3D GEOMETRIC MOMENT INVARIANTS FROM THE POINT OF VIEW OF THE CLASSICAL INVARIANT THEORY

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ABSTRACT. The aim of this paper is to clear up the problem of the connection between the 3D geometric moments invariants and the invariant theory, considering a problem of describing of the 3D geometric moments invariants as a problem of the classical invariant theory. Using the remarkable fact that the groups $SO(3)$ and $SL(2)$ are locally isomorphic, we reduced the problem of deriving 3D geometric moments invariants to the well-known problem of the classical invariant theory. We give a precise statement of the 3D geometric invariant moments computation, introducing the notions of the algebras of simultaneous 3D geometric moment invariants, and prove that they are isomorphic to the algebras of joint $SL(2)$-invariants of several binary forms. To simplify the calculating of the invariants we proceed from an action of Lie group $SO(3)$ to equivalent action of the Lie algebra $\mathfrak{sl}_2$. The author hopes that the results will be useful to the researchers in the fields of image analysis and pattern recognition.

1. Introduction

The issue of the 3D geometric moments is a generalization of the 2D geometric moment invariants which are widely used as global feature descriptors in the different applications for pattern recognition and image analysis. Notice, that by invariance we mean the invariance with respect to translations, uniform scaling and rotations. In nowadays, the interest to the usage of the 3D moment invariants is stimulated by the rapid growth of the 3D technologies, [1]-[4].

For the first time, the 3D moment invariants of the second order were derived in the paper [5]. In [6], Lo and Don found twelve invariants of the third order, but as it was shown in [7] there are several interdepended among them. In the book [8], the author derived 13 invariants and stated that they generate all 3D geometric moments of the third order. Finally, in [9] a set of one 1185 invariants up to order 16 was presented, but these invariants do not form a minimal generating system. However, finding a minimal generating system of the 3D geometric moment invariants still remains an open problem. This kind of problems turn out to be a purely algebraic questions which were studied widely in the 19th century.

Today, there exists a huge massive of the literature on the 3D geometric moments invariants, but a big amount of it is devoted to the application of the invariants, along with the different ways of their constructions which sometimes are rather elegant and ingenious. For instance, the methods of the quantum mechanics used in [6], [7] and [10] are very impressive.

But, those methods based on the rotation group $SO(3)$ are quite complicated and are not adapted well for the invariants calculations. In this paper, we propose to proceed from the usage of the $SO(3)$ group to the usage of its locally isomorphic group $SL(2)$. As far as the Lie algebras $\mathfrak{so}_3$ and $\mathfrak{sl}_2$ are isomorphic, the problem of finding of $SO(3)$-invariants is equivalent to the problem of finding of $SL(2)$-invariants. The latter one is a well-known problem of the classical invariant theory issues, concequentely, the standard classical invariant theory approaches can be applied.

The aim of this paper is to consider the problem of describing 3D geometric moment invariants precisely as a problem of the classical invariant theory. We formulated the problem of the computation of the 3D geometric moments invariants based on the notion of the algebras of the both rational and polynomial simultaneous invariants of several binary forms. Our goal is not to find new invariants, we just put together some facts about the geometric 3D moments and presented it from a single point of view.

In this article, we proved that the introduced algebras of the 3D geometric moment invariants are isomorphic to the well-known objects of the classical invariant theory, namely, algebras of the joint invariants of the several binary forms. In the rational case, we firstly applied the standard infinitesimal method to the studying of the geometric moments and reduced the problem of calculating the $SO(3)$-invariants to the equivalent problem of calculating the invariants of its Lie algebra $\mathfrak{so}_3$.

The paper is arranged as follows.

In Sect. 2, we review basic concepts of the classical invariant theory and provide the necessary facts regarding the action of the Lie groups $SO(3)$ and $SL(2)$ and their Lie algebras $\mathfrak{so}_3$, and $\mathfrak{sl}_2$, respectively on the vector spaces of binary and ternary forms. We introduce the notions of the algebras of simultaneous rational and polynomial 3D geometric moment invariants and prove that they are isomorphic to the algebras of joint rational and polynomial $\mathfrak{sl}_2$-invariants of several binary forms. Also, we presented a system of partial differential equations concerning those invariants.

In Sect. 3, we recall the basic notions of the representation theory of the Lie algebras and present a minimal generating system for the algebra of the 3D geometric polynomial moments invariants of orders two and three which is expressed in the terms of eigenvectors of the Casimir operator. Also we derive the formula for the corresponding Poincaré series.

In Sect. 4, we count out the number of elements in a minimal generating set of the algebra rational rotation invariants and present such minimal generating set for the rational invariants of second and third orders. Also, we express the explicit form the invariants of the degrees one of arbitrary order.

The article is a continuation of the [12] article, which addresses the similar issues for the 2D geometric moment invariants.

2. Preliminary Concepts

In this section, we briefly review some basic concepts of the classical invariant theory, give the necessary facts about the Lie groups $SO(3), SL(2)$ and their Lie algebras $\mathfrak{so}_3$ and $\mathfrak{sl}_2$. Also, we give the definition of the algebras of simultaneous rational and polynomial 3D geometric moment invariants and then establish an isomorphism between these algebras and the algebras of the joint invariants of several binary forms.

2.1. **Basic notions of the invariant theory.** Let $GL(V)$ be the group of all invertible linear transformations of a finite-dimensional complex vector space V. The natural action of $GL(V)$ on V produces an action on the algebras of polynomial and rational functions $\mathbb{C}[V]$ and $\mathbb{C}(V)$. If $g \in GL(V)$, $F \in \mathbb{C}[V]$ define a new polynomial function $g \cdot F \in \mathbb{C}[V]$ by

$$(g \cdot F)(v) = F\left(g^{-1} v\right).$$

If G is subgroup of $GL(V)$ we say that F is *G-invariant* if $g \cdot F = F$ for all $g \in G$. The G-invariant polynomial functions forms a subalgebra $\mathbb{C}[V]^G$ of $\mathbb{C}[V]$. The algebra $\mathbb{C}[V]^G$ is called the *algebra of the polynomial G-invariants*. In the similar way, we define the algebra of *rational invariants* $\mathbb{C}(V)^G$.

Let us recall that *a derivation* of an algebra R is an additive map L satisfying the Leibniz rule:

$$L(r_1 r_2) = L(r_1) r_2 + r_1 L(r_2), \text{ for all } r_1, r_2 \in R.$$

The subalgebra

$$\ker L := \{f \in R | L(f) = 0\},$$

is called *the kernel* of the derivation L.

Let now G be a simply connected Lie group acting on V and let $\mathfrak{g}$ be its Lie algebra. By an action of $\mathfrak{g}$ we understand its representation by preserving Lie products of linear operators on V. We will extend these operators on $\mathbb{C}[V]$ and $\mathbb{C}(V)$ as derivatives. It is well known that the condition $I \in \mathbb{C}[V]^G$ is equivalent to $L(I) = 0$, $\forall L \in \mathfrak{g}$. Thus,

$$\mathbb{C}[V]^G = \mathbb{C}[V]^{\mathfrak{g}} = \bigcap_{L \in \mathfrak{g}} \ker L.$$

As a linear object, a Lie algebra is often a much easier to work with than working directly with the corresponding Lie group. We will use this fact later to ease the computation of invariants.

The classical invariant theory is focused on the action of the general linear group on homogeneous polynomials, with an emphasis on the forms, mainly binary and ternary ones. Let us consider two important invariant constructions which illustrate a computational advantage of the Lie algebras techniques.

Example 2.1. The space V_d of *binary forms* of degree d is the vector space:

$$V_d = \left\{ \sum_{k=0}^{d} \binom{d}{k} a_k x^{d-k} y^k \mid a_k \in \mathbb{C} \right\}.$$

The goup $SL(2)$ is a group of 2×2 complex matrices with determinant one. The corresponding Lie algebra $\mathfrak{sl}_2$ is generated by the matrices with zero trace

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the following commutation relations

$$[h, e_+] = 2e_+, \quad [h, e_-] = -2e_-, \quad [e_+, e_-] = h.$$

The elements e_-, e_+, h act on V_d by the derivations

$$-y\frac{\partial}{\partial x}, -x\frac{\partial}{\partial y}, -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

and act on $\mathbb{C}(V_d)$ by the derivations

$$D_+ = \sum_{k=0} (d-1) a_{k+1} \frac{\partial}{a_k}, D_- = \sum_{k=1}^{d} k a_{k-1} \frac{\partial}{a_k}, H = \sum_{k=0}^{d} (d-2k) a_k \frac{\partial}{a_k}.$$

The polynomial solutions of the corresponding system of differential equations generate the algebra $\mathbb{C}[V_d]^{\mathfrak{sl}_2}$ of invariants of binary form. Since,

$$[D_+, D_-] = D_+D_- - D_-D_+ = H$$

it follows that

$$\mathbb{C}[V_d]^{\mathfrak{sl}_2} = \ker D_+ \cap \ker D_-.$$

The *minimal generating systems* of $\mathbb{C}[V_d]^{\mathfrak{sl}_2}$ were a major object of research in classical invariant theory of the 19th century. At present, such generators have been found only for $d \leq 10$.

In the similar manner we define an action of $SL(2)$ and $\mathfrak{sl}_2$ on the direct sum

$$W = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_n}.$$

The corresponding algebras of polynomial and rational invariants are called the algebras of *joint invariants* (polynomial or rational) of binary forms and denoted by $\mathbb{C}[W]^{\mathfrak{so}_2}$ and $\mathbb{C}(W)^{\mathfrak{so}_2}$, respectively. At the present time, the algebras of the joint invariants are only known for a few values of $k_1, k_2, \ldots, k_n$, see [13].

Example 2.2. The 3D rotation group $SO(3)$ is the group of all rotations about the origin of three-dimensional Euclidean space. It is a three-parameters group with the following matrix realization

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \psi, \theta, \varphi \in [0, 2\pi].$$

where the parameters ψ, θ, φ are the *Euler angles.*

The associated tree-dimensional *complex* Lie algebra $\mathfrak{so}_3$ is generated by the matrix

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

and the Lie brackets are given by commutator, i.e.,

$$[e_1, e_2] = -e_3, \ [e_1, e_3] = e_2, \ [e_2, e_3] = -e_1.$$

Let us recall that the space of *ternary forms* of degree d is the vector space:

$$T_d = \left\{ \sum_{j+k+l=d} \binom{d}{j, k, l} a_{j,k,l} x^j y^k z^l \mid a_{j,k,l} \in \mathbb{C} \right\},$$

where $\displaystyle\binom{d}{j, k, l} = \frac{d!}{j!k!l!}$ denotes the multinomial coefficient. The linear functions

$$\sum_{j+k+l=d} \binom{d}{j, k, l} a_{j,k,l} x^j y^k z^l \mapsto a_{j,k,l},$$

form a basis of the dual vector space T_d^*. For convenience, it is useful to equal the functions and the corresponding coefficients $a_{j,k,l}$.

It is a well-known, see, for example, [14], that $\mathfrak{so}_3$ acts on T_d by derivations

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Wherefrom, it follows that $\mathfrak{so}_3$ acts on the dual space T_d^* by the derivations:

Theorem 1.

$$\begin{aligned} E_1(a_{j,k,l}) &= k a_{j+1,k-1,l} - j a_{j-1,k+1,l}, \\ E_2(a_{j,k,l}) &= l a_{j+1,k,l-1} - j a_{j-1,k,l+1}, \\ E_3(a_{j,k,l}) &= l a_{j,k+1,l-1} - k a_{j,k-1,l+1}. \end{aligned}$$

Proof. Using a definition of the $\mathfrak{so}_3$-action on the dual space. □

In the similar manner, we define an action of $SO(3)$ and $\mathfrak{so}_3$ on the direct sum

$$U = T_{k_1} \oplus T_{k_2} \oplus \cdots \oplus T_{k_3}.$$

The corresponding algebras of polynomial and rational invariants are called the algebras of *joint 3D rotation invariants* and denoted by $\mathbb{C}[U]^{\mathfrak{so}_3}$ and $\mathbb{C}(U)^{\mathfrak{so}_3}$, respectively. More details about 3D rotations can be found , e.g., in [14], [15].

An important detail that plays a crucial role in this article is a well-known fact that the complex Lie algebras $\mathfrak{so}_3$ and $\mathfrak{sl}_3$ are isomorphic, although the corresponding Lie groups are not isomorphic. To establish the isomorphism, we introduce new matrices

$$\mathcal{D}_+ = ie_1 + e_2 = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathcal{D}_- = ie_1 - e_2 = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathcal{H} = 2ie_3 = 2i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

By direct calculations of their commutators, we obtain

$$[\mathcal{H}, \mathcal{D}_+] = 2\mathcal{D}_+, \quad [\mathcal{H}, \mathcal{D}_-] = -2\mathcal{D}_-, \quad [\mathcal{D}_+, \mathcal{D}_-] = \mathcal{H}.$$

The commutators coincide with the corresponding commutators of the basic elements for the algebra $\mathfrak{sl}_2$, which establishes the isomorphism.

Note that the operators act on the basis elements of T_d^* as follows

$$\begin{aligned}
\mathcal{D}_+(a_{j,k,l}) &= i\left(k a_{j+1,k-1,l} - j a_{j-1,k+1,l}\right) + l a_{j+1,k,l-1} - j a_{j-1,k,l+1}, \\
\mathcal{D}_-(a_{j,k,l}) &= i\left(k a_{j+1,k-1,l} - j a_{j-1,k+1,l}\right) - \left(l a_{j+1,k,l-1} - j a_{j-1,k,l+1}\right), \\
\mathcal{H}(a_{j,k,l}) &= 2i\left(l a_{j,k+1,l-1} - k a_{j,k-1,l+1}\right).
\end{aligned}$$

As we will see later, this isomorphism allows us reduce the problem of finding the 3D rotation invariants to the problem of calculating the invariants of binary forms which is a classical invariant theory problem.

2.2. **Algebras of 3D rotation invariants.** In the sequel, we will work with the similarity transformation group G which is widely used in 3D image analysis and pattern recognition. The group is the semi-direct product of the space translation group $TR(3)$, the direct product of the space rotation group $SO(3)$ and the uniform scaling group $\mathbb{R}^*$:

$$G = (\mathbb{R}^* \times SO(3)) \rtimes TR(3).$$

The introduction of the notion of 2D image moment invariants by Hu in the significant paper [11] is a vivid example of the application of the classical invariant theory to the pattern recognition. A way of the generalization of this approach for 3D images was suggested in [5], [6]. Let $\boldsymbol{F}$ be a set of real finite piece-wise continuous functions that can have nonzero values only in a compact subset of $\mathbb{R}^3$.

Let us consider *the geometric moments* of $f \in \boldsymbol{F}$

$$m_{pqr}(f(x,y,z)) = m_{pqr} = \iiint_{\Omega} x^p y^q z^r f(x,y,z)\,dxdydz, \Omega \subset \mathbb{R}^3,$$

and the *central geometric moment*

$$\mu_{pqr}(f(x,y,z)) = \mu_{pqr} = \iiint_{\Omega} (x-\bar{x})^p (y-\bar{y})^q (z-\bar{z})^r f(x,y,z)\,dxdydz,$$

where

$$\bar{x} = \frac{m_{100}}{m_{0,0,0}}, \bar{y} = \frac{m_{010}}{m_{0,0,0}}, \bar{y} = \frac{m_{001}}{m_{0,0,0}}$$

The central geometric moments are already invariants under the translation group. After the normalization

$$\eta_{p,q,r} = \frac{\mu_{p,q,r}}{\mu_{0,0}^{1+\frac{p+q+r}{3}}}, p+q+r \geq 2,$$

they become invariants of the scaling group. Therefore, the problem of determining of the 3D geometric image moment invariants can be reduced to the problem of finding $SO(3)$-invariants as functions of the normalized central geometric moments. Therefore, in this paper we will deal only with the normalized $SO(3)$-invariant functions.

We will consider two types of such functions, specifically, polynomials and rational ones. Let $\mathbb{C}[\eta]$ and $\mathbb{C}(\eta)$ be the polynomial and rational algebras in countably many variables $\{\eta_{p,q,r}\}_{p+q+r=2}^{\infty}$ considered with the natural action of the group $SO(3)$. Denote by $\mathbb{C}[\eta]^{SO(3)}$ and $\mathbb{C}(\eta)^{SO(3)}$ the corresponding algebras of *polynomial* and *rational moment* invariants, respectively. Since these algebras are not finitely generated, then a complete set of invariants consists of infinitely many invariants. However, these algebras can be approximated by the finitely generated algebras $\mathbb{C}[\eta]_d^{SO(3)}$ and $\mathbb{C}(\eta)_d^{SO(3)}$where $[\eta]_d = \{\eta_{p,q,r}, 2 \leq p+q+r \leq d\}$. The elements of these algebras are called the *simultaneous* 3D geometric moment (polynomial or rational) invariants of *order* up to d. For instance, the invariant

$$\eta_{2,0,0} + \eta_{0,2,0} + \eta_{0,0,2},$$

belong to $\mathbb{C}[\eta]_2^{SO(3)}$ and $\mathbb{C}(\eta)_2^{SO(3)}$.

Remarkably, in general case, the problem of describing the algebras of the simultaneous 3D geometric moment invariants can be reduced to the well-known problems of the classical invariant theory. It turns out that the algebras $\mathbb{C}(\eta)_d^{SO(3)}$ and $\mathbb{C}(\eta)_d^{SO(3)}$ are isomorphic to the algebras of *joint polynomial and rational* $SL(2)$*-invariants* of some system of binary forms.

The *locally isomorphism* of $SO(3)$ and $SL(2)$ implies the following theorem.

Theorem 2. *The algebras of polynomial and rational simultaneous 3D geometric moment invariants* $\mathbb{C}[\eta]_d^{SO(3)}$ *and* $\mathbb{C}(\eta)_d^{SO(3)}$ *are isomorphic to the algebras of invariants* $\mathbb{C}[U_d]^{\mathfrak{sl}_2}$ *and* $\mathbb{C}(U_d)^{\mathfrak{sl}_2}$*, respectively. Here*

$$U_d = T_2 \oplus T_3 \oplus \cdots \oplus T_d,$$

and T_k *is the vector space of ternary forms of order* k*.*

Proof. It is sufficient to check that the algebras $\mathfrak{so}_3$ and $\mathfrak{sl}_2$ act by identical derivatives. Let us consider the action of the element

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

on the normalized moment $\eta_{j,k,l}$. By the definition, we have

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \eta_{p,q,r} = \iiint\limits_{\Omega} (x\cos\theta - y\sin\theta)^p (x\sin\theta + y\cos\theta)^q z^r f(x,y,z)dxdydz =$$

$$= \iiint\limits_{\Omega} \sum_{k=0}^{p}\sum_{j=0}^{q} (-1)^{p-k}\binom{p}{k}\binom{q}{j}(\cos\theta)^{p-k+j}(\sin\theta)^{q+k-j}x^{p-k+q-j}y^{k+j}z^r f(x,y,z)dxdydz =$$

$$= \sum_{k=0}^{p}\sum_{j=0}^{q} (-1)^{p-k}\binom{p}{k}\binom{q}{j}(\cos\theta)^{p-k+j}(\sin\theta)^{q+k-j}\eta_{p-k+q-j,k+j,r}.$$

To get the action of the Lie algebra $\mathfrak{so}_3$ we differentiate it by θ and, after simplification, we obtain:

$$\frac{d}{d\theta}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \eta_{p,q,r}\Big|_{\theta=0} = q\eta_{p+1,q-1,r} - p\eta_{p-1,q+1,r}.$$

It is easy to see that this action is identical to the derivation E_1, as it described in Example 2.2. In the same manner, we can show that the following elements of $SO(3)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

act like the derivations E_2 and E_3. Thus, the normalized 3D geometric moment invariants and the joint $\mathfrak{so}_3$-invariants of the binary forms are defined by the same system of the partial differential equation. It implies that $\mathbb{C}(\eta)_d^{SO(3)} \cong \mathbb{C}(U_d)^{\mathfrak{so}_3}$ and $\mathbb{C}[\eta]_d^{SO(3)} \cong \mathbb{C}[U_d]^{\mathfrak{so}_3}$. Since, $\mathfrak{so}_3 \cong \mathfrak{sl}_2$ we get that $\mathbb{C}(\eta)_d^{SO(3)} \cong \mathbb{C}(U_d)^{\mathfrak{sl}_2}$ and $\mathbb{C}[\eta]_d^{SO(3)} \cong \mathbb{C}[U_d]^{\mathfrak{sl}_2}$ as required.

The isomorphism has the simple form: $a_{j,k,r} \mapsto \eta_{j,k,r}$. □

Thus, from the point of view of the classical invariant theory, the problem of the description of the algebras 3D geometric image moment invariants $\mathbb{C}[\eta]_d^{SO(3)}$, $\mathbb{C}(\eta)_d^{SO(3)}$ can be reduced to the following two problems.

- **Problem 1.** What is a minimal generating set of the algebra polynomial joint invariants $\mathbb{C}[U_d]^{\mathfrak{sl}_2}$?
- **Problem 2.** What is a minimal generating set of the algebra rational joint invariants $\mathbb{C}(U_d)^{\mathfrak{sl}_2}$?

Besides, the problem of deriving of 3D geometric moment invariants can be reduced to a system of differential equations. The last result of Subsest. 2.1 implies the theorem:

Theorem 3. *The algebra $\mathbb{C}(U_d)^{\mathfrak{sl}_2}$ coincides with the algebra of rational solutions of the first order simultaneous partial differential equations:*

$$\begin{cases} \sum\limits_{2\le j+k+l\le d} (k\eta_{j+1,k-1,l} - j\eta_{j-1,k+1,l})\dfrac{\partial U}{\partial \eta_{j,k,l}} = 0, \\ \sum\limits_{2\le j+k+l\le d} (l\eta_{j+1,k,l-1} - j\eta_{j-1,k,l+1})\dfrac{\partial U}{\partial \eta_{j,k,l}} = 0. \end{cases}$$

In the next section we will deal with the algebras $\mathbb{C}[U_d]^{\mathfrak{sl}_2}$ and $\mathbb{C}(U_d)^{\mathfrak{sl}_2}$.

3. The algebra of polynomial invariants $\mathbb{C}[U_d]^{\mathfrak{sl}_2}$.

Let us recall some facts about representations of the Lie algebra $\mathfrak{sl}_2$.

3.1. **Representations of $\mathfrak{sl}_2$.** Let V be a finite-dimensional complex vector space equipped with non-trivial linear operators $D_+, D_-, H : V \to V$, which satisfy the following commutation relations

$$[H, D_+] = HD_+ - D_+H = 2D_+, \quad [H, D_-] = -2D_-, \quad [D_+, D_-] = H$$

Then V is called the *linear representation* of the Lie algera $\mathfrak{sl}_2$ or $\mathfrak{sl}_2$*-module.* The vector spaces T_k^*, U_d defined above are the samples of $\mathfrak{sl}_2$-modules. The modules 0 and V are called *trivial* modules. A $\mathfrak{sl}_2$-module V is called *irreducible* if V has no non-trivial $\mathfrak{sl}_2$*-submodule.* All irreducible $\mathfrak{sl}_2$-modules, up to isomorphism, can be described with the following construction.

Let $\mathcal{V}_n = \langle a_0, a_1, \ldots a_n\rangle$ be a $n{+}1$-dimension complex vector space and let the linear operators $D_-, D_+, H : V_n \to V_n$ act on elements of the basis as follows :

$$D_-(a_k) = ka_{k-1}, D_+(a_k) = (n-k)a_{k+1}, H(a_k) = (n-2k)a_k.$$

Let us check that the commutation relation for $\mathfrak{sl}_2$ are fulfilled. In fact, we have

$$\begin{aligned}
&[D_-, D_+](a_k) = D_-(D_+(a_k)) - D_+(D_-(a_k)) = D_-((d-k)a_{k+1}) - D_+(ka_{k-1}) = \\
&= (d-k)(k+1)a_k - k(d-(k-1))a_k = (d-2k)a_k = H(a_k), \\
&[H, D_-](a_k) = H(D_-(a_k)) - D(H(a_k)) = H(ka_{k-1}) - D((d-2k)a_k) = \\
&= k(d-2(k-1))a_{k-1} - (d-2k)ka_{k-1} = 2ka_{k-1} = 2D(a_k), \\
&[H, D_+](a_k) = H(D_+(a_k)) - D_+(H(a_k)) = H((d-k)a_{k+1}) - D_+((d-2k)a_k) = \\
&= (d-k)(d-2(k+1))a_{k+1} - (d-2k)(d-k)a_{k+1} = -2(d-k)a_{k+1} = -2D_+(a_k).
\end{aligned}$$

Therefore, $\mathcal{V}_n$ is an representation of $\mathfrak{sl}_2$. The vector space $\mathcal{V}_n$ considered together with the indicated action of the operators D_-, D_+, H is called *the standard irreducible* sl_2-module. It is well known, see [18], the an arbitrary $\mathfrak{sl}_2$-module can be decomposed into an direct sum of the irreducible standard $\mathfrak{sl}_2$-modules. Next, we present an algorithm of decomposing an arbitrary $\mathfrak{sl}_2$-module into the irreducible submodules. We use the algorithm later to construct invariants.

Let W be an arbitrary $\mathfrak{sl}_2$-module. For any element $w \in W$ the smallest natural number, denoted $\operatorname{ord}(w)$, such that

$$D_+^{\operatorname{ord}(z)}(w) \neq 0, \text{ but } D_+^{\operatorname{ord}(z)+1}(w) = 0.$$

is called *the order* of w. Since D_+ is a *nilpotent* operator, the order $\operatorname{ord}(w)$ is defined correctly.

A vector $z \in W$ is called *the lowest weight vector* if the following conditions holds: $D_-(z) = 0$ and $H(z) = \operatorname{ord}(z)z$. Any lowest weight vector defines an irreducible sl_2-module which is isomorphic to the standard sl_2-module. The following theorem holds.

Theorem 4. *Suppose $z \in W$ is a lowest weight vector. Then the vector space*

$$\mathcal{V}_s(z){:=}\langle v_0(z), v_1(z), \ldots v_s(z)\rangle, s = \operatorname{ord}(z),$$

where

$$v_k(z) = \frac{(s-k)!}{s!}\, D_+^k(z), v_0(z) := z,$$

is $\mathfrak{sl}_2$-module isomorphic to the standard sl_2-module $\mathcal{V}_s$.

Proof. It is easy to verify by direct calculations that the relations

$$H(D_+^k(z)) = (s-2k)\,D_+^k(z),$$
$$D_-(D_+^k(z)) = k(s-k+1)\,D_+^{k-1}(z),$$

hold for all $k \le s$. Let us construct the standard $\mathfrak{sl}_2$-module $\mathcal{V}_s$ with the basis vectors of the form

$$v_k = \alpha_k D_+^k(z), k = 0, \dots, s,$$

for some unknown constants $\alpha_k \in \mathbb{C}$.

In order the vectors form a basis of $\mathcal{V}_s$, the following two conditions must be satisfied:

$$D_-(v_k) = k v_{k-1}, D_+(v_k) = (s-k)v_{k+1},$$

for all $k = 0, \dots, s$. Since

$$D_-(v_k) = D_-(\alpha_k D_+^k(z)) = \alpha_k D_-(D_+^k(z)) = \alpha_k k(s-k+1)\,D_+^{k-1}(z),$$

and

$$D_-(v_k) = k v_{k-1} = k\alpha_{k-1} D_+^{k-1}(z),$$

we obtain the recurrence equation for α_k :

$$\alpha_k(s-k+1) = \alpha_{k-1}, \alpha_0 = 1.$$

It follows immediately that

$$\alpha_k = \frac{1}{s(s-1)\dots(s-k+1)}\,\alpha_0 = \frac{(s-k)!}{s!}.$$

Let us make sure that the second relation

$$D_+(v_k) = (s-k)v_{k+1},$$

also holds. We have

$$D_+(v_k) = D_+(\alpha_k D_+^k(v_0)) = \frac{(s-k)!}{s!}\frac{s!}{(s-(k+1))!}\frac{(s-(k+1))!}{s!} D_+^{k+1}(v_0) = (s-k)v_{k+1},$$

as required which ends the proof. □

The theorem below determines a structure of the sl_2-modules T_d^* and U_d up to isomorphism:

Theorem 5. *The following decompositions hold:*

$$T_d^* \cong \mathcal{V}_{2d} \oplus \mathcal{V}_{2d-4} \oplus \mathcal{V}_{2d-8} \oplus \dots \oplus \mathcal{V}_{2d-4\left[\frac{d}{2}\right]},$$
$$U_d^* \cong l_0^{(d)}\mathcal{V}_0 \oplus l_1^{(d)}\mathcal{V}_2 \oplus l_2^{(d)}\mathcal{V}_4 \oplus \dots \oplus l_d^{(d)}\mathcal{V}_{2d},$$

where

$$l_k^{(d)} = \begin{cases} 0, \text{ if } k > d, \\ \left[\dfrac{d-k}{2}\right], \text{ if } k = 0, 1, \\ \left[\dfrac{d-k}{2}\right] + 1, \text{ if } k \ge 2. \end{cases}$$

Since the proof requires some advanced results of the Lie algebras representation theory, we omit the proof.

Example 3.1. *For small d we have*

$$\begin{aligned} U_2^* &= T_2^* \cong \mathcal{V}_0 \oplus \mathcal{V}_4, \\ U_3^* &= T_2^* + T_3^* \cong \mathcal{V}_0 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6, \\ U_4^* &= T_2^* + T_3^* + T_4^* \cong 2\mathcal{V}_0 \oplus \mathcal{V}_2 \oplus 2\mathcal{V}_4 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8. \end{aligned}$$

Example 3.2. Theorem 5 implies that the invariant of degree one exist only in the case of even d. We can write an explicit form for these invariants. For any $d = 2m$, we consider the element

$$I_d = \sum_{j+k+l=m} \binom{m}{j,k,l} a_{2j,2k,2l}.$$

It is an invariant if the following conditions hold

$$E_1(I_d) = E_2(I_d) = E_1(I_d) = 0.$$

Let us prove that $E_1(I_d) = 0$. We have

$$E_1(I_d) = \sum_{j+k+l=d} \binom{d}{j,k,l} E_1(a_{2j,2k,2l}) = \sum_{j+k+l=d} \binom{d}{j,k,l} (2k\, a_{2j+1,2k-1,2l} - 2j\, a_{2j-1,2k+1,2l}).$$

Then

$$\sum_{j+k+l=n} k \binom{n}{j,k,l} a_{2j+1,2k-1,2l} = \sum_{\substack{j+k+l=n \\ k>0}} k \binom{n}{j,k,l} a_{2j+1,2k-1,2l} =$$

$$\sum_{\substack{j+k+l=n \\ k>0}} n \binom{n-1}{j,k-1,l} a_{2j+1,2k-1,2l} \overset{s=k-1}{=} \sum_{j+s+l=n-1} n \binom{n-1}{j,s,l} a_{2j+1,2s+1,2l} =$$

$$\overset{m=j+1}{=} \sum_{\substack{m+s+l=n \\ m>0}} n \binom{n-1}{m-1,s,l} a_{2m-1,2s+1,2l} = \sum_{m+s+l=n} m \binom{n}{m,s,l} a_{2m-1,2s+1,2l} =$$

$$\overset{\substack{j=m, \\ k=s}}{=} \sum_{j+k+l=n} j \binom{n}{j,k,l} a_{2j-1,2k+1,2l}\,.$$

Thus

$$\sum_{j+k+l=n} k \binom{n}{j,k,l} a_{2j+1,2k-1,2l} = \sum_{j+k+l=n} j \binom{n}{j,k,l} a_{2j-1,2k+1,2l},$$

and $E_1(I_d) = 0$. In the same way, we can show that $E_2(I_d) = 0$ and $E_3(I_d) = 0$.

For small d we have

$$\begin{gathered} I_2 = a_{0,0,2} + a_{0,2,0} + a_{2,0,0}, \\ I_4 = a_{0,0,4} + 2\,a_{0,2,2} + a_{0,4,0} + 2\,a_{2,0,2} + 2\,a_{2,2,0} + a_{4,0,0}, \\ I_6 = 3\,a_{4,0,2} + 3\,a_{4,2,0} + a_{6,0,0} + 3\,a_{0,4,2} + a_{0,6,0} + 3\,a_{2,0,4} + 6\,a_{2,2,2} + 3\,a_{2,4,0} + a_{0,0,6} + 3\,a_{0,2,4}, \\ I_8 = 6\,a_{4,4,0} + 4\,a_{6,0,2} + 4\,a_{6,2,0} + a_{8,0,0} + 12\,a_{2,4,2} + 4\,a_{2,6,0} + 6\,a_{4,0,4} + 12\,a_{4,2,2} + a_{0,8,0} + 4\,a_{2,0,6} + \\ +12\,a_{2,2,4} + a_{0,0,8} + 4\,a_{0,2,6} + 6\,a_{0,4,4} + 4\,a_{0,6,2} \end{gathered}$$

Theorem 5 implies

Theorem 6. *The following decompositions hold:*

$$\begin{array}{ll}(i) & \mathbb{C}[\eta]_d^{SO(3)} \cong \mathbb{C}[l_0^{(d)} V_0 \oplus l_1^{(d)} V_2 \oplus l_2^{(d)} V_4 \oplus \cdots \oplus l_d^{(d)} V_{2d}]^{\mathfrak{sl}_2},\\ (ii) & \mathbb{C}(\eta)_d^{SO(3)} \cong \mathbb{C}(l_0^{(d)} V_0 \oplus l_1^{(d)} V_2 \oplus l_2^{(d)} V_4 \oplus \cdots \oplus l_d^{(d)} V_{2d})^{\mathfrak{sl}_2}\end{array}$$

Therefore, it implies that the problem of determining of the algebra 3D geometric polynomial and rational moment invariants is equivalent to the problem of of determining of the algebras joint $\mathfrak{sl}_2$-invariants. It appears to be a very difficult problem in terms of performing calculations and it is quite a challenge to find a minimal generating set for $d > 5$.

3.2. **The algebra of 3D polynomial moment invariants $\mathbb{C}[\eta]_2^{SO(3)}$.** Let us illustrate the above with the references to the algebra of 3D polynomial moment invariants of order two. Since Theorem 5 implies that $T_2^* \cong \mathcal{V}_0^* \oplus \mathcal{V}_4^*$, the algebra of 3D polynomial moment invariants $\mathbb{C}[\eta]_2^{SO(3)}$ $\mathbb{C}[\eta]_2^{SO(3)}$ is equal to the algebra of $\mathfrak{sl}_2$-invariants $\mathbb{C}[\mathcal{V}_0(u_0) \oplus \mathcal{V}_4(v_0)]^{\mathfrak{sl}_2}$, where u_0, v_0 are the lowest weight vectors in the T_2^*-*realizations* of the standard $\mathfrak{sl}_2$-modules $\mathcal{V}_0$ and $\mathcal{V}_4$.

To find such a realization, firstly we need to find the realization of the standard basis of $\mathcal{V}_0$ and $\mathcal{V}_4$ on T_2^* and, then, substitute it into the expressions for the generating invariants of the algebra $\mathbb{C}[\mathcal{V}_0(u_0) \oplus \mathcal{V}_4(v_0)]^{\mathfrak{sl}_2}$. Since $\mathcal{V}_0$ is a trivial $\mathfrak{sl}_2$-module, it is enough to find the generating elements of the alebra $\mathbb{C}[V_4(v_0)]^{\mathfrak{sl}_2}$. But $\mathbb{C}[V_4(v_0)]^{\mathfrak{sl}_2}$ which is isomorphic to the classical algebra of invariants of binary form of degree four

$$a_0 x^4 + 4\, a_1 x^3 y + 6\, a_2 x^2 y^2 + 4\, a_3 x y^3 + a_4 y^4.$$

It is well-known, that the latter is generated by the following two invariants of degree two and three:

$$S_1 = a_0 a_4 + 3 a_2^2 - 4 a_1 a_3,$$

$$S_2 = a_0 a_2 a_4 + 2 a_1 a_2 a_3 - a_2^3 - a_0 a_3^2 - a_1^2 a_4 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

In terms of the classical invariant theory, the invariant S_1 is called the *apolar invariant* and the invariant S_2 is known as the the *catalecticant* or the *Hankel determinant.*

The six-dimensional sl_2-module T_2^* is generated by the following elements:

$$T_2^* = \langle a_{0,0,2}, a_{0,1,1}, a_{0,2,0}, a_{1,0,1}, a_{1,1,0}, a_{2,0,0} \rangle.$$

The operators $\mathcal{D}_+, \mathcal{D}_-, \mathcal{H}$ act on the basis as follows (see Theorem 1):

$$\begin{aligned} \mathcal{D}_+(a_{j,k,l}) &= i\,(k a_{j+1,k-1,l} - j a_{j-1,k+1,l}) + l a_{j+1,k,l-1} - j a_{j-1,k,l+1},\\ \mathcal{D}_-(a_{j,k,l}) &= i\,(k a_{j+1,k-1,l} - j a_{j-1,k+1,l}) - (l a_{j+1,k,l-1} - j a_{j-1,k,l+1})\,,\\ \mathcal{H}(a_{j,k,l}) &= 2i\,(l a_{j,k+1,l-1} - k a_{j,k-1,l+1})\,. \end{aligned}$$

The lowest weight vectors u_0, v_0 of the $\mathfrak{sl}_2$-modules $\mathcal{V}_0(u_0)$ and $\mathcal{V}_4(v_0)$ are the solutions of the following two systems of linear equations: $\begin{cases} \mathcal{E}_-(z) = 0, \\ \mathcal{H}(z) = 0 \end{cases}$ and $\begin{cases} \mathcal{E}_-(z) = 0, \\ \mathcal{H}(z) = 4z \end{cases}$, respectively. Thus, we obtain

$$\begin{aligned} u_0 &= I_1 = a_{0,0,2} + a_{0,2,0} + a_{2,0,0},\\ v_0 &= 2\, a_{0,1,1} + i(a_{0,0,2} - a_{0,2,0}). \end{aligned}$$

The element u_0 is already an invariant.

Using Theorem 4, we get the standard basis $\mathcal{V}_4(v_0)$:

$$
\begin{aligned}
v_0 &= v_0 = 2\, a_{0,1,1} + i(a_{0,0,2} - a_{0,2,0}),\\
v_1 &= \frac{1}{4} D_+(x_0) = i a_{1,0,1} + a_{1,1,0},\\
v_2 &= \frac{1}{12} D_+^2(x_0) = -\frac{i}{3}\left(a_{0,0,2} + a_{0,2,0} - 2\, a_{2,0,0}\right),\\
v_3 &= \frac{1}{24} D_+^3(x_0) = a_{1,1,0} - i a_{1,0,1},\\
v_4 &= \frac{1}{24} D_+^4(x_0) = -2\, a_{0,1,1} + i\left(-a_{0,2,0} + a_{0,0,2}\right)
\end{aligned}
$$

Substituting v_i for a_i in S_1, S_2 we find invariants I_2 and I_3 :

$$
\begin{aligned}
I_2 =& {a_{0,0,2}}^2 - a_{0,0,2} a_{0,2,0} - a_{0,0,2} a_{2,0,0} + 3\, {a_{0,1,1}}^2 + {a_{0,2,0}}^2 - a_{0,2,0} a_{2,0,0} + 3\, {a_{1,0,1}}^2 + 3\, {a_{1,1,0}}^2 + {a_{2,0,0}}^2,\\
I_3 =& 2\, {a_{0,0,2}}^3 - 3\, {a_{0,0,2}}^2 a_{2,0,0} + 9\, a_{0,0,2} {a_{0,1,1}}^2 - 3\, {a_{0,2,0}}^2 a_{0,0,2} + 12\, a_{0,0,2} a_{2,0,0} a_{0,2,0} +\\
&+ 9\, a_{0,0,2} {a_{1,0,1}}^2 - 18\, a_{0,0,2} {a_{1,1,0}}^2 - 3\, a_{0,0,2} {a_{2,0,0}}^2 + 9\, {a_{0,1,1}}^2 a_{0,2,0} - 18\, {a_{0,1,1}}^2 a_{2,0,0} +\\
&+ 54\, a_{0,1,1} a_{1,1,0} a_{1,0,1} - 3\, {a_{0,2,0}}^2 a_{2,0,0} - 18\, a_{0,2,0} {a_{1,0,1}}^2 + 9\, a_{0,2,0} {a_{1,1,0}}^2 - 3\, a_{0,2,0} {a_{2,0,0}}^2 +\\
&+ 9\, {a_{1,0,1}}^2 a_{2,0,0} + 9\, {a_{1,1,0}}^2 a_{2,0,0} + 2\, {a_{2,0,0}}^3 - 3\, {a_{0,0,2}}^2 a_{0,2,0} + 2\, {a_{0,2,0}}^3.
\end{aligned}
$$

Thus, we have proved the following theorem.

Theorem 7. *The algebras of polynomial and rational invariants* $\mathbb{C}[T_2]^{\mathfrak{sl}_2}$, $\mathbb{C}(T_2)^{\mathfrak{sl}_2}$ *are generated by the invariants* I_1, I_2 *and* I_3 :

$$
\begin{aligned}
\mathbb{C}[T_2]^{sl_2} &= \mathbb{C}[I_1, I_2, I_3],\\
\mathbb{C}(T_2)^{sl_2} &= \mathbb{C}(I_1, I_2, I_3).
\end{aligned}
$$

In order to obtain the 3D moment invariant it is sufficient to replace $a_{j,k,l}$ by the normalized moments $\eta_{j,k,l}$ in I_1, I_2 and I_3.

We should admit that the obtained result is confirmed by the result of [5], [6], [8] obtained by a different method.

As far as the obtained expressions for the invariants are quite cumbersome, we are interested in finding a simpler representation for them. Let us consider *the Laplace operator:*

$$
\mathcal{L} = \mathcal{D}_+\mathcal{D}_- + \mathcal{D}_-\mathcal{D}_+ + \frac{1}{2}\mathcal{H}^2 = E_1^2 + E_2^2 + E_3^2,
$$

which belongs to the *enveloping algebra* of $\mathfrak{sl}_2$. It can be proved that $\mathcal{L}$ commutes with the operators $\mathcal{D}_+, \mathcal{D}_-, \mathcal{H}$. Therefore, $\mathcal{L}$ is *diagonalizable* on every standard $\mathfrak{sl}_2$-module.

Let us express the invariants in terms of the eigenvectors of the Laplace operator $\mathcal{L}$. The operator $\mathcal{L}$ acts on the basis of T_2^* as follows:

$$
\begin{aligned}
\mathcal{L}(a_{0,0,2}) &= -4\, a_{2,0,0} + 8\, a_{0,0,2} - 4\, a_{0,2,0}, \mathcal{L}(a_{0,1,1}) = 12\, a_{0,1,1},\\
\mathcal{L}(a_{0,2,0}) &= 8\, a_{0,2,0} - 4\, a_{2,0,0} - 4\, a_{0,0,2}, \mathcal{L}(a_{1,0,1}) = 12\, a_{1,0,1},\\
\mathcal{L}(a_{1,1,0}) &= 12\, a_{1,1,0}, \mathcal{L}(a_{2,0,0)}) = -4\, a_{0,2,0} + 8\, a_{2,0,0} - 4\, a_{0,0,2}.
\end{aligned}
$$

Since $T_2^* = \mathcal{V}_0(u_0) \oplus \mathcal{V}_4(v_0)$, then there exists one eigenvector, let us denote it by e_0 , associated with the zero eigevalue and five eigenvectors e_1, e_2, e_3, e_4, e_5 associated with the eigevalue twelve.The eigenvectors could be found by the standard linear algebra algorithm:

$$
\begin{aligned}
&e_0 = a_{0,0,2} + a_{0,2,0} + a_{2,0,0},\\
&e_1 = a_{0,1,1}, e_2 = a_{0,2,0} - a_{0,0,2}, e_3 = a_{1,0,1}, e_4 = a_{1,1,0}, e_5 = a_{2,0,0} - a_{0,0,2}.
\end{aligned}
$$

Then the invariants I_1, I_2 and I_3 are expressed in a much more compact form:

$$
\begin{aligned}
&I_1 = e_0,\\
&I_2 = 3\,e_1^2 + e_2^2 - e_5\,e_2 + 3\,e_3^2 + 3\,e_4^2 + e_5^2,\\
&I_3 = 9e_1^2e_2 - 18e_1^2e_5 + 54e_1e_4e_3 + 2e_2^3 - 3e_5e_2^2 - 18e_3^2e_2 + 9e_4^2e_2 - 3e_5^2e_2 + 9e_3^2e_5 + 9e_4^2e_5 + 2e_5^3.
\end{aligned}
$$

3.3. **The algebra of polynomial invariants $\mathbb{C}[U_3]^{\mathfrak{sl}_2}$.** We note, that the case $d = 3$ is much more complicated than the case $d = 2$. For $d = 3$ we have the following decomposition of the $\mathfrak{sl}_2$-module U_3 :

$$U_3 = T_2^* \oplus T_3^* \cong \mathcal{V}_0(v_0) \oplus \mathcal{V}_2(x_0) \oplus \mathcal{V}_4(y_0) \oplus \mathcal{V}_6(u_0).$$

Suppose the $\mathfrak{sl}_2$-modules $\mathcal{V}_0, \mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_6$ are given by their standard bases:

$$
\begin{aligned}
&\mathcal{V}_0(v_0) = \langle v_0 \rangle,\\
&\mathcal{V}_2(x_0) = \langle x_0, x_1, x_2 \rangle,\\
&\mathcal{V}_4(y_0) = \langle y_0, y_1, y_2, y_3, y_4 \rangle,\\
&\mathcal{V}_6(u_0) = \langle u_0, u_1, u_2, u_3, u_4, u_5, u_6 \rangle.
\end{aligned}
$$

Proceeding as above, we again find the lowest weight vectors v_0, x_0, y_0, u_0 by solving systems of linear equations. Further, by using Theorem 4 we obtain the following realization of all these $\mathfrak{sl}_2$-modules in U_3 :

$$
\begin{aligned}
&v_0 = a_{0,0,2} + a_{0,2,0} + a_{2,0,0},\\
&x_0 = a_{0,0,3} + a_{0,2,1} + a_{2,0,1} - i\left(a_{0,1,2} + a_{0,3,0} + a_{2,1,0}\right), x_1 = a_{1,0,2} + a_{1,2,0} + a_{3,0,0}, x_2 = -\overline{x}_0,\\
&y_0 = 2a_{0,1,1} + i\left(a_{0,0,2} - a_{0,2,0}\right), y_1 = a_{1,1,0} + ia_{1,0,1}, y_2 = \frac{i}{3}\left(2\,a_{2,0,0} - a_{0,0,2} - a_{0,2,0}\right), y_3 = \overline{y}_1, y_4 = -\overline{y}_0,\\
&u_0 = a_{0,0,3} - 3\,a_{0,2,1} + i\left(a_{0,3,0} - 3\,a_{0,1,2}\right), u_1 = a_{1,0,2} - a_{1,2,0} - 2\,ia_{1,1,1},\\
&u_2 = \frac{1}{5}(4\,a_{2,0,1} - a_{0,0,3} - a_{0,2,1} + i\left(a_{0,1,2} + a_{0,3,0} - 4\,a_{2,1,0}\right)),\\
&u_3 = \frac{1}{5}(2\,a_{3,0,0} - 3\,a_{1,0,2} - 3\,a_{1,2,0}), u_4 = -\overline{u}_2, u_5 = \overline{u}_1, u_6 = -\overline{u}_0.
\end{aligned}
$$

Here $\overline{}$ indicates the complex conjugate.

Recently, the minimal generating set of polynomial invariants for the algebra $\mathbb{C}[\mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6]^{\mathfrak{sl}_2}$ was calculated, see [13], in the symbolic form. The minimal generating set consists of 195 invariants and their degrees grow up to fifteen. Therefore, a minimal generating set of polynomial invariants of the algebra $\mathbb{C}[T_3]^{\mathfrak{sl}_2}$ consists of 196 invariants. These invariants can be calculated explicitly using author's Maple package [16] or by expanding the transvectants listed in the paper [13]. Below we present only the first thirteen invariants:

deg	Invariants	#
1	v_0	1
2	$x_0x_2-x_1^2, y_0y_4-4\,y_1y_3+3\,y_2^2, u_0u_6-6\,u_1u_5+15\,u_2u_4-10\,u_3^2$	3
3	$y_0y_2y_4-y_0y_3^2-y_1^2y_4+2\,y_1y_2y_3-y_2^3,$ $x_0^2y_4-4\,x_0x_1y_3+2\,x_0x_2y_2+4\,x_1^2y_2-4\,x_1x_2y_1+x_2^2y_0,$ $u_0u_4y_4-2\,u_0u_5y_3-4\,u_1u_3y_4+6\,u_1u_4y_3-2\,u_1u_6y_1+3\,u_2^2y_4-4\,u_2u_3y_3-$ $-9\,u_2u_4y_2+6\,u_2u_5y_1+u_2u_6y_0+8\,u_3^2y_2-4\,u_3u_4y_1-4\,u_3u_5y_0+3\,u_4^2y_0+u_0u_6y_2,$ $u_0x_2y_4-2\,u_1x_1y_4-4\,u_1x_2y_3+8\,u_2x_1y_3+6\,u_2x_2y_2-4\,u_3x_0y_3-12\,u_3x_1y_2-$ $-4\,u_3x_2y_1+6\,u_4x_0y_2+8\,u_4x_1y_1+u_4x_2y_0-4\,u_5x_0y_1-2\,u_5x_1y_0+u_6x_0y_0+u_2x_0y_4$	4
4	$u_0x_2^3-6\,u_1x_1x_2^2+3\,u_2x_0x_2^2+12\,u_2x_1^2x_2-12\,u_3x_0x_1x_2-8\,u_3x_1^3+$ $+3\,u_4x_0^2x_2+12\,u_4x_0x_1^2-6\,u_5x_0^2x_1+u_6x_0^3,$ $u_0u_4x_2^2-2\,u_0u_5x_1x_2+u_0u_6x_1^2-4\,u_1u_3x_2^2+6\,u_1u_4x_1x_2+2\,u_1u_5x_0x_2-$ $-2\,u_1u_5x_1^2-2\,u_1u_6x_0x_1+3\,u_2^2x_2^2-4\,u_2u_3x_1x_2-8\,u_2u_4x_0x_2-u_2u_4x_1^2+$ $+6\,u_2u_5x_0x_1+u_2u_6x_0^2+6\,u_3^2x_0x_2+2\,u_3^2x_1^2-4\,u_3u_4x_0x_1-4\,u_3u_5x_0^2+3\,u_4^2x_0^2,$ $u_0y_1y_4^2-3\,u_0y_2y_3y_4+2\,u_0y_3^3-u_1y_0y_4^2-2\,u_1y_1y_3y_4+9\,u_1y_2^2y_4-6\,u_1y_2y_3^2+$ $+5\,u_2y_0y_3y_4-15\,u_2y_1y_2y_4+10\,u_2y_1y_3^2-10\,u_3y_0y_3^2+10\,u_3y_1^2y_4-5\,u_4y_0y_1y_4+$ $+15\,u_4y_0y_2y_3-10\,u_4y_1^2y_3+u_5y_0^2y_4+2\,u_5y_0y_1y_3-9\,u_5y_0y_2^2+6\,u_5y_1^2y_2-u_6y_0^2y_3+$ $+3\,u_6y_0y_1y_2-2\,u_6y_1^3,$ $x_0^2y_2y_4-x_0^2y_3^2-2\,x_0x_1y_1y_4+2\,x_0x_1y_2y_3+2\,x_0x_2y_1y_3-2\,x_0x_2y_2^2+x_1^2y_0y_4-$ $-x_1^2y_2^2-2\,x_1x_2y_0y_3+2\,x_1x_2y_1y_2+x_2^2y_0y_2-x_2^2y_1^2,$ $u_0u_2u_4u_6-u_0u_2u_5^2-u_0u_3^2u_6+2\,u_0u_3u_4u_5-u_0u_4^3-u_1^2u_4u_6+u_1^2u_5^2+$ $+2\,u_1u_2u_3u_6-2\,u_1u_2u_4u_5-2\,u_1u_3^2u_5+2\,u_1u_3u_4^2-u_2^3u_6+2\,u_2^2u_3u_5+$ $+u_2^2u_4^2-3\,u_2u_3^2u_4+u_3^4.$	5

Substituting the realizations of the standard $\mathfrak{sl}_2$-modules in the invariants expressions, we get the explicit expressions for the invariants of the algebra $\mathbb{C}[T_3]^{\mathfrak{sl}_2}$. In order to obtain the 3D moment invariant it is sufficient to replace $a_{j,k,l}$ by the normalized moments $\eta_{j,k,l}$. For example, the 3D geometric moment invariants of low degrees have the form

$$\begin{aligned}
B_0 =&\eta_{0,0,2}+\eta_{0,2,0}+\eta_{2,0,0},\\
B_1 =&\eta_{0,0,2}{}^2-\eta_{0,0,2}\eta_{0,2,0}-\eta_{0,0,2}\eta_{2,0,0}+3\,\eta_{0,1,1}{}^2+\eta_{0,2,0}{}^2-\eta_{0,2,0}\eta_{2,0,0}+3\,\eta_{1,0,1}{}^2+3\,\eta_{1,1,0}{}^2+\eta_{2,0,0}{}^2,\\
B_2 =&\eta_{0,0,3}{}^2+2\,\eta_{0,0,3}\eta_{0,2,1}+2\,\eta_{0,0,3}\eta_{2,0,1}+\eta_{0,1,2}{}^2+2\,\eta_{0,1,2}\eta_{0,3,0}+2\,\eta_{0,1,2}\eta_{2,1,0}+\eta_{0,2,1}{}^2+\\
&+2\,\eta_{0,2,1}\eta_{2,0,1}+\eta_{0,3,0}{}^2+2\,\eta_{0,3,0}\eta_{2,1,0}+\eta_{1,0,2}{}^2+2\,\eta_{1,0,2}\eta_{1,2,0}+2\,\eta_{1,0,2}\eta_{3,0,0}+\eta_{1,2,0}{}^2+\\
&+2\,\eta_{1,2,0}\eta_{3,0,0}+\eta_{2,0,1}{}^2+\eta_{2,1,0}{}^2+\eta_{3,0,0}{}^2,\\
B_3 =&\eta_{0,0,3}{}^2-3\,\eta_{0,0,3}\eta_{0,2,1}-3\,\eta_{0,0,3}\eta_{2,0,1}+6\,\eta_{0,1,2}{}^2-3\,\eta_{0,1,2}\eta_{0,3,0}-3\,\eta_{0,1,2}\eta_{2,1,0}+6\,\eta_{0,2,1}{}^2-\\
&-3\,\eta_{0,2,1}\eta_{2,0,1}+\eta_{0,3,0}{}^2-3\,\eta_{0,3,0}\eta_{2,1,0}+6\,\eta_{1,0,2}{}^2-3\,\eta_{1,0,2}\eta_{1,2,0}-3\,\eta_{1,0,2}\eta_{3,0,0}+15\,\eta_{1,1,1}{}^2+\\
&+6\,\eta_{1,2,0}{}^2-3\,\eta_{1,2,0}\eta_{3,0,0}+6\,\eta_{2,0,1}{}^2+6\,\eta_{2,1,0}{}^2+\eta_{3,0,0}{}^2.
\end{aligned}$$

All of the 196 invariants can be obtained in a similar way as above.

In the book [8], the 3D moment invariants $\Phi_1,\ldots,\Phi_{13}$ were presented, in particular, the first degree invariant Φ_1 and the invariants Φ_2,Φ_4,Φ_5 of degree two. These invariants could be expressed in terms of the invariants B_0,B_1,B_2,B_3 as follows:

$$\Psi_1=B_0, \Phi_2=\frac{B_0^2+2B_1}{3}, \Phi_4=\frac{3B_2+2B_3}{5}, \Phi_5=B_2.$$

The Poincaré series of the algebra $\mathbb{C}[T_3]^{\mathfrak{sl}_2}$ calculated by using Maple package (see [17]) has the form:

$$\mathcal{P}(\mathbb{C}[T_3]^{sl_2}, z) = \frac{p_{0246}(z)}{(1-z)(1-z^6)(1-z^5)^2(1-z^4)^3(1-z^3)^3(1-z^2)^3} =$$
$$= 1 + z + 4\,z^2 + 8\,z^3 + 26\,z^4 + 53\,z^5 + 146\,z^6 + 305\,z^7 + 704\,z^8 + 1417\,z^9 + \cdots$$

where

$$p_{0246}(z) = z^{28} + z^{25} + 9\,z^{24} + 13\,z^{23} + 37\,z^{22} + 51\,z^{21} + 91\,z^{20} + 119\,z^{19} + 181\,z^{18} + 208\,z^{17} +$$
$$+277\,z^{16} + 283\,z^{15} + 311\,z^{14} + 283\,z^{13} + 277\,z^{12} + 208\,z^{11} + 181\,z^{10} + 119\,z^9 + 91\,z^8 + 51\,z^7 +$$
$$+37\,z^6 + 13\,z^5 + 9\,z^4 + z^3 + 1$$

Therefore, the algebra $\mathbb{C}[\eta]_3^{\mathfrak{sl}_2}$ consists of one invariant of degree 1, namely B_0. Also, there exists four linearly independent invariants of degree two, namely B_0^2, B_1, B_2, B_3, eight linearly independent invariants of degree three etc.

4. The algebra of rational invariants $\mathbb{C}(U_d)^{\mathfrak{sl}_2}$.

Concidering applications, the rational invariants are more interesting applications than the polynomial ones. In the paper [9], a set of 1185 of the 3D rotational moment invariants up to the sixteenth order was presented. However, these invariants do not form a minimal generating system and setting a minimal generating system is still remaining an open problem.

In the following theorem we find the cardinality of a minimal generating set of the algebra of 3D rational rotation invariants.

Theorem 8. *The number of elements in a minimal generating set of the algebra of the rational invariants* $\mathbb{C}(U_d)^{\mathfrak{sl}_2}, d \geq 2$ *is equal to*

$$\binom{d+3}{3} - 7.$$

Proof. Since the group $SL(2)$ as an affine variety is three-dimensional one, then, the transcendence degree of the field extension $\mathbb{C}(U_d)^{SO(3)}/\mathbb{C}$ equals to

$$\operatorname{tr}\deg_{\mathbb{C}} \mathbb{C}(U_d)^{SO(3)} = \dim U_d - \dim SO(3).$$

Thus, the algebra $\mathbb{C}(U_d)^{\mathfrak{sl}_2}$ consists of exactly $\dim U_d - 3$ algebraically independent elements. Taking into account that

$$\dim U_d - 3 = \sum_{k=2}^{d} \dim T_d - 3 = \sum_{k=2}^{d} \binom{k+2}{2} - 3 = \binom{d+3}{3} - 7,$$

which is equal to that to be proved. $\square$

In particular, for $d = 2, 3$ we have three and thirteen invariants, respectively. These results are confirmed by the results in [8]. For $d = 2$, it implies that the algebra $\mathbb{C}(U_2)^{\mathfrak{sl}_2}$ is generated by the invariants I_1, I_2 and I_3.

A system of 13 invariants of the algebra $\mathbb{C}(U_3)^{\mathfrak{sl}_2}$ was presented in [8].

The authors claim, without proof, that these invariants are *independent*. Below we present another system of thirteen invariants for $\mathbb{C}(U_3)^{\mathfrak{sl}_2}$ and prove that all these invariants form a minimal generating set of the algebra rational invariants$\mathbb{C}(U_3)^{\mathfrak{sl}_2}$.

In the Sect. 3.3 we found an explicit form for each of the thirteen polynomial invariants of the algebra $\mathbb{C}[U_3]^{\mathfrak{sl}_2}$.Though, the expressions for the invariants are quite cumbersome, we will express them in terms of the eigenvectors of the Laplace operator $\mathcal{L}$. The operator $\mathcal{L}$ acts on the basis of T_3^* as follows:

$$
\begin{aligned}
&\mathcal{L}(a_{0,0,3}) = -12\,a_{2,0,1} + 12\,a_{0,0,3} - 12\,a_{0,2,1}, \mathcal{L}(a_{0,1,2}) = 20\,a_{0,1,2} - 4\,a_{2,1,0} - 4\,a_{0,3,0},\\
&\mathcal{L}(a_{0,2,1}) = 20\,a_{0,2,1} - 4\,a_{2,0,1} - 4\,a_{0,0,3}, \mathcal{L}(a_{0,3,0}) = -12\,a_{2,1,0} + 12\,a_{0,3,0} - 12\,a_{0,1,2},\\
&\mathcal{L}(a_{1,0,2}) = 20\,a_{1,0,2} - 4\,a_{3,0,0} - 4\,a_{1,2,0}, \mathcal{L}(a_{1,1,1}) = 24\,a_{1,1,1},\\
&\mathcal{L}(a_{1,2,0}) = 20\,a_{1,2,0} - 4\,a_{3,0,0} - 4\,a_{1,0,2}, \mathcal{L}(a_{2,0,1}) = -4\,a_{0,2,1} + 20\,a_{2,0,1} - 4\,a_{0,0,3},\\
&\mathcal{L}(a_{2,1,0}) = -4\,a_{0,3,0} + 20\,a_{2,1,0} - 4\,a_{0,1,2}, \mathcal{L}(a_{3,0,0}) = -12\,a_{1,2,0} + 12\,a_{3,0,0} - 12\,a_{1,0,2}
\end{aligned}
$$

Let us recall that $T_3^* = \mathcal{V}_2(y_0) \oplus \mathcal{V}_6(u_0)$. Let c_1, c_2, c_3 and $b_1, b_2, b_3, b_4, b_4, b_5, b_7$ denote the eigenvectors of $\mathcal{L}$ in the vector spaces $\mathcal{V}_2(y_0)$ and $\mathcal{V}_6(u_0)$, respectively. We find the eigenvectors by the standard linear algebra algorithm:

$$
\begin{aligned}
&c_1 = a_{0,0,3} + a_{0,2,1} + a_{2,0,1}, c_2 = a_{0,1,2} + a_{0,3,0} + a_{2,1,0}, c_3 = a_{1,0,2} + a_{1,2,0} + a_{3,0,0}\\
&b_1 = a_{0,0,3} - 3\,a_{0,2,1}, b_2 = -3\,a_{0,1,2} + a_{0,3,0}, b_3 = a_{1,1,1},\\
&b_4 = a_{1,2,0} - a_{1,0,2}, b_5 = a_{2,0,1} - a_{0,2,1}, b_6 = a_{2,1,0} - a_{0,1,2}, b_7 = -3\,a_{1,0,2} + a_{3,0,0}.
\end{aligned}
$$

The eigenvectors for the spaces $\mathcal{V}_0(u_0)$ and $\mathcal{V}_4(x_0)$ we already found in Subsect. 3.2. Now, let us express the above thirteen invariants in terms of the eigenvectors. We have

$$
\begin{gathered}
\boldsymbol{od} = e_0,\\
\boldsymbol{dv}_1 = {c_1}^2 + {c_2}^2 + {c_3}^2,\\
\boldsymbol{dv}_2 = 3\,{e_1}^2 + {e_2}^2 - e_5 e_2 + 3\,{e_3}^2 + 3\,{e_4}^2 + {e_5}^2,\\
\boldsymbol{dv}_3 = {b_1}^2 - 3\,b_5 b_1 + {b_2}^2 - 3\,b_6 b_2 + 15\,{b_3}^2 + 6\,{b_4}^2 - 3\,b_4 b_7 + 6\,{b_5}^2 + 6\,{b_6}^2 + {b_7}^2,\\
\boldsymbol{tr}_1 = 9\,{e_1}^2 e_2 - 18\,{e_1}^2 e_5 + 54\,e_1 e_4 e_3 + 2\,{e_2}^3 - 3\,e_5 {e_2}^2 - 18\,{e_3}^2 e_2 + 9\,{e_4}^2 e_2 - 3\,{e_5}^2 e_2 + 9\,{e_3}^2 e_5 +\\
+9\,{e_4}^2 e_5 + 2\,{e_5}^3,\\
\boldsymbol{tr}_2 = {c_1}^2 e_2 + {c_1}^2 e_5 - 6\,e_1 c_1 c_2 - 6\,e_3 c_3 c_1 - 2\,{c_2}^2 e_2 + {c_2}^2 e_5 - 6\,e_4 c_3 c_2 + {c_3}^2 e_2 - 2\,{c_3}^2 e_5,\\
\boldsymbol{tr}_3 = 2\,{b_1}^2 e_2 + 2\,{b_1}^2 e_5 + 3\,e_1 b_2 b_1 + 60\,e_4 b_3 b_1 + 3\,e_3 b_1 b_4 - 21\,b_1 b_5 e_2 + 9\,b_1 b_5 e_5 +\\
+3\,e_1 b_6 b_1 + 3\,e_3 b_1 b_7 - 4\,{b_2}^2 e_2 + 2\,{b_2}^2 e_5 + 60\,b_2 b_3 e_3 - 12\,e_4 b_2 b_4 + 3\,e_1 b_2 b_5 + 12\,b_2 b_6 e_2 +\\
+9\,b_2 b_6 e_5 + 3\,e_4 b_2 b_7 - 90\,b_3 e_1 b_4 - 90\,e_4 b_3 b_5 - 90\,b_3 b_6 e_3 + 60\,b_3 e_1 b_7 - 18\,{b_4}^2 e_2 - 9\,{b_4}^2 e_5 +\\
+63\,e_3 b_5 b_4 - 27\,e_4 b_6 b_4 + 9\,b_4 b_7 e_2 + 12\,b_4 b_7 e_5 + 27\,{b_5}^2 e_2 - 18\,{b_5}^2 e_5 - 72\,e_1 b_6 b_5 - 12\,e_3 b_5 b_7 -\\
-9\,{b_6}^2 e_2 - 18\,{b_6}^2 e_5 - 12\,e_4 b_6 b_7 + 2\,{b_7}^2 e_2 - 4\,{b_7}^2 e_5,\\
\boldsymbol{tr}_4 = c_1 b_1 e_2 + c_1 b_1 e_5 + 2\,e_1 c_2 b_1 + 2\,e_3 c_3 b_1 + 2\,e_1 c_1 b_2 - 2\,c_2 b_2 e_2 + c_2 b_2 e_5 + 2\,e_4 c_3 b_2 -\\
-10\,e_4 b_3 c_1 - 10\,b_3 c_2 e_3 - 10\,b_3 e_1 c_3 + 2\,e_3 c_1 b_4 - 8\,e_4 c_2 b_4 - 4\,b_4 c_3 e_2 + 3\,b_4 c_3 e_5 + b_5 c_1 e_2 -\\
-4\,b_5 c_1 e_5 + 2\,e_1 c_2 b_5 - 8\,e_3 c_3 b_5 + 2\,e_1 c_1 b_6 + 3\,b_6 c_2 e_2 - 4\,b_6 c_2 e_5 -\\
-8\,e_4 c_3 b_6 + 2\,e_3 c_1 b_7 + 2\,e_4 c_2 b_7 + c_3 b_7 e_2 - 2\,c_3 b_7 e_5,\\
\boldsymbol{ch}_1 = 2\,b_1 {c_1}^3 - 3\,{c_2}^2 c_1 b_1 - 3\,b_1 {c_3}^2 c_1 - 3\,c_2 b_2 {c_1}^2 + 2\,b_2 {c_2}^3 - 3\,b_2 {c_3}^2 c_2 + 30\,b_3 c_2 c_3 c_1 -\\
-3\,{c_1}^2 b_4 c_3 + 12\,{c_2}^2 b_4 c_3 - 3\,b_4 {c_3}^3 - 3\,b_5 {c_1}^3 - 3\,{c_2}^2 c_1 b_5 + 12\,b_5 {c_3}^2 c_1 - 3\,b_6 c_2 {c_1}^2 - 3\,b_6 {c_2}^3 +\\
+12\,b_6 {c_3}^2 c_2 - 3\,{c_1}^2 c_3 b_7 - 3\,{c_2}^2 c_3 b_7 + 2\,{c_3}^3 b_7,
\end{gathered}
$$

$$\boldsymbol{ch}_2 = {b_1}^2{c_1}^2 - 3\,{c_2}^2{b_1}^2 - 3\,{b_1}^2{c_3}^2 - 2\,c_2b_2c_1b_1 - 40\,b_3c_2c_3b_1 - 2\,b_1b_4c_3c_1 - 3\,b_1b_5{c_1}^2 +$$
$$+19\,{c_2}^2b_1b_5 - b_1{c_3}^2b_5 - 2\,b_6c_2c_1b_1 - 2\,b_1c_3b_7c_1 - 3\,{b_2}^2{c_1}^2 + {b_2}^2{c_2}^2 - 3\,{b_2}^2{c_3}^2 - 40\,b_3b_2c_3c_1 +$$
$$+8\,b_2b_4c_3c_2 - 2\,c_2b_2c_1b_5 + 19\,b_6b_2{c_1}^2 - 3\,b_2b_6{c_2}^2 - b_2{c_3}^2b_6 - 2\,b_2c_3b_7c_2 - 25\,{b_3}^2{c_1}^2 -$$
$$-25\,{b_3}^2{c_2}^2 - 25\,{b_3}^2{c_3}^2 + 60\,b_3c_2b_4c_1 + 60\,b_3c_2c_3b_5 + 60\,b_3b_6c_3c_1 - 40\,b_3c_2b_7c_1 - 28\,{b_4}^2{c_1}^2 +$$
$$+2\,{c_2}^2{b_4}^2 - 4\,{b_4}^2{c_3}^2 - 42\,b_5b_4c_3c_1 + 18\,b_6b_4c_3c_2 + 19\,b_4b_7{c_1}^2 - {c_2}^2b_4b_7 - 3\,b_4{c_3}^2b_7 - 4\,{b_5}^2{c_1}^2 -$$
$$-28\,{c_2}^2{b_5}^2 + 2\,{b_5}^2{c_3}^2 + 48\,b_6c_2b_5c_1 + 8\,b_5c_3b_7c_1 - 28\,{b_6}^2{c_1}^2 - 4\,{b_6}^2{c_2}^2 + 2\,{b_6}^2{c_3}^2 +$$
$$+8\,b_6c_3b_7c_2 - 3\,{b_7}^2{c_1}^2 - 3\,{c_2}^2{b_7}^2 + {c_3}^2{b_7}^2,$$

$$\boldsymbol{ch}_3 = b_1{e_1}^2e_4 - b_1e_1e_2e_3 + b_1e_1e_3e_5 - b_1{e_3}^2e_4 - b_2{e_1}^2e_3 - b_2e_1e_4e_5 + b_2e_3{e_4}^2 -$$
$$- b_3{e_2}^2e_5 + b_3e_2{e_4}^2 + b_3e_2{e_5}^2 + b_3{e_3}^2e_5 - b_3{e_4}^2e_5 + b_4{e_1}^3 + b_4e_1e_2e_5 - 2\,b_4e_1{e_3}^2 + b_4e_1{e_4}^2 -$$
$$- b_4e_1{e_5}^2 - 2\,b_4e_2e_3e_4 + b_4e_3e_4e_5 - 2\,b_5{e_1}^2e_4 + b_5e_1e_2e_3 - 2\,b_5e_1e_3e_5 - b_5e_2e_4e_5 +$$
$$+ b_5{e_3}^2e_4 + b_5{e_4}^3 + 2\,b_6{e_1}^2e_3 - b_6e_1e_2e_4 + 2\,b_6e_1e_4e_5 + b_6{e_2}^2e_3 - b_6e_2e_3e_5 - b_6{e_3}^3 +$$
$$+ b_7e_1{e_3}^2 - b_7e_1{e_4}^2 + b_7e_2e_3e_4 - b_6e_3{e_4}^2,$$

$$\boldsymbol{ch}_4 = 2\,{c_1}^2{e_2}^2 - 5\,{c_1}^2e_2e_5 + 9\,{c_1}^2{e_4}^2 + 2\,{c_1}^2{e_5}^2 - 6\,e_1c_1c_2e_2 + 12\,e_1c_1c_2e_5 - 18\,c_1c_2e_3e_4 -$$
$$- 18\,c_1c_3e_1e_4 + 12\,e_3c_3c_1e_2 - 6\,e_3c_3c_1e_5 - {c_2}^2{e_2}^2 + {c_2}^2e_2e_5 + 9\,{c_2}^2{e_3}^2 + 2\,{c_2}^2{e_5}^2 -$$
$$- 18\,c_2c_3e_1e_3 - 6\,e_4c_3c_2e_2 - 6\,e_4c_3c_2e_5 + 9\,{c_3}^2{e_1}^2 + 2\,{c_3}^2{e_2}^2 + {c_3}^2e_2e_5 - {c_3}^2{e_5}^2,$$

$$\boldsymbol{ch}_5 = {b_1}^4 - 6\,{b_1}^3b_5 + 7\,{b_2}^2{b_1}^2 - 36\,b_6b_2{b_1}^2 - 50\,{b_3}^2{b_1}^2 + 57\,{b_4}^2{b_1}^2 - 36\,b_4b_7{b_1}^2 + {b_5}^2{b_1}^2 +$$
$$+57\,{b_6}^2{b_1}^2 + 7\,{b_7}^2{b_1}^2 - 36\,{b_2}^2b_5b_1 + 60\,b_3b_2b_4b_1 - 40\,b_3b_2b_7b_1 + 128\,b_6b_2b_5b_1 + 150\,{b_3}^2b_5b_1 +$$
$$+160\,b_3b_6b_4b_1 + 60\,b_3b_6b_7b_1 - 206\,{b_4}^2b_5b_1 + 88\,b_4b_7b_5b_1 + 24\,{b_5}^3b_1 - 136\,{b_6}^2b_5b_1 - 6\,{b_7}^2b_5b_1 +$$
$$+{b_2}^4 - 6\,b_6{b_2}^3 - 50\,{b_3}^2{b_2}^2 + 12\,{b_2}^2{b_4}^2 - 6\,{b_2}^2b_4b_7 + 57\,{b_2}^2{b_5}^2 + {b_6}^2{b_2}^2 + 7\,{b_2}^2{b_7}^2 +$$
$$+150\,{b_3}^2b_2b_6 - 340\,b_3b_2b_4b_5 + 60\,b_3b_2b_7b_5 + 4\,b_2{b_4}^2b_6 - 52\,b_2b_4b_7b_6 - 136\,b_6b_2{b_5}^2 +$$
$$+24\,b_2{b_6}^3 - 6\,b_2{b_7}^2b_6 + 625\,{b_3}^4 + 200\,{b_3}^2{b_4}^2 + 150\,{b_3}^2b_7b_4 + 200\,{b_3}^2{b_5}^2 + 200\,{b_3}^2{b_6}^2 -$$
$$-50\,{b_3}^2{b_7}^2 - 240\,b_3b_6b_4b_5 + 160\,b_3b_6b_7b_5 + 16\,{b_4}^4 + 24\,{b_4}^3b_7 + 137\,{b_4}^2{b_5}^2 + 17\,{b_6}^2{b_4}^2 +$$
$$+{b_4}^2{b_7}^2 - 76\,b_4b_7{b_5}^2 + 4\,{b_6}^2b_4b_7 - 6\,{b_7}^3b_4 + 16\,{b_5}^4 + 32\,{b_6}^2{b_5}^2 + 12\,{b_7}^2{b_5}^2 +$$
$$+16\,{b_6}^4 + 12\,{b_6}^2{b_7}^2 + {b_7}^4$$

Theorem 9. *The set of the following thirteen invariants*

$$od, dv_1, dv_2, dv_3, tr_1, tr_2, tr_3, tr_4, ch_1, ch_2, ch_3, ch_4, ch_5$$

is a minimal generating set of the algebra $\mathbb{C}(U_3)^{\mathfrak{sl}_2}$.

Proof. It is enough to prove that the elements are algebraically independed. Let us consider the Jacobian 13×16-matrix of the polynomial set:

$$\begin{pmatrix} \dfrac{\partial\, od}{\partial e_0} & \dfrac{\partial\, od}{\partial e_1} & \dots & \dfrac{\partial\, od}{\partial b_6} & \dfrac{\partial\, od}{\partial b_7} \\ \dfrac{\partial\, dv_1}{\partial e_0} & \dfrac{\partial\, dv_1}{\partial e_1} & \dots & \dfrac{\partial\, dv_1}{\partial b_6} & \dfrac{\partial\, dv_1}{\partial b_7} \\ \dots & \dots & \dots & \dots & \\ \dfrac{\partial\, ch_4}{\partial e_0} & \dfrac{\partial\, ch_4}{\partial e_1} & \dots & \dfrac{\partial\, ch_4}{\partial b_6} & \dfrac{\partial\, ch_4}{\partial b_7} \\ \dfrac{\partial\, ch_5}{\partial e_0} & \dfrac{\partial\, ch_5}{\partial e_1} & \dots & \dfrac{\partial\, ch_5}{\partial b_6} & \dfrac{\partial\, ch_5}{\partial b_7} \end{pmatrix}$$

It is sufficient to show that the rank of the matrix is equal to 13. After substituting the following expressions

$$\begin{aligned} & e_0 = 1, e_1 = 1, e_2 = 23, e_3 = 53, e_4 = 97, e_5 = 151, b_1 = 541, b_2 = 661, b_3 = 827, \\ & b_4 = 1009, b_5 = 1193, b_6 = 1427, b_7 = 1619, c_1 = 227, c_2 = 311, c_3 = 419, \end{aligned}$$

into the Jacobian matrix, we get a matrix whose entries are all numbers. Then, by direct calculation, we obtain that its rankis equal to thirteen. It implies that the Jacobian matrix has the maximal rank equal to thirteen which proves the theorem. □

Applying the same scheme, we can find the minimal generating sets of higher orders, for instance, the minimal generating sets of order four consists of 28 algebraically independent invariants.

5. Conclusion

In this article, we reviewed the 3D geometric moment invariants in the terms of the classical invariant theory. We divided all invariants into two types by introducing the notions of the algebras of simultaneous rational and polynomial rotation invariants $\mathbb{C}[\eta]_d^{SO(3)}$ and $\mathbb{C}(\eta)_d^{SO(3)}$ up to order d where η is a set of normalized moments which are already invariants under the scaling and translations. In addition, we proved that these algebras are isomorphic to some classical object of the invariant theory, that is, to the algebras of join invariants of binary forms $\mathbb{C}[U_d]^{SL(2)}$ and $\mathbb{C}(U_d)^{SL(2)}$. Further on, we used Lie infinitesimal method and reduced the problem of calculating the invariants of the group $SO(3)$ to the equivalent one of calculating the invariants of the Lie algebra $\mathfrak{sl}_2$. From the computational point of view, it is much more simpler problem dealing with polynomial derivations.

In the rational case we count out the cardinality of the minimal generating set of the algebra $\mathbb{C}(U_d)^{SL(2)}$ and present such minimal generating set for invariants of the degrees two and three. Also we found the explicit form of the series of the invariants of the degree one of an arbitrary order, which plays an important role in different applications as a low-order moments which are less sensitive to noise than the higher-order ones.

The author hopes that the results will be useful to the researchers in the fields of image analysis and pattern recognition. Though, the geometric moments are not as effective as the orthogonal ones are, the obtained results are of independent theoretical interest.

As we have seen, in contrast to the 2D case, there is no satisfactory description of 3D rotational invariants of arbitrary order, and the problem of finding the basis of such invariants is hopeless. In our forthcoming researches, we are going to present another invariant constructions, which seems to be an effective way of describing of 3D image moments.

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