On the p-reinforcement and the complexity*

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Abstract

Let $G = (V, E)$ be a graph and p be a positive integer. A subset $S \subseteq V$ is called a p-dominating set if each vertex not in S has at least p neighbors in S. The p-domination number $\gamma_p(G)$ is the size of a smallest p-dominating set of G. The p-reinforcement number $r_p(G)$ is the smallest number of edges whose addition to G results in a graph G' with $\gamma_p(G') < \gamma_p(G)$. In this paper, we give an original study on the p-reinforcement, determine $r_p(G)$ for some graphs such as paths, cycles and complete t-partite graphs, and establish some upper bounds of $r_p(G)$. In particular, we show that the decision problem on $r_p(G)$ is NP-hard for a general graph G and a fixed integer $p \geq 2$.

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1 Induction

For notation and graph-theoretical terminology not defined here we follow [21]. Specifically, let $G = (V, E)$ be an undirected graph without loops and multi-edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set, where $E \neq \emptyset$.

For $x \in V$, the *open neighborhood*, the *closed neighborhood* and the *degree* of x are denoted by $N_G(x) = \{y \in V : xy \in E\}$, $N_G[x] = N_G(x) \cup \{x\}$ and $deg_G(x) = |N_G(x)|$, respectively. $\delta(G) = \min\{deg_G(x) : x \in V\}$ and $\Delta(G) = \max\{deg_G(x) : x \in V\}$ are the minimum degree and the maximum degree of G, respectively. For any $X \subseteq V$, let $N_G[X] = \cup_{x\in X} N_G[x]$.

For a subset $D \subseteq V$, let $\overline{D} = V \setminus D$. The notation G^c denotes the complement of G, that is , G^c is the graph with vertex-set $V(G)$ and edge-set $\{xy : xy \notin E(G)$ for any $x, y \in V(G)\}$. For $B \subseteq E(G^c)$, we use $G + B$ to denote the graph with vertex-set V and edge-set $E \cup B$. For convenience, we denote $G + \{xy\}$ by $G + xy$ for an $xy \in E(G^c)$.

A nonempty subset $D \subseteq V$ is called a *dominating set* of G if $|N_G(x) \cap D| \geq 1$ for each $x \in \overline{D}$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of all dominating sets in G. The domination is a classical concept in graph theory. The early literature on the domination with related topics is, in detail, surveyed in the two books by Haynes, Hedetniemi, and Slater [14, 15].

In 1985, Fink and Jacobson [12] introduced the concept of a generalization domination in a graph. Let p be a positive integer. A subset $D \subseteq V$ is a *p-dominating set* of G if $|N_G(x) \cap D| \geq p$ for each $x \in \overline{D}$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of all p-dominating sets in G. A p-dominating set with cardinality $\gamma_p(G)$ is called a γ_p-set of G. For $S, T \subseteq V$, the set S can p-dominate T in G if $|N_G(x) \cap S| \geq p$ for every $x \in T \setminus S$. Clearly, the 1-dominating set is the classical dominating set, and so $\gamma_1(G) = \gamma(G)$. The p-domination is investigated by many authors (see, for example, [1, 2, 4, 5, 11]). Very recently, Chellali *et al.*[4] have given an excellent survey on this topics. The following are two simple observations.

Observation 1.1 *If G is a graph with $|V(G)| \geq p$, then $\gamma_p(G) \geq p$.*

Observation 1.2 *Every p-dominating set of a graph contains all vertices of degree at most $p - 1$.*

Clearly, addition of some extra edges to a graph could result in decrease of its domination number. In 1990, Kok and Mynhardt [19] first investigated this problem and proposed the concept of the reinforcement number. The *reinforcement number* $r(G)$ of a graph G is defined as the smallest number of edges whose addition to G results in a graph G' with $\gamma(G') < \gamma(G)$. By convention $r(G) = 0$ if $\gamma(G) = 1$.

The reinforcement number has received much research attention (see, for example, [3, 7, 18]), and its many variations have also been well described and studied in graph theory, including total reinforcement [16, 20], independence reinforcement [22], fractional reinforcement [6, 8] and so on. In particular, Blair *et al.* [3], Hu and Xu [17], independently, showed that the problem determining $r(G)$ for a general graph G is NP-hard.

Motivated by the work of Kok and Mynhardt [19], in this paper, we introduce the p-reinforcement number, which is a natural extension of the reinforcement number. The *p-reinforcement number* $r_p(G)$ of a graph G is the smallest number of edges of G^c that have to be added to G in order to reduce $\gamma_p(G)$, that is

$$r_p(G) = \min\{|B| : B \subseteq E(G^c) \text{ with } \gamma_p(G+B) < \gamma_p(G)\}.$$

It is clear that $r_1(G) = r(G)$. By Observation 1.1, we can also make a convention, $r_p(G) = 0$ if $\gamma_p(G) \leq p$. Thus $r_p(G)$ is well-defined for any graph G and integer $p \geq 1$. In this paper, we always assume $\gamma_p(G) > p$ when we consider the p-reinforcement number for a graph G.

The rest of this paper is organized as follows. In Section 2 we present an equivalent parameter for calculating the p-reinforcement number of a graph. As its applications, we determine the values of the p-reinforcement numbers for special classes of graphs such as paths, cycles and complete t-partite graphs in Sections 3, and show that the decision problem on p-reinforcement is NP-hard for a general graph and a fixed integer $p \geq 2$ in Section 4. Finally, we establish some upper bounds for the p-reinforcement number of a graph G by terms of other parameters of G in Section 5.

2 Preliminary

Let G be a graph with $\gamma(G) > 1$ and $B \subseteq E(G^c)$ with $|B| = r(G)$ such that $\gamma(G+B) < \gamma(G)$. Let X be a γ-set of $G+B$. Then $|B| \geq |V(G) \setminus N_G[X]|$. On the other hand, given any set $X \subseteq V(G)$, we can always choose a subset $B \subseteq E(G^c)$ with $|B| = |V(G) \setminus N_G[X]|$ such that X dominates $G+B$. It is a simple observation that, to calculate $r(G)$, Kok and Mynhardt [19] proposed the following parameter

$$\eta(G) = \min\{|V(G) \setminus N_G[X]| : \ X \subseteq V(G), |X| < \gamma(G)\}, \tag{2.1}$$

and showed $r(G) = \eta(G)$. We can refine this technique to deal with the p-reinforcement number $r_p(G)$.

Let G be a graph with $\gamma_p(G) > p$. For any $X \subseteq V(G)$, let

$$X^* = \{x \in \overline{X} : |N_G(x) \cap X| < p\}. \tag{2.2}$$

Let $B \subseteq E(G^c)$ with $|B| = r_p(G)$ such that $\gamma_p(G+B) < \gamma_p(G)$, and let X be a γ_p-set of $G+B$. Then

$$|B| \geq \sum_{x \in X^*} (p - |N_G(x) \cap X|).$$

On the other hand, given any set $X \subseteq V(G)$ with $|X| \geq p$, we can always choose a subset $B \subseteq E(G^c)$ with

$$|B| = \sum_{x \in X^*} (p - |N_G(x) \cap X|)$$

such that X can p-dominate $G + B$. Motivated by this observation, we introduce the following notations. For a subset $X \subseteq V(G)$,

$$\begin{aligned}
\eta_p(x, X, G) &= \begin{cases} p - |N_G(x) \cap X| & \text{if } x \in X^* \\ 0 & \text{otherwise} \end{cases} \quad \text{for } x \in V(G), && (2.3)\\
\eta_p(S, X, G) &= \sum_{x \in S} \eta_p(x, X, G) \quad \text{for } S \subseteq V(G), \text{ and} && (2.4)\\
\eta_p(G) &= \min\{\eta_p(V(G), X, G) : |X| < \gamma_p(G)\}. && (2.5)
\end{aligned}$$

A subset $X \subseteq V(G)$ is called an *η_p-set* of G if $\eta_p(G) = \eta_p(V(G), X, G)$. Clearly, for any two subsets $S', S \subseteq V(G)$ and two subsets $X', X \subseteq V(G)$,

$$\begin{array}{ll}
\eta_p(S', X, G) \leq \eta_p(S, X, G) & \text{if } S' \subseteq S, \\
\eta_p(S, X, G) \leq \eta_p(S, X', G) & \text{if } |X'| \leq |X|.
\end{array}$$

Thus, we have the following simple observation.

Observation 2.1 *If X is an η_p-set of a graph G, then $|X| = \gamma_p(G) - 1$.*

The following result shows that computing $r_p(G)$ can be referred to computing $\eta_p(G)$ for a graph G with $\gamma_p(G) \geq p + 1$.

Theorem 2.2 *For any graph G and positive integer p, $r_p(G) = \eta_p(G)$ if $\gamma_p(G) > p$.*

Proof. Let X be an η_p-set of G. Then $|X| = \gamma_p(G) - 1$ by Observation 2.1. Let $Y = \{y \in V(G) : \eta_p(y, X, G) > 0\}$. Then $Y = X^*$ is contained in $\overline{X}$, where X^* is defined in (2.2). Thus, $\eta_p(G) = \eta_p(X^*, X, G)$. We construct a new graph G' from G, for each $y \in X^*$, by adding $\eta_p(y, X, G)$ edges of G^c to G joining y to $\eta_p(y, X, G)$ vertices in X. Clearly, X is a p-dominating set of G', that is, $\gamma_p(G') \leq |X|$. Let $B = E(G') - E(G)$. Then

$$\gamma_p(G) = |X| + 1 > |X| \geq \gamma_p(G') = \gamma_p(G + B),$$

which implies $r_p(G) \leq |B|$. It follows that

$$r_p(G) \leq |B| = \sum_{y \in X^*} \eta_p(y, X, G) = \eta_p(X^*, X, G) = \eta_p(G). \tag{2.6}$$

On the other hand, let B be a subset of $E(G^c)$ such that $|B| = r_p(G)$ and $\gamma_p(G + B) = \gamma_p(G) - 1$. Let $G' = G + B$ and X' be a γ_p-set of G'. For every $xy \in B$, X' cannot p-dominate the graph $G' - xy$ by the minimality of B. This fact means that only one of x and y is in X'. Without loss of generality, assume $y \in \overline{X'}$. Since X' cannot p-dominate y in $G'-xy$ and so in G, $|N_G(y) \cap X'| < p$. Let Z be all end-vertices of edges in B and $Y = \overline{X'} \cap Z$. Since X' is a γ_p-set of G', $|N_{G'}(u) \cap X'| \geq p$ for any $u \in \overline{X'}$. In other words, any $u \in \overline{X'}$ with $|N_G(u) \cap X'| < p$ must be in Y. It follows that

$$\sum_{u \in \overline{X'}} \eta_p(u, X', G) = \sum_{y \in Y} (p - |N_G(y) \cap X'|) = |B|. \tag{2.7}$$

By (2.7), we immediately have that

$$\eta_p(G) \leq \eta_p(V(G), X', G) = \sum_{u \in \overline{X'}} \eta_p(u, X', G) = |B| = r_p(G).$$

Combining this with (2.6), we obtain $r_p(G) = \eta_p(G)$, and so the theorem follows. ∎

Note that when $p = 1$, X^* defined in (2.2) is $V(G) \setminus N_G[X]$. This fact means that $\eta(G)$ defined in (2.1) is a special case of $p = 1$ in (2.5), that is, $\eta_1(G) = \eta(G)$. Thus, the following corollary holds immediately.

Corollary 2.1 (Kok and Mynhardt [19]) *$r(G) = \eta(G)$ if $\gamma(G) > 1$.*

Using Observation 1.2 and Theorem 2.2, the following corollary is obvious.

Corollary 2.2 *Let $p \geq 1$ be an integer and G be a graph with $\gamma_p(G) > p$. If $\Delta(G) < p$, then*

$$r_p(G) = p - \Delta(G).$$

3 Some Exact Values

In this section we will use Theorem 2.2 to calculate the p-reinforcement numbers for some classes of graphs.

We first determine the p-reinforcement numbers for paths and cycles. Let P_n and C_n denote, respectively, a path and a cycle with n vertices. When $p = 1$, Kok and Mynhardt [19] proved that $r(P_n) = r(C_n) = i$ if $n = 3k+i \geq 4$, where $i \in \{1, 2, 3\}$. We will give the exact values of $r_p(P_n)$ and $r_p(C_n)$ for $p \geq 2$. The following observation is simple but useful.

Observation 3.1 *For integer $p \geq 2$,*

$$\gamma_p(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1 & \text{if } p = 2 \\ n & \text{if } p \geq 3 \end{cases} \quad \text{and} \quad \gamma_p(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } p = 2 \\ n & \text{if } p \geq 3. \end{cases}$$

Theorem 3.2 *Let $p \geq 2$ be an integer. If $\gamma_p(P_n) > p$ then*

$$r_p(P_n) = \begin{cases} 2 & \text{if } p = 2 \text{ and } n \text{ is odd} \\ 1 & \text{if } p = 2 \text{ and } n \text{ is even} \\ p - 2 & \text{if } p \geq 3. \end{cases}$$

Proof. Let $P_n = x_1x_2\cdots x_n$ and X be an η_p-set of P_n. By Theorem 2.2 and $\gamma_p(P_n) > p$, $r_p(P_n) = \eta_p(P_n) = \eta_p(V(P_n), X, P_n) \geq 1$. For $p \geq 3$, it is easy to see that $r_p(P_n) = p-2$ by Corollary 2.2. Assume that $p = 2$ below.

If n is even, then by Observation 3.1, $\gamma_2(P_n) - \gamma_2(C_n) = 1$, which implies that $r_2(P_n) \leq 1$. Furthermore, $r_2(P_n) = 1$.

If n is odd, then $\gamma_2(P_n) = \frac{n+1}{2}$ by Observation 3.1, and so $n \geq 5$ since $\gamma_2(P_n) > 2$. Let

$$X' = \bigcup_{i=1}^{\frac{n-1}{2}} \{x_{2i}\}.$$

Clearly, $|X'| = \frac{n-1}{2} = \gamma_2(P_n) - 1$. So

$$\eta_2(V(P_n), X, P_n) \leq \eta_2(V(P_n), X', P_n) = \eta_2(x_1, X', P_n) + \eta_2(x_n, X', P_n) = 2.$$

Suppose that $\eta_2(V(P_n), X, P_n) = 1$. Then X can 2-dominate either $V(P_n) \setminus \{x_1\}$ or $V(P_n) \setminus \{x_n\}$. In both cases, we have

$$|X| \geq \gamma_2(P_{n-1}) = \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n-1}{2} + 1,$$

which contradicts with $|X| = \frac{n-1}{2}$. Hence $r_2(P_n) = \eta_2(V(P_n), X, P_n) = 2$. ∎

Theorem 3.3 *Let $p \geq 2$ be an integer. If $\gamma_p(C_n) > p$ then*

$$r_p(C_n) = \begin{cases} 2 & \text{if } p = 2 \text{ and } n \text{ is odd} \\ 4 & \text{if } p = 2 \text{ and } n \text{ is even} \\ p-2 & \text{if } p \geq 3. \end{cases}$$

Proof. Let $C_n = x_1x_2\cdots x_nx_1$. If $p \geq 3$ then the result holds obviously by Corollary 2.2. In the following, we only need to calculate the values of $r_p(C_n)$ for $p = 2$. Let X be an η_2-set of C_n. Then $r_2(C_n) = \eta_2(C_n) = \eta_2(V(C_n), X, C_n)$ by Theorem 2.2. Note that $n \geq 5$ since $\gamma_2(C_n) = \lceil \frac{n}{2} \rceil > 2$.

If n is odd, then let

$$X' = \bigcup_{i=1}^{\frac{n-1}{2}} \{x_{2i-1}\}.$$

Clearly, $|X'| = \frac{n-1}{2} = \gamma_2(C_n) - 1$ by Observation 3.1, and $\eta_2(V(C_n), X', C_n) = \eta_2(x_{n-1}, X', C_n) + \eta_2(x_n, X', C_n) = 2$. So

$$r_2(C_n) = \eta_2(V(C_n), X, C_n) \leq \eta_2(V(C_n), X', C_n) = 2.$$

Since X is not a 2-dominating set of C_n, there must be two adjacent vertices, denoted by x_i and x_{i+1}, of C_n not in X. This fact means that $\eta_2(x_i, X, C_n) \geq 1$ and $\eta_2(x_{i+1}, X, C_n) \geq 1$. So

$$r_2(C_n) = \eta_2(V(C_n), X, P_n) \geq \eta_2(x_i, X, C_n) + \eta_2(x_{i+1}, X, C_n) \geq 2.$$

Hence $r_2(C_n) = 2$.

If n is even, then $n \geq 6$. Deleting X and all vertices 2-dominated by X from C_n, we can obtain a result graph, denoted by H, each of whose components is a path with length at least 2. Denote all components of H by $H_1, \cdots, H_h$, where $h \geq 1$. In the case that $h = 1$ and the length of H_1 is equal to one, X can 2-dominate a subgraph of C_n that is isomorphic to P_{n-2}. By Observation 3.1,

$$|X| \geq \gamma_2(P_{n-2}) = \lfloor \frac{n-2}{2} \rfloor + 1 = \frac{n}{2},$$

which contradicts that $|X| = \gamma_2(C_n) - 1 = \lceil \frac{n}{2} \rceil - 1 = \frac{n}{2} - 1$. In other cases, we can find that

$$r_2(C_n) = \eta_2(V(C_n), X, C_n) \geq 4.$$

Let

$$X'' = \bigcup_{i=1}^{\frac{n}{2}-1} \{x_{2i-1}\}.$$

It is easy to check that $|X''| = \frac{n}{2} - 1 = \gamma_2(C_n) - 1$ and $\eta_2(V(C_n), X'', C_n) = 4$. So

$$r_2(C_n) = \eta_2(V(C_n), X, C_n) \leq \eta_2(V(C_n), X'', C_n) = 4.$$

Hence $r_2(C_n) = 4$ and so the theorem is true. ▮

Next we consider the p-reinforcement number for a complete t-partite graph $K_{n_1,\cdots,n_t}$. To state our results, we need some symbols. For any subset $X = \{n_{i_1}, \cdots, n_{i_r}\}$ of $\{n_1, \cdots, n_t\}$, define

$$|X| = r \quad \text{and} \quad f(X) = \sum_{j=1}^{r} n_{i_j}.$$

For convenience, let $|X| = 0$ and $f(X) = 0$ if $X = \emptyset$. let

$$\mathscr{X} = \{X : X \text{ is a subset of } \{n_1, \cdots, n_t\} \text{ with } f(X) \geq \gamma_p(G)\}$$

and, for every $X \in \mathscr{X}$, define

$$f^*(X) = \max\{f(Y) : Y \text{ is a subset of } X \text{ with } |Y| = |X| - 1 \text{ and } f(Y) < p\}.$$

Theorem 3.4 *For any integer $p \geq 1$ and a complete t-partite graph $G = K_{n_1,\cdots,n_t}$ with $t \geq 2$ and $\gamma_p(G) > p$,*

$$r_p(G) = \min\{(p - f^*(X))(f(X) - \gamma_p(G) + 1) : X \in \mathscr{X}\}.$$

Proof. Let $N = \{n_1, \cdots, n_t\}$ and $V(G) = V_1 \cup \cdots \cup V_t$ be the vertex-set of G such that $|V_i| = n_i$ for each $i = 1, \cdots, t$. Let

$$m = \min\{(p - f^*(X))(f(X) - \gamma_p(G) + 1) : X \in \mathscr{X}\}.$$

We first prove that $r_p(G) \le m$. Let $X \subseteq \mathscr{X}$ (without loss of generality, assume $X = \{n_1, \cdots, n_k, n_{k+1}\}$ for some $0 \le k \le t-1$) such that

$$f^*(X) = n_1 + \cdots + n_k \text{ and } (p - f^*(X))(f(X) - \gamma_p(G) + 1) = m.$$

By $X \subseteq \mathscr{X}$, we know that $n_{k+1} = f(X) - f^*(X) \ge \gamma_p(G) - f^*(X)$. So we can pick a vertex-subset V'_{k+1} from V_{k+1} such that $|V'_{k+1}| = \gamma_p(G) - f^*(X) - 1$. Let

$$D = V_1 \cup \cdots \cup V_k \cup V'_{k+1}.$$

Clearly, $|D| = \gamma_p(G) - 1$. Since $\gamma_p(G) > p$, $|D| \ge p$ and so D can p-dominate $\cup_{i=k+2}^t V_i$. Hence by the definition of $\eta_p(V(G), D, G)$,

$$\begin{aligned}
\eta_p(V(G), D, G) &= \eta_p(V(G) \setminus D, D, G) \\
&= \sum_{v \in V_{k+1} \setminus V'_{k+1}} \eta_p(v, D, G) + \sum_{i=k+2}^{t} \eta_p(V_i, D, G) \\
&= |V_{k+1} \setminus V'_{k+1}|(p - f^*(X)) + 0 \\
&= (p - f^*(X))[n_{k+1} - (\gamma_p(G) - f^*(X) - 1)] \\
&= (p - f^*(X))(f(X) - \gamma_p(G) + 1) \\
&= m.
\end{aligned}$$

By Theorem 2.2, we have $r_p(G) = \eta_p(G) \le \eta_p(V(G), D, G) = m$.

On the other hand, we will show that $r_p(G) \ge m$. For any subset M of N, we use $I(M)$ to denote the subindex-sets of all elements in M, that is,

$$I(M) = \{i : n_i \in M\}.$$

Let S be an η_p-set of G and let

$$\begin{aligned}
Y &= \{n_i : \quad\ |V_i \cap S| = |V_i| \text{ for } 1 \le i \le t\}, \text{ and} \\
A &= \{n_i : 0 < |V_i \cap S| < |V_i| \text{ for } 1 \le i \le t\}.
\end{aligned}$$

Thus

$$f(Y \cup A) = f(Y) + f(A) = \sum_{i \in I(Y)} |V_i| + \sum_{i \in I(A)} |V_i| \ge |S| = \gamma_p(G) - 1 \tag{3.1}$$

by Observation 2.1. Since $\cup_{i \in I(Y)} V_i$ $(\subseteq S)$ cannot p-dominate G,

$$f(Y) = \sum_{i \in I(Y)} n_i = |\cup_{i \in I(Y)} V_i| < p. \tag{3.2}$$

Hence, by (3.1) and $\gamma_p(G) > p$,

$$f(A) \ge \gamma_p(G) - 1 - f(Y) > \gamma_p(G) - p - 1 \ge 0,$$

which implies that $|A| \ge 1$.

Claim. $|A| = 1$.

Proof of Claim. Suppose that $|A| \geq 2$. Then we can choose i and j from $I(A)$ such that $i \neq j$. By the definition of A, we have $0 < |V_i \cap S| < |V_i|$ and $0 < |V_j \cap S| < |V_j|$. Therefore, we can pick two vertices x and y from $V_i \cap S$ and $V_j \setminus S$, respectively. Let

$$S' = (S \setminus \{x\}) \cup \{y\}.$$

Obviously, $|S'| = |S| = \gamma_p(G) - 1$, $|V_i \cap S'| = |V_i \cap S| - 1$ and $|V_j \cap S'| = |V_j \cap S| + 1$.

Note that G is a complete t-partite graph. For any $v \in V(G)$, we can easily find the value of $\eta_p(v, S', G) - \eta_p(v, S, G)$ by the definitions of $\eta_p(v, S', G)$ and $\eta_p(v, S, G)$ as follows:

$$\eta_p(v, S', G) - \eta_p(v, S, G) = \begin{cases} (p - |S| + |V_i \cap S| - 1) - 0 & \text{if } v = x \\ -1 & \text{if } v \in V_i \setminus S \\ 0 - (p - |S| + |V_j \cap S|) & \text{if } v = y \\ 1 & \text{if } v \in (V_j \setminus S) \setminus \{y\} \\ 0 & \text{otherwise.} \end{cases}$$

Since S is an η_p-set of G and $|S'| = |S|$, we have

$$\begin{aligned} 0 &\leq \eta_p(V(G), S', G) - \eta_p(V(G), S, G) \\ &= \sum_{v \in V(G)} (\eta_p(v, S', G) - \eta_p(v, S, G)) \\ &= (p - |S| + |V_i \cap S| - 1) - |V_i \setminus S| - (p - |S| + |V_j \cap S|) + |(V_j \setminus S) \setminus \{y\}| \\ &= (|V_i \cap S| - |V_i \setminus S|) - (|V_j \cap S| - |V_j \setminus S|) - 2. \end{aligned}$$

This means that

$$(|V_i \cap S| - |V_i \setminus S|) \geq (|V_j \cap S| - |V_j \setminus S|) + 2.$$

However, by the symmetry of V_i and V_j, we can also obtain

$$(|V_j \cap S| - |V_j \setminus S|) \geq (|V_i \cap S| - |V_i \setminus S|) + 2$$

by applying the similar discussion. This is a contradiction, and so the claim holds. $\square$

By **Claim**, we can assume that $I(A) = \{h\}$. From the definitions of Y and A, we have $|Y \cup A| = |Y| + 1$ and

$$f(Y \cup A) = \sum_{i \in I(Y)} |V_i| + |V_h| \geq \sum_{i \in I(Y)} |V_i| + (|V_h \cap S| + 1) = |S| + 1 = \gamma_p(G).$$

It follows that $Y \cup A \in \mathscr{X}$. Thus, by (3.2) and the definition of $f^*(Y \cup A)$, we have $f(Y) \leq f^*(Y \cup A)$. Since $\gamma_p(G) > p$, $|S| = \gamma_p(G) - 1 \geq p$, and so S p-dominates

$V(G) \setminus (\cup_{i \in I(Y \cup A)} V_i)$. Therefore, by Theorem 2.2,

$$\begin{aligned}
r_p(G) = \eta_p(G) = \eta_p(V(G), S, G) &= \eta_p(V(G) \setminus S, S, G) \\
&= \sum_{v \in V_h \setminus S} \eta_p(v, S, G) \\
&= (p - f(Y))|V_h \setminus S| \\
&= (p - f(Y))[|V_h| - (|S| - f(Y))] \\
&= (p - f(Y))(f(Y \cup A) - \gamma_p(G) + 1) \\
&\geq (p - f^*(Y \cup A))(f(Y \cup A) - \gamma_p(G) + 1) \\
&\geq m.
\end{aligned}$$

This completes the proof of the theorem. ∎

For example, let $G = K_{2,2,10,17}$ and $p = 11$. Then $\gamma_{11}(G) = 12$, and so

$$\mathscr{X} = \{\{17\}, \{2,10\}, \{2,17\}, \{10,17\}, \{2,2,10\}, \{2,2,17\}, \{2,10,17\}, \{2,2,10,17\}\}.$$

By Theorem 3.4, for any $X \in \mathscr{X}$, we have that

$$f^*(X) = \begin{cases} 0 & \text{if } X = \{17\}, \{2,10,17\} \text{ or } \{2,2,10,17\}; \\ 2 & \text{if } X = \{2,17\}; \\ 4 & \text{if } X = \{2,2,10\} \text{ or } \{2,2,17\}; \\ 10 & \text{if } X = \{2,10\} \text{ or } \{10,17\}. \end{cases}$$

Hence

$$\begin{aligned}
r_{11}(G) &= \min\{(11 - f^*(X))(f(X) - \gamma_{11}(G) + 1) : X \in \mathscr{X}\} \\
&= \min\{(11 - f^*(X))(f(X) - 11) : X \in \mathscr{X}\} \\
&= (11 - f^*(\{2,10\}))(f(\{2,10\}) - 11) \\
&= 1.
\end{aligned}$$

4 Complexity

Blair et al. [3], Hu and Xu [17], independently, showed that the 1-reinforcement problem is NP-hard. Thus, for any positive integer p, the p-reinforcement problem is also NP-hard since the 1-reinforcement is a sub-problem of the p-reinforcement problem.

For each fixed p, p-dominating set is polynomial-time computable (see Downey and Fellows [9, 10] for definitions and discussion). However, the p-reinforcement number problem is hard even for specific values of the parameters. In this section, we will consider the following decision problem.

p-Reinforcement

Instance: A graph G, p (≥ 2) is a fixed integer.

Question: Is $r_p(G) \leq 1$?

We will prove that p-**Reinforcement** ($p \geq 2$) is also NP-hard by describing a polynomial transformation from the following NP-hard problem (see [13]).

3-Satisfiability (3SAT)

Instance: A set $U = \{u_1, \ldots, u_n\}$ of variables and a collection $\mathscr{C} = \{C_1, \ldots, C_m\}$ of clauses over U such that $|C_i| = 3$ for $i = 1, 2, \ldots, m$. Furthermore, every literal is used in at least one clause.

Question: Is there a satisfying truth assignment for C?

Theorem 4.1 *For a fixed integer $p \geq 2$, p-**Reinforcement** is NP-hard.*

Proof. Let $U = \{u_1, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, \ldots, C_m\}$ be an arbitrary instance I of **3SAT**. We will show the NP-hardness of p-**Reinforcement** by reducing **3SAT** to it in polynomial time. To this aim, we construct a graph G as follows:

a. For each variable $u_i \in U$, associate a graph H_i, where H_i can be obtained from a complete graph K_{2p+2} with vertex-set $\{u_i, \overline{u}_i\} \cup (\cup_{j=1}^{p}\{v_{i_j}, \overline{v}_{i_j}\})$ by deleting the edge-subset $\cup_{j=1}^{p-1}\{u_i\overline{v}_{i_j}, \overline{u}_i v_{i_j}\}$;

b. For each clause $C_j \in \mathscr{C}$, create a single vertex c_j and join c_j to the vertex u_i (resp. $\overline{u}_i$) in H_i if and only if the literal u_i (resp. $\overline{u}_i$) appears in clause C_j for any $i \in \{1, \ldots, n\}$;

c. Add a complete graph T ($\cong K_p$) and join all of its vertices to each c_j.

For convenience, let $X_i = \cup_{j=1}^{p}\{v_{i_j}\}$ and $\overline{X}_i = \cup_{j=1}^{p}\{\overline{v}_{i_j}\}$. Then $V(H_i) = \{u_i, \overline{u}_i\} \cup X_i \cup \overline{X}_i$. Use H_0 to denote the induced subgraph by $\{c_1, \cdots, c_m\} \cup V(T)$.

It is clear that the construction of G can be accomplished in polynomial time. To complete the proof of the theorem, we only need to prove that $\mathscr{C}$ is satisfiable if and only if $r_p(G) = 1$. We first prove the following two claims.

Claim 1. *Let D be a γ_p-set of G. Then $|D| = p(n+1)$, moreover, $|V(H_i) \cap D| = p$ and $|\{u_i, \overline{u}_i\} \cap D| \leq 1$ for each $i \in \{1, 2, \ldots, n\}$.*

Proof of Claim 1. Suppose there is some $i \in \{1, 2, \cdots, n\}$ such that $|V(H_i) \cap D| < p$. Then there must be a vertex, say x, of $V(H_i) \setminus D$ such that $N_G(x) \subseteq V(H_i)$. And so $|N_G(x) \cap D| \leq |V(H_i) \cap D| < p$, which contradicts that D is a γ_p-set of G. Thus $|V(H_i) \cap D| \geq p$ for each $i \in \{0, 1, \cdots, n\}$, and so

$$\gamma_p(G) = |D| = \sum_{i=0}^{n} |V(H_i) \cap D| \geq p(n+1). \tag{4.1}$$

On the other hand, let

$$D' = \bigcup_{i=1}^{n}[(X_i - \{v_{i_p}\}) \cup \{\overline{u}_i\}] \cup V(T).$$

Clearly, $|D'| = p(n+1)$ and D' is a p-dominating set of G. Hence by (4.1),

$$p(n+1) \leq \sum_{i=0}^{n} |V(H_i) \cap D| = \gamma_p(G) \leq |D'| = p(n+1),$$

which implies that $\gamma_p(G) = p(n+1)$ and $|V(H_i) \cap D| = p$ for each $0 \leq i \leq n$. Furthermore, if $|\{u_i, \overline{u}_i\} \cap D| = 2$ then $|(X_i \cup \overline{X}_i) \cap D| = p - 2$. So we can choose a vertex from $X_i \cup \overline{X}_i$ that is not p-dominated by D. This is impossible since D is a γ_p-set of G, and so $|\{u_i, \overline{u}_i\} \cap D| \leq 1$. The claim holds. $\square$

Claim 2. *If there is an edge $e = xy \in G^c$ such that $\gamma_p(G+e) < \gamma_p(G)$, then any γ_p-set D_e of $G+e$ satisfies the following properties.*

- *(i)* $|V(H_i) \cap D_e| = p$ *and* $|\{u_i, \overline{u}_i\} \cap D_e| \leq 1$ *for each* $i \in \{1, \cdots, n\}$;
- *(ii)* $\{c_1, \cdots, c_m\} \cap D_e = \emptyset$, *and so* $|V(T) \cap D_e| = p - 1$;
- *(iii) One of x and y belongs to $V(T) \setminus D_e$ and the other belongs to $H \cap D_e$, where $H = \cup_{i=1}^{n} V(H_i)$.*

Proof of Claim 2. Because D_e is a γ_p-set of $G+e$ and $\gamma_p(G+e) < \gamma_p(G)$, one of x and y is not in D_e but the other is in D_e. Without loss of generality, say $x \notin D_e$ and $y \in D_e$. It is clear that $|N_G(x) \cap D_e| = p - 1$. Since vertex x is the unique vertex not be p-dominated by D_e, we have

$$\eta_p(V(G), D_e, G) = \eta_p(x, D_e, G) = p - (p-1) = 1. \tag{4.2}$$

Let

$$D = D_e \cup \{x\}.$$

Then D is a p-dominating set of G and $|D| = |D_e| + 1 = \gamma_p(G+e) + 1 \leq \gamma_p(G)$. That is, D is a γ_p-set of G. By Claim 1,

$$|V(H_i) \cap D| = p \text{ for each } i = 0, 1, \cdots, n, \tag{4.3}$$

and $|\{u_i, \overline{u}_i\} \cap D_e| \leq |\{u_i, \overline{u}_i\} \cap D| \leq 1$ for $1 \leq i \leq n$.

Suppose that there exists some $i \in \{1, \cdots, n\}$ such that $|V(H_i) \cap D_e| \neq p$. Then by (4.3), $x \in V(H_i)$ and $|V(H_i) \cap D_e| = p - 1$. Thus every vertex in $(X_i \cup \overline{X}_i) \setminus (D_e \cup \{x\})$ is dominated by at most $p - 1$ vertices of D_e. Hence by $|X_i \cup \overline{X}_i| = 2p$,

$$\eta_p(V(G), D_e, G) \geq \eta_p(X_i \cup \overline{X}_i, D_e, G) \geq |(X_i \cup \overline{X}_i) \setminus D_e| - 1 \geq 2p - (p-1) - 1 > 1,$$

which contradicts with (4.2). Hence (i) holds.

Suppose that there is some $j \in \{1, \cdots, m\}$ such that $c_j \in D_e$. By (i) and (4.3), $x \in V(H_0)$ and so $|V(H_0) \cap D_e| = |V(H_0) \cap D| - 1 = p - 1$. Hence $|V(T) \cap D_e| \leq p - 2$ by $V(H_0) = \{c_1, \cdots, c_m\} \cup V(T)$. Since each vertex of T ($\cong K_p$) has exact $p - 1$ neighbors in D_e,

$$\eta_p(V(G), D_e, G) \geq \eta_p(V(T), D_e, G) = |V(T) \setminus D_e| = p - |V(T) \cap D_e| \geq 2.$$

This contradicts with (4.2). Thus $\{c_1, \cdots, c_m\} \cap D_e = \emptyset$, and so $|V(T) \cap D_e| = |V(H_0) \cap D_e| = p - 1$. Hence (ii) holds.

By (ii), T has a unique vertex, say z, not in D_e. From $|N_G(z) \cap D_e| = |V(H_0) \cap D_e| = p - 1$, the vertex z is not p-dominated by D_e. However, x is the unique vertex not be p-dominated by D_e in G by (4.2). Thus $z = x$, and so $x = z \in V(T) \setminus D_e$. By the construction of G and $xy \in G^c$, it is clear that $y \in (\cup_{i=1}^{n} V(H_i)) \cap D_e$. Hence (iii) holds. $\Box$

We now show that $\mathscr{C}$ is satisfiable if and only if $r_p(G) = 1$.

If $\mathscr{C}$ is satisfiable, then $\mathscr{C}$ has a satisfying truth assignment $t : U \to \{T, F\}$. According to this satisfying assignment, we can choose a subset S from $V(G)$ as follows:

$$S = S_0 \cup S_1 \cup \cdots \cup S_n,$$

where S_0 consists of $p - 1$ vertices of T and

$$S_i = \left\{ \begin{array}{ll} u_i \cup (\overline{X}_i - \{\overline{v}_{i_p}\}) & \text{if } t(u_i) = T \\ \overline{u}_i \cup (X_i - \{v_{i_p}\}) & \text{if } t(u_i) = F \end{array} \right. \text{ for each } i \in \{1, \cdots, n\}.$$

It can be verified easily that $|S| = p(n + 1) - 1 = \gamma_p(G) - 1$ and $\cup_{i=1}^{n} V(H_i)$ can be p-dominated by S. Since t is a satisfying true assignment for $\mathscr{C}$, each clause $C_j \in \mathscr{C}$ contains at least one true literal. That is, the corresponding vertex c_j has at least one neighbor in $\{u_1, \bar{u}_1 \cdots, u_n, \bar{u}_n\} \cap S$ by the definitions of G and S, and so every $c_j \in \{c_1, \cdots, c_m\}$ has at least p neighbors in S since $S_0 \subseteq N_G(c_j)$. Note that the unique vertex in $V(T) \setminus S_0$ has exact $p - 1$ neighbors in S. By Theorem 2.2 and $|S| = \gamma_p(G) - 1$,

$$r_p(G) = \eta_p(G) \leq \eta_p(V(G), S, G) = \eta_p(V(T) \setminus S_0, S, G) = p - (p - 1) = 1.$$

Furthermore, we have $r_p(G) = 1$ since $\gamma_p(G) > p$ by Claim 1.

Conversely, assume $r_p(G) = 1$. That is, there exists an edge $e = xy$ in G^c such that $\gamma_p(G + e) < \gamma_p(G)$. Let D_e be a γ_p-set of $G + e$. Define $t : U \to \{T, F\}$ by

$$t(u_i) = \left\{ \begin{array}{ll} T & \text{if vertex } u_i \in D_e \\ F & \text{if vertex } u_i \notin D_e \end{array} \right. \text{ for } i = 1, \cdots, n. \tag{4.4}$$

We will show that t is a satisfying truth assignment for $\mathscr{C}$. Let C_j be an arbitrary clause in $\mathscr{C}$. By (ii) and (iii) of Claim 2, the corresponding vertex c_j is not in D_e and $|N_G(c_j) \cap D_e| \geq p$ since $c_j \notin \{x, y\}$. Then there must be some $i \in \{1, \cdots, n\}$ such that

$$|\{u_i, \overline{u}_i\} \cap N_G(c_j) \cap D_e| = 1, \tag{4.5}$$

since T contains exact $p-1$ vertices of D_e by (i) and (ii) of Claim 2. If $u_i \in N_G(c_j) \cap D_e$, then $u_i \in C_j$ and $t(u_i) = T$ by the construction of G and (4.4). If $\overline{u}_i \in N_G(c_j) \cap D_e$, then the literal $\overline{u}_i$ belongs to C_j by the construction of G. Note that $u_i \notin D_e$ from $\overline{u}_i \in D_e$ and (i) of Claim 2. This means that $t(u_i) = F$ by (4.4). Hence $t(\overline{u}_i) = T$. The arbitrariness of C_j with $1 \le j \le m$ shows that all the clauses in $\mathscr{C}$ is satisfied by t. That is, $\mathscr{C}$ is satisfiable.

The theorem follows. ∎

5 Upper Bounds

For a graph G and $p = 1$, Kok and Mynhardt [19] provided an upper bound for $r(G)$ in terms of the smallest private neighborhood of a vertex in some γ-set of G. Let $X \subseteq V(G)$ and $x \in X$. The *private neighborhood* of x with respect to X is defined as the set

$$PN(x, X, G) = N_G[x] \setminus N_G[X \setminus \{x\}]. \tag{5.1}$$

Set

$$\mu(X, G) = \min\{|PN(x, X, G)| : \ x \in X\}$$

and

$$\mu(G) = \min\{\mu(X, G) : \ X \text{ is a } \gamma\text{-set of } G\}. \tag{5.2}$$

Using this parameter, Kok and Mynhardt [19] showed that $r(G) \le \mu(G)$ if $\gamma(G) \ge 2$ with equality if $\gamma(G) = 1$. We generalize this result to any positive integer p.

In order to state our results, we need some notations. Let $X \subseteq V(G)$ and $x \in X$. A vertex $y \in \overline{X}$ is called a *p-private neighbor* of x with respect to X if $xy \in E(G)$ and $|N_G(y) \cap X| = p$. The *p-private neighborhood* of x with respect to X is defined as

$$PN_p(x, X, G) = \{y : \ y \text{ is a } p\text{-private neighbor of } x \text{ with respect to } X\}. \tag{5.3}$$

Let

$$\begin{aligned} \mu_p(x, X, G) &= |PN_p(x, X, G)| + \max\{0, p - |N_G(x) \cap X|\}, & (5.4)\\ \mu_p(X, G) &= \min\{\mu_p(x, X, G) : x \in X\}, \quad \text{and} & (5.5)\\ \mu_p(G) &= \min\{\mu_p(X, G) : \ X \text{ is a } \gamma_p\text{-set of } G\}. & (5.6) \end{aligned}$$

Theorem 5.1 *For any graph G and positive integer p,*

$$r_p(G) \le \mu_p(G)$$

with equality if $r_p(G) = 1$.

Proof. If $\gamma_p(G) \le p$, then $r_p(G) = 0 \le \mu_p(G)$ by our convention. Assume that $\gamma_p(G) \ge p + 1$ below. Let X be a γ_p-set of G and $x \in X$ such that

$$\mu_p(G) = \mu_p(X, G) = \mu_p(x, X, G).$$

Since $|X| = \gamma_p(G) \geq p+1$, we can choose a vertex, say u_y, from $X \setminus N_G(y)$ for each $y \in PN_p(x, X, G)$, and a subset X' with $|X'| = \max\{0, p - |N_G(x) \cap X|\}$ from $X \setminus N_G[x]$. Let

$$G' = G + \{yu_y : \ y \in PN_p(x, X, G)\} + \{xv : \ v \in X'\}.$$

Obviously, $X \setminus \{x\}$ is a p-dominating set of G', which implies that

$$r_p(G) \leq |PN_p(x, X, G)| + |X'| = \mu_p(x, X, G) = \mu_p(G).$$

Assume $r_p(G) = 1$. Then $\gamma_p(G) \geq p+1$ and there exists an edge $xy \in E(G^c)$ such that $\gamma_p(G + xy) = \gamma_p(G) - 1$. Let $G' = G + xy$ and X be a γ_p-set of G'. Without loss of generality, assume that $x \in X$ and $y \in \overline{X}$. Clearly, y is a p-private neighbor of x with respect to X in G and $X \cup \{y\}$ is a γ_p-set of G, which implies

$$PN_p(y, X \cup \{y\}, G) = \emptyset \text{ and } p - |N_G(y) \cap (X \cup \{y\})| = 1,$$

that is, $\mu_p(y, X \cup \{y\}, G) = 1$. It follows that

$$r_p(G) \leq \mu_p(G) \leq \mu_p(X \cup \{y\}, G) \leq \mu_p(y, X \cup \{y\}, G) = 1.$$

Thus, $r_p(G) = \mu_p(G) = 1$. The theorem follows. ∎

Note that $|PN_p(x, X, G)| \leq deg_G(x)$ for any $X \subseteq V(G)$ and $x \in X$. By Theorem 5.1, we obtain the following corollary immediately.

Corollary 5.1 *For any graph G with maximum degree $\Delta(G)$ and positive integer p, $r_p(G) \leq \Delta(G) + p$.*

Corollary 5.2 *Let p be a positive integer and G be a graph with minimum degree $\delta(G)$. If $\delta(G) < p$, then $r_p(G) \leq \delta(G) + p$.*

Proof. Let X be a γ_p-set of G and $x \in V(G)$ with degree $\delta(G)$. Since $deg_G(x) = \delta(G) < p$, $x \in X$ by Observation 1.2. Note that $|PN_p(x, X, G)| \leq deg_G(x) = \delta(G)$ and $p - |N_G(x) \cap X| \leq p$. By Theorem 5.1,

$$\begin{array}{rcl} r_p(G) & \leq & \mu_p(G) \\ & \leq & \mu_p(x, X, G) \\ & = & |PN_p(x, X, G)| + \max\{0, p - |N_G(x) \cap X|\} \\ & \leq & \delta(G) + p. \end{array}$$

The corollary follows. ∎

Consider $p = 1$. Let $X \subseteq V(G)$ and $x \in X$. If x is not an isolated vertex of the induced subgraph $G[X]$, then $PN(x, X, G)$ defined in (5.1) does not contain x and $\max\{0, 1 - |N_G(x) \cap X|\} = 0$ in (5.4). Otherwise, $PN(x, X, G)$ contains x and $\max\{0, 1 - |N_G(x) \cap X|\} = 1$. Notice that $PN_1(x, X, G)$ defined in (5.3) does not contain x. Hence, by (5.5),

$$\mu_1(x, X, G) = PN_1(x, X, G) + \max\{0, 1 - |N_G(x) \cap X|\} = |PN(x, X, G)|.$$

This fact means that $\mu(G)$ defined in (5.2) is a special case of $p = 1$ in (5.6), that is, $\mu_1(G) = \mu(G)$. Thus, by Theorem 5.1, the following corollary holds immediately.

Corollary 5.3 (Kok and Mynhardt [19]) *For any graph G with $\gamma(G) \geq 2$, $r(G) \leq \mu(G)$, with equality if $r(G) = 1$.*

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