standard form for the generator matrices of PMDS codes and use this form to give an algebraic description of PMDS generator matrices. This implies that over a sufficiently large finite field a randomly chosen generator matrix in PMDS standard form generates a PMDS code with high probability. This also provides sufficient conditions on the field size for the existence of PMDS codes.

1 Introduction

In a distributed storage system a file $x \in \mathbb{F}_q^k$ (where \mathbb{F}_q denotes the finite field of cardinality q), is encoded and stored as some codeword $c \in \mathbb{F}_q^n$, over several storage nodes. Each of these nodes is assumed, for simplicity, to store exactly one coordinate of c. In case some of the nodes fail, we want to be able to recover the lost information using as little effort as possible. The locality of a code plays an important role in this context, and it denotes the number of nodes one has to contact for repairing a lost node. We call the set of nodes one has to contact if a given node fails, the locality group of that node. For this work we assume that the locality groups are distinct. Partial MDS codes are maximally recoverable codes in this setting, i.e., any erasure pattern that is information theoretically correctable is correctable with such a code.

It is known that maximally recoverable codes in general, and PMDS codes in particular, exist for any locality configuration if the field size is large enough [1]. Furthermore, some constructions of PMDS codes are known, e.g. [2–6].

In this paper we describe a random construction of PMDS codes, by prescribing a generator matrix of the respective code in a specific form, which we will call *PMDS standard form*. If we fill the non-prescribed coordinates of this generator matrix with random values, by high probability, the resulting code is PMDS, if the underlying field size is large enough. We derive a lower bound on this probability (depending on the field size). This gives rise to a lower bound on the necessary field size for PMDS codes to exist. With some final adjustments on the random construction we get a lower bound that improves the one of [1].

2 Preliminaries

2.1 PMDS codes

Consider a distributed storage system with m disjoint locality groups, where the i-th group is of size n_i ($i=1,\ldots,m$) and can correct any r_i erasures. First we set the locality for the code to be $\ell \in \mathbb{N}$. We can divide the coordinates of the code into blocks of length n_1,\ldots,n_m , where $n_i=\ell+r_i$, such that each block represents a locality group.

We denote an MDS code of length n and dimension k by [n, k]-MDS code. We use the definition of PMDS codes given in [7], which generalizes the definition of [2].

Definition 1. Let $\ell, m, r_1, \ldots, r_m \in \mathbb{N}$. Define $n := \sum_{i=1}^m (r_i + \ell)$ and let $C \subseteq \mathbb{F}_q^n$ be a linear code of dimension k < n with generator matrix

$$G = (B_1 \mid \dots \mid B_m) \in \mathbb{F}_q^{k \times n} \tag{1}$$

such that $B_i \in \mathbb{F}_q^{k \times (r_i + \ell)}$. Then C is a $[n, k, \ell; r_1, \dots, r_m]$ -partial-MDS (PMDS) code (with locality ℓ) if

- for $i \in \{1, ..., m\}$ the row space of B_i is a $[r_i + \ell, \ell]$ -MDS code, and
- for any r_i erasures in the *i*-th block (i = 1, ..., m), the remaining code (after puncturing the coordinates of the erasures) is a $[m\ell, k]$ -MDS code.

The erasure correction capability of PMDS codes is as follows:

Lemma 2. [7, Lemma 3] A $[n, k, \ell; r_1, \ldots, r_m]$ -PMDS code can correct any r_i erasures in the i-th block (simultaneously) plus $s := m\ell - k$ additional erasures anywhere in the code.

We can see that the definition of PMDS codes given makes sense only for $k \geq \ell$. In case of equality, or in the case that m=1 there exist only trivial PMDS codes, i.e. the only PMDS codes are MDS codes.

It was shown in [7] that a code is a $[n, k, \ell; r_1, \ldots, r_m]$ -PMDS code if and only if it is maximally recoverable (for the respective locality group configuration). The same results had previously been shown in [6, Lemma 4] for the case $r_1 = r_2 = \cdots = r_m$.

Now we give a summary on known results about PMDS codes.

Proposition 3. [1] Maximally recoverable (MR) codes of length n and dimension k exist for any locality configuration over any finite field of size $q > \binom{n-1}{k-1}$.

MR codes are PMDS codes for disjoint locality blocks. Therefore, Proposition 3 implies that PMDS codes exist for any set of parameters when the field size is large enough.

A construction of PMDS codes based on rank-metric and MDS codes was given in [8], when $r_1 = r_2 = \cdots = r_m$. This gives the following existence result:

Proposition 4. [8] $[n, k, \ell; r, ..., r]$ -PMDS codes with m locality blocks of the same length exist over a finite field of size q^{n-mr} .

Furthermore, some specific constructions of PMDS codes, for particular values of s or of the r_i , are given in [2–4,6].

In particular, a general construction for PMDS codes with s=1 was given in [7]. This construction is based on the concatenation of several MDS codes as building blocks.

Proposition 5. [7, Corollary 14]

- 1. For any integers $m \geq 2$ and $\ell, r_1, \ldots, r_m \geq 1$ there exists a $[n, k = m\ell 1, \ell; r_1, \ldots, r_m]$ -PMDS code over any field \mathbb{F}_q with $q \geq \max_i \{r_i\} + \ell$.
- 2. If there exists $h \in \mathbb{N}$ such that $\ell \in \{3, 2^h 1\}$ and $\max_i \{r_i\} + \ell = 2^h + 1$, then there exists a $[n, k = m\ell 1, \ell; r_1, \dots, r_m]$ -PMDS code over \mathbb{F}_q with $q = 2^h = \max_i \{r_i\} + \ell 1$.

In [7] the authors also show that this construction is basically the only one possible, i.e., every PMDS with s=1 is of this form, giving thus a characterization for this set of parameters. However, for $s \geq 2$ there is no characterization yet for PMDS codes.

2.2 Zarisky topology over finite fields

Let \mathbb{F} be a field, and $\mathbb{F}[x_1,\ldots,x_N]$ be the polynomial ring over \mathbb{F} . Denote by $\overline{\mathbb{F}}$ the algebraic closure of \mathbb{F} . For a subset $S\subseteq \mathbb{F}[x_1,\ldots,x_N]$ we define the algebraic set

$$V(S) := \{ \boldsymbol{\alpha} \in \bar{\mathbb{F}}_q^r \mid f(\boldsymbol{\alpha}) = 0, \forall f \in S \}.$$

The Zariski topology on $\bar{\mathbb{F}}^N$ is defined as the topology whose closed sets are the algebraic sets, while the complements of the Zariski-closed sets are the Zariski-open sets [9, Ch. I, Sec. 1].

Definition 6. A subset $A \subset \overline{\mathbb{F}}^N$ is called a *generic set* if A contains a non-empty Zariski-open set.

In classical geometry one studies the Zariski topology over the complex numbers. In this framework, a generic set inside \mathbb{C}^N is dense and its complement is contained in an algebraic set of dimension at most N-1.

If one wants to consider generic sets restricted to a finite field \mathbb{F}_q , the situation is slightly different. Here, for a subset $T\subseteq \mathbb{F}_q^N$ one can always find a set of polynomials $S\subseteq \mathbb{F}_q[x_1,\ldots,x_N]$ such that

$$T = \{ \boldsymbol{\alpha} \in \mathbb{F}_q^N \mid f(\boldsymbol{\alpha}) = 0, \forall f \in S \}.$$

and therefore the Zariski topology restricted to \mathbb{F}_q^N is the discrete topology. This means that it is not useful to extend the notion of generic sets to finite fields since it would not give any information.

However, given a set of polynomials $S \subseteq \mathbb{F}_q[x_1, \dots, x_N]$, we can define the set of \mathbb{F}_q -rational points as

$$V(S;\mathbb{F}_q) := \{ \boldsymbol{\alpha} \in \mathbb{F}_q^N \mid f(\boldsymbol{\alpha}) = 0, \forall f \in S \}.$$

In this setting the Schwartz-Zippel Lemma implies an analog result to the one of generic sets, as explained in the following.

Lemma 7 (Schwartz-Zippel). [10, Lemma 1.1] Let $f \in \mathbb{F}_q[x_1, \ldots, x_N]$ be a non-zero polynomial of total degree $d \geq 0$. Let $T \subseteq \overline{\mathbb{F}}$ be a finite set and let $\alpha_1, \ldots, \alpha_N$ be selected at random independently and uniformly from T. Then

$$\Pr\left(f(\alpha_1,\ldots,\alpha_N)=0\right) \leq \frac{d}{|T|}.$$

As a consequence of this result we have that, in case the size of S and the total degrees of the polynomials in S do not depend on the finite field, the proportion between the cardinality of $V(S; \mathbb{F}_q)$ and the cardinality of the whole space \mathbb{F}_q^N goes to 0 as q grows. Vice versa, for growing q the probability that a random point is in the complement of $V(S; \mathbb{F}_q)$ tends to 1. This result will be crucial in Section 5 for our random construction of PMDS codes.

2.3 Rank-metric codes

We now give some known facts about rank-metric codes. Recall that \mathbb{F}_{q^N} is isomorphic to \mathbb{F}_q^N as an \mathbb{F}_{q^-} vector space. From this it easily follows that $\mathbb{F}_{q^N}^n \cong \mathbb{F}_q^{N \times n}$. Then we can give the following definition.

Definition 8. The rank distance d_R on $\mathbb{F}_q^{N \times n}$ is defined by

$$d_R(U,V) := \operatorname{rank}(U-V), \quad U,V \in \mathbb{F}_q^{N \times n}.$$

Analogously, if $u, v \in \mathbb{F}_{q^N}^n$, then $d_R(u, v)$ is the rank of the difference of the respective matrix representations in $\mathbb{F}_q^{N \times n}$.

Observe that the definition of rank distance in the case of vectors in $\mathbb{F}_{q^N}^n$ does not depend on the choice of the basis. Moreover it can be shown that the function $d_R: \mathbb{F}_{q^N}^n \times \mathbb{F}_{q^N}^n \to \mathbb{R}_{\geq 0}$ is a metric.

Definition 9. An \mathbb{F}_{q^N} -linear rank-metric code \mathcal{C} of length n and dimension k is a k-dimensional subspace of $\mathbb{F}_{q^N}^n$ equipped with the rank distance. The minimum distance of \mathcal{C} is defined as

$$d_R(\mathcal{C}) := \min \{ d_R(\boldsymbol{u}, \boldsymbol{v}) \mid \boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}, \boldsymbol{u} \neq \boldsymbol{v} \}.$$

Theorem 10 (Singleton-like Bound). [11, Theorem 1] Let $\mathcal{C} \subseteq \mathbb{F}_{q^N}^n$ be an \mathbb{F}_{q^N} -linear rank-metric code of dimension k. Then

$$d_R(\mathcal{C}) \le n - k + 1.$$

Definition 11. Codes attaining the Singleton-like bound are called *Maximum Rank Distance (MRD) Codes*.

A necessary and sufficient condition for the existence of MRD codes is that $n \leq N$. In this framework, a characterization for \mathbb{F}_{q^N} -linear MRD codes in terms of their generator matrices was given in [12, Corollary 2.12], which in turn is based on a result given in [13]. For this we define the set

$$\mathcal{E}_q(k,n) := \left\{ E \in \mathbb{F}_q^{k \times n} \mid \operatorname{rank}(E) = k \right\}.$$

Proposition 12 (MRD criterion). Let $G \in \mathbb{F}_{q^m}^{k \times n}$ be a generator matrix of a rank-metric code $\mathcal{C} \subseteq \mathbb{F}_{q^m}^n$. Then \mathcal{C} is an MRD code if and only if

$$\operatorname{rank}(GE^T) = k$$

for all $E \in \mathcal{E}_q(k, n)$.

3 General construction using rank metric codes

In this section we generalize the construction given in [8]. In that work the authors use Gabidulin codes in order to build $[n, k, \ell, r, \ldots, r]$ -PMDS codes. We will show that this construction also works for different r_i , and that Gabidulin codes can be replaced by any linear MRD codes.

Fix $n, k, \ell, r_1, \ldots, r_m$, and let $G \in \mathbb{F}_{q^N}^{k \times m\ell}$ be the generator matrix of a MRD code. For the existence of an MRD code we need $N \geq m\ell$. Moreover, for every $i = 1, \ldots, m$, we consider a $[\ell + r_i, \ell]$ -MDS code over \mathbb{F}_q with generator matrix M_i , and define

$$M := \begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & M_m \end{pmatrix} \in \mathbb{F}_q^{m\ell \times n}.$$
 (2)

We can now formulate our PMDS construction.

Theorem 13. Let $\widetilde{G} \in \mathbb{F}_{q^N}^{k \times m\ell}$ be the generator matrix of a MRD code and let M be the matrix defined in (2). Then the matrix $\widetilde{G}M$ is a generator matrix for a $[n, k, \ell, r_1, \ldots, r_m]$ -PMDS code over \mathbb{F}_{q^N} .

Proof. Let $G := \widetilde{G}M$ and let $S \in \mathcal{T}_{k,\ell}(G)$ be the submatrix obtained by selecting columns h_1, \ldots, h_{k_j} from the jth block for $j = 1, \ldots, m$, where $k_i \leq \ell$ and $k_1 + \ldots + k_m = k$. S is equal to $\widetilde{G}\widetilde{M}$, where

$$\widetilde{M} = \begin{pmatrix} N_1 & 0 & \dots & 0 \\ 0 & N_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & N_m \end{pmatrix},$$

and N_j is the $\ell \times k_j$ submatrix of M_j obtained by the respective selected columns. Since M_i generates an $[\ell + r_i, \ell]$ -MDS code, any ℓ columns of M_i are linearly independent. Thus, $\mathrm{rank}(N_i) = k_i$ and $\mathrm{rank}(\widetilde{M}) = k_1 + \ldots + k_m = k$. By Proposition 12 we have that $\det(\widetilde{G}\widetilde{M}) \neq 0$, and we conclude the proof using Proposition 16.

Corollary 14. Let $m \geq 2$ and $\ell, r_1, \ldots, r_m \geq 1$, $k \geq \ell$ be positive integers. Then, for every prime p and every positive integer $L \geq n_0 m \ell$ there exists a $[n, k, \ell, r_1, \ldots, r_m]$ -PMDS code over \mathbb{F}_{p^L} , where

$$n_0 = \min\{j \in \mathbb{N} \mid p^j \ge \ell + r_i - 1, \text{ for } i = 1, \dots, m\}.$$

Proof. A MRD code in $\mathbb{F}_{q^N}^{m\ell}$ exists if $N \geq m\ell$. Suitable MDS codes over \mathbb{F}_q for the matrix in (2) exist if $q \geq \max\{\ell + r_i - 1\}$. The statement follows from Theorem 13 with $q = p^{n_0}$ and $L = n_0 N$.

4 Algebraic description of PMDS codes

We will now define a standard form for generator matrices of PMDS codes. This standard form is the main tool for the random construction of PMDS codes.

Theorem 15 (PMDS standard form). Let $m \geq 2$ and $s, \ell, r_1, \ldots, r_m \geq 1$ and let \mathcal{C} be a $[n, k = m\ell - s, \ell; r_1, \ldots, r_m]$ -PMDS code over a field \mathbb{F}_q . Then \mathcal{C} has a generator matrix of the form

$$G = (B_1 \mid \dots \mid B_m), \tag{3}$$

where

- $B_i = (C_i \mid D_i), C_i \in \mathbb{F}_q^{k \times \ell}$ and $D_i \in \mathbb{F}_q^{k \times r_i}$ for $i = 1, \dots, m$, and
- the submatrix $G_C = (C_1 \mid \cdots \mid C_m)$ is of the form

$$G_C = [I_k \mid A],$$

with A being superregular.

Proof. Let \widetilde{G} be a generator matrix for \mathcal{C} of the form (1), i.e.

$$\widetilde{G} = \left(\widetilde{B}_1 \mid \dots \mid \widetilde{B}_m\right).$$

Puncturing every block \widetilde{B}_i in the last r_i columns, we get that the submatrix \widetilde{G}_C is the generator matrix of a $[m\ell,k]$ -MDS code. Operating on the rows of such a submatrix we can transform it to a matrix $G_C = [I_k \mid A]$, with A superregular. I.e., there exists an invertible matrix $P \in \mathrm{GL}_k(\mathbb{F}_q)$ such that $P\widetilde{G}_C = [I_k \mid A]$, and therefore the matrix $G := P\widetilde{G}$ is a generator matrix of \mathcal{C} of the required form.

We now consider the entries $a_{w,z}$ of A as variables $x_{w,z}$ for $w=1,\ldots,k$ amd $z=1,\ldots,s$. We know that the column space of D_i is inside the column space of C_i , by the parameters of the block MDS codes. This means that every column in D_i is a linear combination of the columns of C_i . If we denote by $D_i^{(j)}$ the jth column of D_i , then

$$D_i^{(j)} = \sum_{t=1}^{\ell} y_{t,i,j} C_i^{(t)} \tag{4}$$

for some $y_{t,i,j}$, which we also consider variable. This way we can consider a $k \times n$ generator matrix as a matrix in $\mathbb{F}_q[x_{w,z}, y_{t,i,j}]^{k \times n}$ (where $\mathbb{F}_q[x_{w,z}, y_{t,i,j}]$ denotes the polynomial ring in all $x_{w,z}, y_{t,i,j}$).

denotes the polynomial ring in all $x_{w,z}, y_{t,i,j}$). Let $R = \sum_{i=1}^{m} r_i$. We denote $\boldsymbol{\alpha} := (\alpha_{w,z})_{w,z} \in \mathbb{F}_q^{sk}$ and $\boldsymbol{\beta} := (\beta_{t,i,j})_{t,i,j} \in \mathbb{F}_q^{\ell R}$. If we replace the variables $x_{w,z}, y_{t,i,j}$ described above in a matrix in PMDS standard form by the values $\alpha_{w,z}, \beta_{t,i,j}$, we denote the corresponding generator matrix by

$$G(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

Analogously we will denote the variable form by G(x, y).

However, a general matrix of this form is not necessarily a generator matrix of a PMDS code for any values α, β . The following proposition shows what needs to be fulfilled to generate a PMDS code:

Proposition 16. A matrix $G \in \mathbb{F}_q^{k \times n}$ generates a $[n, k = m\ell - s, \ell; r_1, \dots, r_m]$ -PMDS code if and only if, every submatrix in the set

$$\mathcal{T}_{k,\ell}(G) := \left\{ S \in \mathbb{F}^{k \times k} \mid \begin{array}{c} S \ \textit{is a submatrix of G with} \\ \textit{at most ℓ columns per block B_i} \end{array} \right\}$$

has non-zero determinant.

Proof. This follows from the definition of PMDS, cf. also [7].

The above results give an algebraic description of the generator matrix of a $[n, k = m\ell - s, \ell; r_1, \dots, r_m]$ -PMDS code over \mathbb{F}_q , as follows. If we consider the variable form of a generator matrix G as above, and the polynomial

$$p(\boldsymbol{x}, \boldsymbol{y}) := \operatorname{lcm} \{ \det S \mid S \in \mathcal{T}_{k,\ell}(G) \} \in \mathbb{F}_q[x_{w,z}, y_{t,i,j}],$$
 (5)

then, we have that $G(\alpha, \beta)$ generates a $[n, k = m\ell - s, \ell; r_1, \dots, r_m]$ -PMDS code over \mathbb{F}_q if and only if $p(\alpha, \beta)$ is non-zero.

5 Topological and probability results

In this section we first deal with the algebraic description of the generator matrix of a PMDS code in the algebraic closure of the finite field where we want our code to be built. After that, we analyze the probability that a code whose generator matrix is of the form $G(\alpha, \beta)$ is PMDS. Moreover, we also study the existence of PMDS codes for given parameters $n, k, \ell, s, r_1, \ldots, r_m$ and $R = \sum_{i=1}^m r_i$, giving sufficient conditions on the field size. Although for s=1 this problem was completely solved in [7], for $s \geq 2$ this is still an open problem.

We denote the set of valid entries for PMDS generator matrices over the algebraic closure of the finite field \mathbb{F}_q by

$$\mathcal{A}_{\text{PMDS}} := \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \bar{\mathbb{F}}_q^{sk} \times \bar{\mathbb{F}}_q^{\ell R} \mid \text{rowspace}(G(\boldsymbol{\alpha}, \boldsymbol{\beta})) \text{ is PMDS} \right\},$$

Then the following result holds.

Theorem 17. A_{PMDS} is a generic set.

Proof. By Proposition 16 we have that

$$\mathcal{A}_{\mathrm{PMDS}} := \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \bar{\mathbb{F}}_q^{sk} \times \bar{\mathbb{F}}_q^{\ell R} \mid p(\boldsymbol{\alpha}, \boldsymbol{\beta}) \neq 0 \right\},$$

and therefore $\mathcal{A}_{\text{PMDS}}$ is a Zariski open set.

Concerning the non-emptiness, let $q = p^{t_0}$. From Corollary 14 there exists a $[n, k, \ell, r_1, \ldots, r_m]$ -PMDS code \mathcal{C} over \mathbb{F}_{p^L} , for some L multiple of t_0 . By Theorem 15, \mathcal{C} has a generator matrix of the form $G(\alpha, \beta)$, therefore $(\alpha, \beta) \in \mathcal{A}_{\text{PMDS}}$.

This means that over the algebraic closure, by probability 1, for randomly chosen α, β the matrix $G(\alpha, \beta)$ generates a PMDS code. For underlying *finite* fields, this implies that for growing field size this probability will tend to 1. We now derive a probability formula depending on the field size.

We can easily observe that the entries of G(x, y) are polynomials of total degree 0, 1 or 2. In particular, if $t := \lceil \frac{s}{\ell} \rceil$ and we write G(x, y) as in (3), then the entries of the blocks D_i are polynomials of degree at most 1 for $i = 1, \ldots, m-t$, and of degree at most 2 for the last t blocks.

To estimate the degree of p(x, y) we need the following lemma.

Lemma 18 (Derivation of Vandermonde's identity).

$$\sum_{j=0}^{r} (r-j) \binom{m}{j} \binom{n}{r-j} = n \binom{m+n-1}{r-1}.$$

Lemma 19. The total degree of the polynomial p(x, y), defined as in (5), satisfies the inequality

 $\deg p(\boldsymbol{x}, \boldsymbol{y}) \le 2(n-k) \binom{n-1}{k-1}.$

Proof. It holds that

$$\mathcal{T}_{k,\ell}(G) \subset \mathcal{M}_k(G) := \{ S \in \mathbb{F}_q^{k \times k} \mid S \text{ is a submatrix of } G \},$$

hence the polynomial p(x, y) divides the polynomial

$$q(\boldsymbol{x}, \boldsymbol{y}) := \operatorname{lcm} \{ \det S \mid S \in \mathcal{M}_k(G) \}.$$

Observe that the entries of the first k columns of the submatrix G_C have degree 0. Let $t := \lceil \frac{s}{\ell} \rceil$. Then the entries of the columns corresponding to the blocks D_i for $i = 1, \ldots, m-t$ have degree at most 1, as well as the last $m\ell - k$ columns of G_C . Finally, the columns of the blocks D_i for $i = m - t + 1, \ldots, m$, have degree at most 2. In particular, all the entries of the blocks D_i and the last $m\ell - k$ columns of G_C have degree at most 2. Therefore,

$$\deg q(\boldsymbol{x}, \boldsymbol{y}) \leq \sum_{S \in \mathcal{M}_k(G)} \deg \det S$$

$$\leq \sum_{j_0=0}^k 2(k - j_0) \binom{k}{j_0} \binom{n - k}{k - j_0}$$

$$= 2(n - k) \binom{n - 1}{k - 1}$$

where the last equality follows from Lemma 18. Since $\deg p(\boldsymbol{x}, \boldsymbol{y}) \leq \deg q(\boldsymbol{x}, \boldsymbol{y})$ we conclude the proof.

We can now formulate a lower bound for the probability that a randomly chosen generator matrix in PMDS standard form generates a PMDS code over a finite field \mathbb{F}_q :

Theorem 20. Let the entries of α and β be uniformly and independently chosen at random in \mathbb{F}_q . Then

$$\Pr\{\text{rowspace}(G(\boldsymbol{\alpha},\boldsymbol{\beta})) \text{ is } PMDS \} \ge 1 - \frac{2(n-k)\binom{n-1}{k-1}}{q}.$$

Proof. We have

$$\begin{split} & \operatorname{Pr}\{\operatorname{rowspace}(G(\boldsymbol{\alpha},\boldsymbol{\beta})) \text{ is PMDS } \} \\ = & \operatorname{Pr}\{(\boldsymbol{\alpha},\boldsymbol{\beta}) \notin V(p(\boldsymbol{x},\boldsymbol{y});\mathbb{F}_q)\} \\ = & 1 - \operatorname{Pr}\{p(\boldsymbol{\alpha},\boldsymbol{\beta}) = 0\} \\ \geq & 1 - \frac{\deg p(\boldsymbol{x},\boldsymbol{y})}{q} \geq 1 - \frac{2(n-k)\binom{n-1}{k-1}}{q}, \end{split}$$

where the last two inequalities follow from Lemmas 7 and 19, respectively.

From this we can deduce an existence result for PMDS codes over finite fields of a given minimal size.

Corollary 21. If $q > 2(n-k)\binom{n-1}{k-1}$ then there exists a $[n, k, \ell, r_1, \ldots, r_m]$ -PMDS code over the finite field \mathbb{F}_q .

One notices that this is not an improvement over the known existence result from Proposition 3.

However, we can improve the above result, considering a step-by-step construction. We will again consider a generator matrix in PMDS standard form as in (3). We start with an $[m\ell, k]$ -MDS code over a finite field \mathbb{F}_q and write its generator matrix as $(C_1 \mid \cdots \mid C_m)$. For this purpose it is sufficient that $q \geq m\ell - 1$. Then we construct the first column $D_1^{(1)}$ of the block D_1 as in (4). Every entry will be a degree 1 polynomial in the variables $y_{t,1,1}$ for $t = 1, \ldots, \ell$. Imposing that every $k \times k$ minor of

$$G' := \left(C_1 \mid D_1^{(1)} \mid C_2 \mid \dots \mid C_m \right)$$

is non-zero, we get the condition $p'(y_{t,1,1}) \neq 0$, where

$$p'(y_{t,1,1}) = \text{lcm}\{\det S \mid S \in \mathcal{T}_{k,\ell}(G')\}.$$

Using Lemma 7, we obtain

$$\Pr\{p'(\beta_{t,1,1}) = 0\} \le \frac{\deg p'}{q}.$$

In this situation deg $p' \leq {m\ell \choose k-1}$, therefore for $q > {m\ell \choose k-1}$ we have that there exists at least one evaluation of p' that is non-zero and such that $G'(\beta_{t,1,1})$ generates a $[n, k, \ell, 1, 0, \ldots, 0]$ -PMDS code.

Repeating this construction step by step, we get a $[n, k, \ell, r_1, \ldots, r_{m-1}, r_m - 1]$ -PMDS code. From that code we build the last column $D_m^{(r_m)}$ of the block D_m again as in (4):

$$D_m^{(r_m)} = \sum_{t=1}^{\ell} y_{t,m,r_m} C_m^{(t)}.$$

In the end we get the matrix

$$G(y_{t,m,r_m}) = (C_1 \mid D_1 \mid \dots \mid C_m \mid D_m),$$
 (6)

where the matrix $G(y_{t,m,r_m})$ without the last column generates a $[n,k,\ell,r_1,\ldots,r_{m-1},r_m-1]$ -PMDS code, and the entries of the last column are polynomials of total degree at most 1 in the variables y_{t,m,r_m} , for $t=1,\ldots,\ell$.

Definition 22. Let $m, n, k, n_1, \ldots, n_m, f_1, \ldots, f_m$ be positive integers such that $n = \sum_i n_i$. Let $N_0 := 0$, $N_i := \sum_{j=1}^i n_j$ and $J_i = \{N_{i-1} + 1, \ldots, N_i\}$ for $i = 1, \ldots, m-1$. We define the set

$$\mathcal{M}(k; n_1 \dots, n_m; f_1, \dots, f_m) = \{I \subset \{1, \dots, n\} \mid |I| = k, |I \cap J_i| \le f_i\}$$

and $M(k; n_1, \ldots, n_m; f_1, \ldots, f_m)$ as its cardinality.

Proposition 23. Let $G(y_{t,m,r_m})$ be as in (6). The total degree of the polynomial

$$\tilde{p}(y_{t,m,r_m}) := \operatorname{lcm} \{ \det S \mid S \in \mathcal{T}_{k,\ell}(G(y_{t,m,r_m})) \}$$

is less or equal to

$$M(k-1; \ell+r_1, \ldots, \ell+r_{m-1}, \ell+r_m-1; \ell, \ldots, \ell, \ell-1) =: M^*.$$

Proof. The polynomial $\tilde{p}(y_{t,m,r_m})$ has degree less or equal to $\sum \deg \det S$. By assumption all the determinants $\det S$ for S not containing the last column are non zero elements in \mathbb{F}_q . The only polynomials with degree 1 are the determinants of $k \times k$ submatrices involving the last column, and they are exactly M^* many.

Corollary 24. If $q > M^*$ then there exists a $[n, k, \ell, r_1, \dots, r_m]$ -PMDS code over the finite field \mathbb{F}_q .

To our knowledge there is no closed formula for M^* . However, it is easy to see that $M^* \leq \binom{n-1}{k-1}$ and that the inequality is strict if any of the conditions $|I \cap J_i| \leq f_i$ is non-empty. Hence, Corollary 24 improves upon Proposition 3.

6 Conclusion

We gave a generalization of a known PMDS code construction based on rankmetric codes. Furthermore, we investigated a random construction of PMDS codes by prescribing a PMDS standard form. We derived a lower bound on the probability that a randomly filled matrix in PMDS standard form generates a PMDS code. This probability implies a lower bound on the field size needed for such codes to exist. In the end we gave a step-by-step construction of such a generator matrix to improve this lower bound on the necessary field size.

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