

# THE LEVEL SET METHOD FOR THE TWO-SIDED MAX-PLUS EIGENPROBLEM

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**ABSTRACT.** We consider the max-plus analogue of the eigenproblem for matrix pencils,  $Ax = \lambda Bx$ . We show that the spectrum of  $(A, B)$  (i.e., the set of possible values of  $\lambda$ ) is a finite union of intervals, which can be computed by a pseudo-polynomial number of calls to an oracle that computes the value of a mean payoff game. The proof relies on the introduction of a spectral function, which we interpret in terms of the least Chebyshev distance between  $Ax$  and  $\lambda Bx$ . The spectrum is obtained as the zero level set of this function.

## 1. INTRODUCTION

Max-plus algebra is the analogue of linear algebra developed over the max-plus semiring which is the set  $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$  equipped with the operations of “addition” “ $a + b$ ” :=  $a \vee b = \max(a, b)$  and “multiplication” “ $ab$ ” :=  $a + b$ . Zero of this semiring is  $-\infty$ , and unity of this semiring is 0. Note that “ $a^{-1}$ ” =  $-a$ . The operations of the semiring are extended to matrices and vectors over  $\mathbb{R}_{\max}$ . That is if  $A = (a_{ij})$ ,  $B = (b_{ij})$  and  $C = (c_{ij})$  are matrices of compatible sizes with entries from  $\mathbb{R}_{\max}$ , we write  $C = A \vee B$  if  $c_{ij} = a_{ij} \vee b_{ij}$  for all  $i, j$  and  $C = AB$  if  $c_{ij} = \bigvee_k (a_{ik} + b_{kj})$  for all  $i, j$ .

We investigate the two-sided eigenproblem in max-plus algebra: for two matrices  $A, B \in \mathbb{R}_{\max}^{n \times m}$ , find scalars  $\lambda \in \mathbb{R}_{\max}$  called *eigenvalues* and vectors  $x \in \mathbb{R}_{\max}^n \setminus \{-\infty\}$  called *eigenvectors* such that

$$(1) \quad “Ax = \lambda Bx”,$$

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where the operations have max-plus algebraic sense. In the conventional notation this reads

$$(2) \quad \max_{j=1}^m (a_{ij} + x_j) = \lambda + \max_{j=1}^m (b_{ij} + x_j), \quad \forall i = 1, \dots, n.$$

The set of eigenvalues will be called the *spectrum* of  $(A, B)$  and denoted by  $\text{spec}(A, B)$ .

When  $B$  is the max-plus identity matrix  $I$  (all diagonal entries equal 0 and all off-diagonal entries equal  $-\infty$ ), problem (1) is the max-plus spectral problem. The latter spectral problem, as well as its continuous extension for max-plus linear operators, is of fundamental importance for a wide class of problems in discrete event systems theory, dynamic programming, optimal control and mathematical physics [BCOQ92, HOvdW05, KM97].

Problem (1) is related to the Perron-Frobenius theory for the two-sided eigenproblem in the conventional linear algebra, as studied in [MOS<sup>+</sup>98, MNV08]. When both matrices are nonnegative and depend on large parameter, it can be shown following the lines of [ABG98, Theorem 1] that the asymptotics of an eigenvalue with nonnegative eigenvector is controlled by an eigenvalue of (1). This argument calls for the development of two-sided analogue of the tropical eigenvalue perturbation theory presented in [ABG06a, ABG04].

A specific motivation to study the two-sided max-plus eigenproblem arises from discrete event systems. In particular, systems of the form “ $Ax = Bx$ ” or “ $Ax \leq Bx$ ” appear in scheduling. Indeed, when  $\lambda = 0$ , System (2) can be interpreted as a *rendez-vous* constraint. Here,  $x_j$  represents the starting time of a certain task  $j$  (for instance, the availability of a part in a manufacturing systems). The expression  $\max_{j=1}^n (a_{ij} + x_j)$  represents the earliest completion time of a task which needs at least  $a_{ij}$  time units to be completed after task  $j$  started. Thus, the system “ $Ax = Bx$ ” requires to find starting times such that two different sets of tasks are completed at the earliest exactly at the same times. In many situations, such systems cannot be solved exactly, and it is natural to solve perturbed problems like “ $Ax = \lambda Bx$ ”, which amounts to computing time separation between events. Motivations of this nature arose for instance in the work of Burns, Hulgaard, Amon, and Borriello [BHAB95], following the work of Burns on the checking of asynchronous digital circuits [Bur91]. Systems of the form “ $Ax \leq Bx$ ” have also been studied in relation with scheduling problems with both AND and OR precedence constraints, as in the work by Möhring, Skutella, and Stork [MSS04].

Similar motivations led to the study of min-max functions by Olsder [Ols91] and Gunawardena [Gun94]. Such functions can be written as finite infima of max-plus linear maps, or finite suprema of min-plus linear maps. They also arise as dynamic programming programming operators of zero-sum deterministic games. In particular, the fixed points

and invariant halflines of min-max functions studied in [CTGG99, DG06] can be also used to compute values of zero-sum deterministic games with mean payoff [DG06, ZP96]. Intimate relations between the computation of the value of mean payoff games and two-sided linear systems in max-plus algebra have been established in [AGG09].

In max-plus algebra, a special form of min-max functions appears in Cuninghame-Green [CG79], under the name of  $AA^*$ -products. The same functions appear as nonlinear projectors on max-plus cones playing essential role in the max-plus analogue of Hahn-Banach theorem [CGQS05, LMS01]. The compositions of nonlinear projectors are more general min-max functions, and they appear when one approaches two-sided systems  $Ax = By$  and  $Ax = Bx$  [CGB03], and intersections of max-plus cones [GS08, Ser09]. It is immediate to see that (1) is a parametric version of  $Ax = Bx$ .

In max-plus algebra, partial results for Problem (1) have been obtained by Binding and Volkmer [BV07], and Cuninghame-Green and Butkovič [CGB08]. In particular, Cuninghame-Green and Butkovič [CGB08] give an interval bound on the spectrum of (1) in the case when the entries of both matrices are real. Besides that, both papers treat interesting special cases, for instance when  $A$  and  $B$  square, or one of them is a multiple of the other.

In the present paper, we first show that (1) can be viewed as a fixed-point problem for a family of parametric min-max functions  $h_\lambda$ . Based on this observation, we introduce a spectral function  $s(\lambda)$  of (1), defined as the spectral radius of  $h_\lambda$ . The zero level set of  $s(\lambda)$  is precisely  $\text{spec}(A, B)$ . More generally,  $s(\lambda)$  has a natural geometric sense, being equal to the least Chebyshev distance between  $Ax$  and  $\lambda Bx$ .

The function  $s(\lambda)$  is piecewise-linear and Lipschitz continuous, and it has a linear asymptotics at large and small  $\lambda$ . In an important special case, the asymptotics is just  $\lambda + \alpha_1$  (at small  $\lambda$ ) and  $-\lambda + \alpha_2$  (at large  $\lambda$ ). We also give bounds on the spectrum of two-sided eigenproblem, which improve and generalize the bound of Cuninghame-Green and Butkovič [CGB08]. In the case when the entries of  $A$  and  $B$  are integer or  $-\infty$ , this allows us to show that all linear pieces of  $s(\lambda)$  can be identified in a pseudopolynomial number of calls to an oracle which identifies  $s(\lambda)$  at a given point. Importantly,  $s(\lambda)$  can be interpreted as the value of associated parametric mean-payoff game and it can be computed by the policy iteration algorithm of [CTGG99, DG06]. This leads to a procedure for computing the whole spectrum of (1). To our knowledge, no such general algorithm for computing the whole spectrum of (1) was known previously. We also believe that the level set method used here, relying on the introduction of the spectral function, is of independent interest and may have other applications.

In some cases the spectral function can be computed analytically. In particular, we will consider an example of [Ser], where it is shown that any finite system of intervals and points on the real line can be represented as spectrum of (1).

The paper is organized as follows. In Section 2 we consider two-sided systems “ $Ax = By$ ” and “ $Ax = Bx$ ”. We relate the systems “ $Ax = Bx$ ” to certain min-max functions and show that the spectral radii of these functions are equal to the least Chebyshev distance between “ $Ax$ ” and “ $Bx$ ”. In Section 3, we introduce the spectral function of two-sided eigenproblem as the spectral radius of a natural parametric extension of the min-max functions studied in Section 2. We give bounds on the spectrum of two-sided eigenproblem and investigate the asymptotics of  $s(\lambda)$ . We reconstruct the spectral function and hence the whole spectrum in a pseudopolynomial number of calls to the oracle.

## 2. TWO-SIDED SYSTEMS AND MIN-MAX FUNCTIONS

**2.1. Max-plus linear systems and nonlinear projectors.** Consider the  $n$ -fold Cartesian product  $\mathbb{R}_{\max}^n$  equipped with operations of taking supremum  $u \vee v$  and scalar “multiplication” (i.e., addition) “ $\lambda v$ ” =  $\lambda + v$ . This structure is an example of semimodule over the semiring  $\mathbb{R}_{\max}$  defined in the introduction. The subsets of  $\mathbb{R}_{\max}^n$  closed under these two operations are its subsemimodules. We will call them *max-plus cones* or just *cones*, by abuse of language. Indeed, there are important analogies and links between max-plus cones and convex cones [CGQS05, DS04, GK09, Ser09]. We also need the operation of taking infimum which we denote by  $\inf$ .

With a max-plus cone  $\mathcal{V} \subseteq \mathbb{R}_{\max}^n$  we can associate an operator  $P_{\mathcal{V}}$  defined by its action

$$(3) \quad P_{\mathcal{V}}z = \bigvee \{y \in \mathcal{V} \mid y \leq z\}.$$

Consider the case when  $\mathcal{V} \subseteq \mathbb{R}_{\max}^n$  is *generated* by a set  $S \in \mathbb{R}_{\max}^n$ , which means that it is the set of bounded max-plus linear combinations

$$(4) \quad z = \bigvee_{y \in S} \lambda_y + y.$$

In this case

$$(5) \quad \begin{aligned} P_{\mathcal{V}}z &= \bigvee_{y \in S} z/y + y, \text{ where} \\ z/y &= \max\{\gamma \mid \gamma + y \leq z\} = \bigwedge_{j \in \text{supp}(y)} (z_j - y_j) = \bigwedge_{j=1}^n (z_j - y_j), \end{aligned}$$

with the convention  $(-\infty) + (+\infty) = +\infty$ . Here and in the sequel  $\text{supp}(y) := \{i \mid y_i \neq -\infty\}$  denotes the support of  $y$ . Note that  $z/y = \infty$  if and only if  $y = -\infty$ .

Further we are interested only in the case when  $\mathcal{V}$  is finitely generated, i.e.,  $S$  is finite. Let  $T_i$  denote the set of indices where the minimum in  $z/y^i$  is attained. The following result is classical.

**Proposition 1** ([BCOQ92, But03, HOvdW05]). *Let a cone  $\mathcal{V} \subseteq \mathbb{R}_{\max}^n$  be generated by  $y^1, \dots, y^m$  and let  $z \in \mathbb{R}_{\max}^n$ . The following statements are equivalent.*

1.  $z \in \mathcal{V}$ .
2.  $P_{\mathcal{V}}z = z$ .
3.  $\bigcup_{i=1}^m T_i = \text{supp } z$ .

We note that the set covering condition 3. has been generalized to the case of Galois connections [AGK05].

By this proposition, operator  $P_{\mathcal{V}}$  is a projector onto  $\mathcal{V}$ . It is an isotonic and  $+$ -homogeneous operator, meaning that  $z^1 \leq z^2$  implies  $P_{\mathcal{V}}z^1 \leq P_{\mathcal{V}}z^2$ , and that  $P_{\mathcal{V}}(\lambda + z) = \lambda + P_{\mathcal{V}}z$ . However, in general it is neither  $\vee$ - nor  $\wedge$ -linear.

A finitely generated cone can be described as a max-plus column span of a matrix  $A \in \mathbb{R}_{\max}^{n \times m}$ :

$$(6) \quad \text{span}(A) := \left\{ \bigvee_{i=1}^m \lambda_i + A_{\cdot i} \mid \lambda_i \in \mathbb{R}_{\max}, i = 1, \dots, m \right\}.$$

In this case we denote  $P_A := P_{\text{span}(A)}$ , and there is an explicit expression for this operator which we recall below.

We denote  $\overline{\mathbb{R}}_{\max} := \mathbb{R}_{\max} \cup \{+\infty\}$  and view  $A \in \overline{\mathbb{R}}_{\max}^{n \times m}$  as an operator from  $\overline{\mathbb{R}}_{\max}^n$  to  $\overline{\mathbb{R}}_{\max}^m$ . The *residuated operator*  $A^\sharp$  from  $\overline{\mathbb{R}}_{\max}^m$  to  $\overline{\mathbb{R}}_{\max}^n$  is defined by

$$(7) \quad (A^\sharp y)_j = y/A_{\cdot j} = \bigwedge_{i=1}^m (-a_{ij} + y_i),$$

with the convention  $(-\infty) + (+\infty) = +\infty$ . Note that this operator, also known as *Cuninghame-Green inverse*, sends  $\mathbb{R}_{\max}^m$  to  $\mathbb{R}_{\max}^n$  whenever  $A$  does not have columns equal to  $-\infty$ . The term “residuated” refers to the property

$$(8) \quad Ax \leq y \Leftrightarrow x \leq A^\sharp y,$$

where  $\leq$  is the partial order on  $\mathbb{R}_{\max}^m$  or  $\mathbb{R}_{\max}^n$ . Using (5) we obtain

$$(9) \quad P_A(z) = \bigvee_{i=1}^n (z/A_{\cdot i}) + A_{\cdot i} = AA^\sharp z.$$

In this form (9), the nonlinear projectors were studied by Cuninghame-Green [CG79] (as  $AA^*$ -products).

Finitely generated cones are closed in the topology induced by the metric

$$(10) \quad d(x, y) = \max_i |e^{x_i} - e^{y_i}|,$$

which coincides with Birkhoff's order topology. It is known [CGQS05, Theorem 3.11] that the projectors onto such cones are continuous.

The intersection of two finitely generated cones can be expressed in terms of two-sided max-plus linear systems with separated variables  $Ax = By$ , by the following proposition.

**Proposition 2.** *Let  $A \in \mathbb{R}_{\max}^{n \times m_1}$  and  $B \in \mathbb{R}_{\max}^{n \times m_2}$ .*

1. *If  $(x, y)$  satisfies  $Ax = By \neq -\infty$  then  $z = Ax = By$  belongs to  $\text{span}(A) \cap \text{span}(B)$ . Equivalently,  $P_A P_B z = P_B P_A z = z$ .*
2. *If  $P_A P_B z = z \neq -\infty$  then there exist  $x$  and  $y$  such that  $Ax = By = z$ .*

This approach to two-sided systems is also useful in the case of systems with non-separated variables  $Ax = Bx$ , which is of greater importance for us here. This system is equivalent to

$$(11) \quad \begin{aligned} &Cx = Dy, \text{ where} \\ &C = \begin{pmatrix} A \\ B \end{pmatrix}, \quad D = \begin{pmatrix} I_n \\ I_n \end{pmatrix}, \end{aligned}$$

and  $I_n = (\delta_{ij}) \in \mathbb{R}_{\max}^{n \times n}$  denotes the max-plus  $n \times n$  identity matrix with entries

$$(12) \quad \delta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ -\infty, & \text{if } i \neq j. \end{cases}$$

In this case we have the following version of Proposition 2.

**Proposition 3.** *Let  $A, B \in \mathbb{R}_{\max}^{n \times m}$ .*

1. *If  $x$  satisfies  $Ax = Bx \neq -\infty$ , then  $v = (z \ z)^T$ , where  $z = Ax = Bx$ , belongs to  $\text{span}(C) \cap \text{span}(D)$ . Equivalently,  $P_C P_D v = P_D P_C v = P_C v = v$ .*
2. *If  $v = (z \ z)^T \neq -\infty$  and  $P_C v = v$ , then there exist  $x$  such that  $Ax = Bx = v$ .*

Pairs  $(x, y) \neq -\infty$  such that  $Ax = By = -\infty$  are described by:  $x_i \neq -\infty \Leftrightarrow A_{\cdot i} = -\infty$  and  $y_j \neq -\infty \Leftrightarrow B_{\cdot j} = -\infty$ . Analogously, vectors  $x \neq -\infty$  such that  $Ax = Bx = -\infty$  are described by  $x_i \neq -\infty \Leftrightarrow A_{\cdot i} = B_{\cdot i} = -\infty$ . Any such pair of vectors can be added to any other pair  $(x', y')$  or, respectively, vector  $x'$ , and the resulting pair of vectors will satisfy the system if and only if so does  $(x', y')$  or, respectively,  $x'$ . Therefore, we can assume in the sequel without loss of generality that there are no such solutions, i.e., that 1)  $A$  and  $B$  do not have  $-\infty$  columns in the case of separated variables, 2)  $A$  and  $B$  do not have common  $-\infty$  columns in the case of non-separated variables.

**2.2. Projectors and Perron-Frobenius theory.** Suppose that a function  $f : \mathbb{R}_{\max}^n \mapsto \mathbb{R}_{\max}^n$  is homogeneous, isotone and continuous in the topology induced by (10). As  $x \mapsto \exp(x)$  yields a homeomorphism with  $\mathbb{R}_+^n$  endowed with the usual Euclidean topology, we can use spectral theory for homogeneous, isotone and continuous functions in  $\mathbb{R}_+^n$ . We will use the following important identities, which follow from the results of Nussbaum [Nus86], see [AGG09, Lemma 2.8] for a detailed proof.

**Theorem 4** (Coro. of [Nus86],[AGG09, Lemma 2.8]). *Let  $f$  denote an order-preserving, additively homogeneous and continuous map from  $(\mathbb{R} \cup \{-\infty\})^n$  to itself. Then it has the largest eigenvalue  $r(f)$  which equals*

$$(13) \quad r(f) = \max\{\lambda \mid \exists x \in \mathbb{R}_{\max}^n \lambda + x \leq f(x)\},$$

$$(14) \quad r(f) = \inf\{\lambda \mid \exists x \in \mathbb{R}^n, \lambda + x \geq f(x)\}.$$

Note that (14) is nonlinear generalization of the classical Collatz-Wielandt formula [Min88]. Equations (13) and (14) are useful in max-plus algebra, since they work for max-plus matrix multiplication as well as for compositions of nonlinear projectors. For (14) it is essential that it is taken over vectors with real entries, and that the infimum may not be reached. Using (14) we obtain that the spectral radius of such functions is isotone:  $f(x) \leq g(x)$  for all  $x \in \mathbb{R}^n$  implies  $r(f) \leq r(g)$ . We next recall an application of (14) to the metric properties of compositions of projectors, which is due to [GS08]. The *Hilbert distance* between  $u, v \in \mathbb{R}_{\max}^n$  such that  $\text{supp}(u) = \text{supp}(v)$  is defined by

$$(15) \quad d_H(u, v) = \max_{i,j \in \text{supp}(v)} (u_i - v_i + v_j - u_j).$$

If  $\text{span}(u) \neq \text{span}(v)$  then we set  $d_H(u, v) = +\infty$ . Using (15) we define the Hilbert distance between cones  $\text{span}(A)$  and  $\text{span}(B)$ , for  $A \in \mathbb{R}_{\max}^{n \times m_1}$  and  $B \in \mathbb{R}_{\max}^{n \times m_2}$ :

$$(16) \quad d_H(A, B) := \min\{d_H(u, v) \mid u \in \text{span}(A), v \in \text{span}(B), \text{supp}(u) = \text{supp}(v)\}.$$

**Theorem 5** (cp. [GS08], Theorem 25). *Let  $A \in \mathbb{R}_{\max}^{n \times m_1}$  and  $B \in \mathbb{R}_{\max}^{n \times m_2}$ . Then*

$$(17) \quad r(P_A P_B) = r(P_B P_A) = -d_H(A, B).$$

*If  $d_H(A, B)$  is finite then it is attained by any eigenvector  $\bar{u}$  of  $P_A P_B$  with eigenvalue  $r(P_A P_B)$ , and its image  $\bar{v}$  by  $P_B$ .*

*Proof.* As  $\text{supp}(P_A P_B u) \subseteq \text{supp}(P_B u) \subseteq \text{supp}(u)$ , it follows that  $P_A P_B$  and also  $P_B P_A$  may have finite eigenvalue only if  $\text{span}(A)$  and  $\text{span}(B)$  have vectors with common support. This shows the claim for the case  $d_H(A, B) = +\infty$ .

Now let  $d_H(A, B)$  be finite. We show that  $-d_H(\bar{u}, \bar{v}) = -d_H(A, B) = r(P_A P_B)$ . Take arbitrary vectors  $u \in \text{span}(A)$  and  $v \in \text{span}(B)$  with  $\text{supp}(u) = \text{supp}(v)$ , and let  $P_u$ ,

resp.  $P_v$ , be projectors onto the rays  $U = \{\lambda u, \lambda \in \mathbb{R}_{\max}\}$ , resp.  $V = \{\lambda v, \lambda \in \mathbb{R}_{\max}\}$ . As  $U \subseteq \text{span}(A)$  and  $V \subseteq \text{span}(B)$ , we have that  $P_u \leq P_A$  and  $P_v \leq P_B$ , hence  $P_u P_v \leq P_A P_B$  and, by the monotonicity of the spectral radius,  $r(P_u P_v) \leq r(P_A P_B)$ . But  $r(P_u P_v) = -d_H(u, v)$ , as this is the only finite eigenvalue of  $P_u P_v$ , and hence  $-d_H(u, v) \leq r(P_A P_B)$  and  $-d_H(A, B) \leq r(P_A P_B)$ . Now observe that  $-d_H(\bar{u}, \bar{v})$  is equal to the eigenvalue  $r(P_A P_B)$ , like in the case of  $P_u P_v$  discussed above. This completes the proof.  $\square$

In the case of the systems with non-separated variables, we will be more interested in Chebyshev distance. For  $u, v \in \mathbb{R}_{\max}^n$  with  $\text{supp}(u) = \text{supp}(v)$  it is defined by

$$(18) \quad d_\infty(u, v) = \max_{i \in \text{supp}(v)} |u_i - v_i|.$$

There is an important special case when Hilbert and Chebyshev distances coincide.

**Lemma 6.** *Let  $u, v \in \mathbb{R}_{\max}^n$  be such that  $u \geq v$  and  $u_i = v_i$  for some  $i \in \{1, \dots, n\}$ . Then  $d_H(u, v) = d_\infty(u, v)$ .*

**Theorem 7.** *Let  $A, B \in \mathbb{R}_{\max}^{n \times m}$ , and let  $C$  and  $D$  be defined as in (11). Then*

$$(19) \quad r(P_C P_D) = r(P_D P_C) = - \min_{x \in \mathbb{R}_{\max}^m} d_\infty(Ax, Bx).$$

*Proof.* Theorem 5 implies that

$$(20) \quad r(P_C P_D) = - \min\{d_H(u, v) \mid u \in \text{span}(C), v \in \text{span}(D)\}.$$

Let  $u \in \text{span}(C)$  and denote by  $P_u$  the projector onto  $U := \{\lambda u \mid \lambda \in \mathbb{R}_{\max}\}$ . Then  $u$  is an eigenvector of  $P_u P_D$  which corresponds to the spectral radius of this operator, and applying Theorem 5 to the max cones  $U$  and  $\text{span}(D)$  we see that

$$(21) \quad d_H(u, P_D u) = \min\{d_H(u, v) \mid v \in \text{span}(D)\}.$$

Note that (21) also holds if there is no  $v \in \text{span}(D)$  with  $\text{supp}(u) = \text{supp}(v)$ , in which case  $d_H(u, P_D u) = +\infty$ . This implies

$$(22) \quad r(P_C P_D) = - \min\{d_H(u, P_D u) \mid u \in \text{span}(C)\}.$$

Observe that

$$(23) \quad u = \begin{pmatrix} Ax \\ Bx \end{pmatrix}, \quad P_D u = \begin{pmatrix} Ax \wedge Bx \\ Ax \wedge Bx \end{pmatrix}$$

for some  $x \in \mathbb{R}_{\max}^m$ , and also that  $u$  and  $P_D u$  satisfy the conditions of Lemma 6 unless  $P_D u = -\infty$ . Hence  $d_H(u, P_D u) = d_\infty(u, P_D u) = d_\infty(Ax, Bx)$ . Conversely,  $d_\infty(Ax, Bx)$  equals  $d_H(u, P_D u)$  for  $u = (Ax \ Bx)^T$ . Hence the r.h.s. of (19) is the same as the r.h.s. of (22), which completes the proof.  $\square$



**2.3. Min-max functions and Chebyshev distance.** Let  $A \in \mathbb{R}_{\max}^{n \times m_1}$  and  $B \in \mathbb{R}_{\max}^{n \times m_2}$ . In order to find a point in the intersection of  $\text{span}(A)$  and  $\text{span}(B)$  (or equivalently, solve  $Ax = By$ ), one can try to compute the action of  $(P_A P_B)^l$ , for  $l = 1, 2, \dots$ , on a vector  $z \in \mathbb{R}_{\max}^n$ . Equivalently, one can start with a vector  $x^0 \in \mathbb{R}_{\max}^{m_1}$  and compute

$$(24) \quad x^k = A^\sharp B B^\sharp A x^{k-1}, \quad k \geq 1.$$

We can assume that  $A$  and  $B$  do not have columns equal to  $-\infty$  so that  $A^\sharp y \in \mathbb{R}_{\max}^{m_1}$  and  $B^\sharp y \in \mathbb{R}_{\max}^{m_2}$  for any  $y \in \mathbb{R}_{\max}^n$ .

If at some stage  $x^k = x^{k-1} \neq -\infty$ , then  $x^k$  is a solution of the system. The details of this simple algorithm called *alternating method* can be found in [CGB03] and [Ser09].

Let  $A, B \in \mathbb{R}_{\max}^{n \times m}$ . A system  $Ax = Bx$  can be written equivalently as  $Cx = Dy$  with  $C$  and  $D$  as in (11). Applying alternating method (24) to this system, we obtain  $x^k = g(x^{k-1})$ , where

$$(25) \quad g(x) = A^\sharp A x \wedge B^\sharp B x \wedge A^\sharp B x \wedge B^\sharp A x.$$

As it is assumed that  $A$  and  $B$  do not have common  $-\infty$  columns and hence  $C$  (and  $D$ ) do not have  $-\infty$  columns,  $g(x) \in \mathbb{R}_{\max}^m$  for all  $x \in \mathbb{R}_{\max}^n$ .

It follows that (see also [CGB03])

$$(26) \quad r(g) = 0 \Leftrightarrow Ax = Bx \text{ is solvable.}$$

In particular, if  $x$  is a fixed point of  $g$  then it satisfies  $Ax = Bx$ . For the function

$$(27) \quad f(x) = x \wedge A^\sharp B x \wedge B^\sharp A x$$

which appears in [DG06], it is also true the other way around, since

$$(28) \quad \begin{aligned} Ax = Bx &\Leftrightarrow Ax \geq Bx \ \& \ Bx \geq Ax \Leftrightarrow \\ &\Leftrightarrow B^\sharp A x \geq x \ \& \ A^\sharp B x \geq x \Leftrightarrow \\ &\Leftrightarrow x \wedge A^\sharp B x \wedge B^\sharp A x = x. \end{aligned}$$

We also introduce the function  $h$ :

$$(29) \quad h(x) := A^\sharp B x \wedge B^\sharp A x.$$

Although  $f$ ,  $g$  and  $h$  are different functions, they have the same spectral radius, equal to the inverse minimal Chebyshev distance between  $Ax$  and  $Bx$ . To show this, we use the following identity.

$$(30) \quad -d_\infty(u, v) = \max\{\lambda: \lambda + u \leq v \ \& \ \lambda + v \leq u\}.$$

**Theorem 8.** *Let  $A, B \in \mathbb{R}_{\max}^{n \times m}$ . For  $C, D$  defined by (11), and  $f, g$  and  $h$  defined by (27) and (25),*

$$(31) \quad r(P_C P_D) = r(P_D P_C) = r(f) = r(g) = r(h) = - \min_{x \in \mathbb{R}_{\max}^m} d_{\infty}(Ax, Bx).$$

*Proof.* If  $v$  is an eigenvector of  $P_D P_C$  with a finite eigenvalue, then  $C^{\sharp}v$  is an eigenvector of  $g$  and  $P_C v$  is an eigenvector  $P_C P_D$ , both with the same eigenvalue. The other way around, if  $x$  is an eigenvector of  $g$  with a finite eigenvalue, then  $(Ax \ Bx)^T$  is an eigenvector of  $P_D P_C$  with the same eigenvalue. This argument shows that 1) either the spectral radii of  $P_D P_C$ ,  $P_C P_D$  and  $g$  are all finite or they all equal  $-\infty$ , 2) the equality  $r(g) = r(P_D P_C) = r(P_C P_D)$  holds true both in finite and in infinite case.

We show the remaining equalities. By (13),  $r(h)$  is the maximum of  $\lambda$  which satisfy

$$(32) \quad \exists x \in \mathbb{R}_{\max}^m : \lambda + x \leq A^{\sharp} Bx \wedge B^{\sharp} Ax.$$

This is equivalent to

$$(33) \quad \exists x \in \mathbb{R}_{\max}^m : \lambda + Ax \leq Bx \quad \& \quad \lambda + Bx \leq Ax$$

Using (30) we obtain

$$(34) \quad r(h) = \max_{x \in \mathbb{R}_{\max}^m} -d_{\infty}(Ax, Bx) = - \min_{x \in \mathbb{R}_{\max}^m} d_{\infty}(Ax, Bx).$$

It follows in particular that  $r(h) \leq 0$  and moreover,  $\lambda \leq 0$  for any  $x$  satisfying (33). Applying (13) to  $f$  and  $g$  we obtain that both  $r(f)$  and  $r(g)$  are equal to the maximum of  $\lambda$  which satisfy

$$(35) \quad \exists x \in \mathbb{R}_{\max}^m : \quad \lambda \leq 0 \quad \& \quad \lambda + Ax \leq Bx \quad \& \quad \lambda + Bx \leq Ax$$

As the first inequality follows from the other two, we obtain  $r(f) = r(g) = r(h)$ . □

Functions  $f, g$  and  $h$  as well as projectors onto finitely generated max-plus cones and their compositions, belong to the class of *min-max functions* taking  $\overline{\mathbb{R}}_{\max}^m$  to  $\overline{\mathbb{R}}_{\max}^m$ . Min-max functions were originally considered by Olsder [Ols91] and Gunawardena [Gun94]. See [CTGG99] for a formal definition. In a nutshell, these are additively homogeneous and order preserving maps, every coordinate of which can be represented as a minimum of a finite number of max-plus linear forms, or as a maximum of a finite number of min-plus linear forms. It is important that any min-max function  $q : \mathbb{R}_{\max}^n \rightarrow \mathbb{R}_{\max}^n$  can be represented as infimum of finite number of max-plus linear maps  $Q^{(p)}$  meaning that

$$(36) \quad q(x) = \bigwedge_p Q^{(p)} x,$$

in such a way that the following *selection property* is satisfied:

$$(37) \quad \forall x \exists p : q(x) = Q^{(p)}x.$$

In connection with the mean payoff games [DG06], each matrix  $Q^{(p)}$  corresponds to a one player game, where the player Min has chosen her strategy and the player Max is trying to win what he can.

In particular,  $f(x)$ ,  $g(x)$  and  $h(x)$ , respectively, are represented as infima of the max-plus linear maps  $F^{(p)}$ ,  $G^{(p)}$  and  $H^{(p)}$ , whose rows are taken from the max-plus linear forms appearing in (27), (25) and (29), respectively, in the following way:

$$(38) \quad F_{i\cdot}^{(p)} = \begin{cases} I_{i\cdot}, \\ -a_{ki} + B_{k\cdot}, \\ -b_{ki} + A_{k\cdot}. \end{cases} \quad G_{i\cdot}^{(p)} = \begin{cases} -a_{ki} + A_{k\cdot}, \\ -b_{ki} + B_{k\cdot}, \\ -a_{ki} + B_{k\cdot}, \\ -b_{ki} + A_{k\cdot}. \end{cases} \quad H_{i\cdot}^{(p)} = \begin{cases} -a_{ki} + B_{k\cdot}, \\ -b_{ki} + A_{k\cdot}. \end{cases}$$

Here  $I_{i\cdot}$  denotes the  $i$ th row of the max-plus identity matrix, and the brackets mean that any possibility, for any  $k = 1, \dots, m$  and  $a_{ki} \neq -\infty$  or  $b_{ki} \neq -\infty$ , can be taken (assumed that  $A$  and  $B$  do not have common  $-\infty$  columns). Applying Collatz-Wielandt formula (14) we obtain the following.

**Proposition 9.** *Suppose that a min-max function  $q : \mathbb{R}_{\max}^m \rightarrow \mathbb{R}_{\max}^m$  is represented as infimum of max-plus linear maps  $Q^{(l)} \in \mathbb{R}_{\max}^{m \times m}$  so that the selection property is satisfied. Then*

$$(39) \quad r(q) = \min_l r(Q^{(l)}).$$

*Proof.* The spectral radius is isotone, hence  $r(q) \leq r(Q^{(l)})$  for all  $l$ . Using (14) we conclude that for any  $\epsilon$  there is  $x \in \mathbb{R}^m$  such that  $q(x) \leq r(q) + \epsilon + x$ . As  $q(x) = Q^{(l)}x$  for some  $l$  and there is only finite number of matrices  $Q^{(l)}$ , there exists  $l$  such that

$$(40) \quad r(q) = \inf\{\mu \mid \exists x \in \mathbb{R}^n, Q^{(l)}x \leq \mu + x\} = r(Q^{(l)}).$$

□

Proposition 9 can be derived alternatively from the duality theorem in [GG98b, Theorem 19] (see also [GG98a]) or from the existence of the value of stochastic games with perfect information [LL69]. Indeed, the spectral radius can be seen to coincide with the value of a game in which Player Max chooses the initial state, see [AGG09] for more information.

The greatest eigenvalue  $r(Q^{(l)})$  of the max-plus matrix  $Q^{(l)} = (q_{ij}^{(l)}) \in \mathbb{R}_{\max}^{m \times m}$  can be computed explicitly. It is equal to the maximum cycle mean of  $Q^{(l)}$  defined by

$$(41) \quad \max_{1 \leq k \leq m} \max_{i_1, \dots, i_k} \frac{q_{i_1 i_2}^{(l)} + q_{i_2 i_3}^{(l)} + \dots + q_{i_k i_1}^{(l)}}{k}.$$

This result is fundamental in max-plus algebra, see [BCOQ92, HOvdW05, ABG06b] for more details.

### 3. THE SPECTRUM AND THE SPECTRAL FUNCTION

**3.1. Construction of the spectral function.** Given  $A \in \mathbb{R}_{\max}^{n \times m}$  and  $B \in \mathbb{R}_{\max}^{n \times m}$ , we consider the *two-sided eigenproblem* which consists in finding *eigenvalues*  $\lambda \in \mathbb{R}_{\max}$  and *eigenvectors*  $x \in \mathbb{R}_{\max}^m \setminus \{-\infty\}$  such that

$$(42) \quad Ax = \lambda + Bx.$$

The set of eigenvalues is called the *spectrum of*  $(A, B)$  and denoted by  $\text{spec}(A, B)$ .

The case of  $\lambda = -\infty$  appears if and only if  $A$  has  $-\infty$  columns, and the corresponding eigenvectors are described by  $x_i \neq -\infty \Leftrightarrow A_{\cdot i} = -\infty$ . In the sequel we assume that  $\lambda$  is finite.

Problem (42) is equivalent to  $C(\lambda)x = Dy$ , where  $C(\lambda) \in \mathbb{R}_{\max}^{2n \times m}$  and  $D \in \mathbb{R}_{\max}^{2n \times n}$  are defined by

$$(43) \quad C(\lambda) = \begin{pmatrix} A \\ \lambda + B \end{pmatrix}, \quad D = \begin{pmatrix} I \\ I \end{pmatrix}.$$

As it follows from Theorem 8,  $\text{spec}(A, B) = \{\lambda : r(P_D P_{C(\lambda)}) = 0\} = \{\lambda : r(h_\lambda) = 0\}$ , where

$$(44) \quad h_\lambda(x) = (\lambda + A^\# Bx) \wedge (-\lambda + B^\# Ax).$$

The function  $h_\lambda$  can be represented as infimum of max-plus linear maps so that the selection property (37) is satisfied. Namely,

$$(45) \quad h_\lambda(x) = \bigwedge_p H_\lambda^{(p)} x,$$

where for  $i = 1, \dots, m$

$$(46) \quad (H_\lambda^{(p)})_{i \cdot} = \begin{cases} \lambda - a_{ki} + B_{k \cdot}, & \text{for } 1 \leq k \leq n, a_{ki} \neq -\infty, \\ -\lambda - b_{ki} + A_{k \cdot}, & \text{for } 1 \leq k \leq n, b_{ki} \neq -\infty, \end{cases}$$

the brackets meaning that any listed choice can be taken.

The greatest eigenvalue of  $H_\lambda$  equals the maximum cycle mean of  $H_\lambda$ . Using formula (41), we observe that  $r(H_\lambda)$  is a piecewise-linear function, meaning that it is composed of a finite number of linear pieces. More precisely, we have the following.

**Proposition 10.** *Either  $r(H_\lambda^{(p)}) = -\infty$  for all  $\lambda$ , or  $r(H_\lambda^{(p)})$  is a finite piecewise-linear convex Lipschitz function of  $\lambda$ .*

*Proof.* Using (41) we observe that  $r(H_\lambda^{(p)}) = -\infty$  if and only if the associated digraph of  $H_\lambda^{(p)}$  is acyclic, and then this holds for all values of  $\lambda$ .

If  $r(H_\lambda^{(p)})$  is finite, then any finite cycle mean of  $H_\lambda^{(p)}$  can be written as  $(k\lambda + a)/l$ , where  $l$  is the length of the cycle and  $k$  is an integer number with modulus not greater than  $l$ , hence this linear function is Lipschitz. The function  $r(H_\lambda^{(p)})$  is pointwise maximum of a finite number of such linear functions, hence it is a convex Lipschitz piecewise-linear function.  $\square$

**Definition 1** (Spectral Function). *We define the spectral function of (42) by*

$$(47) \quad s(\lambda) := r(h_\lambda) = r(P_D P_{C(\lambda)}).$$

It follows from Theorem 8 that  $s(\lambda) \leq 0$  and that  $s(\lambda) = 0$  if and only if  $\lambda \in \text{spec}(A, B)$ . In general,  $s(\lambda)$  is equal to the inverse minimal Chebyshev distance between  $Ax$  and  $\lambda + Bx$ .

By Proposition 9,

$$(48) \quad s(\lambda) = \bigwedge_p r(H_\lambda^{(p)}).$$

As  $r(H_\lambda^{(p)})$  are piecewise-linear and Lipschitz, we conclude the following.

**Corollary 11.** *Either  $s(\lambda) = -\infty$  for all  $\lambda \in \mathbb{R}$  or  $s(\lambda)$  is a finite piecewise-linear Lipschitz function.*

Let us consider the case  $s(\lambda) = -\infty$  in more detail. Define  $A^\circ = (a_{ij}^\circ) \in \mathbb{R}_{\max}^{m \times n}$  and  $B^\circ = (b_{ij}^\circ) \in \mathbb{R}_{\max}^{m \times n}$  by

$$(49) \quad a_{ij}^\circ = \begin{cases} 0, & \text{if } a_{ij} \in \mathbb{R}, \\ -\infty, & \text{if } a_{ij} = -\infty. \end{cases}, \quad b_{ij}^\circ = \begin{cases} 0, & \text{if } b_{ij} \in \mathbb{R}, \\ -\infty, & \text{if } b_{ij} = -\infty. \end{cases},$$

The spectral function of the eigenproblem  $A^\circ x = \lambda + B^\circ x$  will be denoted by  $s^\circ(\lambda)$ .

**Proposition 12.** *The following are equivalent:*

1.  $s(\lambda)$  is finite for all  $\lambda$ ;
2.  $s^\circ(\lambda)$  is finite for all  $\lambda$ ;

3.  $A^\circ x = B^\circ x$  has a nontrivial solution whose entries belong to  $\{0, -\infty\}$ .

*Proof.* 1.  $\Leftrightarrow$  2 :  $s(\lambda) = -\infty$  if and only if there exists  $H_\lambda^{(p)}$  such that  $r(H_\lambda^{(p)}) = -\infty$ . By (41), this just means that the associated digraph of  $H_\lambda^{(p)}$  does not have cycles with finite weight. This property does not depend on the value of finite coefficients in  $H_\lambda^{(p)}$  and hence  $s(\lambda) = -\infty$  if and only if  $s^\circ(\lambda) = -\infty$ .

3.  $\Leftrightarrow$  1 : As  $s(\lambda)$  is equal to the inverse minimal Chebyshev distance between  $Ax$  and  $Bx$ , it is infinite if and only if there is no nontrivial vector  $x$  such that  $\text{supp}(Ax) = \text{supp}(Bx)$ , which is the negation of 3.  $\square$

Condition 3. of Proposition 12 provides a criterion for  $s(\lambda) = -\infty$ , which can be verified in polynomial time.

Let us indicate yet another consequence of the fact that  $r(H_\lambda^{(p)})$  and  $s(\lambda)$  are piecewise-linear.

**Proposition 13.** *If  $\text{spec}(A, B)$  is not empty, then it is a finite system of closed intervals and points.*

Conversely, in [Ser], see also Subsect. 3.5, it is shown that any system of closed intervals and points in  $\mathbb{R}$  can be represented as spectrum of  $(A, B)$ .

**3.2. Bounds on the spectrum of  $(A, B)$ .** Next we recall a bound on the spectrum obtained by Cuninghame-Green and Butkovič [CGB08], extending it to the case when  $A = (a_{ij})$  and  $B = (b_{ij})$  may have infinite entries. Denote

$$(50) \quad \begin{aligned} \underline{D}(A, B) &= \bigvee_{i: A_{i\cdot} \text{ finite}} A_{i\cdot} / B_{i\cdot}, \\ \overline{D}(A, B) &= - \bigvee_{i: B_{i\cdot} \text{ finite}} B_{i\cdot} / A_{i\cdot}. \end{aligned}$$

Since  $A_{i\cdot} / B_{i\cdot} = \max\{\gamma \mid A_{i\cdot} \geq \gamma + B_{i\cdot}\}$  is finite when  $A_{i\cdot}$  is finite and  $B_{i\cdot}$  is not  $-\infty$ , we immediately see the following.

**Lemma 14.**  *$\underline{D}(A, B)$  (resp.  $\overline{D}(A, B)$ ) is finite if and only if there exists  $i = 1, \dots, n$  such that  $A_{i\cdot}$  is finite (resp.  $B_{i\cdot}$  is finite).*

When  $A$  and  $B$  have finite entries only,  $\underline{D}(A, B)$  and  $\overline{D}(A, B)$  are just like the bounds of [CGB08, Theorem 2.1]:

$$(51) \quad \begin{aligned} \underline{D}(A, B) &= \bigvee_i \bigwedge_j (a_{ij} - b_{ij}), \\ \overline{D}(A, B) &= \bigwedge_i \bigvee_j (a_{ij} - b_{ij}). \end{aligned}$$

Note that  $\underline{D}(A, B)$  and  $\overline{D}(A, B)$  defined by (50) take infinite values if  $A$  or  $B$  do not contain finite rows.

**Proposition 15.** *If  $Ax \leq \lambda + Bx$  (resp.  $Ax \geq \lambda + Bx$ ) has solution  $x > -\infty$ , then  $\lambda \geq \underline{D}(A, B)$  (resp.  $\lambda \leq \overline{D}(A, B)$ ).*

*Proof.* If there exists  $i$  such that  $a_{ij} > \lambda + b_{ij}$  for all  $j = 1, \dots, m$ , then  $Ax \leq \lambda + Bx$  cannot have solutions. This condition is equivalent to  $A_{i\cdot}/B_{i\cdot} > \lambda$  plus the finiteness of  $A_{i\cdot}$ . Taking supremum of  $A_{i\cdot}/B_{i\cdot}$  over  $i$  such that  $A_{i\cdot}$  is finite yields  $\underline{D}(A, B)$ . This shows that if  $Ax \leq \lambda + Bx$  then  $\lambda \geq \underline{D}(A, B)$ . The remaining part follows analogously.  $\square$

The next result is an extension of [CGB08, Theorem 2.1].

**Corollary 16.**  $\text{spec}(A, B) \subseteq [\underline{D}(A, B), \overline{D}(A, B)]$ .

We use identity (13) to give a more precise bound. It will be assumed that  $A$  and  $B$  do not have  $-\infty$  columns. Note that this condition is more restrictive than that  $A$  and  $B$  do not have *common*  $-\infty$  columns, and it cannot be assumed without loss of generality.

**Theorem 17.** *Suppose that  $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}_{\max}^{n \times m}$  do not have  $-\infty$  columns. Then*

$$(52) \quad \text{spec}(A, B) \subseteq [-r(A^\sharp B), r(B^\sharp A)] \subseteq [\underline{D}(A, B), \overline{D}(A, B)].$$

*Proof.* Let  $Ax = \lambda Bx$ , then we also have

$$(53) \quad \begin{aligned} Ax \leq \lambda + Bx &\Leftrightarrow -\lambda + x \leq A^\sharp Bx, \\ \lambda + Bx \leq Ax &\Leftrightarrow \lambda + x \leq B^\sharp Ax. \end{aligned}$$

As  $A$  and  $B$  do not have  $-\infty$  columns so that  $A^\sharp Bx$  and  $B^\sharp Ax$  do not have  $+\infty$  entries, we can use (13) to obtain from (53) that  $\lambda \in [-r(A^\sharp B), r(B^\sharp A)]$ . For  $\lambda = r(B^\sharp A)$  we can find  $y \neq -\infty$  such that  $\lambda + y \leq B^\sharp Ay$  and hence  $\lambda + By \leq Ay$ . Using Proposition 15 we obtain  $\lambda \leq \overline{D}(A, B)$ . The remaining inequality  $\lambda \geq \underline{D}(A, B)$  can be obtained analogously.  $\square$

By comparison with the finer bounds  $-r(A^\sharp B)$  and  $r(B^\sharp A)$ , the interest of the bounds of Butkovič and Cuninghame-Green,  $\underline{D}(A, B)$  and  $\overline{D}(A, B)$ , lies in their explicit character. However, these bounds become infinite when the matrices  $A$  and  $B$  do not have finite rows. We next give different explicit bounds, which turns out to be finite as soon as  $A$  and  $B$  do not have identically infinite columns.

**Proposition 18.** *We have*

$$\text{spec}(A, B) \subseteq \bigcup_{1 \leq i \leq n} [-(A^\sharp B0)_i, (B^\sharp A0)_i],$$

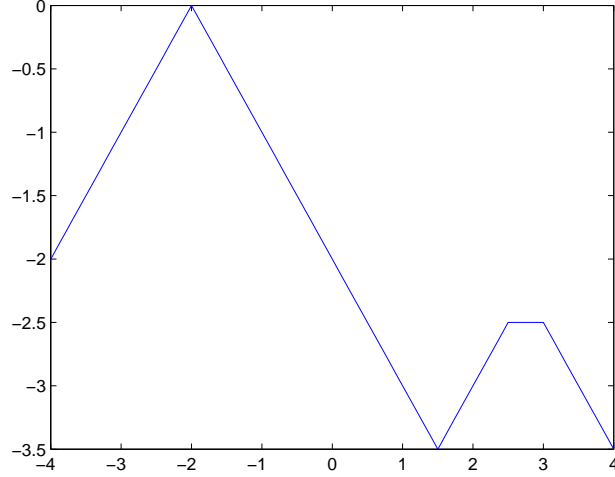


FIGURE 1. Spectral function of (54)

and so

$$\text{spec}(A, B) \subseteq [-\bigvee_i (A^\sharp B 0)_i, \bigvee_i (B^\sharp A 0)_i]$$

*Proof.* Consider  $x := 0$  and  $\mu := \bigvee_i [h_\lambda(0)]_i$ , so that  $h_\lambda(x) \leq \mu + x$ . Then, the non-linear Collatz-Wielandt formula (14) implies that  $r(h_\lambda) \leq \mu$ . If  $\lambda \in \text{spec}(A, B)$ , we have  $0 \leq r(h_\lambda)$ , and so, there exists at least one index  $i \in \{1, \dots, n\}$  such that

$$0 \leq [h_\lambda(0)]_i = (\lambda + (A^\sharp B 0)_i) \wedge (-\lambda + (B^\sharp A 0)_i) .$$

It follows that  $\lambda \leq (B^\sharp A 0)_i$  and  $\lambda \geq -(A^\sharp B 0)_i$ . □

*Remark 1.* It follows readily from the Collatz-Wielandt property (14) that

$$[-r(A^\sharp B), r(B^\sharp A)] \subseteq [-\bigvee_i (A^\sharp B 0)_i, \bigvee_i (B^\sharp A 0)_i]$$

*Example 1.* We next give an example, to compare the bounds of Corollary 16, Theorem 17 and Proposition 18. Consider the following finite matrices of dimension  $3 \times 4$ :

$$(54) \quad A = \begin{pmatrix} -2 & 3 & -3 & -3 \\ -4 & 1 & 2 & -2 \\ 5 & -1 & 5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 5 & -3 & 3 \\ 2 & 0 & -1 & 4 \\ 0 & 2 & -3 & -1 \end{pmatrix}$$

From the graph of spectral function, Figure 1, it follows that the only eigenvalue is  $-2$ . The interval  $[-r(A^\sharp B), r(B^\sharp A)]$  is in this case  $[-2, 0.5]$ . Bounds (51) of [CGB08, Theorem 2.1] yield the interval  $[\underline{D}(A, B), \overline{D}(A, B)] = [-3, 2]$ , which is less precise. Proposition 18 yields the union of intervals  $[3, 0] = \emptyset$ ,  $[-2, -2]$ ,  $[3, 3]$  and  $[-3, -2]$ , thus



$[-3, -2] \cup \{3\}$ . Note that these intervals are incomparable both with  $[-r(A^\sharp B), r(B^\sharp A)]$  and  $[\underline{D}(A, B), \overline{D}(A, B)] = [-3, 2]$ .

We remark that the intervals  $[-\bigvee_i (A^\sharp B 0)_i, \bigvee_i (B^\sharp A 0)_i]$  and  $[\underline{D}(A, B), \overline{D}(A, B)]$  are also in general incomparable. Also, Subsect. 3.5 will provide an example where the bounds  $[-r(A^\sharp B), r(B^\sharp A)]$  are exact.

**3.3. Asymptotics of the spectral function.** If  $A$  and  $B$  do not have  $-\infty$  columns, the functions  $\lambda + A^\sharp B$  and  $-\lambda + B^\sharp A$  are represented as infima of all max-linear mappings  $K_\lambda^{(p)}$  and, respectively,  $M_\lambda^{(s)}$  such that

$$(55) \quad \begin{aligned} (K_\lambda^{(p)})_i &= \lambda - a_{ki} + B_{k\cdot}, \quad 1 \leq k \leq n, \quad a_{ki} \neq -\infty, \\ (M_\lambda^{(s)})_i &= -\lambda - b_{ki} + A_{k\cdot}, \quad 1 \leq k \leq n, \quad b_{ki} \neq -\infty. \end{aligned}$$

This representation satisfies the selection property.

Matrices  $K_\lambda^{(p)}$  and  $M_\lambda^{(s)}$  are both instances of  $H_\lambda^{(p)}$  which represent  $h_\lambda$ . The spectral radii  $r(H_\lambda^{(p)})$  are maximum cycle means expressed by  $\lambda k_1/l_1 + \alpha$ , where  $0 \leq |k_1| \leq l_1 \leq m$ . The coefficient  $k_1/l_1$  will be called *slope* and  $\alpha$  will be called *offset*. In fact, it is only possible that  $k_1 = l_1 - 2t$  for  $t = 1, \dots, l_1$ , since all entries in the representing matrices are weighted with  $\pm\lambda$  and only even numbers of  $\pm\lambda$  terms can cancel out. As the arithmetic mean does not exceed the maximum, it follows that  $\alpha \leq \Delta(A, B)$ , where

$$(56) \quad \Delta(A, B) := \bigvee_{i,j,k: a_{ij} \neq -\infty, b_{ik} \neq -\infty} (a_{ij} - b_{ik}) \vee \bigvee_{i,j,k: b_{ij} \neq -\infty, a_{ik} \neq -\infty} (b_{ij} - a_{ik}).$$

We also define

$$(57) \quad \begin{aligned} \overline{C}(A, B) &:= \bigvee_{i,j,k: a_{ij} \neq -\infty, b_{ik} \neq -\infty} (a_{ij} - b_{ik}), \\ \underline{C}(A, B) &:= \bigwedge_{i,j,k: a_{ik} \neq -\infty, b_{ij} \neq -\infty} (a_{ik} - b_{ij}). \end{aligned}$$

Using these observations and notation we study the asymptotics of  $s(\lambda)$ , both in general case and in some special cases.

**Theorem 19.** *Suppose that  $A, B \in \mathbb{R}_{\max}^{n \times m}$ .*

1. *Either  $s(\lambda) = -\infty$  for all  $\lambda$ , or there exist  $k_1, l_1, k_2, l_2$  such that  $0 \leq l_1 \leq m$ ,  $k_1 = l_1 - 2t_1$  where  $0 \leq t_1 \leq \lfloor l_1/2 \rfloor$ ,  $0 \leq l_2 \leq m$ ,  $k_2 = l_2 - 2t_2$  where  $0 \leq t_2 \leq \lfloor l_2/2 \rfloor$ , and  $\alpha_1, \alpha_2 \in \mathbb{R}$  such that*

$$(58) \quad \begin{aligned} s(\lambda) &= \lambda k_1/l_1 + \alpha_1, \quad \text{if } \lambda \leq -2m^2 \times \Delta(A, B), \\ s(\lambda) &= -\lambda k_2/l_2 + \alpha_2, \quad \text{if } \lambda \geq 2m^2 \times \Delta(A, B). \end{aligned}$$

2. Suppose that  $A$  and  $B$  do not have  $-\infty$  columns. Then either  $s(\lambda) = -\infty$  for all  $\lambda$ , or there exist  $\alpha_1 \leq r(A^\sharp B)$  and  $\alpha_2 \leq r(B^\sharp A)$  such that

$$(59) \quad \begin{aligned} s(\lambda) &= \lambda + \alpha_1, & \text{if } \lambda \leq -2m \times \Delta(A, B), \\ s(\lambda) &= -\lambda + \alpha_2, & \text{if } \lambda \geq 2m \times \Delta(A, B). \end{aligned}$$

3. Suppose that  $A$  and  $B$  are real. Then

$$(60) \quad \begin{aligned} s(\lambda) &= \lambda + r(A^\sharp B), & \text{if } \lambda \leq \underline{C}(A, B), \\ s(\lambda) &= -\lambda + r(B^\sharp A), & \text{if } \lambda \geq \overline{C}(A, B). \end{aligned}$$

*Proof.* 1: For the proof of this part, we observe that for each  $\lambda$ , the function  $s(\lambda)$  is the maximum cycle mean of a certain representing matrix  $H_\lambda^{(p)}$ , so that it equals  $\lambda k/l + \alpha$  where  $0 \leq l \leq m$ ,  $k = l - 2t$  where  $0 \leq t \leq l$ . For any two such terms, difference between coefficients  $k/l$  is not less than  $1/m^2$ , and the difference between the offsets does not exceed  $2\Delta(A, B)$ , which yields that all intersection points must be in the interval  $[-2m^2 \times \Delta(A, B), 2m^2 \times \Delta(A, B)]$ . Thus  $s(\lambda)$  is linear for  $\lambda \leq -2m^2 \times \Delta(A, B)$  and for  $\lambda \geq 2m^2 \times \Delta(A, B)$ . As  $s(\lambda) \leq 0$  for all  $\lambda$ , the left asymptotic slope is nonnegative, and the right asymptotic slope is non-positive.

2: When  $A$  does not have  $-\infty$  columns, some of the matrices  $H_\lambda^{(p)}$  are of the form  $K_\lambda^{(p)}$  and their maximum cycle mean is  $\lambda + \alpha$ . Taking minimum over all  $r(H_\lambda^{(p)})$  of that form yields an offset  $\alpha_1 \leq r(A^\sharp B)$ . The cycle mean  $\lambda + \alpha_1$  will dominate at small  $\lambda$ , and the smallest intersection point may occur with a term  $\lambda(m-1)/m + \alpha'_1$ . Indeed, the difference between coefficients is precisely the smallest possible  $1/m$ , and the difference  $|\alpha_1 - \alpha'_1|$  may be up to  $2\Delta(A, B)$ . This yields the bound  $-2m \times \Delta(A, B)$ . An analogous argument follows when  $\lambda$  is large and  $B$  does not have  $-\infty$  columns.

3: When  $A$  and  $B$  are real and  $\lambda < \underline{C}(A, B)$ , all coefficients in the min-max function  $\lambda + A^\sharp B$  are real negative, and all coefficients in the min-max function  $-\lambda + B^\sharp A$  are real positive. This implies that  $s(\lambda)$  is equal to the minimum over  $r(K_\lambda^{(p)})$ , which is equal to  $\lambda + r(A^\sharp B)$ . An analogous argument follows when  $\lambda > \overline{C}(A, B)$ .  $\square$

In Proposition 23 we will show by an explicit construction that any slope  $k/l$  can be realized as asymptotics of a spectral function.

We next observe that the asymptotics of  $s(\lambda)$  can be read off from  $s^\circ(\lambda)$ , which is the spectral function associated with  $A^\circ$  and  $B^\circ$ . These matrices were defined in (49), they have 0 entries exactly where  $A$ , resp.  $B$ , have finite entries.

**Proposition 20.** *Suppose that  $A, B \in \mathbb{R}_{\max}^{n \times m}$  and that  $\lambda k_1/l_1$  where  $k_1, l_1 \geq 0$  (resp.  $-\lambda k_2/l_2$  where  $k_2, l_2 \geq 0$ ) is the left (resp. the right) asymptotic slope of  $s(\lambda)$ . Then*

$$(61) \quad s^\circ(\lambda) = \begin{cases} \lambda k_1/l_1, & \text{if } \lambda \leq 0, \\ -\lambda k_2/l_2, & \text{if } \lambda \geq 0. \end{cases}$$

*Proof.* Observe that the representing matrices  $H_\lambda^{(p^\circ)}$  of

$$(62) \quad h_\lambda^\circ := (\lambda + (A^\circ)^\sharp B^\circ x) \wedge (-\lambda + (B^\circ)^\sharp A^\circ x)$$

are in one-to-one correspondence with the representing matrices  $H_\lambda^{(p)}$  of  $h_\lambda$ . The finite entries  $H_\lambda^{\circ(p)}$  equal to  $\pm\lambda$ , they are in the same places and with the same sign of  $\lambda$  as in  $H_\lambda^{(p)}$ . Hence the cycle means in  $H_\lambda^{\circ(p)}$  have the same slopes as the corresponding cycle means in  $H_\lambda^{(p)}$ , but with zero offsets. When  $s(\lambda) = r(h_\lambda)$  is computed by (45), the asymptotics at large and small  $\lambda$  is determined by the slopes only and yields the same expression as for  $s^\circ(\lambda) = r(h_\lambda^\circ)$ .  $\square$

**3.4. Reconstructing the spectrum and the spectral function.** Now we consider the problem of identifying all linear pieces that constitute the spectral function and computing the whole spectrum of  $(A, B)$  in the case when  $A$  and  $B$  have integer entries.

The result will be formulated in terms of calls to an *oracle* computing the value of  $s(\lambda)$  at a given point. Essentially, this oracle computes the value of associated mean payoff game parametrized by  $\lambda$ . The role of such an oracle can be played by the policy iteration algorithm of [CTGG99, DG06].

**Theorem 21.** *Let  $A, B \in \mathbb{R}_{\max}^{n \times m}$  have only integer or  $-\infty$  entries.*

1. *All linear pieces that constitute the function  $s(\lambda)$  and hence the spectrum of  $(A, B)$  can be identified in no more than  $\Delta(A, B) \times O(m^6)$  calls to the oracle.*
2. *When  $A$  and  $B$  have no  $-\infty$  columns, the number of calls needed to reconstruct the function  $s(\lambda)$  can be decreased to  $\Delta(A, B) \times O(m^5)$ . When  $A$  and  $B$  are real, it is decreased to  $(\overline{C}(A, B) - \underline{C}(A, B)) \times O(m^4)$ .*

*Proof.* In all cases we have a finite interval of length  $L$ , which is  $L := \Delta(A, B) \times O(m^2)$  in case 1, and  $L = \Delta(A, B) \times O(m)$  or  $L := (\overline{C}(A, B) - \underline{C}(A, B))$  in case 2.

We first compute the asymptotic slopes of  $s(\lambda)$  outside  $L$ . By Proposition 20, we can do this by computing  $s^\circ(\pm 1)$  in just two calls to the oracle. Then the goal is to reconstruct all linear pieces which constitute  $s(\lambda)$  in the interval  $L$ .

The linear pieces of  $s(\lambda)$  correspond to the maximal cycle means in the matrices from the representation of  $h_\lambda(x)$ . The points where such linear pieces may intersect are given

by

$$(63) \quad \frac{a_1 + k_1 \lambda}{n_1} = \frac{a_2 + k_2 \lambda}{n_2},$$

where all parameters are integers and  $1 \leq |k_1|, |k_2|, n_1, n_2 \leq m$ . This implies

$$(64) \quad \lambda = \frac{a_1 n_2 - a_2 n_1}{k_2 n_1 - k_1 n_2}$$

The denominators of these points range from  $-m^2$  to  $m^2$ , hence their number in the interval  $L$  is  $L \times O(m^4)$ . We reconstruct the whole spectral function by calculating  $s(\lambda)$  at these points, since  $s(\lambda)$  is linear between them.  $\square$

Since  $\text{spec}(A, B)$  is the zero set of  $s(\lambda)$ , we can identify  $\text{spec}(A, B)$  by reconstructing  $s(\lambda)$  in the spectral intervals given by Proposition 18. This yields the following

**Corollary 22.** *Let  $A, B \in \mathbb{R}_{\max}^{n \times m}$  have only integer or  $-\infty$  entries and no  $-\infty$  columns. Then the number of calls to the oracle needed to identify  $\text{spec}(A, B)$  does not exceed  $(\bigvee_i (B^\sharp A 0)_i + \bigvee_i (A^\sharp B 0)_i) \times O(m^4)$ .*

**3.5. Examples of analytic computation.** In this section we consider two particular situations when the spectral function can be constructed analytically. The first example shows that any asymptotics  $k/l$ , where  $l = 1, \dots, m$  and  $k = l - 2t$  for  $t = 1, \dots, l$ , can be realized. The second example is taken from [Ser], and it shows that any system of intervals and points on the real line can be represented as spectrum of a max-plus two-sided eigenproblem.

**Asymptotic slopes.** In our first example we consider pairs of matrices  $A^{m,l} \in \mathbb{R}_{\max}^{m \times m}$ ,  $B^{m,l} \in \mathbb{R}_{\max}^{m \times m}$  with  $(0, -\infty)$  entries, where  $0 \leq l \leq \lfloor m \rfloor$ . An intuitive idea is to make a certain “exchange” between the max-plus identity matrix and a certain cyclic permutation matrix. For instance

$$(65) \quad A^{6,2} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}, B^{6,2} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

where the dots denote  $-\infty$  entries.

Formally,  $A^{m,l} = (a_{ij}^{m,l})$  are defined as matrices with  $\{0, -\infty\}$  entries such that  $a_{ij}^{m,l} = 0$  for  $i = 1$  and  $j = m$ , or  $i = j + 1$  where  $2l < i \leq m$ , or  $i = j = 2k$  where  $1 \leq k \leq l$ , or  $i = 2k + 1$  and  $j = 2k$ , where  $1 \leq k < l$ , and  $a_{ij}^{m,l} = -\infty$  otherwise.

Similarly,  $B^{m,l} = (b_{ij}^{m,l})$  are defined as matrices with  $\{0, -\infty\}$  entries such that  $b_{ij}^{m,l} = 0$  for  $i = j$  where  $2l < i \leq m$ , or  $i = j = 2k - 1$  where  $1 \leq k \leq l$ , or  $i = 2k$  and  $j = 2k - 1$ , where  $1 \leq k \leq l$ , and  $b_{ij}^{m,l} = -\infty$  otherwise.

**Proposition 23.** *The spectral function associated with  $A^{m,l}, B^{m,l}$  consists of two linear pieces:  $s(\lambda) = \lambda \times (m - 2l)/m$  for  $\lambda \leq 0$  and  $s(\lambda) = -\lambda \times (m - 2l)/m$  for  $\lambda \geq 0$ .*

*Proof.* Let us introduce yet another matrix  $C^{m,l}(\lambda) = (c_{ij}^{m,l}(\lambda)) \in \mathbb{R}_{\max}^{m \times m}$ . Informally, it is a sum of a  $\{0, -\infty\}$  permutation (circulant) matrix and its inverse, weighted by  $\pm\lambda$ . This pattern corresponds to the above mentioned “exchange” in the construction of  $A^{m,l}$  and  $B^{m,l}$ . In particular, (65) corresponds to

$$(66) \quad C^{6,2} = \begin{pmatrix} \cdot & -\lambda & \cdot & \cdot & \cdot & -\lambda \\ \lambda & \cdot & \lambda & \cdot & \cdot & \cdot \\ \cdot & -\lambda & \cdot & -\lambda & \cdot & \cdot \\ \cdot & \cdot & \lambda & \cdot & \lambda & \cdot \\ \cdot & \cdot & \cdot & -\lambda & \cdot & \lambda \\ \lambda & \cdot & \cdot & \cdot & -\lambda & \cdot \end{pmatrix}.$$

Defining formally,  $c_{1,m}^{m,l} = -\lambda$ ,  $c_{m,1}^{m,l} = \lambda$ , and

$$(67) \quad c_{ij}^{m,l} = \begin{cases} \text{sign}(i, j) \times \lambda, & \text{if } 1 \leq i, j \leq m \text{ and } |j - i| = 1, \\ -\infty, & \text{otherwise,} \end{cases}$$

where

$$(68) \quad \text{sign}(i, j) = \begin{cases} 1, & j - 1 = i \geq 2l \text{ or } j \pm 1 = i = 2k \leq 2l, \\ -1, & i - 1 = j \geq 2l \text{ or } i \pm 1 = j = 2k \leq 2l. \end{cases}$$

Observe that the pairs  $(i, j)$  and  $(j, i)$  for  $j = i + 1$  (and also  $(1, m)$  and  $(m, 1)$ ) have the opposite sign.

It can be shown that each representing max-plus matrix of the min-max function

$$(69) \quad h_{\lambda}^{m,l}(x) = (\lambda + (A^{m,l})^{\#} B^{m,l} x) \wedge (-\lambda + (B^{m,l})^{\#} A^{m,l} x)$$

is choosing one of the two entries in each row of  $C^{m,l}(\lambda)$ . The matrices can be classified according to this choice as follows (see (66) for an illustration):

1. Choose  $(m, 1)$ , and  $(i, i + 1)$  for  $i = 1, \dots, m - 1$ ;
2. Choose  $(1, m)$ , and  $(i, i - 1)$  for  $i = 2, \dots, m$ ;
3. Choose both  $(m, 1)$  and  $(1, m)$ , or both  $(i - 1, i)$  and  $(i, i - 1)$  for some  $i = 2, \dots, n$ .

The first two strategies give just one matrix each, with the (maximum) cycle means  $\lambda \times (m - 2l)/m$  and  $-\lambda \times (m - 2l)/m$ . The rest of the representing matrices are described

by 3., and it follows that their maximum cycle means are always greater than or equal to 0. Hence  $s(\lambda) = \lambda \times (m - 2l)/m \wedge -\lambda \times (m - 2l)/m$ .  $\square$

**The spectrum of two-sided eigenproblem.** Now we consider an example of [Ser]. Let us define  $A \in \mathbb{R}_{\max}^{2 \times 3t}$ ,  $B \in \mathbb{R}_{\max}^{2 \times 3t}$ :

$$(70) \quad \begin{aligned} A &= \begin{pmatrix} \dots & a_i & b_i & c_i & \dots \\ \dots & 2a_i & 2b_i & 2c_i & \dots \end{pmatrix}, \\ B &= \begin{pmatrix} \dots & 0 & 0 & 0 & \dots \\ \dots & a_i & c_i & b_i & \dots \end{pmatrix}, \end{aligned}$$

where  $a_i \leq c_i < a_{i+1}$  for  $i = 1, \dots, t-1$ , where  $b_i := \frac{a_i + c_i}{2}$ . The following result describes  $\text{spec}(A, B)$ .

**Theorem 24** ([Ser]). *With  $A, B$  defined by (70),*

$$(71) \quad \text{spec}(A, B) = \bigcup_{i=1}^t [a_i, c_i].$$

To calculate  $s(\lambda)$ , which is a more general task, one can study the representing matrices like in the previous example. Another way is to guess, for each  $\lambda$ , a finite eigenvector of  $P_D \circ P_{C(\lambda)}$  and then  $s(\lambda)$  is the corresponding eigenvalue. By this method we obtained that:

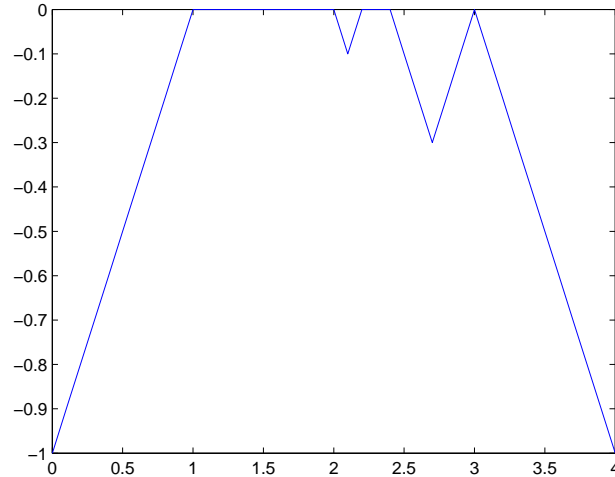
$$(72) \quad s(\lambda) = \begin{cases} \lambda - a_1, & \text{if } \lambda \leq a_1, \\ 0, & \text{if } a_k \leq \lambda \leq c_k, k = 1, \dots, t, \\ \max(c_k - \lambda, \lambda - a_{k+1}), & \text{if } c_k \leq \lambda \leq a_{k+1}, k = 1, \dots, t-1, \\ c_t - \lambda, & \text{if } \lambda \geq c_t. \end{cases}$$

More precisely, it can be shown that the following vectors are eigenvectors of  $P_D P_{C(\lambda)}$ :

$$(73) \quad y^\lambda = \begin{cases} (0 \ a_1 \ 0 \ a_1), & \text{if } \lambda \leq a_1, \\ (0 \ \lambda + b_k - a_k \ 0 \ \lambda + b_k - a_k), & \text{if } a_k \leq \lambda \leq b_k, k = 1, \dots, t, \\ (0 \ c_k \ 0 \ c_k), & \text{if } b_k \leq \lambda \leq c_k, k = 1, \dots, t, \\ (0 \ \lambda \ 0 \ \lambda), & \text{if } c_k \leq \lambda \leq a_{k+1}, k = 1, \dots, t-1, \\ (0 \ c_t \ 0 \ c_t)^T, & \text{if } \lambda \geq c_t, \end{cases}$$

with the eigenvalues expressed by (72).

We can also conclude that in this case  $-r(A^\sharp B) = a_1$  and  $r(B^\sharp A) = c_t$ . Indeed, by (72),  $s(\lambda) = \lambda - a_1$  for  $\lambda \leq a_1$  and  $s(\lambda) = c_t - \lambda$  for  $\lambda \geq c_t$ . Comparing this with the result of Theorem 19, part 3, we get the claim.

FIGURE 2. The spectral function of  $A$  and  $B$  in (74)

As  $a_1$  and  $c_t$  are eigenvalues, the last result shows that the bounds given in Theorem 17 cannot be improved in general.

*Example 2.* In (70), take  $t = 3$ ,  $[a_1, c_1] = [1, 2]$ ,  $[a_2, c_2] = [2.2, 2.4]$  and  $[a_3, c_3] = [3, 3]$ . Then

$$(74) \quad \begin{aligned} A &= \begin{pmatrix} 1 & 1.5 & 2 & 2.2 & 2.3 & 2.4 & 3 \\ 2 & 3 & 4 & 4.4 & 4.6 & 4.8 & 6 \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1.5 & 2.2 & 2.4 & 2.3 & 3 \end{pmatrix} \end{aligned}$$

The spectral function is shown on Figure 2.

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