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ILIGRA: An Efficient Inverse Line Graph Algorithm

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Abstract This paper presents a new and efficient algorithm, ILIGRA, for inverse line graph construction. Given a line graph H, ILIGRA constructs its root graph G with the time complexity being linear in the number of nodes in H. If ILIGRA does not know whether the given graph H is a line graph, it firstly assumes that H is a line graph and starts its root graph construction. During the root graph construction, ILIGRA checks whether the given graph H is a line graph and ILIGRA stops once it finds H is not a line graph. The time complexity of ILIGRA with line graph checking is linear in the number of links in the given graph H. For sparse line graphs of any size and for dense line graphs of small size, numerical results of the running time show that ILIGRA outperforms all currently available algorithms.

Keywords Graph Algorithm · Line Graph · Root Graph

1 Introduction

A simple graph with N nodes (vertices) and L links (edges), denoted by $G(N, L)$, is an unweighted, undirected graph containing no self-loops (links starting and ending at the same node) nor multiple links between the same pair of nodes. The *line graph* $H = l(G)$ of a graph G is a graph [22] in which every node in H corresponds to a link in G and two nodes are adjacent if and only if their corresponding links have a common node in G. The graph G is called the *root graph* of H. The complete graph with three nodes K_3 is a line graph, which has two different root graphs, K_3 and the bipartite graph $K_{1,3}$.

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Except for K_3, Whitney's theorem [24] states that all connected line graphs have only one root graph up to an isomorphism. Whitney's theorem provides the theoretical basis for the inverse line graph conversion. Cvetković *et al.* [5] reviewed the state-of-the-art knowledge about line graphs.

There exist plenty of real-world networks that can be modeled by line graphs [16,17]. A graph is assortative if its low-degree nodes tend to be adjacent with other low-degree nodes and its high-degree nodes tend to be adjacent with other high-degree nodes. Line graphs are assortative and clustered [14, 13,16,11]. If two or more communities overlap on a node, it is not feasible to partition nodes to detect communities. In order to detect the overlapping communities, the links are partitioned. However, the algorithms for partitioning links are less efficient than the algorithms for partitioning the nodes. We can transform the networks into their line graphs and partition the nodes of the line graphs [1,8].

To facilitate the applications of line graphs, the construction of a line graph H from a root graph G and the inverse construction from the line graph $H = l(G)$ to the root graph G are necessary. The root-to-line graph construction follows straightforwardly from the definition of a line graph [22]. However, the line-to-root graph construction is more complex. Two algorithms for line-to-root graph construction were proposed concurrently by Roussopoulos [20] and Lehot [12]. The algorithm of Roussopoulos is based on the theorem of Krausz [10]: *A graph is a line graph if and only if it is possible to find a collection of cliques in the graph, partitioning all the links, such that each node belongs to at most two of the cliques (some of the cliques can be a single node) and two cliques share at most one node.* Lehot's algorithm employs the principles of Van Rooij and Wilf [23]: *A graph is a line graph if and only if it does not have the complete bipartite graph $K_{1,3}$ as an induced sub-graph, and if two odd triangles*[1] *have a common link, the sub-graph induced by their nodes is the complete graph K_4.* Lehot's algorithm first constructs a root graph G from the given graph H, and then compares $l(G)$ and H to determine whether the given H is a line graph, unlike Roussopoulos' algorithm, which determines whether the given graph H is a line graph during the construction of the root graph G. Naor and Novick [18] proposed a parallel algorithm for line-to-root graph construction based on a divide-and-conquer scheme. Motivated by eigenvectors, Simić [21] proposed an algorithm for recognizing generalized line graphs. Simić's algorithm searches for the maximum degree node in each loop. Degiorgi and Simon [6] proposed a constructive algorithm, based on the Ore's proof [19] of Whitney's theorem [24], which states that two connected and edge-isomorphic graphs with more than four nodes are also node isomorphic and there exists exactly one node isomorphism which generates the given edge isomorphism. The original graph construction examines 2-coloring classes in the input graph components. They showed that their algorithm is more time-efficient than the algorithms of Roussopoulos and Lehot for sparse line graphs and non-line graphs.

[1] If every node is adjacent to two or zero nodes of a triangle then it is an even triangle.

In this paper, we propose ILIGRA, an Inverse LIne GRaph Algorithm, for line-to-root graph construction. Unlike previous algorithms, ILIGRA checks the connectivity locally. The root graph G is constructed based on the correspondence between a node in the line graph H and a link in its root graph G. Due to the choice of an arbitrary node in the line graph H and checking the connectivity of its neighbors during the algorithm's execution, ILIGRA is the simplest inverse line graph algorithm proposed so far.

The paper is organized as follows. ILIGRA is presented in Section 2 and additional details are given in Appendix A. Section 3 demonstrates how ILIGRA works on a descriptive example. Numerical comparisons of ILIGRA with the algorithms of Lehot, Roussopoulos, and Degiorgi and Simon for different types of line graphs are presented in Section 4. Finally, we conclude in Section 5. The link density of line graphs is discussed in Appendix B.

2 ILIGRA

2.1 Notation

Table 1 summarizes the notation, which is used in the presentation of ILIGRA. According to the definition of the line graph, each node in a line graph $H\,(N_H, L_H)$ corresponds to a link in its root graph $G\,(N, L)$. Hence, the number of nodes N_H in the line graph H and the number of links L in the root graph G are equal, $N_H = L$. We always use n (or n with subscript) to denote a node in H. The link in G corresponding to node n in H is denoted by l_n. In the remainder of the paper, we use v (or v with subscript) to denote a node in G. Denote by $\mathcal{N}_b\,(n)$ the set of the nodes in H which are adjacent to node n and called the neighbors[2] of node n. Denote by $\mathcal{L}_b\,(l_n)$ the set of the links in G which corresponds to the nodes in $\mathcal{N}_b\,(n)$. Every link in the root graph G has two incident nodes[3]. In order to construct the root graph G from a given line graph H, we have to determine the two incident nodes of every link in G. In the root graph G, we denote by v_{l_n} the incident node of link l_n which is first encountered during the algorithm's execution. The set of the nodes in H, which corresponds to the links in G whose incident nodes are not yet determined, is denoted by $\mathcal{N}_w$. The set of the nodes in H corresponding to the links in G of which one incident node is determined, is denoted by $\mathcal{N}_h$.

2.2 Concept

The nodes in a line graph $H\,(N_H, L_H)$ are denoted by $n_1, n_2, n_3, \cdots, n_{N_H}$, and the corresponding links in the root graph G are denoted by l_{n_1}, l_{n_2}, l_{n_3}, $\cdots$, $l_{n_{N_H}}$. Initially, it is unknown how the links l_{n_1}, l_{n_2}, l_{n_3}, $\cdots$, $l_{n_{N_H}}$ connect the nodes in G, and even the number of nodes N in G is unknown.

[2] A node is the neighbor of another node if they are connected.

[3] An incident node of a link is one of the two nodes of that link.

Table 1 Notation

$G(N, L)$	The root graph with N nodes and L links
$H(N_H, L_H)$	The line graph of G with N_H nodes and L_H links
n	The node n in H
$\mathcal{N}$	The set of all the nodes in H
$\mathcal{N}_w$	The set of the nodes in H, corresponding to the links in G whose incident nodes are not yet determined
$\mathcal{N}_h$	The set of the nodes in H, corresponding to the links in G of which one incident node is determined
$\mathcal{N}_b(n)$	The set of the neighbors of node n in H
l_n	The link in G which corresponds to node n in H
$\mathcal{L}_b(l_n)$	The set of the links in G which correspond to the nodes in $\mathcal{N}_b(n)$
v_{l_n}	The first identified incident node of link l_n in G
$\textsc{Addnode}(G, v)$	The function which adds a node v to G
$\textsc{Addlink}(G, v_1, v_2)$	The function which adds a link $v_1 \sim v_2$ to G

Suppose that link l_{n_1} is incident to v_1 and v_2 in G. From the line graph H, the set $\mathcal{N}_b(n_1)$ of the neighbors of node n_1 in H is known, and the set $\mathcal{L}_b(l_{n_1})$ of the links in G, which corresponds to the nodes in $\mathcal{N}_b(n_1)$, is also known. By the definition of a line graph, the links in $\mathcal{L}_b(l_{n_1})$ are the neighboring links of link l_{n_1}, hence, the links in $\mathcal{L}_b(l_{n_1})$ should be incident to either v_1 or v_2. If the links in $\mathcal{L}_b(l_{n_1})$ which are incident to v_1 are known, the rest of links in $\mathcal{L}_b(l_{n_1})$ must be incident to v_2. Unfortunately, it is unknown which links in $\mathcal{L}_b(l_{n_1})$ are incident to v_1.

When considering links $l_{n_2}, l_{n_3}, \cdots, l_{n_{N_H}}$, the same problem appears. The difficulty in constructing the root graph G lies in partitioning the set of the neighboring links into two complementary subsets of links: the links that are incident to the first incident node of the concerned link, and the other links which are incident to the second incident node of that link.

Without loss of generality, suppose that $\mathcal{L}_b(l_{n_1}) = \{l_{n_2}, l_{n_3}, \cdots, l_{n_k}\}$, where k is an integer. Suppose that the set $\mathcal{L}_b(l_{n_1})$ of the neighboring links of l_{n_1} are partitioned successfully into two subsets: $\mathcal{L}_{b,v_1}(l_{n_1}) = \{l_{n_2}, l_{n_3}, l_{n_4}\}$ where the links are incident to v_1, and $\mathcal{L}_{b,v_2}(l_{n_1}) = \{l_{n_5}, l_{n_6}, \cdots, l_{n_k}\}$ where the links are incident to v_2. Then, the set $\mathcal{L}_b(l_{n_2})$ of the neighboring links of l_{n_2} is automatically partitioned: the links $l_{n_1}, l_{n_3}, l_{n_4}$ are incident to v_1, and the rest of links in $\mathcal{L}_b(l_{n_2})$ are incident to the second incident node of l_{n_2}. Similarly, the sets of the neighboring links of links $l_{n_3}, l_{n_4}, \cdots, l_{n_k}$, are also automatically partitioned. Assuming H is a connected line graph, the sets of the neighboring links of all the links in G can be partitioned by iterating the described process. This is the basic idea of Iligra.

Partitioning the set $\mathcal{L}_b(l_{n_1})$ of the neighboring links of l_{n_1} becomes a crucial task in the root graph construction. The theorems in Section 2.3 and Appendix A provide the theoretical basis for this task.

2.3 Theoretical preliminaries

Theorem 1 *Suppose that two adjacent nodes n_1 and n_2 in H correspond to links l_{n_1} and l_{n_2} in G, respectively, where l_{n_1} is incident to v_1 and v_2 and where v_1 is also incident to l_{n_2}, as shown in Figure 1 (a). Then, for each $n \in \mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$ in H, the corresponding link l_n in G must be incident to v_2, and the nodes in $\mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$ must form a clique in H.*

Proof. For each $n \in \mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$, the corresponding link l_n in G has to be incident to either v_1 or v_2, since n is adjacent to n_1 in H. Because n is not adjacent to n_2, l_n in G can only be incident to v_2. Since the corresponding links of all the nodes $\in \mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$ are incident to v_2, the nodes in $\mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$ must be fully connected with each other and form a clique in H. □

Based on Theorem 1, starting with an initial link with nodes n_1 and n_2 in H, ILIGRA determines the first incident node of the links corresponding to the nodes in $\mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$. The nodes in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$ (i.e. common neighbors of n_1 and n_2 in G) may form a clique in H with the corresponding links being incident to v_1 in G, as shown in Figure 1 (a).

There may also exist a node in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$ which is not adjacent to any other node in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$ and whose corresponding link in G is incident to v_2 and another node v_3, as shown in Figure 1 (b) where the corresponding links of n_1, n_2 and that node form a triangle in G. If there are three or more nodes in the set $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, we can identify the position of the corresponding link in G of that node. The last discussion is formalized in Theorem 2.

Theorem 2 *Suppose that two adjacent nodes n_1 and n_2 in H correspond to links l_{n_1} and l_{n_2} respectively in G, where l_{n_1} is incident to v_1 and v_2 and l_{n_2} is incident to v_1 and v_3. Suppose that $|\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)| \geq 3$. If there exists $n_u \in \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$ such that n_u is not adjacent to any other node in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, then link l_{n_u} must be incident to both v_2 and v_3 in G.*

Proof. Since $n_u \in \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, l_{n_u} can be incident to v_1 or be incident to both v_2 and v_3. If l_{n_u} is incident to v_1, n_u should be adjacent to at least one other node in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, since $|\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)| \geq 3$. Because n_u is not adjacent to any other node in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, l_{n_u} can only be incident to v_2 and v_3, as shown in Figure 1 (b). The links corresponding to nodes in $(\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)) \setminus \{n_u\}$ are incident to v_1. □

If the set of common neighbors of two adjacent nodes n_1 and n_2 have no more than two nodes in H, then for each node in this set that also has at least 3 additional neighbors different from n_1 and n_2, the position of its correspondent link in G can be uniquely determined. This is formalized in Theorem 3.

Theorem 3 *Suppose that two adjacent nodes n_1 and n_2 in H correspond to links l_{n_1} and l_{n_2} in G, respectively, where l_{n_1} is incident to v_1 and v_2 and where l_{n_2} is incident to v_1 and v_3, as shown in Figure 2. If $|\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)| \leq 2$,*

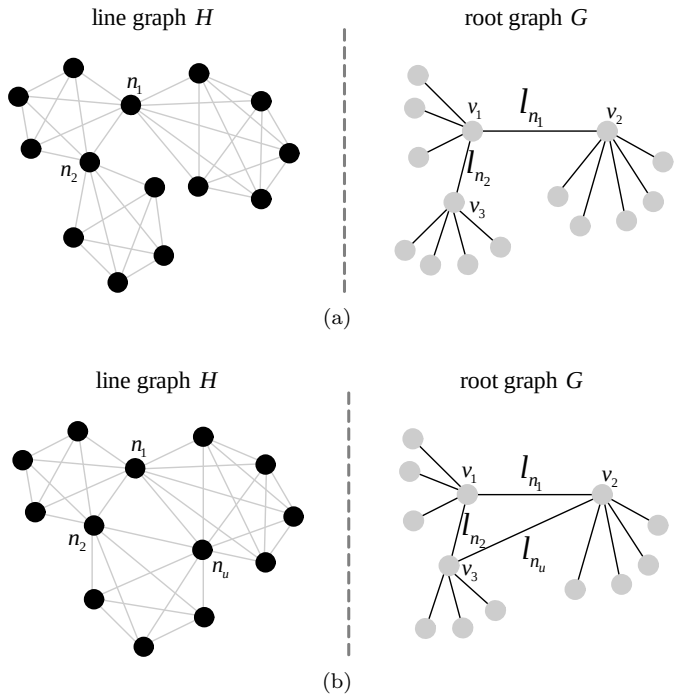


Fig. 1 Scenarios in Theorem 1 and 2. Each node (black) in H corresponds to a link (black) in G.

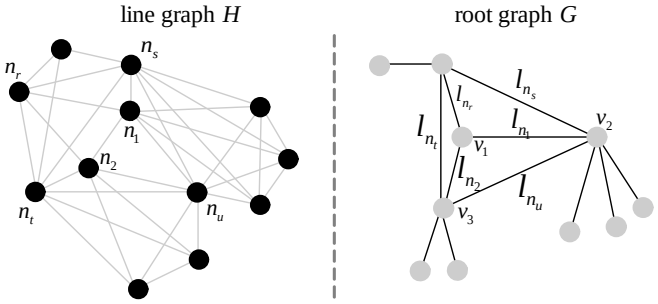


Fig. 2 Illustration of Theorem 3.

then for each $n_u \in \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$*, such that* $|\mathcal{N}_b(n_u) \setminus \{n_1, n_2\}| \geq 3$ *and* $\mathcal{N}_b(n_u) \subseteq \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$*, link* l_{n_u} *must be incident to both* v_2 *and* v_3 *in* G*.*

Proof. Since $n_u \in \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, l_{n_u} can be incident to v_1 or be incident to both v_2 and v_3. Let us first assume l_{n_u} is incident to v_1. Two neighbors n_{x1}, n_{x2} of n_u can be adjacent to n_1 or n_2. However, we have $|\mathcal{N}_b(n_u) \setminus \{n_1, n_2\}| \geq 3$, there must be at least one neighbor of n_u which is not adjacent to either n_1 or n_2, which contradicts with the fact that $\mathcal{N}_b(n_u) \subseteq \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$. Hence, l_{n_u} can only be incident to v_2 and v_3. □

When the set of common nodes of two adjacent nodes n_1 and n_2 in H has no more than two nodes and a node in this set has no more than two neighbors, different from n_1 and n_2, then Theorems 2 and 3 are not applicable. We treat those remaining cases in Appendix A.

For a given node n_1, ILIGRA uses Theorem 1, 2, 3 and Table 3 from Appendix A to determine which links in $\mathcal{L}_b(l_{n_1})$ are incident to v_1 and which else are incident to v_2, where v_1 and v_2 are the nodes of l_{n_1} in G. Then, for each link in $\mathcal{L}_b(l_{n_1})$, the first incident node has been determined.

2.4 Algorithm description

ILIGRA starts by setting G to an empty graph *(line 1)*. Initially, nothing in G is *determined*, hence $\mathcal{N}_w = \mathcal{N}$ and $\mathcal{N}_h = \emptyset$ *(line 2)*, where $\mathcal{N}_w$ and $\mathcal{N}_h$ are defined in Table 1. Then ILIGRA picks an arbitrary node n_1 in the set $\mathcal{N}_w$ and picks an arbitrary neighbor n_2 of n_1 in the set $\mathcal{N}_b(n_1)$ *(lines 3-4)*. Two nodes v_1 and v_2 are added to the root graph G *(line 5)*, and link $l_{n_1} = v_1 \sim v_2$ is added to the root graph G *(line 6)*. Since the incident nodes of link l_{n_1} have been determined in G, node n_1 is removed from $\mathcal{N}_w$ *(line 6)*. Then v_1 is chosen[4] to be incident to link l_{n_2} *(line 7)*. Since the first incident node of link l_{n_2} is determined, node n_2 is moved from $\mathcal{N}_w$ to $\mathcal{N}_h$ *(line 7)*.

According to the definition of the line graph, the links in $\mathcal{L}_b(l_{n_1})$ have a node in common with link l_{n_1} in G. Since l_{n_1} is incident to v_1 and v_2, the links in $\mathcal{L}_b(l_{n_1})$ should also be incident to either v_1 or v_2. By Theorem 1, ILIGRA determines that the links in $\mathcal{L}_b(l_{n_1}) \setminus \mathcal{L}_b(l_{n_2})$, corresponding to the nodes in $n \in \mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$, are incident to v_2. For each node n in $\mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$, ILIGRA sets the first identified incident node v_{l_n} of the corresponding link l_n to v_2, and moves n from $\mathcal{N}_w$ to $\mathcal{N}_h$ *(lines 8-9)*.

ILIGRA sets J to the intersection of $\mathcal{N}_b(n_1)$ and $\mathcal{N}_b(n_2)$ *(line 10)*. If there are only 1 or 2 nodes in J, and if there exists $n_u \in J$ such that any neighbor of n_u is also a neighbor of either n_1 or n_2, and node n_u satisfies $|\mathcal{N}_b(n_u) \setminus \{n_1, n_2\}| \geq 3$, according to Theorem 3, link l_{n_u} should be incident to v_2. ILIGRA sets $v_{l_{n_u}}$ to v_2, and adds n_u to $\mathcal{N}_h$ and removes n_u from $\mathcal{N}_w$ and removes n_u from J *(lines 11-14)*. If $|J| \leq 2$ and $|\mathcal{N}_b(n_u) \setminus \{n_1, n_2\}| \leq 2$, the special cases are handled by the subroutine InitSpecCases *(lines 15-16)*. The subroutine InitSpecCases is stated in Appendix A. If $|J| \geq 3$ and if there exists $n_u \in J$ such that n_u is not adjacent to any other node in J, according to Theorem 2, link l_{n_u} should be incident to v_2. ILIGRA sets $v_{l_{n_u}}$ to v_2, and adds n_u to $\mathcal{N}_h$ and removes n_u from $\mathcal{N}_w$ and removes n_u from J *(lines 17-19)*.

Since node n_u has been removed from J, the rest of links in J should be incident to v_1. For each n in J which is adjacent with both n_1 and n_2, ILIGRA sets v_{l_n} to v_1, and adds n to $\mathcal{N}_h$ and removes n from $\mathcal{N}_w$ *(lines 20-21)*. The nodes in J should be fully connected to each other, since the corresponding links are all incident to v_1. If the nodes in J do not form a clique in H, then

[4] ILIGRA arbitrarily chooses a node from v_1 and v_2 and lets it be incident to l_{n_2}.

Algorithm 1: ILIGRA(H)

Input: A line graph H
Output: The root graph G of H if H is a line graph

1 $G \leftarrow$ an empty graph;
2 $\mathcal{N} \leftarrow$ the set of nodes in H;$\mathcal{N}_w \leftarrow \mathcal{N}$; $\mathcal{N}_h \leftarrow \emptyset$;
3 $n_1 \leftarrow$ an arbitrary node $\in \mathcal{N}_w$;
4 $n_2 \leftarrow$ an arbitrary node $\in \mathcal{N}_b(n_1)$;
5 ADDNODE(G, v_1); ADDNODE(G, v_2);
6 ADDLINK(G, v_1, v_2); $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_1\}$;
7 $v_{l_{n_2}} \leftarrow v_1$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_2\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_2\}$;
8 **for** *each* $n \in \mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2)$ **do**
9 $\quad v_{l_n} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n\}$;
10 $J \leftarrow \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$;
11 **if** $1 \leq |J| \leq 2$ **then**
12 $\quad$ **if** $\exists n_u \in J$ *such that* $\mathcal{N}_b(n_u) \subseteq \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ *and* $|\mathcal{N}_b(n_u) \setminus \{n_1, n_2\}| \geq 3$ **then**
13 $\quad\quad v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$;
14 $\quad\quad \mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
15 $\quad$ **else**
16 $\quad\quad$ INITSPECCASES(H, n_1, n_2, n_u)
17 **else if** $|J| \geq 3$ ***and*** $\exists n_u \in J$ *such that* n_u *is not adjacent to any other node in* J **then**
18 $\quad v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$;
19 $\quad \mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
20 **for** *each* $n \in J$ **do**
21 $\quad v_{l_n} \leftarrow v_1$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n\}$;
22 **if** $J \neq \emptyset$ ***and*** J *is not a* ***clique*** *in* H **then**
23 $\quad H$ **is not** a line graph. **Exit**.
24 **if** $|\mathcal{N}_b(n_1) \setminus J| \neq 0$ ***and*** $\mathcal{N}_b(n_1) \setminus J$ *is not a* ***clique*** **then**
25 $\quad H$ **is not** a line graph. **Exit**.
26 **while** $\mathcal{N}_h \neq \emptyset$ **do**
27 $\quad n \leftarrow$ an arbitrary node in $\mathcal{N}_h$;
28 $\quad$ ADDNODE(G, v);ADDLINK(G, v_{l_n}, v);
29 $\quad \mathcal{N}_h \leftarrow \mathcal{N}_h \setminus \{n\}$; $\mathcal{C} \leftarrow \emptyset$;
30 $\quad$ **for** *each* $n_r \in \mathcal{N}_b(n)$ **do**
31 $\quad\quad$ **if** $n_r \in \mathcal{N}_h$ ***and*** $v_{l_n} \neq v_{l_{n_r}}$ **then**
32 $\quad\quad\quad \mathcal{C} \leftarrow \mathcal{C} \cup \{n_r\}$;
33 $\quad\quad\quad$ ADDLINK$(G, v_{l_{n_r}}, v)$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \setminus \{n_r\}$;
34 $\quad\quad$ **else if** $n_r \in \mathcal{N}_w$ **then**
35 $\quad\quad\quad \mathcal{C} \leftarrow \mathcal{C} \cup \{n_r\}$;$v_{l_{n_r}} \leftarrow v$;
36 $\quad\quad\quad \mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_r\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_r\}$;
37 $\quad$ **if** $\mathcal{C} \neq \emptyset$ ***and*** $\mathcal{C}$ *is not a* ***clique*** *in* H **then**
38 $\quad\quad H$ **is not** a line graph. **Exit**.

H is not a line graph *(lines 22-23)*. The nodes in $\mathcal{N}_b(n_1) \setminus J$ should also be fully connected to each other, since the corresponding links are all incident to v_2. If the nodes in $\mathcal{N}_b(n_1) \setminus J$ do not form a clique in H, then H is not a line graph *(lines 24-25)*.

The loop *(lines 26-38)* runs until $\mathcal{N}_h$ is an empty set. ILIGRA picks an arbitrary node n in $\mathcal{N}_h$ *(line 27)*. ILIGRA adds a node v and a link l_n between v_{l_n} and v to G *(line 28)*, and removes n from $\mathcal{N}_h$ *(line 29)*. ILIGRA sets $\mathcal{C}$ to an empty set *(line 29)*. For each neighbor n_r of n, if $n_r \in \mathcal{N}_h$ and $v_{l_n} \neq v_{l_{n_r}}$, ILIGRA adds link l_{n_r} between $v_{l_{n_r}}$ and v to G, and removes n_r from $\mathcal{N}_h$, and adds n_r to $\mathcal{C}$ *(lines 30-33)*; If $n_r \in \mathcal{N}_w$, ILIGRA sets $v_{l_{n_r}}$ to v, and moves n_r from $\mathcal{N}_w$ to $\mathcal{N}_h$, and adds n_r to $\mathcal{C}$ *(lines 34-36)*. The nodes in $\mathcal{C}$ should be fully connected with each other, since the corresponding links are all incident to v. If the nodes in $\mathcal{C}$ do not form a clique in H, H is not a line graph *(lines 37-38)*. If H is a connected graph, $\mathcal{N}_w$ should be an empty set when $\mathcal{N}_h$ becomes an empty set. While $\mathcal{N}_w \neq \emptyset$, repeat lines 3-38. For each component of a given disconnected line graph, lines 3-38 will be executed once. If the input graphs are line graphs, lines 22-25 and 37-38 can be skipped, which are used to check whether the given graph is a line graph.

2.5 Complexity

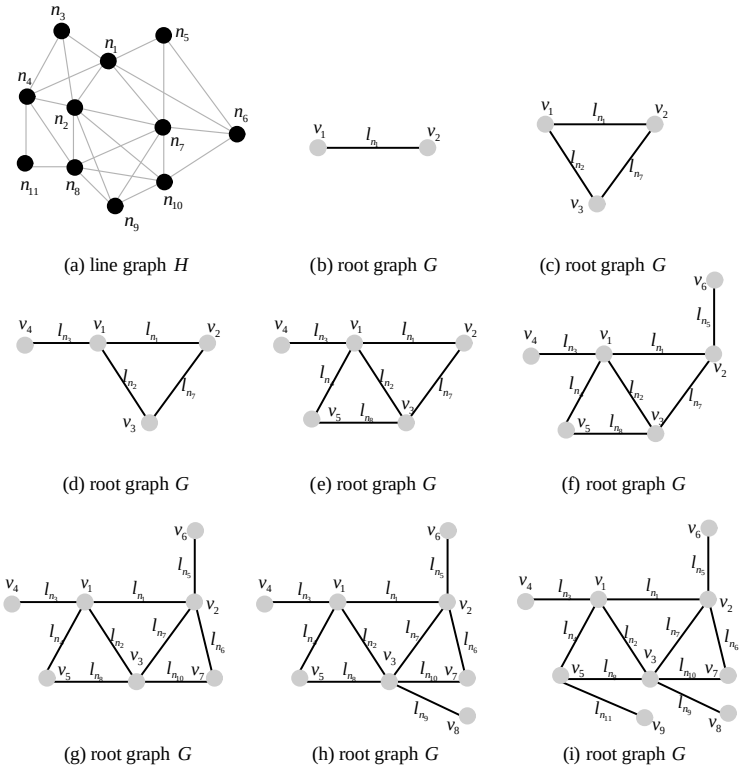
The lines 1-21 of ILIGRA examine all the neighbors of the n_1 in H, with the complexity $O(N_H)$, where N_H is the number of nodes in H. The lines 22-25, which check whether H is a line graph, have the complexity $O(N_L)$, where N_L is the number of links in H. The lines 26-36 have the complexity $O(N_H)$. The lines 37-38 check whether H is a line graph and have the complexity $O(N_L)$. Hence, the overall complexity of ILIGRA with checking if H is a line graph is $O(N_L)$, and the complexity of ILIGRA without checking is $O(N_H)$.

3 An example

In this section, we use an example depicted in Figure 3 to show how ILIGRA works. Given a line graph H shown in Figure 3 (a), ILIGRA constructs its root graph G incrementally as shown in Figure 3 (b) to (i).

Initially, set G to an empty graph. We have $\mathcal{N}_w = \{n_1, n_2, \cdots, n_{11}\}$ and $\mathcal{N}_h = \emptyset$. Add nodes v_1 and v_2 to G, and add link l_{n_1} between v_1 and v_2 to G as shown in Figure 3 (b), and $\mathcal{N}_w = \{n_2, n_3, \cdots, n_{11}\}$. Set $v_{l_{n_2}}$ to v_1, $\mathcal{N}_w = \{n_3, n_4, \cdots, n_{11}\}$ and $\mathcal{N}_h = \{n_2\}$. Since $\mathcal{N}_b(n_1) \setminus \mathcal{N}_b(n_2) = \{n_5, n_6\}$, according to Theorem 1, set $v_{l_{n_5}}$ to v_2 and also set $v_{l_{n_6}}$ to v_2. We have $\mathcal{N}_w = \{n_3, n_4, n_7, n_8, \cdots, n_{11}\}$ and $\mathcal{N}_h = \{n_2, n_5, n_6\}$. Since $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2) = \{n_3, n_4, n_7\}$ and none of n_3 and n_4 is not adjacent to n_7, according to Theorem 2, set $v_{l_{n_7}}$ to v_2. Now $\mathcal{N}_w = \{n_3, n_4, n_8, n_9, n_{10}, n_{11}\}$ and $\mathcal{N}_h = \{n_2, n_5, n_6, n_7\}$. For the two nodes n_3 and n_4 in $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2) \setminus \{n_7\}$, the corresponding links should be incident to v_1. Hence, set both $v_{l_{n_3}}$ and $v_{l_{n_4}}$ to v_1. Now $\mathcal{N}_w = \{n_8, n_9, n_{10}, n_{11}\}$ and $\mathcal{N}_h = \{n_2, n_3, n_4, n_5, n_6, n_7\}$.

Take n_2 from $\mathcal{N}_h$. Add a node v_3 to G and add link l_{n_2} between v_3 and $v_{l_{n_2}}$ ($v_{l_{n_2}}$ has been found to be v_1 previously), as shown in Figure 3 (c). Now $\mathcal{N}_h = \{n_3, n_4, n_5, n_6, n_7\}$. We have $\mathcal{N}_b(n_2) = \{n_1, n_3, n_4, n_7, n_8, n_9, n_{10}\}$. Since



(a) line graph H (b) root graph G (c) root graph G

(d) root graph G (e) root graph G (f) root graph G

(g) root graph G (h) root graph G (i) root graph G

Fig. 3 An example shows how ILIGRA constructs G from a given H.

$n_7 \in \mathcal{N}_h$ and $v_{l_{n_7}} = v_2 \neq v_{l_{n_2}} = v_1$, add l_{n_7} between v_2 and v_3 to G. Now $\mathcal{N}_h = \{n_3, n_4, n_5, n_6\}$. Since n_8, n_9 and n_{10} belong to $\mathcal{N}_w$, set $v_{l_{n_8}}, v_{l_{n_9}}$ and $v_{l_{n_{10}}}$ to v_3. Now $\mathcal{N}_w = \{n_{11}\}$ and $\mathcal{N}_h = \{n_3, n_4, n_5, n_6, n_8, n_9, n_{10}\}$.

Take n_3 from $\mathcal{N}_h$. Add a node v_4 to G and add link l_{n_3} between v_4 and $v_{l_{n_3}}$, which is namely v_1, as shown in Figure 3 (d). Now $\mathcal{N}_h = \{n_4, n_5, n_6, n_8, n_9, n_{10}\}$.

Take n_4 from $\mathcal{N}_h$. Add a node v_5 to G and add link l_{n_4} between v_5 and $v_{l_{n_4}}$, which is also v_1, as shown in Figure 3 (e). Now $\mathcal{N}_h = \{n_5, n_6, n_8, n_9, n_{10}\}$. We have $\mathcal{N}_b(n_4) = \{n_1, n_2, n_3, n_8, n_{11}\}$. Since $n_8 \in \mathcal{N}_h$ and $v_{l_{n_8}} = v_3 \neq v_{l_{n_4}} = v_1$, add l_{n_8} between v_5 and v_3 to G. Now $\mathcal{N}_h = \{n_5, n_6, n_9, n_{10}\}$. Since $n_{11} \in \mathcal{N}_w$, set $v_{l_{n_11}}$ to v_5. Now $\mathcal{N}_w = \emptyset$ and $\mathcal{N}_h = \{n_5, n_6, n_9, n_{10}, n_{11}\}$.

Take n_5 from $\mathcal{N}_h$. Add a node v_6 to G and add link l_{n_5} between v_6 and $v_{l_{n_5}}$, which is also v_2, as shown in Figure 3 (f). Now $\mathcal{N}_h = \{n_6, n_9, n_{10}, n_{11}\}$.

Take n_6 from $\mathcal{N}_h$. Add a node v_7 to G and add link l_{n_6} between v_7 and $v_{l_{n_6}}$, which is also v_2, as shown in Figure 3 (g). Now $\mathcal{N}_h = \{n_9, n_{10}, n_{11}\}$. We have $\mathcal{N}_b(n_6) = \{n_1, n_5, n_7, n_{10}\}$. Since $n_{10} \in \mathcal{N}_h$ and $v_{l_{n_{10}}} = v_3 \neq v_{l_{n_6}} = v_2$, add $l_{n_{10}}$ between v_7 and v_3 to G. Now $\mathcal{N}_h = \{n_9, n_{11}\}$.

Take n_9 from $\mathcal{N}_h$. Add a node v_8 to G and add link l_{n_9} between v_8 and $v_{l_{n_9}}$, which is also v_3, as shown in Figure 3 (h). Now $\mathcal{N}_h = \{n_{11}\}$.

Take the only node n_9 from $\mathcal{N}_h$. Add a node v_9 to G and add link $l_{n_{11}}$ between v_9 and $v_{l_{n_{11}}}$, which is also v_5, as shown in Figure 3 (i). Now $\mathcal{N}_h = \emptyset$. Since $\mathcal{N}_w$ is also an empty set, the construction of G is accomplished.

4 Evaluation

We compare ILIGRA's running time with the running times of three published line graph reconstruction algorithms: Roussopouloss' algorithm [20], Lehot's algorithm [12], and Degiorgi and Simon's algorithm [6]. All algorithms have been implemented[5] in the same programming language (C++) and the same data structures and libraries [15] have been used[6]. The evaluation of all the algorithms has been conducted on the same machine[7].

Table 2 The fastest algorithm for different input line graphs.

graph types	line graphs	
p_H \ L_H	≤ 500	> 500
0.05	ILIGRA	
0.125	ILIGRA	
0.5	ILIGRA	Lehot
0.65	ILIGRA	Lehot

The performances of the above-mentioned algorithms have been compared using the same input graphs H. All the algorithms are able to construct the root graph G if the given graph H is a line graph, and can tell *non-line graph* when H if is not a line graph. The line graphs with link density[8] $p_H = 0.05$ and 0.125 are generated by converting random graphs [7,3] with a fixed link density into line graphs. However, the line graphs of these random graphs can never have high link densities (explained with details in Appendix B). Therefore, the line graphs with $p_H = 0.5$ and 0.65 have been generated by converting the scale-free graphs [2] into line graphs.

Figure 4 reflects the trends for the running times of all the algorithms when the input graphs are line graphs with different link density p_H and different number of links L_H. Figure 4a and Figure 4b show the running times for line graphs with small link density $p_H = 0.05, 0.125$, where ILIGRA performs faster than all the other algorithms. Figures 4c, 4d, 4e and 4f illustrate the trends for the algorithms' running times for line graphs with high link density $p_H = 0.50$ and $p_H = 0.65$. ILIGRA is the fastest algorithm for line graphs with

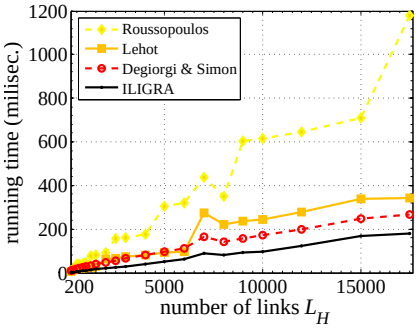
[5] The implementations are available on the authors' web page: http://www.nas.ewi.tudelft.nl/people/Stojan/code/ILIGRA.zip

[6] LEDA: http://www.algorithmic-solutions.com/leda/

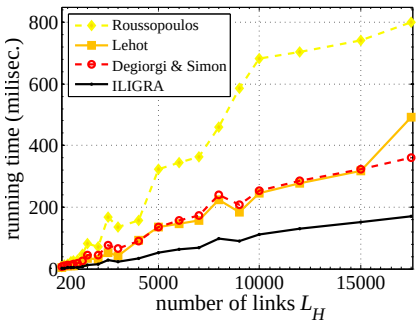
[7] Intel(R) Core(TM) 2 Duo CPU T9600 on 2 x 2.80GHz; 4GB RAM memory

[8] The *link density* of a given line graph $H(L, L_H)$ is defined by $p_H = L_H/\binom{L}{2}$, where L is the number of nodes in H and L_H is the number of links in H.

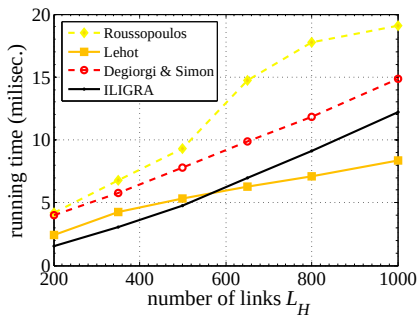
small number of links $200 \leq L_H \leq 500$ and Lehot's algorithm has the shortest running time for line graphs with high number of links $650 \leq L_H \leq 18000$. The best algorithms, reflected by the running time for all the cases, are summarized in Table 2.



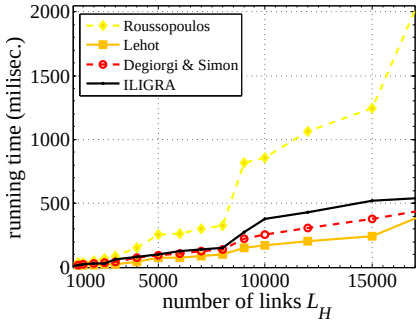
(a) $p_H = 5\%$ and $200 \leq L_H \leq 18000$



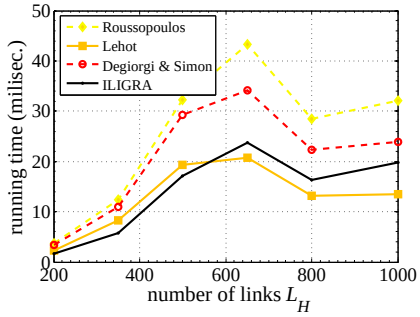
(b) $p_H = 12.5\%$ and $200 \leq L_H \leq 18000$



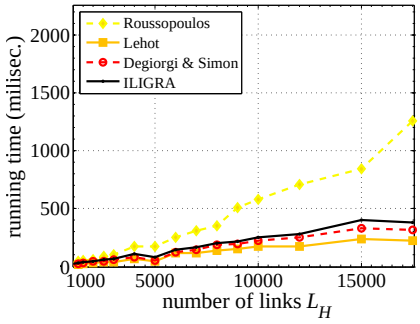
(c) $p_H = 50\%$ and $200 \leq L_H \leq 1000$



(d) $p_H = 50\%$ and $1000 \leq L_H \leq 18000$



(e) $p_H = 65\%$ and $200 \leq L_H \leq 1000$



(f) $p_H = 65\%$ and $1000 \leq L_H \leq 18000$

Fig. 4 Algorithms' running times for **line graphs** with different p_H and L_H.

5 Conclusion

We present ILIGRA algorithm for inverse line graph construction. Given a line graph H, ILIGRA constructs its root graph G and checks whether the given graph is a line graph during the construction. ILIGRA also works for disconnected line graphs by iterating through the connected components of the input line graph. The time complexity of ILIGRA is linear in the number of nodes in the input graph H without checking if the given graph is a line graph. The time complexity of ILIGRA with full functionality is linear in the number of links in the given line graphs. Numerical comparisons with the algorithms of Lehot, Roussopoulos, and Degiorgi and Simon have been demonstrated. Given line graphs with small link density (i.e. sparse graphs), ILIGRA is the fastest algorithm in root graph construction, as shown in Table 2.

6 Acknowledgments

We would like to thank Prof. Dr. Klaus Simon from ETH Zürich for providing us with the initial implementations of the algorithms of Lehot, Roussopoulos, and Degiorgi and Simon. We are grateful to two anonymous reviewers who provided us with valuable comments.

A Special cases

We start with two adjacent nodes n_1 and n_2 in H that correspond to links l_{n_1} and l_{n_2} in G, respectively, where l_{n_1} is incident to v_1 and v_2 and where l_{n_2} is incident to v_1 and v_3. We denote by $J = \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2)$, $C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and L is the number of links in G. For each $n_u \in J$, link l_{n_u} is either incident to v_1, or incident to both v_2 and v_3. For each $n_u \in J$, we denote $Z = \mathcal{N}_b(n_u) \setminus \{n_1, n_2\}$. In the remainder of this appendix and Table 3, we continue with the case analysis for the remaining cases: $|J| \leq 2$ and $|Z| \leq 2$.

1. $Z = \emptyset$
   (a) $|J| = 1$
      i. $L = 3$
      The root graph G is K_3 or $K_{1,3}$, as shown in Figure 5 (a).
      ii. $L \geq 4$
      l_{n_u} is incident to v_1.
      Proof. Let us assume l_{n_u} is incident to v_2 and v_3. Since H has more than 3 nodes and $\mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2) = \{n_u\}$, the extra links must be incident to v_2 or v_3 in G, which means n_u must have other neighbors different from n_1 and n_2, contradicting the fact that $Z = \mathcal{N}_b(n_u) \setminus \{n_1, n_2\} = \emptyset$. Hence, l_{n_u} can only be incident to v_1. □

   (b) $|J| = 2$ ($J = \{n_u, n_r\}$) and $n_r \notin \mathcal{N}_b(n_u)$
      i. $L = 4$
      l_{n_u} is incident to v_1 or v_2, as shown in Figure 5 (b). The resulting root graphs are isomorphic.
      ii. $L \geq 5$ and n_x is the node in H different from n_1, n_2, n_u, and n_r.
      If $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$, then l_{n_u} is incident to v_2 (Figure 5 (c)), otherwise l_{n_u} is incident to v_1.
      Proof. Let us first assume $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to

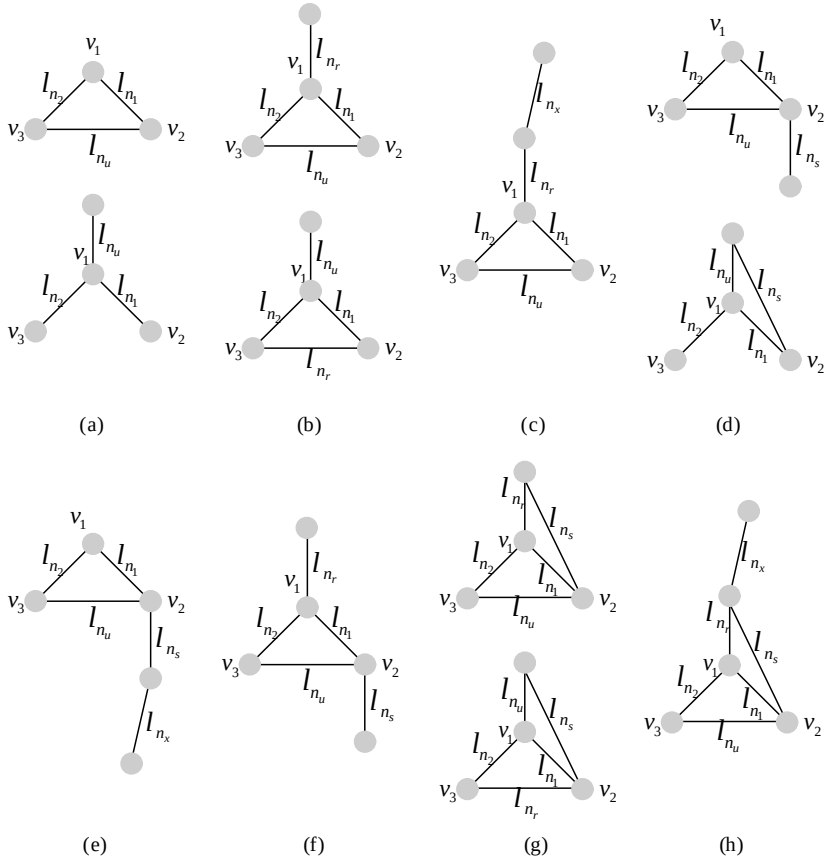


Fig. 5 Scenarios for (a) $|Z| = 0$, $|J| = 1$, $L = 3$; (b) $|Z| = 0$, $|J| = 2$, $L = 4$; (c) $|Z| = 0$, $|J| = 2$, $L = 5$, $n_x \notin C$; (d) $|Z| = 1$, $|J| = 1$, $L = 4$; (e) $|Z| = 1$, $|J| = 1$, $L = 5$, $n_x \notin C$; (f) $|Z| = 1$, $|J| = 2$, $L = 5$, $n_s \notin \mathcal{N}_b(n_r)$, $n_s \in C$; (g) $|Z| = 1$, $|J| = 2$, $L = 5$, $n_s \in \mathcal{N}_b(n_r)$; and (h) $|Z| = 1$, $|J| = 2$, $L = 6$, $n_s \in \mathcal{N}_b(n_r)$, $n_x \notin C$.

v_1. Since $|Z| = 0$, l_{n_x} must be incident to either v_2 or v_3, which contradicts the fact that $n_x \notin C$. Hence, l_{n_u} is incident to v_2.
Let us now assume $n_x \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_2. Since $|Z| = 0$, l_{n_x} must be incident to v_1, which contradicts the fact that $|J| = 2$. Hence, l_{n_u} is incident to v_1. □

2. $|Z| = 1$ ($Z = \{n_s\}$)
   (a) $|J| = 1$
      i. $L = 4$
      l_{n_u} is incident to v_1 or v_2, as shown in Figure 5 (d). The resulting root graphs are isomorphic.
      ii. $L \geq 5$ and n_x is the node in H different from n_1, n_2, n_u, and n_s.
      If $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$, l_{n_u} is incident to v_2 (Figure 5 (e)), otherwise l_{n_u} is incident to v_1.
      Proof. Let us first assume $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_1. Since $|J| = 1$ and $|Z| = 1$, l_{n_x} must be incident to either v_2 or v_3, which contradicts the fact that $n_x \notin C$. Hence, l_{n_u} is incident to v_2.

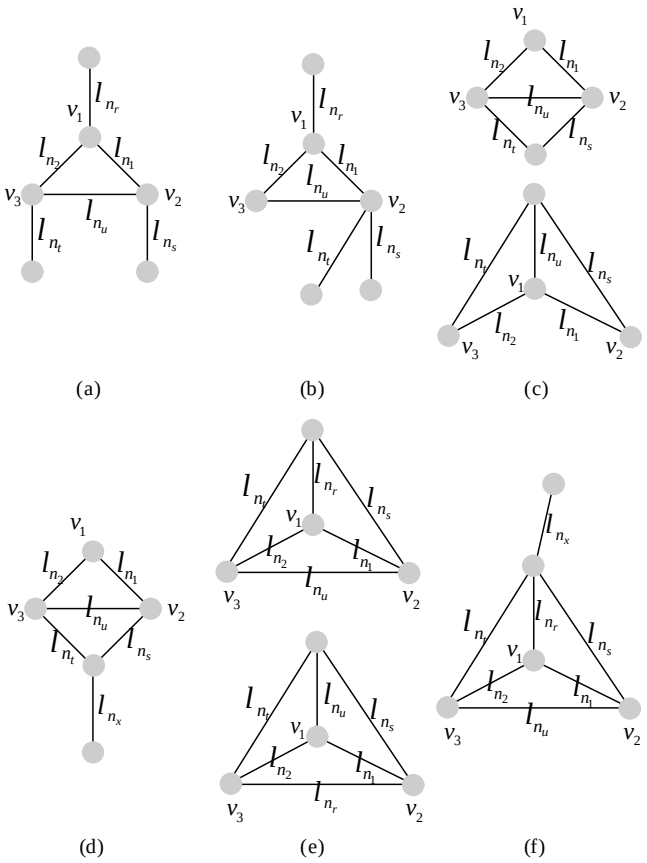


Fig. 6 Scenarios for $|Z| = 2$: (a) $n_s \notin \mathcal{N}_b(n_t)$, $|J| = 2$; (b) $n_s \in \mathcal{N}_b(n_t)$ and n_s, n_t, n_1 (or n_2) are pairwise adjacent; $n_s \in \mathcal{N}_b(n_t)$ and n_s, n_t, n_1 (or n_2) are not pairwise adjacent: (c) $|J| = 1$, $L = 5$; (d) $|J| = 1$, $L = 6$, $n_x \notin C$; (e) $|J| = 2$, $L = 6$; and (f) $|J| = 2$, $L = 7$, $n_x \notin C$.

Let us now assume $n_x \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_2 and v_3. Since $|Z| = 1$, l_{n_x} must be incident to v_1, which contradicts the fact that $|J| = 1$. Hence, l_{n_u} is incident to v_1. □

(b) $|J| = 2$ ($J = \{n_u, n_r\}$) and $n_r \notin \mathcal{N}_b(n_u)$

i. $n_s \notin \mathcal{N}_b(n_r)$

If $n_s \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$, l_{n_u} is incident to v_2 (Figure 5 (f)), otherwise l_{n_u} is incident to v_1.

Proof. Let us first assume $n_s \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_1. Since l_{n_u} is incident to v_1, l_{n_r} must be incident to v_2 and v_3. Since $n_s \notin \mathcal{N}_b(n_r)$, l_{n_s} must be incident to v_1, contradicting the fact that $|J| \leq 2$. Hence, l_{n_u} is incident to v_2.

Let us now assume $n_s \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_2 and v_3. Since $n_s \in \mathcal{N}_b(n_u)$, n_s is incident to either v_2 or v_3, contradicting the fact that $n_s \notin C$. Hence, l_{n_u} is incident to v_1. □

ii. $n_s \in \mathcal{N}_b(n_r)$

A. $L = 5$
l_{n_u} is incident to v_1 or v_2, as shown in Figure 5 (g). The resulting root graphs are isomorphic.

B. $L \geq 6$ and n_x is the node in H different from n_1, n_2, n_u, n_r, and n_s.
If $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$, l_{n_u} is incident to v_2 (Figure 5 (h)), otherwise l_{n_u} is incident to v_1.
Proof. Assume that $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_1. Since $n_r \notin \mathcal{N}_b(n_u)$, l_{n_r} is incident to v_2 and v_3. Since $n_s \in \mathcal{N}_b(n_r)$ and $n_s \in \mathcal{N}_b(n_u)$, l_{n_s} is incident to the node of l_{n_u} different from v_1 and either v_2 or v_3. Since $|J| = 2$ and $|Z| = 1$, l_{n_x} must be incident to either v_2 or v_3, which contradicts the fact that $n_x \notin C$. Hence, l_{n_u} is incident to v_2.
Now, assume that $n_x \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_2 and v_3 and l_{n_r} is incident to v_1. Since $n_s \in \mathcal{N}_b(n_u)$, l_{n_s} is incident to either v_2 or v_3. Since $|Z| = 1$, l_{n_x} must be incident to v_1, which contradicts the fact that $|J| = 2$. Hence, l_{n_u} is incident to v_1. □

3. $|Z| = 2$ ($Z = \{n_s, n_t\}$)

(a) $n_s \notin \mathcal{N}_b(n_t)$

i. $|J| = 1$ ($J = \{n_u\}$)
l_{n_u} is incident to v_2.
Proof. Assume that l_{n_u} is incident to v_1 and v_u. Since $|J| = 1$, both l_{n_s} and l_{n_t} must be incident to v_u, then $n_s \in \mathcal{N}_b(n_t)$, contradicting the assumption that $n_s \notin \mathcal{N}_b(n_t)$. Hence, l_{n_u} is incident to v_2. □

ii. $|J| = 2$ ($J = \{n_u, n_r\}$) and $n_r \notin \mathcal{N}_b(n_u)$
l_{n_u} is incident to v_2, as shown in Figure 6 (a).
Proof. Assume that l_{n_u} is incident to v_1 and v_u. Since l_{n_u} is incident to v_1, l_{n_r} must be incident to v_2. Since $|J| = 2$, both l_{n_s} and l_{n_t} must be incident to v_u, then $n_s \in \mathcal{N}_b(n_t)$, contradicting with the assumption that $n_s \notin \mathcal{N}_b(n_t)$. Hence, l_{n_u} is incident to v_2. □

(b) $n_s \in \mathcal{N}_b(n_t)$ and n_s, n_t, n_1 (or n_2) are pairwise adjacent
l_{n_u} is incident to v_2, as shown in Figure 6 (b).
Proof. If l_{n_u} is incident to v_1, neither n_s, n_t, n_1 nor n_s, n_t, n_2 can be pairwise adjacent, hence l_{n_u} is incident to v_2. □

(c) $n_s \in \mathcal{N}_b(n_t)$ and n_s, n_t, n_1 are not pairwise adjacent

i. $|J| = 1$

A. $L = 5$
l_{n_u} is incident to v_1 or v_2, as shown in Figure 6 (c). The resulting root graphs are isomorphic.

B. $L \geq 6$ and n_x is the node in H different from n_1, n_2, n_u, n_s, and n_t.
If $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$, l_{n_u} is incident to v_2 (Figure 6 (d)), otherwise l_{n_u} is incident to v_1.
Proof. Assume $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_1. Since $|J| = 1$ and $|Z| = 2$, there is no link incident to both v_2 and v_3, hence l_{n_s} and l_{n_t} are incident to the node of l_{n_u} different from v_1 and one of the nodes v_2 and v_3. Now, l_{n_x} cannot be adjacent to v_1 as $|J| = 1$; it cannot be adjacent to none of v_2, v_3 and the node of l_{n_u} different from v_1, because $|Z| = 1$, hence there will not be a space for link l_{n_x}, which contradicts the assumption of the existence of n_x and $L \geq 6$. Hence, l_{n_u} is incident to v_2.
Now, assume $n_x \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_2 and v_3. Since $|J| = 1$, no other link is incident to v_1, therefore l_{n_s}, l_{n_t} and l_{n_x} are all incident to either v_2 or v_3, which contradicts the assumption $|Z| = 2$. Hence, l_{n_u} is incident to v_1. □

ii. $|J| = 2$ ($J = \{n_u, n_r\}$) and $n_r \notin \mathcal{N}_b(n_u)$

A. $L = 6$
l_{n_u} is incident to v_1 or v_2, as shown in Figure 6 (e). The resulting root graphs are isomorphic.

B. $L \geq 7$ and n_x is the node in H different from n_1, n_2, n_u, n_r, n_s, and n_t. If $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$, l_{n_u} is incident to v_2 (Figure 6 (f)), otherwise l_{n_u} is incident to v_1.
Proof. Assume $n_x \notin C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_1. Since $|J| = 2$ and $|Z| = 2$, l_{n_r} is incident to both v_2 and v_3; l_{n_s} and l_{n_t} are incident to the node of l_{n_u} different from v_1 and one of the nodes v_2 and v_3. Since $n_x \notin C$, link l_{n_x} is incident to the node of l_{n_u} different from v_1, which contradicts the assumption of $|Z| = 2$. Hence, l_{n_u} is incident to v_2. Now, assume $n_x \in C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ and l_{n_u} is incident to v_2 and v_3. Since $|J| = 2$, l_{n_r} is incident to v_1. Since $n_x \in C$ and $|J| = 1$, link l_{n_x} cannot be incident to v_1.Finally, l_{n_s}, l_{n_r} and l_{n_x} are all incident to either v_2 or v_3, which contradicts the fact that $|Z| = 2$. Hence, l_{n_u} is incident to v_1. □

Table 3 All cases for identifying the special node in J, whose corresponding link is incident to v_2. Notation: $J = \mathcal{N}_b(n_1) \cap \mathcal{N}_b(n_2), n_u \in J, C = \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2), Z = \mathcal{N}_b(n_u) \setminus \{n_1, n_2\}$.

Conditions					l_{n_u} **is incident to**
$\|J\| \geq 3$					Use Theorem 2.
$\|J\| \leq 2$	$\|Z\| \geq 3$				Use Theorem 3.
	$Z = \emptyset$	$\|J\|$=1	$L = 3$		v_1 or v_2 (G is K_3 or $K_{1,3}$)
			$L \geq 4$		v_1
		$\|J\|$=2 and $n_r \notin \mathcal{N}_b(n_u)$	$L = 4$		v_1 or v_2 (isomorphic)
			$L \geq 5$		v_2 if $n_x \notin C$; v_1 if $n_x \in C$
	$\|Z\|$=1	$\|J\|$=1	$L = 4$		v_1 or v_2 (isomorphic)
			$L \geq 5$		v_2 if $n_x \notin C$; v_1 if $n_x \in C$
		$\|J\|$=2 and $n_r \notin \mathcal{N}_b(n_u)$	$n_s \notin \mathcal{N}_b(n_r)$		v_2 if $n_s \in C$; v_1 if $n_s \notin C$
			$n_s \in \mathcal{N}_b(n_r)$	$L = 5$	v_1 or v_2 (isomorphic)
				$L \geq 6$	v_2 if $n_x \notin C$; v_1 if $n_x \in C$
	$\|Z\|$=2	$n_s \notin \mathcal{N}_b(n_t)$	$\|J\| = 1$		Figure 6 (a)
			$\|J\| = 2$ and		v_2
			$n_r \notin \mathcal{N}_b(n_u)$		v_2
		$n_s \in \mathcal{N}_b(n_t)$, n_s, n_t, n_1 pairwise adjacent			v_2
		$n_s \in \mathcal{N}_b(n_t)$, n_s, n_t, n_1 are not pairwise adjacent	$\|J\| = 1$	$L = 5$	v_1 or v_2 (isomorphic)
				$L \geq 6$	v_2 if $n_x \notin C$; v_1 if $n_x \in C$
			$\|J\| = 2$ and $n_r \notin \mathcal{N}_b(n_u)$	$L = 6$	v_1 or v_2 (isomorphic)
				$L \geq 7$	v_2 if $n_x \notin C$; v_1 if $n_x \in C$

B The link density of line graphs

The link density is an important characteristic for the topology of line graphs. This section discusses the relation between the link density of line graph H, the number of nodes N and the number of links L in the root graph $G(N, L)$.

The number of nodes N_H in the line graph H is equal to the number of links L in the root graph G. For the number of links L_H in the line graph H, we have

$$L_H = \frac{1}{2} \sum_{i=1}^{N} d_i^2 - L \tag{1}$$

where $d = [d_1, d_2, \cdots, d_N]$ is the degree sequence of G.

Algorithm 2: InitSpecCases(H, n_1, n_2, n_u)

if $\mathcal{N}_b(n_u) \setminus \{n_1, n_2\} = \emptyset$ **then**
  if $J = \{n_u\}$ **then**
    if $L = 3$ **then**
      G is $K_{1,3}$ or K_3. **Exit**.
    else if $L \geq 4$ **then**
      $v_{l_{n_u}} \leftarrow v_1$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
  else if $J = \{n_u, n_r\}$ *and* $n_r \notin \mathcal{N}_b(n_u)$ **then**
    if $L = 4$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
    else if $L \geq 5$ *and* $n_x \notin \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
else if $\mathcal{N}_b(n_u) \setminus \{n_1, n_2\} = \{n_s\}$ **then**
  if $J = \{n_u\}$ **then**
    if $L = 4$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
    else if $L \geq 5$ *and* $n_x \notin \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
  else if $J = \{n_u, n_r\}$ *and* $n_r \notin \mathcal{N}_b(n_u)$ **then**
    if $n_s \notin \mathcal{N}_b(n_r)$ *and* $n_s \in \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
    else if $n_s \in \mathcal{N}_b(n_r)$ **then**
      if $L = 5$ **then**
        $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
      else if $L \geq 6$ *and* $n_x \notin \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ **then**
        $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
else if $\mathcal{N}_b(n_u) \setminus \{n_1, n_2\} = \{n_s, n_t\}$ **then**
  if $n_s \notin \mathcal{N}_b(n_t)$ **then**
    if $J = \{n_u\}$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
    else if $J = \{n_u, n_r\}$ *and* $n_r \notin \mathcal{N}_b(n_u)$ **then**
      $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
  else if $n_s \in \mathcal{N}_b(n_t)$ *and* n_s, n_t, n_1 *or* n_2 *form a* K_3 **then**
    $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
  else if $n_s \in \mathcal{N}_b(n_t)$ *and neither* n_s, n_t, n_1 *nor* n_s, n_t, n_2 *form a* K_3 **then**
    if $J = \{n_u\}$ **then**
      if $L = 5$ **then**
        $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
      else if $L \geq 6$ *and* $n_x \notin \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ **then**
        $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
    else if $J = \{n_u, n_r\}$ *and* $n_r \notin \mathcal{N}_b(n_u)$ **then**
      if $L = 6$ **then**
        $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;
      else if $L \geq 7$ *and* $n_x \notin \mathcal{N}_b(n_1) \cup \mathcal{N}_b(n_2)$ **then**
        $v_{l_{n_u}} \leftarrow v_2$; $\mathcal{N}_h \leftarrow \mathcal{N}_h \cup \{n_u\}$; $\mathcal{N}_w \leftarrow \mathcal{N}_w \setminus \{n_u\}$; $J \leftarrow J \setminus \{n_u\}$;

By using (1), the link density p_H of H equals

$$p_H = \frac{L_H}{\binom{L}{2}} = \frac{\frac{1}{2}\sum_{i=1}^{N} d_i^2 - L}{\binom{L}{2}} = \frac{\sum_{i=1}^{N} d_i^2 - 2L}{L^2 - L} \tag{2}$$

Using the basic law of degrees, $\sum_{i=1}^{N} d_i = 2L$, and Cauchy's inequality [4,9]

$$\sum_{i=1}^{N} d_i^2 \geq \frac{(\sum_{i=1}^{N} d_i)^2}{N} = \frac{4L^2}{N} \tag{3}$$

and using (3) into (2), we obtain

$$p_H = \frac{\sum_{i=1}^{N} d_i^2 - 2L}{L^2 - L} \geq \frac{2L(\frac{2L}{N} - 1)}{L^2 - L} = \frac{2}{N}\frac{2L - N}{L - 1} \tag{4}$$

Equality in (4) holds for regular root graphs G, where $d_i = \frac{2L}{N}$, for $i = 1, 2, \cdots, N$. When $L \gg N$, the link density p_H asymptotically tends to $\frac{4}{N}$. Hence, the line graphs of dense root graphs with $L \gg N$ have small link densities.

We derive an upper bound for the link density p_H. Using $L = (\sum_{i=1}^{N} d_i)/2$ and the inequality $(\sum_{i=1}^{N} x_i)^2 \geq \sum_{i=1}^{N} x_i^2$ for $x_i = d_i - 1 \geq 0$, we obtain

$$\begin{aligned}\sum_{i=1}^{N} d_i^2 =& \sum_{i=1}^{N}(d_i - 1)^2 - N + 2\sum_{i=1}^{N} d_i = \sum_{i=1}^{N}(d_i - 1)^2 - N + 4L \\ \leq& 4L - N + \left(\sum_{i=1}^{N}(d_i - 1)\right)^2 = (2L - N + 1)^2 + N - 1\end{aligned} \tag{5}$$

Finally, p_H is bounded by

$$\frac{4L - 2N}{N(L-1)} \leq p_H \leq \frac{(2L - N + 1)^2 + N - 2L - 1}{L^2 - L}$$

Equality in (5) is achieved if and only if $(d_i - 1)(d_j - 1) = 0$ for all $i, j \in 1, 2, \cdots, N$. The star graph $K_{1,N}$ satisfies the condition for equality in (5), indicating that the line graph of $K_{1,N}$ reaches the upper bound of link density p_H. In fact, the line graph of $K_{1,N}$ is complete graph K_{N-1} with maximum link density of 1. In conclusion, dense line graphs can be obtained if the original graph has one node with a high degree and the other nodes have relatively small degrees. On the other hand, the line graph of a regular graph has the minimum link density. Hence, the line graphs with $p_H = 0.5$ and 0.65 in Section 4 are generated by converting the scale-free graphs into line graphs.

References

1. Ahn, Y.Y., Bagrow, J.P., Lehmann, S.: Link communities reveal multiscale complexity in networks. Nature **466**(7307), 761–764 (2010)
2. Barabási, A.L., Albert, R.: Emergence of scaling in random networks. Science **286**(5439), 509–512 (1999)
3. Bollobás, B.: Random Graphs. Cambridge University Press, Cambridge, UK. (2001)
4. Cauchy, A.L.: Cours d'analyse de l'Ecole Royale Polytechnique Vol. 3 (1821). Imprimerie royale, Paris (reissued by Cambridge University Press), Cambridge, UK. (2009)
5. Cvetković, D., Rowlinson, P., Simić, S.: Spectral Generalizations of Line Graphs. Cambridge University Press, Cambridge, UK. (2004)
6. Degiorgi, D.G., Simon, K.: A dynamic algorithm for line graph recognition. In: Proc. of 21st Int. Workshop on Graph-Theoretic Concepts in Computer Science (Lecture Notes in Computer Science 1017), pp. 37–48. Springer-Verlag (1995)

7. Erdős, P., Rényi, A.: On random graphs, I. Publicationes Mathematicae (Debrecen) **6**, 290–297 (1959)
8. Evans, T., Lambiotte, R.: Line graphs, link partitions, and overlapping communitities. Phys. Rev. E **80**(1) (2009)
9. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities, 2 edn. Cambridge University Press (1988)
10. Krausz, J.: Démonstration nouvelle d'un théorème de Whitney sur les réseaux. Mat. Fiz. Lapok **50**, 75–85 (1943)
11. Krawczyk, M.J., Muchnik, L., Manka-Krason, A., Kulakowski, K.: Line graphs as social networks. Physica A **390**, 2611–2618 (2011)
12. Lehot, P.G.H.: An optimal algorithm to detect a line graph and output its root graph. Journal of the ACM **21**, 569–575 (1974)
13. Manka-Krason, A., Kulakowski, K.: Assortativity in random line graphs. Acta Physica Polonica B Proceedings Supplement **3**(2), 259–266 (2010)
14. Manka-Krason, A., Mwijage, A., Kulakowski, K.: Clustering in random line graphs. Computer Physics Communications **181**(1), 118–121 (2010)
15. Mehlhorn, K., Näher, S.: LEDA: A Platform for Combinatorial and Geometric Computing. Cambridge University Press, Cambridge, UK. (1999)
16. Nacher, J.C., Ueda, U., Yamada, T., Kanehisa, M., Akutsu, T.: Line graphs as social networks. BMC Bioinformatics **24**(207), 2611–2618 (2004)
17. Nacher, J.C., Yamada, T., Goto, S., Kanehisa, M., Akutsu, T.: Two complementary representations of a scale-free network. Physica A **349**, 349–363 (2005)
18. Naor, J., Novick, M.B.: An efficient reconstruction of a graph from its line graph in parallel. Journal of algorithms **11**, 132–143 (1990)
19. Ore, O.: Theory of Graphs, vol. 21. American Mathematical Society Colloquium Publications (1962)
20. Roussopoulos, N.D.: A $max\{m, n\}$ algorithm for detecting the graph h from its line graph g. Information Processing Letters **2**, 108–112 (1973)
21. Simić, S.: An algorithm to recognize a generalized line graphs and ouput its root graph. Publ. Math. Inst. (Belgrade) **49**(63), 21–26 (1990)
22. Van Mieghem, P.: Graph Spectra for Complex Networks. Cambridge University Press, Cambridge, UK. (2011)
23. van Rooij, A.C.M., Wilf, H.S.: The interchange graph of a finite graph. Acta Mathematica Academiae Scientiarum Hungaricae **16**, 263–269 (1965)
24. Whitney, H.: Congruent graphs and the connectivity of graphs. American Journal of Mathematics **54**, 150–168 (1932)