



Digital Inquiry Through Games

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Abstract

This paper aims to show how the Logic of Game Theory can facilitate the structuring of games for the learning of mathematical concepts, in a way which is cognitively resonant with students' attitudes and epistemologically sound from the mathematical standpoint. We propose a kind of game, based on an inquiry approach to mathematics, called Digital Inquiry Game (DIG), the aim of which is to foster students' positive beliefs about their mathematical capabilities with regards to problem solving and to improve the way students are able to grasp the epistemic aspects of the mathematical knowledge in question. The main issues surrounding the theoretical background and the inspiring key constructs of the DIG are explained. The design of a DIG is validated through a case study concerning some properties of integers and a general divisibility criterion. Finally, some issues for further researches are considered.

Keywords Online digital games · Inquiry · Mathematics education · Gamification

1 Introduction

Games are becoming more and more pervasive thanks to their availability on modern portable devices. As a result, people working within the educational setting have been using them as a medium through which teaching environments can be designed. However, success is not so frequent, especially in case of educational games in Science and Mathematics. This lack of success has much to do with a dilemma that exists between the game- and teaching- environments one must face: it seems that if a designer chooses in-game tools and representations that make the game enjoyable, they generally do not align visually or epistemologically with the tools and representations typically utilized in the target domain (the discipline).

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We propose to overcome this dilemma proposing a specific method in which the tools and representations of the game (a role-playing game) become the same as those of the mathematical context, without losing the enjoyability of the game itself. In our game the players assume the role of characters within a suitable narrative, which has the same structure as that of proving theorems in mathematics through a questions-answers plot, triggered and sustained by the narrative itself.

This method is possible because of an epistemological revolution in the structure of mathematical proofs, which was brought about by the elaboration of the so-called game-theory-logic (GTL) by J. Hintikka and his school. GTL forwards an unexpected and profound structural commonality between the way people actually reason in (full information) games and that by which mathematical proofs can be built. As a result, we get a design principle for number theory, and possibly for all elementary mathematics, which we call Constructible Authentic Representations for Mathematics (CARM), and in which the alignment between the design principle for number theory and that of the game is guaranteed by the coincidence of the two. This method extends the CAR of Holbert and Wilensky (2014), who instead looked for an alignment and not a full coincidence between the game and the environment to which the game is dedicated. In the paper we will demonstrate how this can be concretely realized. Hence our research essentially answers in the affirmative to the following question: is it possible and, if so, how is it possible to overcome the dilemma of the game/teaching environments for mathematics?

In recent years, many researchers have been involved in research on digital games across various domains (for instance Tomé et al. 2015; Garrett and Young 2019). However, digital games are mainly used in educational, training and academic domains (Albertazzi et al. 2019). A number of authors have pointed out the many pedagogical advantages of using instructional games. Besides being more consonant with students' habits and interests, instructional games are very useful for teaching complex procedures since they (a) use action instead of explanation, (b) create personal motivation and satisfaction, (c) accommodate multiple learning styles and skills, (d) reinforce mastery skills, and (e) provide interactive and decision making contexts (Charles and McAlister 2004; Holland et al. 2003). Educational games provide for the development of students' competencies (Niss and Højgaard 2019), since they are engaged in problem-solving, collaboration, and communication activities (Dicheva et al. 2015). If well designed, they can be seen as problem solving spaces that provide pathways to learning through entertainment and pleasure (Gee 2009). Motivation and engagement are crucial to develop complex problem-solving competencies in game based learning, that should be sustained designing carefully interactivity with the game environment (interface), between the players and the storyline (narrative) and among players (social) (Eseryel et al. 2014).

Also the interest in digital games for mathematics learning is growing more and more, as shown by a recent book by Lowrie and Jorgensen (2015), in which the potential promises and pitfalls of digital games are explored with respect to the mathematics sense developed by the students during their engagement in the game. Game-based learning environments can impact both on cognitive and affective levels. For example, they enable an effective assessment (Ifenthaler et al. 2012; Ninaus et al. 2017) or improving (Vander-cruysse et al. 2016) of students' knowledge in elementary mathematics. Moreover, some game-based environments for the elementary school maths course (for example Hung et al. 2014) can effectively promote student learning outcomes, self-efficacy and motivation towards mathematics. Fokides (2018) reports that the first, fourth, and sixth-grade primary school students using digital games grasp mathematical concepts better than the ones that were taught conventionally. White and McCoy (2019) point out that the attitudes of fifth

grade students in mathematics are improved both towards lessons and towards mathematics in general through game-based learning. Furthermore, Clark et al. (2016) show that digital games have significantly improved the learning of K-16 students involved in their study, compared to non-game conditions. Gros (2015) analyses the use of games in e-learning, where students can learn and interact only by the web. She points out that most games for e-learning adopt a “behaviourist approach of learning and focus mainly on the transmission of content and not on complex learning activities” (Gros 2015, p. 42). Furthermore, she recognises the personalisation feature that games in e-learning can offer thanks to their interactive potential. Besides the introduction of different levels of difficulty according to various levels of skill, fine grained personalised learning paths can be devised which exploit the interactions of the learners with the game.

Although the aspects of engagement and motivation which are promoted by the games is undeniable, a further key point which is highlighted by Gros is that, on the students’ side, such aspects concern fun and not learning and thus the use of just Game does not guarantee learning.

With regards to mathematics education, the book cited above shows the potential of digital games for enhancing mathematical thinking (e.g. problem-solving, decision-making, visual-spatial reasoning) and underlines the need to focus on which embedded theories are needed for high quality instructional design. However, the positive effects of game-based learning are not yet evident despite several meta-analyses which have been carried out. In a similar vein, Tokac et al. (2019) highlight that video games are only a slightly effective teaching strategy for teaching mathematics at PreK-secondary levels. Moreover, Byun and Joung (2018) underline the need for more empirical research studies so as to more accurately determine the effect of electronic games on maths learning. According to Tokac et al. (2019), there is a need for a more attentive approach to game design with respect to specific learning outcomes across various mathematical domains.

This paper addresses this issue by aiming at the theoretical design of a digital game. Our paper moves from a networking of theories. Indeed in mathematics education it is largely assumed the richness of diversity of theories available, and networking of theories allows to actively connect more theories on an empirical, theoretical or methodological level (Prediger et al. 2008; Bikner-Ahsbabs and Prediger 2014). We use a more inquiry-focused approach rather than working from a behaviouristic foundation. Specifically, the design refers to a game which has the purpose of triggering and supporting students learning of mathematical properties and in bridging the gap that, according to the literature, separates their informal argumentations from the formal aspects of mathematical statements, moving towards proofs. The point of departure in our design work can be found in the contrast between the logic of *discovering* and that of *justifying*, a widely debated subject across the whole history of mathematics (Arzarello et al. 1998). In solving a problem, in conjecturing a hypothesis, in proving (or disproving) a result, a crucial point consists in the dialectic between an explorative, groping phase and an organising strategy which proceeds towards some piece of validated knowledge. Our paper faces this contrast by suitably adapting a model of Holbert and Wilensky (2014) to the design of a game aiming at the learning of a piece of elementary number theory.

Holbert and Wilensky point out a dilemma that emerges when trying to support scientific learning through games: namely, popular video games are fun but not apt for educational aims; on the other side educational games cannot compete with those in out-of-classroom contexts and are possibly boring for the students (Holbert and Wilensky 2014, p. 55). As a solution to the first problem they propose to design educational games so that they “look and feel like traditional video games, and refrain from adopting a didactic approach

or obvious educational agenda” (p. 55). As a solution to the second problem, they propose what they call Constructible Authentic Representations (CAR) design principle: it consists in providing “players with opportunities to engage in meaningful construction with components that integrate relevant concepts to create in-game representations that visually and epistemologically align with tools and representations utilized in the target domain” (p. 55). The resulting game can therefore align “visually and epistemologically” with the educational tools outside of the game.

Our proposal aligns with CAR, with two differences: (1) it concerns mathematics, while Holbert and Wilensky considered science; (2) it does not only concern representations that align with the components of the target domain but also those which are the core of the target domain itself. The second claim depends on two specific features of mathematics, which make this discipline different from physics. First, the core business of mathematics consists in working with representations of abstract objects (numbers, functions, graphs, etc.) without having an alternate direct access to them. This makes a strong epistemological difference with physics, where for example, there is a modelisation of concrete phenomena. Secondly, epistemologically speaking, mathematics reasonings can be aligned with game logic, as the researches of J. Hintikka have shown and which we will discuss below (Sect. 1). Hence, our proposal is not only aligned with its representations but also with the structure itself of mathematical reasoning. In this way, our game proposal meets the two challenges of game design listed by Holbert and Wilensky (p. 61): (1) designing games that align with youth gaming culture; (2) building game experiences that players see as relevant to problem-solving in non-game contexts. The second clause can be read at an even a deeper level, as hinted above. For this reason we call use the acronym CARM for our approach.

One of the main features of our proposal is not simply a chance and ingenious way of designing some educational game in mathematics, but also represents a possible systematic method for transposing the mathematical knowledge to be taught into the game machinery,¹ according to game-theory. More precisely, we refer to *Backward reasoning* (or backward chaining) that is an inference method that can be described as “reasoning backward from the goals”. To Hintikka, this approach to reasoning as a double-sided process seems analogous with game theory: in a full-information game between two players, one player has to elaborate a winning strategy according to the rules of the game. In short, backward reasoning can be considered as a practice that involves the making of a number of arguments from the bottom of the problem and proceeds through logical correspondence which allows for something known to obtained or reached through other paths. In game-theory this way of reasoning is also called backward induction (it corresponds to what in chess is called retrograde analysis). This method is implemented, for example, in automated theorem provers, and its logical features were put forward by Hintikka, who called it Game Theory Logic (GTL) (1998), and based it essentially on the semantic tableaux (Henkin 1949) to elaborate a sophisticated analysis of the so called *Logic of Inquiry*, typical of the scientific investigation method (Hintikka 1999).

¹ For our aims it is enough to use the following definition of Game:

A game is a structure (S, R, G) , where:

- S is a set (the set of possible states),
- R is a binary relation in S (correctness relation)
- G is a subset of S (set of winning states).

A game cycle is a succession of states s_1, \dots, s_n , such that $s_i R s_{i+1}$; if $s_n \in G$ the round-game is winning. If not, the game cycle is losing.

He identifies two types of rules in the game: *definitory rules*, which are the possible and acceptable moves (the deductive moves), and *strategic rules*, which correspond to questions and answers for investigating which moves are the most convenient for a player in a specific situation (the argumentative moves). The GTL demonstrates a deep interaction between the logic of justification and that of proving: the games are the cognitive and epistemic pivot, which allows this link between these two forms of reasoning to be made palpable. All the mathematical theorems to be taught can be transposed into a game according to the GTL (Hintikka and Kulas 1983). The mathematical theorems, so transformed, embody a form of cognitive and epistemic continuity between the usual deductive logic of justification and the more argumentative logic of inquiry. This continuity constitutes real added value to the educational games from the point of view of mathematics education. Furthermore, as collaborative problem solving (CPS) has been identified as an important skill in education (OECD 2017), we intend to transpose mathematical theorems into a team game. Nevertheless, it is worthwhile to note that productive collaboration does not happen spontaneously and learners' interactions need to be planned in detail within a sequence of tasks, suitably designed in order to reach the desired educational objectives. In order to do that, researchers in cooperative learning started to use a particular instructional support, named *script*, which they borrowed from cognitive psychology (Schank and Abelson 1977). This has been utilised in educational contexts, with the aim of encouraging, structuring and regulating roles and actions expected from students in collaborative setting, in order to foster learning (King 2007). The use of *collaboration scripts* has increased incrementally in computer-supported collaborative environments (Weinberger et al. 2009), where the need for pre-structuring and regulating the social and cognitive processes is becoming ever and more evident.

Our project not only uses games as a pedagogical approach to learning but also as a method which grants entry into the epistemic content of taught mathematics whilst simultaneously appertaining to the motivational aspect of the teaching situations proposed to the students. Hence, we formulate the following research questions:

- RQ 1 How does Hintikka's framework about the Logic of Game Theory facilitate the structuring of games for the learning of mathematical concepts, in a way which is cognitively resonant with students' attitudes and epistemologically sound from the mathematical standpoint?
- RQ 2 How does the CARM approach allow the game/teaching dilemma to be overcome for elementary number theory in a concrete video-game environment to be used in the classroom?

More specifically, RQ2 can be articulated into the following sub-questions: how can the CARM approach allow for

- a. the fostering of students' positive beliefs about their mathematical capabilities in problem solving?
- b. the scaffolding of students' skills in problem solving through games?
- c. improving the way students are able to grasp the epistemic aspects of the mathematical knowledge at stake in the games?

As to the pedagogical and epistemological features of our game, we can summarize them according to the following key constructs:

1. The learning environment should propose situations similar to the ones girls and boys voluntarily engage in outside the school;
2. The learning environment should challenge students by offering them games that require them to win against an antagonist;
3. The game should facilitate the interactions between peers;
4. The game should stimulate the inquiry attitudes of students;
5. The learning environment should, if possible, provide an enjoyable and stimulating experience;
6. The game establishes an epistemological continuity with the mathematical content.

This paper will focus on the design of a game, taking into account the issues discussed above. For this reason, we scripted the game according to the above literature and key constructs which elicit the general design principles of the CARM approach. In order to validate the design, we will present a case study which uses the defined design to transpose a mathematical problem concerning the divisibility criteria into a game.

2 Method

In this section we will first expand upon the general design of the game based on the Hintikka's framework and the CARM approach. Then we use such design to present a case study on the specific content of divisibility criteria. The case study serves as validation of our research hypotheses.

2.1 Game Design

The design of our Digital Inquiry Game (DIG) is based on the key constructs described above. In accordance with the work of Jagušt et al. (2018), we have designed a game mode that combines competition, collaboration, narrative, adaptability and personalization. The idea is that the effectiveness of gamification does not depend on individual game elements, but instead on the balanced combination of those elements. The game is designed to be enjoyed on an online platform and consists of two teams, each consisting of two players, which compete against each other. Within each team, the players have pre-established roles:

- the *spokesperson*, who is responsible for communication with the other team;
- the *blogger*, who takes notes for her team.

Students communicate with each other by means of the tools:

- the *Group Chat*, for communications between spokesperson and blogger within the same team;
- the *General Chat*, for communications between spokespersons of the two different teams;
- the *General Forum*, as “gaming board”, where the two teams write their moves, or as a place to share common conjectures and winning strategies;
- the *Logbook*, one for each team, in which the blogger takes notes, writes down moves and formalizes a winning strategy.

The design foresees seven phases called sets:

Set 0: Play game for winning, is a collaborative problem solving phase. The two teams play against each other in some warm-up matches, with the purpose of winning the game.

Set 1: Looking for a winning strategy, again is a collaborative problem solving phase. At the beginning, the two players of each team play a game with each other to find a winning strategy. Then they share their moves and strategy with the other team's players in order to optimize their winning strategy. Then they play a new game to validate that strategy. Finally, all 4 players work together to draw up a vademecum, that is, an "instruction booklet" that suggests to a player the strategy to win the game.

Set 2: Discovering mathematics, is still a collaborative problem solving phase. An additional "disturbance element" is introduced into the game with respect to the initial phase. The aim is to create situations that can encourage players to reflect about aspects related to the underlying mathematics.

Set 3: Towards first mathematical conjectures, is an individual problem solving phase. Each player still has the collaborative tools at her disposal, but they answer individually. At the beginning, questions with a request for justification are given to each player to verify the correct understanding and application of the rules of the game. Then, other questions for reflection on the mathematical aspects, with a request for justification, are given to each player in order to encourage the emergence of individual conjectures about the relationships between the mathematical elements and to enhance discovery of the scope and limitations of the mathematical properties behind the game.

Set 4: Grasping mathematical scope and limitations, here the players from the same team reflect together to answer stimulus questions related to borderline cases or rather, on the hypotheses of the theorem. Therefore, this is again a collaborative problem solving phase.

Set 5: Launching for proving, is instead an individual problem solving phase. A number of statements are provided to the players, who are expected to agree or disagree, added with proofs or *counterexamples* built using appropriate digital scaffolding toolkit.

Set 6: Institutionalization, players view a short video that helps them to solidify the entire activity, to highlight and make explicit the knowledge developed during the game.

2.2 The Case Study

To validate the DIG design, we implemented a DIG concerning the following problem, called *The game of divisibility*: two positive integer numbers are given: n and p with $p < n$. You play following these two rules:

- R1: If n ends or starts with a group of consecutive digits all equal to 0 then you can delete that group of digits;
- R2: You can add or subtract from consecutive digits of n a multiple of p .

The game ends when you get to a number less than p .

For example, if $n = 510$ and $p = 3$ you can

- apply R1 and eliminate the final zero, obtaining 51;
- apply R2 and subtract 3 to the initial 5, obtaining 21;
- apply R2 again and subtract 21, getting 0, which is less than $p = 3$ and the game is over.

The two rules, R1 and R2, are definitory rules, in the sense of Hintikka (1998, 1999). The game of divisibility is based on the following definition and theorem:

Definition Two numbers n and n' are p -equivalent, and we write $n \equiv_p n'$, if it emerges that n is divisible by $p \iff n'$ is divisible by p .

Theorem Let's consider $p \neq 2, 5$ and multiples of 2 and 5 and let's assume that n ends with 0, then n is p -equivalent to the number n' obtained by deleting that digit.

Therefore, this is a criterion of universal divisibility, hidden from the students. The trajectory of our educational game is to guide the student first to the identification of a winning strategy, then to the discovery of the criterion of divisibility and its boundary cases and, finally, to the construction of proofs and counterexamples based on this criterion.

Then the phases of the design for the specific problem at hand will become as follows.

In *Set 0*, the two teams test the game and can choose among three games.

In *Set 1*, in order to find a winning strategy, the members of each team play a game with n and p assigned in such a way that p divides n ;

In *Set 2*, we introduce the following variation of the game: assign three numbers n , p and k such that $p < n$ and $k < n$, the game ends when you get to k , using the rules R1 and R2 a finite number of times. Fixing $k < p$ means making the student reflect on the possibility of reaching different k (and above all different from 0).

In *Set 3*, we introduce elements of reflection on the criterion of divisibility with respect to values of k . We provide students with tables and examples of games played. The rows indicate the moves made and the rules used in each move. In some cases, we arrived at different k starting from the same values of n and p and, in others, we always arrived at the same $k=0$.

In *Set 4*, we have provided students with tables referring to values of p that divide n but show k , that are not zero when p is a multiple of 2 and 5. Some questions and requests for justifications are posed to the students for fostering reflection.

In *Set 5*, the student is involved in the construction of proofs and counterexamples, such as for instance:

Assuming that A: 13 divides $uvw00$, B: 13 divides uvw . John says that $A \Rightarrow B$. Do you agree? Explain why.

In our implementation, we have used Moodle platform. As Group Chat and General Chat, we used the classic Moodle Chat. As a General Forum we used the Moodle Questions and Answers Forum, which allows everyone to view the posts of other people only after posting their own. As a logbook we used a gdoc document which had been suitably integrated within Moodle through the URL activity. In *Set 5*, to support players in building proofs and counterexamples, we used a digital application, which we called ISQ (Interactive Semi-open Question) (Albano and Iacono 2019b), built by GeoGebra software and integrated within the Moodle platform (Albano and Iacono 2019a). The student is provided with digital tiles containing words, conjunctions or sentence pieces, which they can manipulate appropriately to construct their own argument sentence.

We validated the DIG *The game of divisibility* with 13 Italian students in their fourth year of high school. The students were split into 3 groups: 2 groups of 4 students and a group of 5 students (in this last case, one of the team was composed of 3 students and two of them played the role of blogger). Within each group we formed 2 teams of 2 players each (spokesperson and blogger). The students participated in the game

and after 4 months they answered a questionnaire concerning the positive and negative aspects of the experience. The teacher of the class was also present in almost all the experimentation meetings and was interviewed.

We introduce below a sample of the students' output while working at the game. Each item is headed by a small comment which indicates its main features and relevance for the considerations that will be sketched at the end of this section. We will distinguish the data coming from the Chat and those from the Forum.

Looking at the Group Chat (S1 and S2, two interacting students), we have:

1. Direct actions.

S1. the move I would like to make is to delete 78 first because it is a multiple of 3

S2. ok

S1. therefore we will meet again with the number 653200

S2. then we remove the 00

S1. or do we remove the 6?

S2. we remove the 6

S1. okay, so 053200

2. Looking for a strategy, refinements; interactions between S1 (more technical) and S2 (more strategic).

S1. but we can directly divide the 78 then remove the 6 and we find ourselves 532

S2. see which one you want to use

S1. you decide

S2. the one where the smallest number comes out

3. Anticipatory thinking and backward reasoning: the moves of the others and their consequent moves

S2. now they take the 00 off, surely

S1. then we remove the other 0, and then they find themselves with 532

S2. 32 cannot be divided

S1. no

S2. ok

- S1. However, they can take away the 3, then 502, then we take 3 from 5 and we arrive at 202
4. First conjectures. Refinement of the strategy and its linguistic compression in a more concise and understandable form (a communication aim appears)
- S2. 559 is a multiple of 13: I can subtract it from 565
- S1. So, what is your final number?
- S2. 66560
- S1. but look, you can do everything in a move because 565656 is a multiple of 13
- S2. eh that number is a multiple of 13; then I mark it on the table and say we used R2
- S1. so now you have to write this on the logbook: $565656 - 13 \times 43512$, so then it comes to all zero so we won
5. Winning strategy...first conjectures; strong emotional involvement (let's enjoy it: now we keep them is said with a typical word of Neapolitan dialect, not so easy to translate: PARIAMM)
- S1. the number 5656560
- S2. does 435120 if I divide it by 13
- S1. Therefore, it is multiple. PARIAMM (let's enjoy it: now we keep them)
- S2. then
- S1. if we consider the whole number as 'consecutive digits'
- S2. it's not less than p now
- S1. we can take away the whole number itself
6. First conjectures about the winning strategy. Epistemic action (suggested by S1)
- S2. then
- S1. then, think, we do $n - 435120p$ and we get 0 which is less than p therefore as an initial strategy we can see if n is entirely a multiple of then

Looking at the General Forum, we have:

7. Discovering the divisibility criterion. Impersonal form of verbs means transition from actions to knowledge. Use of the quantifier form "whatever move... we always arrive" to formulate the strategy. Growing levels of formalization. Communicational aspects taken into consideration.

Step	Move	Rule
1	2382	I subtract 35000 (R2)
2	282	I subtract 2100 (R2)
3	72	I subtract 210 (R2)
4	2	I subtract 70 (R2)

Step	Move	Rule
1	37340	I subtract 42 (R2)
2	2340	I subtract 35000 (R2)
3	234	I delete the final zero (R1)
3	206	I subtract 28 (R2)
4	66	I subtract 63 (R2)
5	3	I subtract 63 (R2)

Step	Move	Rule
1	37340	I subtract 42 (R2)
2	2340	I subtract 35000 (R2)
3	234	I delete the final zero (R1)
4	220	I subtract 14 (R2)
5	22	I delete the final zero (R1)
6	1	I subtract 21 (R2)

Step	Move	Rule
1	81060	I add 12 (R2)
2	8106	I delete the final zero (R1)
3	8100	I subtract 6 (R2)
4	81	I delete the final zero (R1)
5	21	I subtract 60 (R2)
6	0	I subtract 21 (R2)

Step	Move	Rule
1	81000	I subtract 48 (R2)
2	81	I delete the final zeroes (R1)
3	0	I subtract 81 (R2)

Step	Move	Rule
1	21048	I subtract 60000 (R2)
2	48	I subtract 21000 (R2)
3	18	I subtract 30 (R2)
4	0	I subtract 18 (R2)

Fig. 1 Tables shown in Set 2

S1 and S2 see the tables of Fig. 1 shown to the students during Set 2 in order to foster reflection on the final number reached at the end of the played game.

Then they report their remarks on the General Forum as follows:

7.1 We have noticed that in the table on the right the number n is a multiple of p , ergo whatever move is made, we always arrive at the same k , whereas in the table on the left it is not possible

7.2 They are all multiples of 3.

On the right all $k=0$ came out because they are all divisible by 3.

In the table on the left you don't get to the same K because the number is not divisible by 7, while in the table on the right the same K always comes out because N is divisible by 3.

7.3 We note that if our initial N is an integer multiple of p we can use rule R2 by performing.

The following operation: $(N/p) * p$ obtaining a multiple of p $n - [(n/p) * p]$ for the rule R2 we get 0 simplifying and therefore $0 < p$ for each p that satisfies the premise.

In the case in which it is not an integer multiple, it is sufficient to subtract from n the maximum number p contained in N .

7.4 I add the wizard:

(n/p) = number with the comma, of which we will take only the integer part
 we calculate or $k * p$ obtaining a multiple of p less than n
 $n - [k * p]$ we certainly get a minor result of p .

There is an evolution in the language used first in the ‘chat’ then in the ‘forum’: first the language uses verbs that indicate actions, then it is formulated in an impersonal form, detached from subject actions. In the end the language becomes formal and the logical relationships between the moves for winning are made explicit. The chat and the forum support the intertwining of these forms of different languages. The tables introduced in the Forum play a particular role in this evolution: they make the difference between the case when a number N is divisible by p and when it is not palpable and explicit for students. In fact, students, commenting on the two types of tables operationalize this difference into actions to perform and to check for the achieved result.

A further evolution in the language is evident: at the beginning (chat modality) the students are linked to the specific examined case and the numbers and rules are seen as particular cases; as soon as they go on with the game the rules to apply acquire a meaning within a strategy. This is successively (forum modality) explained in a more general way, independently from the concrete used numbers, which become ‘generic examples’ to support the reasoning.

The strategy starts from the explorations made at the beginning and develops as it becomes clearer what to do in order to reduce the number of moves required. In this process there are three aspects, which play a crucial role. Firstly, backward reasonings, where it is checked “what could happen if?” (chat modality). Secondly, a sequence of questions that the students in the chat session pose to each other, which have an epistemic feature (Dreyfus and Kidron 2014)² and mark the transition from an empirical phase to a more theoretical side (forum modality). Thirdly, especially the final use of the ISQ platform instruments the transition to a formal transposition of the mathematical meaning of the game.

The involvement of students in the activity was high: they tried to make what they had understood explicit by explaining the developed strategy of the game. This aspect is acknowledged by themselves in the final questionnaire and in the interview by the teacher.

The students individuated the pros of this approach as follows:

² Dreyfus and Kidron (2014), use the notion of epistemic action to define micro processes of abstraction as Actions in Context (AiC): this central theoretical construct of AiC is a theoretical-methodological model, according to which the emergence is described and analyzed of a new construct “through three observable epistemic actions: recognize (R), build with (B) and build (C). Recognize refers to the student seeing the relevance of a specific previous knowledge to the problem in question. Build-with comprises the combination of recognized constructs, to achieve a localized objective, such as updating a strategy, a justification or solving a problem. The model suggests building as the central epistemic action of mathematical abstraction” (Dreyfus and Kidron, p. 89 ss). In our case AiC concerns the way students become aware of a strategy for winning.

- i. the use of these games greatly stimulates the brain compared to those who want to be smart using the phone to do something else or to copy;
- ii. you understand how to use math and learn to think in another way;
- iii. it helps you improve your ability to reason and interact with other people.

However, the students find that the inhibition created by the game with respect to verbal interactions is not so positive. In fact, they comment: “The disadvantage is that by participating in these types of games you can find it difficult to communicate between the participants because the use of the word is denied”.

The teacher’s interview shows that digital technologies can complement the learning objectives of curricula (Camilleri and Camilleri 2017). They are not an added-value per se but, if they help to recover cognitive gaps, then they help to construct a deeper knowledge. In the case of the game in question, she found that this occurred: “In the activity that we carried out, technology was certainly necessary because the activity was aimed at developing in the children a capacity for written argument of what they understood. Therefore, the use of technology, of chat, was somehow necessary in order to verify and to be able to somehow force them to write in a clear and understandable way to the students of the same group”. Furthermore, the teacher claims that the use of technology should be combined with motivation in order to produce learning: “When I let them discover some regularity through technology, I also created in them the need to understand why certain events happen, why certain things are true”. According to the teacher, technology can also help on the affective level of learning, since the students very often have prejudices about what they can or cannot do with this discipline (mathematics). This work goes in the direction of earning them sufficient levels of self-esteem to get involved with the discipline.

However, the teacher highlights that technology can also have disadvantages or be an obstacle when the student loses control.

3 Results

The game design and the case study allow us to draw some answers to the research questions of our work.

Concerning the RQ1, we have envisaged how to transpose any mathematical content within gamification by means of a collaborative script (Sect. 2.1) consisting of the following phases based on the Hintikka’s *Logic of Inquiry* (Sect. 1): Play game for winning, Looking for a winning strategy, Discovering mathematics, Towards first mathematical conjectures, Grasping mathematical scope and limitation, Launching for proving, Institutionalisation. The transposition is made possible by the Hintikka’s Game Theory Logic which guarantees the conversion any mathematical theorem into a game (Sect. 1).

Concerning the RQ2, item a), students’ answers to the questionnaire and the interview, conducted 4 months after the experience, show that such games stimulate their ingenuity and let them experience a different way of thinking and learning mathematics, so fostering positive beliefs about their mathematical capabilities in problem solving. This is also shown by the use of some typical Neapolitan expressions (Sect. 1, item 5) testifying not only great engagement but also the pleasure of having been able to grasp the underlying mathematics which allowed them to win.

Concerning the RQ2, item b), the case study (Sect. 2.2) has given evidence of how the digital game allows the students to develop anticipatory thinking and backward reasoning,

to produce conjectures, to elaborate strategies and their refinements through linguistic compression, to discover mathematical properties with growing levels of formalization.

Concerning the RQ2, item c), in respect to the way students are able to grasp the epistemic aspects of the mathematical knowledge at play in the games, in the case study (Sect. 2.2) we observed different changes:

1. *Change in the language.* At the beginning language is used to describe actions while in the end it refers to the properties of mathematical objects.
2. *Change in the way students approach the game.* This change allows them to elaborate a winning strategy, marked by putting together backward and forward forms of reasoning.
3. *Epistemic actions.* The three phases of Dreyfus and Kidron model (2014) correspond to the above evolution and can give a solid basis to the claim that the students have been able to grasp the mathematical content behind the game they played.

4 Discussion

In the paper we have introduced a methodology to design a mathematical digital game, named DIG, as a learning activity for students, supported by a technological platform. To be effective, an educational game has to be both pedagogically sound and engaging. This is difficult to achieve, as Denham (2016) points out. In fact, educational game designers often have pedagogical skills, but are not skilled designers of motivating games or, vice versa, are experts in creating engaging games, but not experts in educational and pedagogical aspects. The game we designed and implemented seems to have characteristics of being both motivating and engaging, but with well-defined educational objectives. This seems to be due to the involvement of the design team that combines digital, educational and knowledge domain experts.

The general aim of the DIG is to change students' positive beliefs about their mathematical capabilities in problem solving and to improve the way students are able to grasp the epistemic aspects of the mathematical knowledge in question. The DIG is designed starting from six principles (listed in Sect. 1):

1. The learning environment proposes situations similar to the ones girls and boys voluntarily engage in outside the school.

This is one of the two main constituents which, thanks to the CARM approach, can be amalgamated into a single component. This merger is guaranteed by the structure of the DIG, which is designed according to a typical challenging situation among teams who are engaged in a new digital game. Thus, at the beginning they have some kind of tutorial session in which they start to gain confidence with the new game. Then, as they are asked to compete against another, they are steered in a natural way towards searching for the winning strategy. The two dimensions of the game and of the challenge make the learning situation attractive for the students, giving motivation for their engagement at an affective, cognitive, and metacognitive level. This fact constitutes a pivot, around which mathematics learning can be built.

2. The learning environment challenges students by offering them games that require them to win against an antagonist, and where interactions between peers are facilitated.

This aspect is underlined in the description of the DIG itself. The competitive and collaborative features of the designed game let the students be oriented to the mastery goal, thus being focused on learning the subject and finding the best strategies (Plass et al. 2013). Individual and social skills come into play. In fact, good outcomes of the game require learners to be able to track the shared group knowledge. This issue requires that players take notes about what has been done, which actions derived conclusions, which was the winning strategy, and why. Moreover, it stimulates the reporting the right information within the team and with other teams. This form of collaboration is not natural, even in face-to-face setting: so the digital design scaffolds interactions among students scripting the collaboration phases, specifying tasks step-by-step, as well students' roles and actions for each task. However, it is important to underline that the game can improve students' motivation and performance and lead to optimal challenges if the competition is balanced, that is, players are equally skilled (Liu et al. 2013).

3. The game stimulates the inquiry attitudes of students and establishes an epistemological continuity with its mathematical content.

The DIG has been designed according to the game-theoretic semantics developed by J. Hintikka, which turns the usual way according to which mathematical theorems are proved upside down. The DIG is based on the idea of Hintikka and Kulas (1983), according to which all mathematical theorems to be taught can be transposed into a game. This fundamental principle, which embodies a form of cognitive and epistemic continuity between the deductive logic of justification and the logic of investigation, makes the DIG endogenous in the sense that it originates from mathematics and Game Theory Logic. Experience is also endogenous because there is a continuous relationship between the context, the educational contents (Denham 2016) and the students, with their participation in the discussions strongly influencing the interpretations of the game experience. The last part of the activity is aimed at making students reflect on the game they have played in order to enter progressively into its rationale. The students enter into the properties of divisibility through the game: playing the game they elaborate their own strategies. After doing so, they are asked to reflect on the played game: they are pushed by the questions in the final part to reflect "from the outside" on what they have done. At the end, they are supported with the combination game of the "tiles" to elaborate the mathematical meaning of the game in a more formal way.

A key word, frequently used in this last part of the paper, is "reflective". It is not a case: it must be contrasted with "played". A game can (must) be played, but to understand it, one must also reflect on what has happened during play, a step back from the moment of playing. This issue has been discussed in Soldano and Arzarello (2016), who introduced the notion of the *reflected game*:

[A reflected game] consists of playing the game in a detached way, in order to analyse and make judgments about what has happened in the played game [...]. During the reflected game a student plays for both and his/her aim is different according to the situation. (p. 21).

The two aspects of DIG discussed above constitute its logical-mathematical constituent: it is amalgamated into a single component through the narrative flow described in point 4.

4. The learning environment provides an enjoyable and stimulating experience as possible: the proposed game.

The narrative flow allows for the bridging of the gap between the agreeableness that a game environment offers and the pedagogical aims of mathematics teaching, in this case some subtle aspects of the divisibility notion. It guarantees the amalgamation of the components described in the previous points according to the CARM approach. In fact, on the one hand, the episodes of the narration describe its temporal evolution. On the other hand, they are used as a way of letting the learners proceed towards deeper and deeper understanding of the logic behind the game and of the underlying mathematical facts, possibly until they arrive at proof of some statements. Therefore the episodic dimension, which is configured as a challenge in consecutive levels, allows for the creation of an exciting adventure for the learners engaged in the game.

Moreover, the learning environment captivates players' attention exploiting the features of the computer-mediated communication.

An instance of the DIG has been implemented and experimented with high school students, so that they can

- discover an interesting mathematical property concerning a general divisibility criterion between integers;
- make explicit the mathematical reasons why and when such a criterion can be applied;
- produce statements that express such reasons, as canonical mathematical statements, with a correct use of the logical connectives.

More precisely, a usual test for divisibility has been rebuilt according to our theorem (see Sect. 2.2), where a number n is transformed through a p -chain in numbers that save the p -divisibility property until a final number, which can be directly checked for its divisibility by p . Hence the “effective” divisibility checking is postponed until the end.

This paper shows how to design a collaborative script apt to transpose any mathematical content into a digital inquiry game (DIG). Experimentation in a case study has shown first results on how engaging students in a DIG can encourage deeper student thinking, allow them to enter into the epistemic content of the mathematics underlying the game, enhancing mathematical discourse and building their positive beliefs about their self-efficacy in mathematics.

The analysis has given evidence of improvement in students:

1. at mathematical formal level: moving from procedural rules to more formal and compact expressions of the rules;
2. at logical level: moving from explaining the strategy from very practical, imprecise and incomplete expressions to a more general formulation with quantifiers.

The items 1. and 2. mean students' growing formalization capabilities in mathematics. Thus the students transformed the concepts explored during the game into formal learning, so allowing them to transfer the constructed knowledge in a new broader contexts (Gros 2007).

This paper could provide useful guidance on how to design digital games in mathematics and enable teachers to use all the skills of the twenty first century (Liu et al. 2013) to build games integrated within the curriculum (De Grove et al. 2012). The careful design of the DIG might stimulate further educational research. On the mathematics hand, it would

be interesting to investigate the application of the DIG to the case of geometry (Arzarelo and Soldano 2019). Following the students' feedback, video-communication can be integrated into the design, and studies can concern themselves with how to modulate oral and written communication in order to foster effective mathematical discourse. On the gamification hand, although our DIG already contains some elements of gamification, such as levels/stages and storylines (Nah et al. 2014), we will try to add further elements, such as badges, prizes and rewards, progress bars and more feedback, to investigate their effects on students' learning.

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References

- Albano, G., & Dello Iacono, U. (2019a). GeoGebra in e-learning environments: A possible integration in mathematics and beyond. *Journal of Ambient Intelligence and Humanized Computing*, 10(11), 4331–4343. <https://doi.org/10.1007/s12652-018-1111-x>.
- Albano, G., & Dello Iacono, U. (2019b). A scaffolding toolkit to foster argumentation and proofs in mathematics: Some case studies. *International Journal of Educational Technology in Higher Education*, 16(4), 1–12. <https://doi.org/10.1186/s41239-019-0134-5>.
- Albertazzi, D., Ferreira, M. G. G., & Forcellini, F. A. (2019). A wide view on gamification. *Technology, Knowledge and Learning*, 24(2), 191–202. <https://doi.org/10.1007/s10758-018-9374-z>.
- Arzarelo, F., Andriano, V., Olivero, F., & Robutti, O. (1998). Abduction and conjecturing in mathematics. *Philosophica*, 61(1), 77–94.
- Arzarelo, F., & Soldano, C. (2019). Approaching proof in the classroom through the logic of inquiry. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 221–243). Berlin: Springer. https://doi.org/10.1007/978-3-030-15636-7_10.
- Bikner-Ahsbahs, A., & Prediger, S. (Eds.). (2014). *Networking of theories as a research practice. Advances in mathematics education series*. New York: Springer.
- Byun, J., & Joung, E. (2018). Digital game-based learning for K-12 mathematics education: A meta-analysis. *School Science and Mathematics*, 118(3–4), 113–126. <https://doi.org/10.1111/ssm.12271>.
- Camilleri, M. A., & Camilleri, A. C. (2017). Digital learning resources and ubiquitous technologies in education. *Technology, Knowledge and Learning*, 22(1), 65–82.
- Charles, D., & McAlister, M. (2004). Integrating ideas about invisible playgrounds from play theory into online educational digital games. In Rauterberg, M. (Ed.) *ICEC 2004, LNCS 3166* (pp. 598–601). https://doi.org/10.1007/978-3-540-28643-1_79.
- Clark, D. B., Tanner-Smith, E. E., & Killingsworth, S. S. (2016). Digital games, design, and learning: A systematic review and meta-analysis. *Review of educational research*, 86(1), 79–122.
- De Grove, F., Bourgonjon, J., & Van Looy, J. (2012). Digital games in the classroom? A contextual approach to teachers' adoption intention of digital games in formal education. *Computers in Human Behavior*, 28(6), 2023–2033.
- Denham, A. R. (2016). Improving the design of a learning game through intrinsic integration and play-testing. *Technology, Knowledge and Learning*, 21(2), 175–194. <https://doi.org/10.1007/s10758-016-9280-1>.

- Dicheva, D., Dichev, C., Agre, G., & Angelova, G. (2015). Gamification in education: A systematic mapping study. *Educational Technology & Society*, 18(3), 75–88.
- Dreyfus, T., & Kidron, I. (2014). Introduction to abstraction in context (AiC). In *Networking of theories as a research practice in mathematics education* (pp. 85–96). Springer, Cham. https://doi.org/10.1007/978-3-319-05389-9_6.
- Eseryel, D., Law, V., Ifenthaler, D., Ge, X., & Miller, R. (2014). An investigation of the interrelationships between motivation, engagement, and complex problem solving in game-based learning. *Educational Technology & Society*, 17(1), 42–53.
- Fokides, E. (2018). Digital educational games and mathematics. Results of a case study in primary school settings. *Education and Information Technologies*, 23(2), 851–867.
- Garett, R., & Young, S. D. (2019). Health care gamification: A study of game mechanics and elements. *Technology, Knowledge and Learning*, 24(3), 341–353. <https://doi.org/10.1007/s10758-018-9353-4>.
- Gee, J. P. (2009). Deep learning properties of good digital games: How far can they go? In U. Ritterfeld, M. Cody, & P. Vorderer (Eds.), *Serious games: Mechanisms and effects* (pp. 67–82). New York, NY: Routledge.
- Gros, B. (2007). Digital games in education: The design of games-based learning environments. *Journal of Research on Technology in Education*, 40(1), 23–38. <https://doi.org/10.1080/15391523.2007.10782494>.
- Gros, B. (2015). Integration of digital games in learning and E-learning environments: Connecting experiences and context. In T. Lowrie & R. Jorgensen (Eds.), *Digital games and mathematics learning: Potential, promises and pitfalls* (pp. 35–54). Berlin: Springer. https://doi.org/10.1007/978-94-017-9517-3_3.
- Henkin, L. (1949). The completeness of the first-order functional calculus. *Journal of Symbolic Logic*, 14, 159–166. <https://doi.org/10.2307/2267044>.
- Hintikka, J. (1998). A game theory of logic—a logic of game theory. In W. Leinfellner & E. Köhler (Eds.), *Game theory, experience, rationality. Vienna circle institute yearbook [1997] (Institut 'Wiener Kreis' society for the advancement of the Scientific World conception)* (Vol. 5). Dordrecht: Springer. https://doi.org/10.1007/978-94-017-1654-3_25.
- Hintikka, J. (1999). *Inquiry as inquiry: A logic of scientific discovery*. Dordrecht: Springer.
- Hintikka, J., & Kulas, J. (1983). *The game of language: Studies in game-theoretical semantics and its applications. Synthese language library*. Berlin: Springer.
- Holbert, N. R., & Wilensky, U. (2014). Constructible authentic representations: Designing video games that enable players to utilize knowledge developed in-game to reason about science. *Technology, Knowledge and Learning*, 19(1–2), 53–79. <https://doi.org/10.1007/s10758-014-9214-8>.
- Holland, W., Jenkins, H., & Squire, K. (2003). Theory by design. In M. J. P. Wolf & B. Perron (Eds.), *The video game theory reader* (pp. 25–46). New York: Routledge.
- Hung, C. M., Huang, I., & Hwang, G. J. (2014). Effects of digital game-based learning on students' self-efficacy, motivation, anxiety, and achievements in learning mathematics. *Journal of Computers in Education*, 1(2–3), 151–166. <https://doi.org/10.1007/s40692-014-0008-8>.
- Ifenthaler, D., Eseryel, D., & Ge, X. (Eds.). (2012). *Assessment in game-based learning: Foundations, innovations, and perspectives*. New York, NY: Springer.
- Jagušt, T., Botički, I., & So, H. J. (2018). Examining competitive, collaborative and adaptive gamification in young learners' math learning. *Computers & Education*, 125, 444–457. <https://doi.org/10.1016/j.compedu.2018.06.022>.
- King, A. (2007). Scripting collaborative learning processes: A cognitive perspective. In F. Fischer, I. Kollar, H. Mandl, & J. Haake (Eds.), *Scripting computer-supported collaborative learning: Cognitive, computational and educational perspectives* (pp. 13–37). New York: Springer. https://doi.org/10.1007/978-0-387-36949-5_2.
- Klock, A. C. T., da Cunha, L. F., de Carvalho, M. F., Eduardo Rosa, B., Jaqueline Anton, A., & Gasparini, I. (2015). Gamification in E-learning systems: A conceptual model to engage students and its application in an adaptive E-learning system. In P. Zaphiris & A. Ioannou (Eds.), *Learning and collaboration technologies. LCT 2015. Lecture notes in computer science* (Vol. 9192). Cham: Springer. https://doi.org/10.1007/978-3-319-20609-7_56.
- Liu, D., Li, X., & Santhanam, R. (2013). Digital games and beyond: What happens when players compete? *Mis Quarterly*, 37(1), 111–124.
- Lowrie, T., & Jorgensen, R. (Eds.). (2015). *Digital games and mathematics learning: Potential, promises and pitfalls*. Berlin: Springer. <https://doi.org/10.1007/978-94-017-9517-3>.
- Nah, F. F. H., Zeng, Q., Telaprolu, V. R., Padmanabhuni Ayyappa, A., & Eschenbrenner, B. (2014). Gamification of education: A review of literature. In F. F. H. Nah (Ed.), *HCI in business. Lecture notes in computer science* (Vol. 8527). Cham: Springer. https://doi.org/10.1007/978-3-319-07293-7_39.

- Ninaus, M., Kiili, K., McMullen, J., & Moeller, K. (2017). Assessing fraction knowledge by a digital game. *Computers in Human Behavior*, 70, 197–206. <https://doi.org/10.1016/j.chb.2017.01.004>.
- Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics*. <https://doi.org/10.1007/s10649-019-09903-9>.
- OECD. (2017). *Pisa 2015 Collaborative Problem-Solving Framework*. Retrieved from <https://www.oecd.org/pisa/pisaproducts/Draft%20PISA%202015%20Collaborative%20Problem%20Solving%20Framework%20.pdf>.
- Plass, J. L., O'Keefe, P. A., Homer, B. D., Case, J., Hayward, E. O., Stein, M., et al. (2013). The impact of individual, competitive, and collaborative mathematics game play on learning, performance, and motivation. *Journal of Educational Psychology*, 105(4), 1050.
- Prediger, S., Bikner-Ahsbals, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM—The International Journal on Mathematics Education*, 40(2), 165–178.
- Schank, R. C., & Abelson, R. P. (1977). *Scripts, plans, goals and understandings*. Hillsdale, NJ: Erlbaum.
- Soldano, C., & Arzarello, F. (2016). Learning with touchscreen devices: Game strategies to improve geometric thinking. *Mathematics Education Research Journal*, 28(1), 9–30. <https://doi.org/10.1007/s13394-015-0166-7>.
- Tokac, U., Novak, E., & Thompson, C. G. (2019). Effects of game-based learning on students' mathematics achievement: A meta-analysis. *Journal of Computer Assisted learning*, 35, 407–420. <https://doi.org/10.1111/jcal.12347>.
- Vandercruysse, S., ter Vrugte, J., de Jong, T., Wouters, P., van Oostendorp, H., Verschaffel, L., et al. (2016). The effectiveness of a math game: The impact of integrating conceptual clarification as support. *Computers in Human Behavior*, 64, 21–33. <https://doi.org/10.1016/j.chb.2016.06.004>.
- Weinberger, A., Kollar, I., Dimitriadis, Y., Mäkitalo-Siegl, K., & Fischer, F. (2009). Computer-supported collaboration scripts: Perspectives from educational psychology and computer science. In N. Balachef, S. Ludvigsen, T. De Jong, A. Lazonder, & S. Barnes (Eds.), *Technology-enhanced learning: Principles and products* (pp. 155–174). Dordrecht: Springer. https://doi.org/10.1007/978-1-4020-9827-7_10.
- White, K., & McCoy, L. P. (2019). Effects of game-based learning on attitude and achievement in elementary mathematics. *Networks: An Online Journal for Teacher Research*, 21(1), 5. <https://doi.org/10.4148/2470-6353.1259>.

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