The domination number of Cartesian product of two directed paths

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Abstract

Let $\gamma(P_m \Box P_n)$ be the domination number of the Cartesian product of directed paths P_m and P_n for $m, n \geq 2$. In [13] Liu and al. determined the value of $\gamma(P_m \Box P_n)$ for arbitrary n and $m \leq 6$. In this work we give the exact value of $\gamma(P_m \Box P_n)$ for any m, n and exhibit minimum dominating sets.

AMS Classification[2010]:05C69,05C38.

Keywords: Directed graph, digraph, Cartesian product, Domination number, Paths.

1 Introduction and definitions

Let $G = (V, E)$ be a finite directed graph (digraph for short) without loops or multiple arcs.

A vertex u *dominates* a vertex v if $u = v$ or $uv \in E$. A set $S \subset V$ is a *dominating* set of G if any vertex of G is dominated by at least a vertex of S. The *domination number* of G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set. The set V is a dominating set thus $\gamma(G)$ is finite. These definitions extend to digraphs the classical domination notion for undirected graphs.

The determination of domination number of a directed or undirected graph is, in general, a difficult question in graph theory. Furthermore this problem has connections with information theory. For example the domination number of Hypercubes is linked to error-correcting codes. Among the lot of related works ([7], [8]) mention the special case of domination of Cartesian product of undirected paths or cycles ([1] to [6], [9], [10]).

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For two digraphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *Cartesian product* $G_1 \Box G_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \Box G_2)$ if and only if $x_1 y_1 \in E_1$ and $x_2 = y_2$ or $x_2 y_2 \in E_2$ and $x_1 = y_1$. Note that $G \Box H$ is isomorphic to $H \Box G$.

The domination number of Cartesian product of two directed cycles have been recently investigated ([11], [12], [14], [15]). Even more recently, Liu and al.([13]) began the study of the domination number of the Cartesian product of two directed paths P_m and P_n. They proved the following result

Theorem 1 *Let $n \geq 2$. Then*

- $\gamma(P_2 \Box P_n) = n$
- $\gamma(P_3 \Box P_n) = n + \lceil \frac{n}{4} \rceil$
- $\gamma(P_4 \Box P_n) = n + \lceil \frac{2n}{3} \rceil$
- $\gamma(P_5 \Box P_n) = 2n + 1$
- $\gamma(P_6 \Box P_n) = 2n + \lceil \frac{n+2}{3} \rceil$.

In this paper we are able to give a complete solution of the problem. In Theorem 2 we determine the value of $\gamma(P_m \Box P_n)$ for any $m, n \geq 2$. When m grows, the cases approach appearing in the proof of Theorem 1 seems to be more and more complicated. We proceed by a different and elementary method, but will assume that Theorem 1 is already obtained (at least for $m \leq 5$ and arbitrary n). In the next section we describe three dominating sets of $P_m \Box P_n$ corresponding to the different values of m modulo 3. In the last section we prove that these dominating sets are minimum and deduce our main result:

Theorem 2 *Let $n \geq 2$. Then*

- $\gamma(P_{3k} \Box P_n) = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$ *for $k \geq 2$ and $n \neq 3$*
- $\gamma(P_{3k+1} \Box P_n) = k(n+1) + \lceil \frac{2n-3}{3} \rceil$ *for $k \geq 1$ and $n \neq 3$*
- $\gamma(P_{3k+2} \Box P_n) = k(n+1) + n$ *for $k \geq 0$ and $n \neq 3$*
- $\gamma(P_3 \Box P_n) = \gamma(P_n \Box P_3) = n + \lceil \frac{n}{4} \rceil$.

We will follow the notations used by Liu and al. and refer to their paper for a more complete description of the motivations. Let us recall some of these notations.

We denote the vertices of a directed path P_n by the integers $\{0, 1, \ldots, n-1\}$. For any i in $\{0, 1, \ldots, n-1\}$, P_m^i is the subgraph of $P_m \Box P_n$ induced by the vertices $\{(k, i) \,/\, k \in \{0, 1, \ldots, m-1\}\}$. Note that P_m^i is isomorphic to P_m. Notice also that $P_m \Box P_n$ is isomorphic to $P_n \Box P_m$ thus $\gamma(P_m \Box P_n) = \gamma(P_n \Box P_m)$. A vertex $(a, b) \in P_m^b$ can be dominated by (a, b), $(a-1, b) \in P_m^b$ (if $a \geq 1$), $(a, b-1) \in P_m^{b-1}$ (if $b \geq 1$).

2 Three Dominating sets

We will first study $P_{3k} \Box P_n$ for $k \geq 1$ and $n \geq 2$. Consider the following sets of vertices of P_{3k}.

- $X = \{0, 1, 3, 4, \ldots, 3k-3, 3k-2\} = \{3i/i \in \{0, 1, \ldots k-1\}\} \cup \{3i+1/i \in \{0, 1, \ldots k-1\}\}$
- $Y = \{2, 5, 8, \ldots, 3k-1\} = \{3i+2/i \in \{0, 1, \ldots k-1\}\}$
- $I = \{0, 3, 6, \ldots, 3k-3\} = \{3i/i \in \{0, 1, \ldots k-1\}\}$
- $J = \{1, 4, 7, \ldots, 3k-2\} = \{3i+1/i \in \{0, 1, \ldots k-1\}\}$
- $K = \{0, 2, 5, 8 \ldots, 3k-1\} = \{0\} \cup \{3i+2/i \in \{0, 1, \ldots k-1\}\}$.

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Let D_n (see Figure 1) be the set of vertices of $P_{3k} \Box P_n$ consisting of the vertices

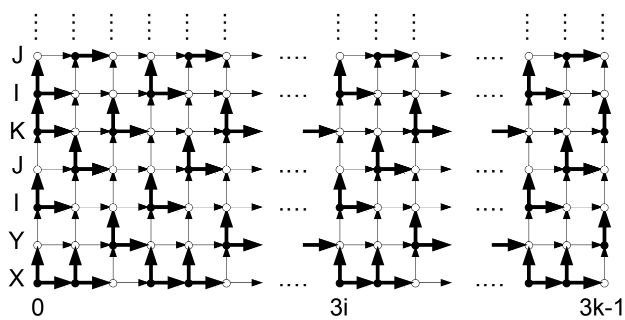

Figure 1: The dominating set D_n

- $(a, 0)$ for $a \in X$
- $(a, 1)$ for $a \in Y$
- (a, b) for $b \equiv 2 \bmod 3$ $(2 \leq b < n)$ and $a \in I$
- (a, b) for $b \equiv 0 \bmod 3$ $(3 \leq b < n)$ and $a \in J$
- (a, b) for $b \equiv 1 \bmod 3$ $(4 \leq b < n)$ and $a \in K$.

Lemma 3 *For any $k \geq 1$, $n \geq 2$ the set D_n is a dominating set of $P_{3k} \Box P_n$ and* $|D_n| = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$.

Proof : It is immediate to verify that

- All vertices of P_{3k} are dominated by the vertices of X
- The vertices of P_{3k} not dominated by some of Y are $\{0, 1, 4, \ldots, 3k-2\} \subset X$
- The vertices of P_{3k} not dominated by some of I are $\{2, 5, \ldots, 3k-1\} = Y \subset K$
- The vertices of P_{3k} not dominated by some of J are $\{0, 3, 6, \ldots, 3k-3\} \subset I$

- The vertices of P_{3k} not dominated by some of K are $\{4, 7, \ldots, 3k-2\} \subset J$.

Therefore any vertex of some P_{3k}^{i} is dominated by a vertex in $P_{3k}^{i} \cap D_n$ or in $P_{3k}^{i-1} \cap D_n$ (if $i \geq 1$). Furthermore $|X| = 2k$, $|Y| = |I| = |J| = k$, and $|K| = k+1$ thus $|D_n| = k(n+1) + \lfloor \frac{n-2}{3} \rfloor$. $\square$

Let us study now $P_{3k+1} \square P_n$ for $k \geq 1$ and $n \geq 2$. Consider the following sets of vertices of P_{3k+1}.

- $X = \{0, 2, 4, 5, 7, 8, \ldots, 3k-2, 3k-1\} = \{0\} \cup \{3i+2/i \in \{0, 1, \ldots k-1\}\} \cup \{3i+1/i \in \{1, \ldots k-1\}\}$
- $I = \{0, 3, 6, \ldots, 3k\} = \{3i/i \in \{0, 1, \ldots k\}\}$
- $J = \{1, 4, 7, \ldots, 3k-2\} = \{3i+1/i \in \{0, 1, \ldots k-1\}\}$
- $K = \{0, 2, 5, 8, \ldots, 3k-1\} = \{0\} \cup \{3i+2/i \in \{0, 1, \ldots k-1\}\}$.

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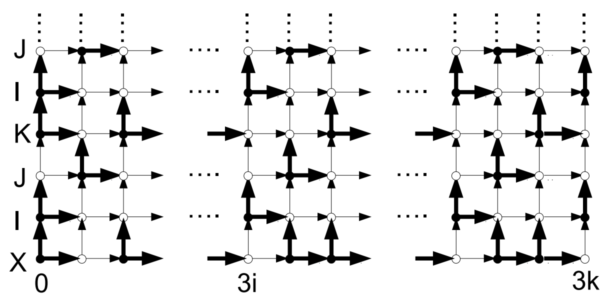

Figure 2: The dominating set E_n

Let E_n (see Figure 2) be the set of vertices of $P_{3k+1} \square P_n$ consisting of the vertices

- $(a, 0)$ *for* $a \in X$
- (a, b) *for* $b \equiv 1 \bmod 3$ $(1 \leq b < n)$ *and* $a \in I$
- (a, b) *for* $b \equiv 2 \bmod 3$ $(2 \leq b < n)$ *and* $a \in J$
- (a, b) *for* $b \equiv 0 \bmod 3$ $(3 \leq b < n)$ *and* $a \in K$.

Lemma 4 *For any $k \geq 1$, $n \geq 2$ the set E_n is a dominating set of $P_{3k+1} \square P_n$ and $|E_n| = k(n+1) + \lceil \frac{2n-3}{3} \rceil$.*

Proof : It is immediate to verify that

- All vertices of P_{3k+1} are dominated by the vertices of X

- The vertices of P_{3k+1} not dominated by some of I are $\{2, 5, \ldots, 3k-1\} \subset K \subset X$
- The vertices of P_{3k+1} not dominated by some of J are $\{0, 3, 6, \ldots, 3k\} = I$
- The vertices of P_{3k+1} not dominated by some of K are $\{4, 7, \ldots, 3k-2\} \subset J$.

Therefore any vertex of some P^i_{3k+1} is dominated by a vertex in $P^i_{3k+1} \cap E_n$ or in $PP^{i-1}_{3k+1} \cap E_n$ (if $i \geq 1$). Furthermore $|X| = 2k$, $|I| = |K| = k+1$, and $|J| = k$ thus $|E_n| = k(n+1) + \lceil \frac{2n-3}{3} \rceil$. $\square$

The last case will be $P_{3k+2} \square P_n$ for $k \geq 0$ and $n \geq 2$. Consider the following sets of vertices of P_{3k+2}.

- $X = \{0, 1, 3, 4, \ldots, 3k, 3k+1\} = \{3i/i \in \{0, 1, \ldots k\}\} \cup \{3i+1/i \in \{0, 1, \ldots k\}\}$
- $Y = \{2, 5, 8, \ldots, 3k-1\} = \{3i+2/i \in \{0, 1, \ldots k-1\}\}$
- $I = \{0, 3, 6, \ldots, 3k\} = \{3i/i \in \{0, 1, \ldots k\}\}$
- $J = \{1, 4, 7, \ldots, 3k+1\} = \{3i+1/i \in \{0, 1, \ldots k\}\}$
- $K = \{0, 2, 5, 8 \ldots, 3k-1\} = \{0\} \cup \{3i+2/i \in \{0, 1, \ldots k-1\}\}$.

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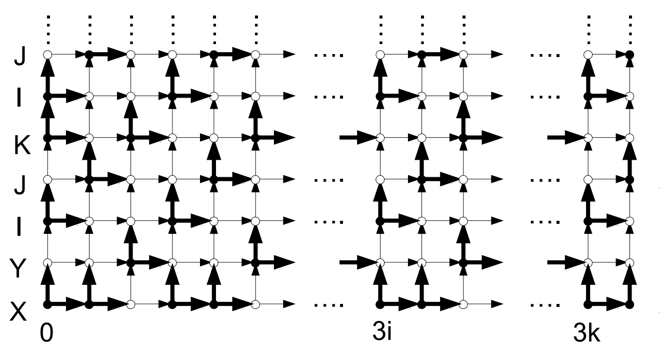

Figure 3: The dominating set F_n

Let F_n (see Figure 3) be the set of vertices of $P_{3k+2} \square P_n$ consisting of the vertices

- $(a, 0)$ for $a \in X$
- $(a, 1)$ for $a \in Y$
- (a, b) for $b \equiv 2 \, mod \, 3$ $(2 \leq b < n)$ and $a \in I$
- (a, b) for $b \equiv 0 \, mod \, 3$ $(3 \leq b < n)$ and $a \in J$
- (a, b) for $b \equiv 1 \, mod \, 3$ $(4 \leq b < n)$ and $a \in K$.

Lemma 5 *For any $k \geq 0$, $n \geq 2$, the set F_n is a dominating set of $P_{3k+2} \square P_n$ and $|F_n| = k(n+1) + n$.*

Proof : It is immediate to verify that

- All vertices of P_{3k+2} are dominated by the vertices of X
- The vertices of P_{3k+2} not dominated by some of Y are $\{0, 1, 4, \ldots, 3k+1\} \subset X$
- The vertices of P_{3k+2} not dominated by some of I are $\{2, 5, \ldots, 3k-1\} = Y \subset K$
- The vertices of P_{3k+2} not dominated by some of J are $\{0, 3, \ldots, 3k\}$=I
- The vertices of P_{3k+2} not dominated by some of K are $\{4, 7, \ldots, 3k+1\} \subset J$.

Therefore any vertex of some P^i_{3k+2} is dominated by a vertex in $P^i_{3k+2} \cap F_n$ or in $P^{i-1}_{3k+2} \cap F_n$ (if $i \geq 1$). Furthermore $|X| = 2k+2$, $|Y| = k$ and $|I| = |J| = |K| = k+1$, thus $|F_n| = k(n+1) + n$. $\square$

3 Optimality of the three sets

The structure of $P_m \square P_n$ implies the following strong property.

Proposition 6 *Let S be a dominating set of $P_m \square P_n$. For any $n' \leq n$ consider*

$$S_{n'} = \bigcup_{i=0,..,n'-1} P^i_m \cap S.$$

Then $S_{n'}$ is a dominating set of $P_m \square P_{n'}$.

Notice that the three sets D_n, E_n, F_n satisfy, for example, $(D_n)_{n'}$= $D_{n'}$ therefore we can use the same notation without ambiguity.

If S is a dominating set of $P_m \square P_n$, for any i in {0,1,...,n-1} let $s_i = |P^i_m \cap S|$. We have thus $|S| = \sum_{i=0}^{n-1} s_i$.

Proposition 7 *Let S be a dominating set of $P_m \square P_n$. Let $i \in \{1, 2, \ldots, n-1\}$ then $s_{i-1} + 2s_i \geq m$.*

Proof : Any vertex of P^i_m must be dominated by some vertex of $P^i_m \cap S$ or of $P^{i-1}_m \cap S$. A vertex in $P^i_m \cap S$ dominates at most two vertices of P^i_m and a vertex in $P^{i-1}_m \cap S$ dominates a unique vertex of P^i_m. $\square$

Lemma 8 *Let $k \geq 0$ and $n \geq 2$, $n \neq 3$, then $\gamma(P_{3k+2} \square P_n) = k(n+1) + n$.*

Proof : The case $n = 2$ is immediate by Theorem 1.
Let S be a dominating set of $P_{3k+2} \square P_n$ with $n \geq 4$.

By Proposition 7, $s_i \leq k$ implies $s_{i-1} + s_i \geq m - s_i \geq 2k+2$. Therefore for any $i \in \{2, \ldots, n-1\}$ we get $s_i \geq k+1$ or $s_{i-1} + s_i \geq 2(k+1)$.
Apply the following algorithm:

$I := \emptyset$; $J := \emptyset$; $i := n-1$;
while $i \geq 5$ **do**
  if $s_i \geq k+1$ **then**
    $I := I \cup \{i\}$; $i := i-1$
  else
    $J := J \cup \{i, i-1\}$; $i := i-2$
  end if
end while

If $n = 4$ or $n = 5$ the algorithm only sets I and J to $\emptyset$. In the general case, the algorithm stop when $i = 3$ or $i = 4$ and we get two disjoint sets I, J with $\{0, 1, \ldots, n-1\} = \{0, 1, 2, 3\} \cup I \cup J$ or $\{0, 1, \ldots, n-1\} = \{0, 1, 2, 3, 4\} \cup I \cup J$. Furthermore $\sum_{i \in I} s_i \geq |I|(k+1)$ and $\sum_{i \in J} s_i \geq |J|(k+1)$. We have thus one of the two inequalities

$$|S| - (s_0 + s_1 + s_2 + s_3) \geq (n-4)(k+1)$$

or

$$|S| - (s_0 + s_1 + s_2 + s_3 + s_4) \geq (n-5)(k+1).$$

In the first case by Proposition 6 and Theorem 1 we get $s_0 + s_1 + s_2 + s_3 \geq \gamma(P_{3k+2} \Box P_4) = \gamma(P_4 \Box P_{3k+2}) = 3k + 2 + \lceil \frac{6k+4}{3} \rceil = 5k + 4$. Thus $|S| \geq (n+1)k + n$. In the second case we get $s_0 + s_1 + s_2 + s_3 + s_4 \geq \gamma(P_5 \Box P_{3k+2}) = 6k + 5$. Thus again $|S| \geq (n+1)k + n$.
Therefore for any $n \geq 4$ we have $\gamma(P_{3k+2} \Box P_n) \geq k(n+1) + n$ and the equality occurs by Lemma 5. $\square$
Notice that, by Theorem 1, $\gamma(P_{3k+2} \Box P_3) = 3k + 2 + \lceil \frac{3k+2}{4} \rceil \neq 4k + 3$ for $k \geq 1$.

Lemma 9 *Let $k \geq 1$ and $n \geq 2$, $n \neq 3$, then $\gamma(P_{3k+1} \Box P_n) = k(n+1) + \lceil \frac{2n-3}{3} \rceil$.*

Proof : Consider some fixed $k \geq 1$. Notice first that by Theorem 1, $\gamma(P_{3k+1} \Box P_2) = 3k + 1$, $\gamma(P_{3k+1} \Box P_4) = 5k + 2$ and $\gamma(P_{3k+1} \Box P_5) = 6k + 3$ thus the result is true for $n \leq 5$.
We knows, by Lemma 4, that for any $n \geq 2$ the set E_n is a dominating set of $P_{3k+1} \Box P_n$ and $|E_n| = (n+1)k + \lceil \frac{2n-3}{3} \rceil$.
We will prove now that E_n is a minimum dominating set .
If this is not true consider n minimum, $n \geq 2$, such that there exists a dominating set S of $P_{3k+1} \Box P_n$ with $|S| < |E_n|$. We knows that $n \geq 6$.
For $n' \leq n$ let $S_{n'} = \cup_{i=0,..,n'-1} P^i_{3k+1} \cap S$ and $s_{n'} = |P^{n'}_{3k+1} \cap S|$.

Case 1 $n = 3p$, $p \geq 2$.
Notice first that $|E_n| - |E_{n-1}| = k$ and $|E_n| - |E_{n-2}| = 2k + 1$. We have also by hypothesis $|S| \leq |E_n| - 1$. By minimality of n, E_{n-1} is minimum thus $|S_{n-1}| \geq |E_{n-1}|$. Therefore $s_{n-1} = |S| - |S_{n-1}| \leq |E_n| - 1 - |E_{n-1}| = k - 1$. On the other hand, by Proposition 7, $s_{n-2} + 2s_{n-1} \geq 3k+1$ thus $s_{n-2} + s_{n-1} \geq (3k+1) - (k-1) = 2k+2$. This implies $|S_{n-2}| \leq |S_n| - 2k - 2 \leq |E_n| - 2k - 3 < |E_n| - 2k - 1 = |E_{n-2}|$, thus E_{n-2} is not minimum in contradiction with n minimum.

Case 2 $n = 3p + 1$, $p \geq 2$.
In this case we have $|E_n| - |E_{n-1}| = k + 1$ and $|E_n| - |E_{n-2}| = 2k + 1$. We have also by hypothesis $|S| \leq |E_n| - 1$. By minimality of n, E_{n-1} is minimum thus $|S_{n-1}| \geq |E_{n-1}|$. Therefore $s_{n-1} = |S| - |S_{n-1}| \leq |E_n| - 1 - |E_{n-1}| = k$. On the other hand, by Proposition 7, $s_{n-2} + 2s_{n-1} \geq 3k + 1$ thus $s_{n-2} + s_{n-1} \geq 2k + 1$. This implies $|S_{n-2}| \leq |S_n| - 2k - 1 < |E_n| - 2k - 1 = |E_{n-2}|$, thus E_{n-2} is not minimum in contradiction with n minimum.

Case 3 $n = 3p + 2$, $p \geq 2$.
In this case, $|E_n| - |E_{n-2}| = 2k + 2$ and we cannot proceed like case 1 and case 2. Hopefully, by Lemma 8, $\gamma(P_{3k+1} \Box P_{3p+2}) = \gamma(P_{3p+2} \Box P_{3k+1}) = p(3k+2) + 3k + 1 = k(3p + 3) + 2p + 1$. Therefore, since $n + 1 = 3p + 3$ and $\lceil \frac{2n-3}{3} \rceil = 2p + 1$, E_n is minimum. $\Box$

Lemma 10 *Let $k \geq 2$ and $n \geq 2$, $n \neq 3$ then $\gamma(P_{3k} \Box P_n) = k(n + 1) + \lfloor \frac{n-2}{3} \rfloor$.*

Proof :
Case 1 $n = 3p + 1$, $p \geq 1$.
By Lemma 9, $\gamma(P_{3k} \Box P_{3p+1}) = \gamma(P_{3p+1} \Box P_{3k}) = p(3k+1) + 2k - 1 = k(3p+2) + p - 1$. We obtain the conclusion since $3p + 2 = n + 1$ and $\lfloor \frac{n-2}{3} \rfloor = p - 1$.

Case 2 $n = 3p + 2$, $p \geq 0$.
By Lemma 8, $\gamma(P_{3k} \Box P_{3p+2}) = \gamma(P_{3p+2} \Box P_{3k}) = p(3k + 1) + 3k = k(3p + 3) + p$. We obtain again the conclusion since $3p + 3 = n + 1$ and $\lfloor \frac{n-2}{3} \rfloor = p$.

Case 3 $n = 3p$, $p \geq 2$.
We knows, by Lemma 3, that the set D_n is a dominating set of $P_{3k} \Box P_n$ and $|D_n| = k(n + 1) + \lfloor \frac{n-2}{3} \rfloor$.

If D_n is not a minimum dominating set let S be a dominating set with $|S| < |D_n|$. For $n' \leq n$ let $S_{n'} = \cup_{i=0,..,n'-1} P^i_{3k} \cap S$ and $s_{n'} = |P^{n'}_{3k} \cap S|$.
Because $n = 3p$ and $p \geq 2$ we get $|D_n| - |D_{n-1}| = k$ and $|D_n| - |D_{n-2}| = 2k + 1$. We have also by hypothesis $|S| \leq |D_n| - 1$. Notice that, by Lemma 8, $\gamma(P_{3k} \Box P_{n-1}) = \gamma(P_{3p-1} \Box P_{3k}) = (p-1)(3k+1) + 3k = kn + \lfloor \frac{n-3}{3} \rfloor = |D_{n-1}|$ thus D_{n-1} is minimum and $|S_{n-1}| \geq |D_{n-1}|$.

Therefore $s_{n-1} = |S| - |S_{n-1}| \leq |D_n| - 1 - |D_{n-1}| = k - 1$. By Proposition 7, $s_{n-2} + 2s_{n-1} \geq 3k$ thus $s_{n-2} + s_{n-1} \geq 2k + 1$. This implies $|S_{n-2}| \leq |S| - 2k - 1 < |D_n| - 2k - 1 = |D_{n-2}|$. On the other hand, by Lemma 9, $\gamma(P_{3k} \Box P_{3p-2}) = \gamma(P_{3p-2} \Box P_{3k}) = (p-1)(3k+1) + 2k - 1 = k(n-1) + \lfloor \frac{n-4}{3} \rfloor = |D_{n-2}|$ thus D_{n-2} is minimum, a contradiction. $\Box$

Notice that, by Theorem 1, $\gamma(P_{3k} \Box P_3) = 3k + \lceil \frac{3k}{4} \rceil \neq 4k$ for $k \geq 3$.

4 Conclusions

Putting together Lemma 8, Lemma 9, Lemma 10 and the case $m = 3$ or $n = 3$, we obtain $\gamma(P_m \Box P_n)$ for any m, n (Theorem 2).
As a conclusion, notice that the minimum dominating sets we build for $P_5 \Box P_n$ and

$P_6\Box P_n$ are different than those proposed by Liu and al.([13]). An open problem would be to characterize all minimum dominating sets of $P_m\Box P_n$.

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