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Approximation for Maximizing Monotone Non-decreasing Set Functions with a Greedy Method

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Abstract We study the problem of maximizing a monotone non-decreasing function f subject to a matroid constraint. Fisher, Nemhauser and Wolsey have shown that, if f is submodular, the greedy algorithm will find a solution with value at least $\frac{1}{2}$ of the optimal value under a general matroid constraint and at least $1 - \frac{1}{e}$ of the optimal value under a uniform matroid ($\mathcal{M} = (X, \mathcal{I})$, $\mathcal{I} = \{S \subseteq X : |S| \leq k\}$) constraint. In this paper, we show that the greedy algorithm can find a solution with value at least $\frac{1}{1+\mu}$ of the optimum value for a general monotone non-decreasing function with a general matroid constraint, where $\mu = \alpha$, if $0 \leq \alpha \leq 1$; $\mu = \frac{\alpha^K(1-\alpha^K)}{K(1-\alpha)}$ if $\alpha > 1$; here α is a constant representing the "elemental curvature" of f, and K is the cardinality of the largest maximal independent sets. We also show that the greedy algorithm can achieve a $1 - (\frac{\alpha+\cdots+\alpha^{k-1}}{1+\alpha+\cdots+\alpha^{k-1}})^k$ approximation under a uniform matroid

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constraint. Under this unified α-classification, submodular functions arise as the special case $0 \leq \alpha \leq 1$.

Keywords monotone submodular set function · matroid · approximation algorithm

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1 Introduction

1.1 Background

In this work, we deal with the following combinatorial optimization problem. Given a ground set X of n elements and a monotone non-decreasing set function $f : 2^X \to \Re_+$, we wish to find

$$\max_{S \in \mathcal{I}} f(S) \tag{1}$$

where $\mathcal{I}$ is an independent family of subsets of X. Without loss of generality, let $X = \{1, 2, \ldots, n\}$. The family $\mathcal{I} \subseteq 2^X$ is *independent* if, for all $B \in \mathcal{I}$, any set $A \subseteq B$ is also in $\mathcal{I}$. Furthermore, if for all $A \in \mathcal{I}, B \in \mathcal{I}, |A| < |B|$, there exists $j \in B \setminus A$ such that $A \cup \{j\} \in \mathcal{I}$, where $|S|$ denotes the cardinality of set S, then $\mathcal{M} = (X, \mathcal{I})$ is called a *matroid*.

A set function f is *non-decreasing* if $f(A) \leq f(B)$ for all $A \subseteq B$. We assume $f(\emptyset) = 0$, otherwise f can be replaced by $f - f(\emptyset)$. The function f is called *submodular* if, for all $A \subseteq B \subseteq X$ and $j \in X \setminus B$, $f(A \cup \{j\}) - f(A) \geq f(B \cup \{j\}) - f(B)$. The problem of maximizing a submodular set function subject to independence constraints is NP-hard and a practical, computationally feasible solution can often be found using a *greedy algorithm*. The greedy algorithm starts with the empty set $S = \emptyset$, and then incrementally adds to the current solution set S (according to f) an element j which most improves $f(S)$ while maintaining the condition $S \in \mathcal{I}$.

Based on classical results in linear programming, Nemhauser et al. (1978) proved that the greedy algorithm yields a $(1 - \frac{1}{e})$-approximation for the Problem (1) with a value oracle[1] if f is a monotone non-decreasing submodular function under a uniform matroid, i.e., $\mathcal{I} = \{S \subseteq X : |S| \leq k\}$. Moreover, this is the best possible performance guarantee in polynomial time with a value oracle model(Nemhauser and Wolsey 1978). Feige (1998) proved that if f is an explicitly given coverage function, then the best possible performance guarantee is $1 - 1/e$ unless $P = NP$. For a general matroid $\mathcal{M}$ in Problem (1) with a monotone non-decreasing submodular function f, Fisher et al. (1978) proved that the greedy algorithm provides a $\frac{1}{2}$-approximation of the optimum.

Conforti and Cornuéjols (1984) generalized the results given by Nemhauser, Wolsey and Fisher (Nemhauser et al. 1978; Fisher et al. 1978) and proposed a $\frac{1}{1+c}$-approximation to Problem (1) with a value oracle when a monotone

[1] Given a set $S \in \mathcal{I}$, return $f(S)$.

non-decreasing submodular function f has a total curvature c, defined as $c = \max_{j \in X}\{1 - \frac{f(X)-f(X\setminus\{j\})}{f(\{j\})-f(\emptyset)}\}$. In the case of uniform matroid constraints, the approximation factor is $\frac{1}{c}(1-e^{-c})$. Note that if $c=1$, the result is the same as that obtained by Nemhauser et al. (1978) and Fisher et al. (1978).

Recently, using the idea of a multilinear extension of submodular functions, Vondrák (2010) has proposed a continuous greedy algorithm that together with the pipage rounding technique (Ageev and Sviridenko 2004) and which can achieve a $\frac{1}{c}(1-e^{-c})$ approximation of the optimal value for any matroid, where $c \in [0,1]$ is the curvature with respect to the optimum. This is optimal in the value oracle model in the sense that any better approximation would require an exponential number of value queries.

In Vondrák's approach, a continuous greedy algorithm is used to approximate $\max\{F(\boldsymbol{y}) : \boldsymbol{y} \in P(\mathcal{M})\}$ within a factor of $\frac{1}{c}(1-e^{-c})$, where $P(\mathcal{M}) = \{\boldsymbol{x} \geq 0 : \forall S \subseteq X; \sum_{j \in S} x_j \leq r_{\mathcal{M}}(S)\}$ is the matroid polytope, and $F(\boldsymbol{y}) = \sum_{S \subseteq X} f(S) \prod_{i \in S} y_i \prod_{j \notin S}(1-y_j)$ is a multilinear extension of submodular function f. Inspired by the pipage rounding technique described in Ageev and Sviridenko (2004), a variant of that technique is used by Calinescu et al. (2011) to convert a fractional solution $\boldsymbol{y} \in P(\mathcal{M})$ to a discrete solution S, $f(S) \geq F(y) \geq \frac{1}{c}(1-e^{-c})OPT$. This method has been applied to the Submodular Welfare Problem (SWP) (Vondrák 2008) and the Generalized Assignment Problem (GAP) (Calinescu et al. 2011). Using the same continuous greedy process, Kulik et al. (2009) have extended the $(1-\frac{1}{e})$-approximation to a single knapsack constraint on the Max-Coverage Problem given by Sviridenko (2004), and obtained a $(1-\epsilon)(1-\frac{1}{e})$-approximation for $\epsilon > 0$ under a constant number of knapsack constraints.

Based on *non-oblivious local search* (Alimonti 1994), Filmus and Ward (2012) proposed an optimal, combinatorial $(1-\frac{1}{e})$-approximation algorithm that requires no rounding for monotone submodular optimization over a matroid constraint.

Fisher et al. (1978) proved the greedy algorithm yields a $\frac{1}{1+p}$-approximation for maximizing a submodular function subject to a p-system[2] independence constraint (e.g. the intersection of p matroids). A recent result by Lee et al. (2010) shows that, for any $p \geq 2$ and any $\epsilon > 0$, there is a natural local-search algorithm that has approximation guarantee of $\frac{1}{p+\epsilon}$ for the problem of maximizing a monotone submodular function subject to p matroid constraints.

In the setting of non-negative non-monotone submodular maximization, based on an adaptation of the greedy approach which exploits certain symmetry properties, Buchbinder et al. (2012) gave a simple randomized linear time algorithm achieving a tight approximation guarantee of $\frac{1}{2}$ for the unconstrained problem. Lee et al. (2009, 2010) considered the multiple matroid and knapsack constraints, attaining a $\frac{1}{p+1+\frac{1}{p-1}+\epsilon}$-approximation for $p \geq 2$

[2] Given an independence family $\mathcal{I}$ and a set $Y \subseteq X$, let $\mathcal{B}(Y)$ be the set of maximal independent sets of $\mathcal{I}$ included in Y. Then $\mathcal{I}$ is a p-system if, for all $Y \subseteq X$, $\frac{\max_{A \in \mathcal{B}(Y)} |A|}{\min_{A \in \mathcal{B}(Y)} |A|} \leq p$. See the definition in Korte and Hausmann (1978) and Calinescu et al. (2011).

matroid constraints, and a $(\frac{1}{5}-\epsilon)$-approximation under the p knapsack constraints ($\epsilon > 0$ is any constant) using local search. Chekuri et al. (2011) gave non-monotone counterparts of the continuous greedy algorithm (Vondrák 2010) and defined a contention resolution rounding scheme which allows one to obtain approximations for different combinations of constraints, deriving a number of new results for maximizing a non-negative submodular set function subject to a variety of packing type constraints including (multiple) matroid constraints, knapsack constraints, and their intersections.

In the case of the demand oracle model[3], which is stronger than the value oracle model since a demand oracle can simulate a value oracle in polynomial time, better results for some special submodular functions have been obtained. A $(1-\frac{1}{e})$-approximation for SWP has been given by Dobzinski and Schapira (2006). Based on configuration linear programming and a contention resolution technique, Feige and Vondrák (2006, 2010) presented a $(1-\frac{1}{e}+\epsilon)$-approximation for some small fixed $\epsilon > 0$ for SWP and GAP. Chakrabarty and Goel (2008) proved that it is NP-hard to approximate SWP within a ratio better than $\frac{15}{16}$ and the GAP within a ratio better than $\frac{10}{11}$. Badanidiyuru et al. (2011) presented a $\frac{8}{9}$-approximation algorithm for the problem of maximizing a monotone submodular function subject to a cardinality constraint and a $(\frac{8}{9}-\epsilon)$-approximation for the problem of maximizing a monotone submodular function subject to a knapsack constraint for an arbitrary small constant $\epsilon > 0$.

1.2 Main Contributions

In this paper, we propose a unified approximation bound $\frac{1}{1+\mu}$ for the solution of (1) using a greedy algorithm under a general matroid constraint (in the Theorem 1). The constant μ depends on the elemental curvature α of the underlying non-decreasing monotone function f and the cardinality K of the largest maximal independent sets. To be precise, $\mu = \alpha$, if $0 < \alpha \leq 1$, and $\mu = \frac{\alpha^K(1-\alpha^K)}{K(1-\alpha)}$, if $\alpha > 1$. Under such a framework, the aforementioned existing approximation bound appears as a special case where $\alpha = 1$. In addition, we show that for a uniform matroid the greedy algorithm finds a solution with a $1-(\frac{\alpha+\ldots+\alpha^{k-1}}{1+\alpha+\ldots+\alpha^{k-1}})^k$ approximation, where k is the cardinality of the solution. Using an example, we demonstrate that if the bound of α is known, which maybe considered as a stronger oracle, it can be used to obtain a tighter approximation bound for a submodular function.

[3] Given prices $p_1, \ldots, p_n$, return a bundle $S \in \arg\max_{T, T\subseteq X} f(T) - \sum_{i\in T} p_i$.

1.3 Preliminaries

Submodular functions. A set-function $f : 2^X \to \Re_+$ is submodular if, for all subsets $A \subseteq B \subseteq X$ and $j \in X \setminus B$,

$$f_j(A) \geq f_j(B)$$

where $f_j(S) \triangleq f(S \cup \{j\}) - f(S)$ is the "marginal value "for given $S \subset X$. A submodular function has a natural diminishing returns property. It may also be defined in the following two equivalent forms:

1. The set-function f is submodular if for all subsets $A, B \subseteq X$,
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$
2. The set-function f is submodular if for all subsets, $A \subset X$ and $i, j \in X \setminus A$,
$$f_i(A) \geq f_i(A \cup \{j\}).$$

In the following, we assume that $f(\emptyset) = 0$ and f is monotone non-decreasing.
Curvature. The total curvature of a monotone non-decreasing submodular function f is, (Conforti and Cornuéjols 1984),

$$c = \max_{j \in X} \Big\{ 1 - \frac{f(X) - f(X \setminus \{j\})}{f(\{j\}) - f(\emptyset)} \Big\}.$$

Note that $c \in [0, 1]$. The readers may refer to paper (Conforti and Cornuéjols 1984) for the analysis of c dependent greedy algorithms.
Elemental Curvature. The elemental curvature of a monotone non-decreasing function f is defined as

$$\alpha = \max_{S \subset X, i,j \in X \setminus S, i \neq j} \frac{f_i(S \cup \{j\})}{f_i(S)}.$$

In other words, α is the smallest value such that, for every $S \subset X$ and $i, j \in X \setminus S$,

$$f(S \cup \{j\} \cup \{i\}) - f(S \cup \{j\}) \leq \alpha[f(S \cup \{i\}) - f(S)].$$

Note that $\alpha \in [0, +\infty)$. If $\alpha = 0$, f is a constant valued set function; if $0 \leq \alpha \leq 1$, then f is a submodular function. In this work, we assume that $\alpha \neq 0$.
Matroids. A matroid is a pair $\mathcal{M} = (X, \mathcal{I})$, where $\mathcal{I} \subseteq 2^X$ and 1) $\forall B \in \mathcal{I}, A \subset B \to A \in \mathcal{I}$; 2) $\forall A, B \in \mathcal{I}; |A| < |B| \to \exists j \in B \setminus A; A \cup \{j\} \in \mathcal{I}$. A matroid is a structure which captures and generalizes the notion of linear independence in vector spaces. In a matroid, all maximal sets have the same cardinality.
Uniform Matroid. A matroid $\mathcal{M} = (X, \mathcal{I})$ is a uniform matroid if $\mathcal{I} = \{A \subseteq X : |A| \leq k\}$, where k is a given integer with $0 \leq k \leq |X|$.
Partition Matroid. A matroid $\mathcal{M} = (X, \mathcal{I})$ is a partition matroid if X is partitioned into l sets $X_1, X_2, \dots, X_l$ with associated integers $k_1, k_2, \dots, k_l$, and a set $S \subseteq X$ is independent if and only if $|S \cap X_i| \leq k_i$. If $l = 1$, then $\mathcal{M} = (X, \mathcal{I})$ is the uniform matroid.

2 The Bound $1/(1+\mu)$

Lemma 1 *Let X be a finite set and $f : 2^X \to \Re_+$ be a monotone non-decreasing set function with $f(\emptyset) = 0$ and elemental curvature α. Then for any set $S \subset T \subseteq X$, $f(T) - f(S) \leq \beta \sum_{j \in T\setminus S} f_j(S)$, where*

$$\beta = \begin{cases} \frac{1-\alpha^r}{r(1-\alpha)}, & \alpha \neq 1, r = |T \setminus S|; \\ 1, & \alpha = 1. \end{cases}$$

Proof For arbitrary $S \subset T$ with $T \setminus S = \{j_1, j_2, \ldots, j_r\}$, we have,

$$\begin{aligned} f(T) - f(S) &= f(S \cup \{j_1, j_2, \ldots, j_r\}) - f(S) \\ &= \sum_{t=1}^{r} [f(S \cup \{j_1, \ldots, j_t\}) - f(S \cup \{j_1, \ldots, j_{t-1}\})] \\ &= \sum_{t=1}^{r} f_{j_t}(S \cup \{j_1, \ldots, j_{t-1}\}). \end{aligned} \tag{2}$$

From the definition of elemental curvature, (2) can be written as

$$\begin{aligned} f(T) - f(S) &\leq f_{j_1}(S) + \alpha f_{j_2}(S) + \ldots + \alpha^{r-1} f_{j_r}(S) \\ &= \sum_{t=1}^{r} \alpha^{t-1} f_{j_t}(S). \end{aligned} \tag{3}$$

We note that the value of the above inequality does not depend on the order of $j_1, j_2, \ldots, j_r$. It is possible to choose a set P of r permutations π from the group of all permutations, such that if $\pi, \pi' \in P$, then $\pi'\pi^{-1}$ is fixed point free. That is, $\pi(i) \neq \pi'(i)$ for all i. It can easily be seen that $|P| = r$. The above process can be repeated for each choice of permutation; that is, $\{j_1, j_2, \ldots, j_r\}$ is replace by $\{j_{\pi(1)}, j_{\pi(2)}, \ldots, j_{\pi(r)}\}$. If all of these r inequalities are summed, the following inequality follows.

$$r(f(T) - f(S)) \leq \sum_{t=1}^{r} \alpha^{t-1} \sum_{t=1}^{r} f_{j_t}(S) = \sum_{t=1}^{r} \alpha^{t-1} \sum_{j \in T\setminus S} f_j(S). \tag{4}$$

Therefore, when $\alpha \neq 1$,

$$f(T) - f(S) \leq \frac{1-\alpha^r}{r(1-\alpha)} \sum_{j \in T\setminus S} f_j(S), \tag{5}$$

and, when $\alpha = 1$,

$$f(T) - f(S) \leq \sum_{j \in T\setminus S} f_j(S). \tag{6}$$

Lemma 2 *Let X be a finite set and $f : 2^X \to \Re_+$ be a monotone non-decreasing set function with $f(\emptyset) = 0$ and elemental curvature α. Then for all $S \subset T \subseteq X, j \in X \setminus T$, $f_j(S) \geq \frac{1}{\alpha^r} f_j(T)$, where $r = |T \setminus S|$.*

Proof For arbitrary $S \subset T$ with $T \setminus S = \{j_1, j_2, \ldots, j_r\}$, we have,

$$
\begin{aligned}
f_j(S) &= f(S \cup \{j\}) - f(S) \\
&\geq \frac{1}{\alpha}[f(S \cup \{j\} \cup \{j_1\}) - f(S \cup \{j_1\})] \\
&\geq \frac{1}{\alpha^r}[f(S \cup \{j\} \cup \{j_1\} \cup \ldots \cup \{j_r\}) - f(S \cup \{j_1\} \cup \ldots \cup \{j_r\})] \\
&= \frac{1}{\alpha^r}[f(T \cup \{j\}) - f(T)].
\end{aligned} \tag{7}
$$

Lemma 3 *Let $r \in \{z \geq 1 : z \in \mathcal{Z}_+\}$, $\beta(r) = \frac{1-\alpha^r}{r(1-\alpha)}$. Then $\beta(r)$ is decreasing with respect to r if $\alpha \in (0, 1)$ and increasing with respect to r if $\alpha \in (1, +\infty)$.*

Proof For all $r \in \{z \geq 1 : z \in \mathcal{Z}_+\}$ and $\alpha \in (0, +\infty) \setminus \{1\}$,

$$
\begin{aligned}
\beta(r+1) - \beta(r) &= \frac{1-\alpha^{r+1}}{(r+1)(1-\alpha)} - \frac{1-\alpha^r}{r(1-\alpha)} \\
&= \frac{1}{1-\alpha}\frac{\alpha^r(1+r-r\alpha)-1}{r(r+1)} \\
&= \frac{1}{1-\alpha}\frac{\gamma(r)}{r(r+1)},
\end{aligned} \tag{8}
$$

where $\gamma(r) = \alpha^r(1 + r - r\alpha) - 1$. For all $r \in \{z \geq 1 : z \in \mathcal{Z}_+\}$ take the first derivative of γ with respect to α to obtain

$$\gamma\prime(\alpha) = r(r+1)(1-\alpha)\alpha^{r-1}. \tag{9}$$

If $0 \leq \alpha \leq 1$, then $\gamma\prime(\alpha) \geq 0$ and $\gamma(\alpha)$ is non-decreasing with respect to α. Similarly, if $\alpha > 1$, then $\gamma\prime(\alpha) < 0$ and $\gamma(\alpha)$ is non-increasing with respect to α. Therefore, $\forall r, \gamma(r) \leq \max\gamma = \gamma(1) = 0$. Then $\beta(r+1) - \beta(r) \leq 0$ for $\alpha \in (0, 1)$ and $\beta(r+1) - \beta(r) \geq 0$ for $\alpha \in (1, +\infty)$.

Assume $\mathcal{M} = (X, \mathcal{I})$ is a matroid. The greedy algorithm commences by setting $\mathcal{G}_0 = \emptyset$, then selects a set in $\mathcal{I}$. At each step $i = \{1, 2, \ldots\}$, the new element selected is:

$$g_i = \arg\max_{j \in X \setminus \mathcal{G}_{i-1}} f_j(\mathcal{G}_{i-1}), \tag{10}$$

where $\mathcal{G}_i = \mathcal{G}_{i-1} \cup \{g_i\}$ and $\mathcal{G}_i \in \mathcal{I}$.

Theorem 1 *Let X be a finite set and $f : 2^X \rightarrow \Re_+$ a monotone non-decreasing set function with $f(\emptyset) = 0$ and elemental curvature α. Then the greedy algorithm achieves a $\frac{1}{1+\mu}$ approximation for any matroid $\mathcal{M} = (X, \mathcal{I})$, where $\mu = \alpha$ if $0 < \alpha \leq 1$, and $\mu = \frac{\alpha^K(1-\alpha^K)}{K(1-\alpha)}$ if $\alpha > 1$, and K is the cardinality of the largest maximal independent sets.*

Proof Let $\mathcal{O}$ be an optimal solution and $\mathcal{G}$ the greedy solution. Since $\mathcal{M} = (X, \mathcal{I})$ is a matroid and f is non-decreasing, without loss of generality, we can assume that $\mathcal{O}$ is a maximal independent set such that $|\mathcal{O}| = |\mathcal{G}| = K$ ($\mathcal{G}$ is a maximal independent set in $\mathcal{I}$ and all maximal independent sets have the same cardinality). Write $\mathcal{G} = \{g_1, g_2, \ldots, g_K\}$ and let $r = |\mathcal{O} \setminus \mathcal{G}|$, so that $\mathcal{G} = \mathcal{O}$ if $r = 0$. Now suppose that $r \neq 0$ and define $\mathcal{O}_K = \mathcal{O}$. Then, from the exchange property of matroids and the fact that $\mathcal{G}_{K-1} = \{g_1, g_2, \ldots, g_{K-1}\}$ is in $\mathcal{I}$, there exists a $o_K \in \mathcal{O}_K \setminus \mathcal{G}_{K-1}$ such that $\mathcal{G}_{K-1} \cup \{o_K\} \in \mathcal{I}$. Here $\mathcal{O}_{K-1}$, defined to be $\mathcal{O}_K \setminus \{o_K\}$, is also in $\mathcal{I}$ by the hereditary property of matroids. We now repeat this argument with $\mathcal{O}_{K-1}$ in place of $\mathcal{O}_K$ and $\mathcal{G}_{K-1}$ by $\mathcal{G}_{K-2}$ and continue to define $\mathcal{O}_i$ $(i = K-1, \ldots, 1)$ in the same way. This procedure matches the elements of $\mathcal{G}$ and $\mathcal{O}$. Note that this guarantees that $\mathcal{G}_{i-1} \cup \{o_i\}$ is in $\mathcal{I}$. Let $\mathcal{O} \setminus \mathcal{G} = \{o_{i_1}, o_{i_2}, \ldots, o_{i_r}\}$ and $\mathcal{O} \cap \mathcal{G} = \{o_{i_{r+1}}, o_{i_{r+2}}, \ldots, o_{i_K}\}$. By Lemma 1 and the non-decreasing property of f,

$$\begin{aligned} f(\mathcal{O}) \leq f(\mathcal{G} \cup \mathcal{O}) &\leq f(\mathcal{G}) + \beta(r) \sum_{j \in \mathcal{O} \setminus \mathcal{G}} f_j(\mathcal{G}) \\ &= f(\mathcal{G}) + \beta(r) \sum_{s=1}^{r} f_{o_{i_s}}(\mathcal{G}) \end{aligned} \tag{11}$$

where $\beta(r) = \begin{cases} \frac{1-\alpha^r}{r(1-\alpha)}; \alpha \neq 1 \\ 1; \qquad \alpha = 1 \end{cases}$. According to Lemma 2, we have,

$$f(\mathcal{O}) \leq f(\mathcal{G}) + \beta(r) \sum_{s=1}^{r} \alpha^{K-i_s+1} f_{o_{i_s}}(\mathcal{G}_{i_s-1}) \tag{12}$$

$$\leq f(\mathcal{G}) + \beta(r) \sum_{s=1}^{r} \alpha^{K-i_s+1} f_{o_{i_s}}(\mathcal{G}_{i_s-1}) + \beta(r) \sum_{s=r+1}^{K} \alpha^{K-i_s+1} f_{o_{i_s}}(\mathcal{G}_{i_s-1}) \tag{13}$$

$$\leq \begin{cases} f(\mathcal{G}) + \alpha\beta(r) \sum_{s=1}^{K} f_{o_{i_s}}(\mathcal{G}_{i_s-1}); & \alpha \leq 1 \\ f(\mathcal{G}) + \alpha^K \beta(r) \sum_{s=1}^{K} f_{o_{i_s}}(\mathcal{G}_{i_s-1}); & \alpha > 1 \end{cases} \tag{14}$$

(13) is derived from the non-negative property of $\beta(r)$, α and $f_{o_{i_s}}(\mathcal{G}_{i_s-1})$.

In view of Lemma 3 and the definition of the greedy algorithm, we obtain

$$\begin{aligned} f(\mathcal{O}) &\leq \begin{cases} f(\mathcal{G}) + \alpha\beta(1) \sum_{s=1}^{K} f_{o_{i_s}}(\mathcal{G}_{i_s-1}); \alpha \leq 1 \\ f(\mathcal{G}) + \alpha^K \beta(K) \sum_{s=1}^{K} f_{o_{i_s}}(\mathcal{G}_{i_s-1}); \alpha > 1 \end{cases} \\ &\leq \begin{cases} f(\mathcal{G}) + \alpha\beta(1) \sum_{s=1}^{K} f_{g_{i_s}}(\mathcal{G}_{i_s-1}); \alpha \leq 1 \\ f(\mathcal{G}) + \alpha^K \beta(K) \sum_{s=1}^{K} f_{g_{i_s}}(\mathcal{G}_{i_s-1}); \alpha > 1 \end{cases} \\ &= \begin{cases} f(\mathcal{G}) + \alpha\beta(1) f(\mathcal{G}); \alpha \leq 1 \\ f(\mathcal{G}) + \alpha^K \beta(K) f(\mathcal{G}); \alpha > 1. \end{cases} \end{aligned} \tag{15}$$

Therefore,

$$f(\mathcal{G}) \geq \frac{1}{1+\mu} f(\mathcal{O}), \tag{16}$$

where

$$\mu = \begin{cases} \alpha; 0 < \alpha \leq 1 \\ \frac{\alpha^K(1-\alpha^K)}{K(1-\alpha)}; \alpha > 1. \end{cases} \tag{17}$$

Corollary 1 *$f(\mathcal{G}) \geq \frac{1}{2} f(\mathcal{O})$ if f is a monotone non-decreasing submodular function with matroid constraint (Fisher et al. 1978).*

Remark 1

1. It is worth emphasizing that the proposed approximation bound $\frac{1}{1+\mu}$ obtained by the greedy algorithm is general in the sense that f does not have to be a submodular function. The result provides a worst-case guarantee for the solution of the greedy algorithm with a general monotone non-decreasing function under a general matroid constraint.
2. $\mu \to \infty$ when $\alpha \to \infty$. Additionally, $\lim\limits_{\alpha \to 1^+} \mu = \lim\limits_{\alpha \to 1^-} \mu = \mu(1) = 1$ and μ is continuous with respect to $\alpha \in (0, +\infty)$.
3. For a given application, if f is a submodular function and the bound of its elemental curvature α is known, our approximation ratio $\frac{1}{1+\alpha}$ is tighter than the existing result $\frac{1}{2}$. The latter occurs as the worst case (that is, $\alpha = 1$), for which point the elemental curvature is undefined.

3 The Bound $1 - (\frac{\alpha+\ldots+\alpha^{k-1}}{1+\alpha+\ldots+\alpha^{k-1}})^k$

Theorem 2 *Let X be a finite set and $f : 2^X \to \Re_+$ a monotone non-decreasing set function with $f(\emptyset) = 0$ and elemental curvature α. Then the greedy algorithm achieves a $1 - (\frac{\alpha+\ldots+\alpha^{k-1}}{1+\alpha+\ldots+\alpha^{k-1}})^k$ approximation under a uniform matroid $\mathcal{M} = (X, \mathcal{I})$, where k is the cardinality of the solution.*

Proof Let $\mathcal{O} = \{o_1, o_2, \ldots, o_k\}$ be the optimal k-element subset, and $\mathcal{G} = \{g_1, g_2, \ldots, g_k\}$ the set chosen by the greedy algorithm. At the ith stage of the greedy algorithm, with $i \in \{0, 1, \ldots, k-1\}$, according to Lemma 1 we may write

$$f(\mathcal{O}) \leq f(\mathcal{O} \cup \mathcal{G}_i) \leq f(\mathcal{G}_i) + \beta(r_i) \sum_{j \in \mathcal{O} \setminus \mathcal{G}_i} f_j(\mathcal{G}_i), \tag{18}$$

where

$$\beta(r_i) = \begin{cases} \frac{1-\alpha^{r_i}}{r_i(1-\alpha)}; \alpha \neq 1 \\ 1; \qquad \alpha = 1 \end{cases}, \quad r_i = |\mathcal{O} \setminus \mathcal{G}_i|, \text{ and } \mathcal{G}_0 = \emptyset.$$

By definition of g_{i+1}, $f_{g_{i+1}}(\mathcal{G}_i) \geq f_j(\mathcal{G}_i)$, $(j \in X \setminus \mathcal{G}_i,\ \mathcal{G}_i \cup \{j\} \in \mathcal{I})$. Hence

$$\begin{aligned} f(\mathcal{O}) &\leq f(\mathcal{G}_i) + \beta(r_i) \sum_{j \in \mathcal{O} \setminus \mathcal{G}_i} f_{g_{i+1}}(\mathcal{G}_i) \\ &= f(\mathcal{G}_i) + r_i \beta(r_i) f_{g_{i+1}}(\mathcal{G}_i). \end{aligned} \tag{19}$$

For $\alpha \in (0, +\infty)$, the function

$$r\beta(r) = \begin{cases} \frac{1-\alpha^r}{1-\alpha}, & \alpha \neq 1, \\ r, & \alpha = 1 \end{cases}$$

is increasing in $r \in \{1, \ldots, k\}$. As a result, we obtain,

$$f(\mathcal{O}) \le f(\mathcal{G}_i) + \beta(k) k f_{g_{i+1}}(\mathcal{G}_i). \tag{20}$$

Let $\mu = \beta(k)$. It follows from the definition of $f_{g_{i+1}}(\mathcal{G}_i)$ that

$$f(\mathcal{G}_i) = \sum_{j=1}^{i} f_{g_j}(\mathcal{G}_{j-1}), \tag{21}$$

and, for all $i \in \{0, 1, \ldots, k-1\}$,

$$f(\mathcal{O}) \le \sum_{j=1}^{i} f_{g_j}(\mathcal{G}_{j-1}) + \mu k f_{g_{i+1}}(\mathcal{G}_i). \tag{22}$$

Multiplying both sides of (22) by $(1 - \frac{1}{\mu k})^{k-1-i}$ ($i \in \{0, 1, \ldots, k-1\}$) and summing, we obtain

$$[1 - (\frac{\mu k - 1}{\mu k})^k] f(\mathcal{O}) \le \sum_{i=1}^{k} f_{g_i}(\mathcal{G}_{i-1}) = f(\mathcal{G}), \tag{23}$$

where the coefficient of $f(\mathcal{O})$ follows from

$$\sum_{i=0}^{k-1} (1 - \frac{1}{\mu k})^{k-1-i} = \sum_{i=0}^{k-1} (1 - \frac{1}{\mu k})^i = \frac{1 - (1 - \frac{1}{\mu k})^k}{1 - (1 - \frac{1}{\mu k})} = \mu k \big(1 - (\frac{\mu k - 1}{\mu k})^k\big), \tag{24}$$

and the coefficient $f_{g_i}(\mathcal{G}_{i-1})$ over $\{1, \ldots, k\}$ is achieved by

$$\mu k (1 - \frac{1}{\mu k})^{k-i} + \sum_{j=i}^{k-1} (1 - \frac{1}{\mu k})^{k-1-j} = \mu k (1 - \frac{1}{\mu k})^{k-i} + \mu k \big(1 - (1 - \frac{1}{\mu k})^{k-i}\big) = \mu k. \tag{25}$$

Finally, we note that

$$[1 - (\frac{\mu k - 1}{\mu k})^k] = 1 - (\frac{\alpha + \cdots + \alpha^{k-1}}{1 + \alpha + \cdots + \alpha^{k-1}})^k,$$

yielding

$$f(\mathcal{G}) \ge [1 - (\frac{\alpha + \cdots + \alpha^{k-1}}{1 + \alpha + \cdots + \alpha^{k-1}})^k] f(\mathcal{O}). \tag{26}$$

Corollary 2 $f(\mathcal{G}) \ge (1 - \frac{1}{e}) f(\mathcal{O})$ *if* f *is a monotone non-decreasing submodular function with uniform matroid constraint (Nemhauser et al. 1978).*

Proof If f is a monotone non-decreasing submodular function, then $0 < \alpha \leq 1$. For any fixed k, $1-(\frac{\alpha+\cdots+\alpha^{k-1}}{1+\alpha+\cdots+\alpha^{k-1}})^k$ is decreasing as a function of $\alpha \in (0,1]$. Therefore, $1-(\frac{\alpha+\cdots+\alpha^{k-1}}{1+\alpha+\cdots+\alpha^{k-1}})^k \geq 1-(\frac{k-1}{k})^k$, $\forall k > 1$; $[1-(\frac{k-1}{k})^k] \rightarrow 1-\frac{1}{e}$, $\forall\, k \rightarrow \infty$.

4 Application examples

Two application examples are presented in this section to demonstrate the usefulness of the proposed approximation bounds in different matroid constraints.

Example 1 concerns the case for $\alpha \leq 1$. Let the ground set be $X = \{1, \ldots, n\}$. $\forall S \subseteq X$, define

$$f(S) = \frac{1}{n}\sum_{i=1}^{n}\Big(1-\prod_{j\in S}(1-p_{ij})\Big), \tag{27}$$

where $p_{ij} \in [0,1], \forall i,j \in X$. Since for all $S \subset X, s \in X \setminus S$, $f_s(S) = \frac{1}{n}\sum_{i=1}^{n} p_{is}\prod_{j\in S}(1-p_{ij})$. It is easily verified that (27) is a monotone non-decreasing submodular function. For all $S \subset X$, $s,t \in X \setminus S$,

$$\begin{aligned}\frac{f_s(S\cup\{t\})}{f_s(S)} &= \frac{\sum_{i=1}^{n} p_{is}\prod_{j\in S}(1-p_{ij})(1-p_{it})}{\sum_{i=1}^{n} p_{is}\prod_{j\in S}(1-p_{ij})}\\ &\leq \max_{i,t\in X}(1-p_{it}).\end{aligned} \tag{28}$$

Therefore, the elemental curvature $\alpha = 1-\min_{i,j\in X} p_{ij}$. By Theorem 2, a greedy algorithm obtains a $1-(\frac{\alpha+\ldots+\alpha^{k-1}}{1+\alpha+\ldots+\alpha^{k-1}})^k$ approximation of the optimal objective value if a uniform matroid constraint is imposed. Potential applications of the submodular function (27) include the coverage-aware self-scheduling in sensor networks (Lu and Suda 2003) and social network inference problem (Kempe et al. 2005).

Example 2 concerns the case for $\alpha \geq 1$. This example deals with the application of a greedy algorithm to the solution of a static Weapon Target Assignment (WTA) problem. The WTA problem arises in the modeling of combat operations where the total value of one's own assets to be protected is maximized subject to a constraint on the number of weapons. Specifically, given a set of offense targets N with the value of each target $v_n \geq 0$, $n = 1, \ldots, |N|$ assume that a set of defensive weapons M is available to protect a set of assets G with the value of each asset w_k, $k = 1, 2, \ldots, |G|$. The set of targets aimed at each asset k is $H_k, k = 1, 2, \ldots, |G|$. The probability that the nth target destroys an asset to which it is aimed is assumed to be π_n, $n = 1, 2, \ldots, |N|$ and the probability that the mth weapon destroys the nth target is assumed to be $0 \leq p_{nm} \leq 1$, $n = 1, \ldots, |N|$, $m = 1, \ldots, |M|$. The objective is to seek a solution which can maximize the expected value of the surviving assets.

Therefore, the objective function is considered to be the sum over all assignment of one or more weapons to each target of the product of each asset value and the survival probability, so that the underlying problem is the following constrained maximization problem:

$$\max_{M_1,M_2,\ldots,M_{|N|}} \sum_{k=1}^{|G|} w_k \prod_{n\in H_k} \Big(1-\pi_n \prod_{m\in M_n}(1-p_{nm})\Big), \qquad (29)$$
$$s.t. \quad M_n \cap M_l = \emptyset, \forall n \neq l, n, l \in N.$$

where M_n, $n = 1, 2, \ldots, |N|$ signifies the set of defense weapons assigned to target n. The constraints in (29) imply that a weapon can assigned to at most one target. It has been demonstrated by Lloyd and Witsenhausen (1986) that the WTA is an NP-complete problem.

Define a ground set $X = \{(m,n)|n \in N, m \in M\}$, $\mathcal{I} = \{S \subseteq X : |S| \leq |M| \wedge (\nexists m, n_1 \neq n_2 : (m,n_1) \in S \wedge (m,n_2) \in S)\}$, and a function f on a subset $S \in \mathcal{I}$ as

$$f(S) = \sum_{k=1}^{|G|} w_k \left(\prod_{n\in H_k} \Big(1-\pi_n \prod_{\{m:(m,n)\in S\}} (1-p_{nm})\Big) - \prod_{n\in H_k}(1-\pi_n) \right). \quad (30)$$

Firstly, we note that $\mathcal{M} = (X, \mathcal{I})$ is a partition matroid. To see this, let $X_i = \{(i,1),\ldots,(i,|N|)\},\ i = 1,\ldots,|M|$. Clearly, $X = X_1 \cup X_2 \cup \ldots, \cup X_{|M|}$. According to the constraints in (29) that a weapon be assigned to at most one target we conclude that $\forall S \in \mathcal{I}$ and X_i, $|S \cap X_i| \leq 1$. It follows that $\mathcal{I}$ is an independent set family and $\mathcal{M}$ is a partition matroid.

Secondly, there is a one-to-one correspondence between the sets S and the feasible solutions $M_1, M_2, \ldots, M_{|N|}$ for (29). Furthermore, the corresponding solutions have identical values. Hence, (29) is equivalent to $\max_{S\in\mathcal{I}} f(S)$. Next, we show that (30) is a monotone non-decreasing function and its elemental curvature $\alpha \geq 1$.

We note that $f(\emptyset) = \sum_{k=1}^{|G|} w_k(\prod_{n\in H_k}(1-\pi_n) - \prod_{n\in H_k}(1-\pi_n)) = 0$. Let $q_{nm} = 1 - p_{nm}$ and $\forall S \in \mathcal{I}$, $s = (m_1, n_1) \in X \setminus S$. Suppose that the target n_1

attacks the asset k_1. We have

$$
\begin{aligned}
&f(S\cup\{s\})-f(S)\\
&=\sum_{k=1}^{|G|} w_k\Big(\prod_{n\in H_k}(1-\pi_n\prod_{\{m:(m,n)\in S\cup\{s\}\}} q_{nm})-\prod_{n\in H_k}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})\Big)\\
&=w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})(1-\pi_{n_1}\prod_{\{m:(m,n_1)\in S\cup\{s\}\}} q_{n_1m})\\
&-w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})(1-\pi_{n_1}\prod_{\{m:(m,n_1)\in S\}} q_{n_1m})\\
&=w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})\pi_{n_1}p_{n_1m_1}\prod_{\{m:(m,n_1)\in S\}} q_{n_1m}\\
&\geq 0
\end{aligned}
\tag{31}
$$

The result of (31) indicates that f is a monotone non-decreasing function with $f(\emptyset)=0$.

The elemental curvature α of f is calculated as follows. For each $S\in\mathcal{I}$, $s=(m_1,n_1)\in X\setminus S$, $t=(m_2,n_2)\in X\setminus S$, and $s\neq t$. Suppose that the target n_1 attacks the asset k_1 and the target n_2 attacks the asset k_2, respectively. The marginal difference ratio for computing α is derived as three cases:
(1) If $n_1=n_2$, i.e., n_1,n_2 are the same one target, then $k_1=k_2$,

$$
\begin{aligned}
&\frac{f(S\cup\{s\}\cup\{t\})-f(S\cup\{t\})}{f(S\cup\{s\})-f(S)}\\
&=\frac{w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})\pi_{n_1}p_{n_1m_1}\prod_{\{m:(m,n_1)\in S\cup\{t\}\}} q_{n_1m}}{w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})\pi_{n_1}p_{n_1m_1}\prod_{\{m:(m,n_1)\in S\}} q_{n_1m}}\\
&=1-p_{n_2m_2}.
\end{aligned}
\tag{32}
$$

(2) If $n_1\neq n_2$ (the different two targets) and $k_1=k_2$ (these two targets attack the same asset),

$$
\begin{aligned}
&\frac{f(S\cup\{s\}\cup\{t\})-f(S\cup\{t\})}{f(S\cup\{s\})-f(S)}\\
&=\frac{w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\cup\{t\}\}} q_{nm})\pi_{n_1}p_{n_1m_1}\prod_{\{m:(m,n_1)\in S\}} q_{n_1m}}{w_{k_1}\prod_{n\in H_{k_1}\setminus\{n_1\}}(1-\pi_n\prod_{\{m:(m,n)\in S\}} q_{nm})\pi_{n_1}p_{n_1m_1}\prod_{\{m:(m,n_1)\in S\}} q_{n_1m}}\\
&=\frac{1-\pi_{n_2}\prod_{\{m:(m,n_2)\in S\cup\{t\}\}} q_{n_2m}}{1-\pi_{n_2}\prod_{\{m:(m,n_2)\in S\}} q_{n_2m}}\\
&=\frac{1-\pi_{n_2}\prod_{\{m:(m,n_2)\in S\}} q_{n_2m}q_{n_2m_2}}{1-\pi_{n_2}\prod_{\{m:(m,n_2)\in S\}} q_{n_2m}}\\
&\geq 1.
\end{aligned}
\tag{33}
$$

Note that by (33), f is not submodular.
(3) If $n_1 \neq n_2$ (the different two targets) and $k_1 \neq k_2$ (these two targets attack two different two assets),

$$\begin{aligned}
&\frac{f(S \cup \{s\} \cup \{t\}) - f(S \cup \{t\})}{f(S \cup \{s\}) - f(S)} \\
&= \frac{w_{k_1} \prod_{n \in H_{k_1} \setminus \{n_1\}} (1 - \pi_n \prod_{\{m:(m,n) \in S\}} q_{nm}) \pi_{n_1} p_{n_1 m_1} \prod_{\{m:(m,n_1) \in S\}} q_{n_1 m}}{w_{k_1} \prod_{n \in H_{k_1} \setminus \{n_1\}} (1 - \pi_n \prod_{\{m:(m,n) \in S\}} q_{nm}) \pi_{n_1} p_{n_1 m_1} \prod_{\{m:(m,n_1) \in S\}} q_{n_1 m}} \\
&= 1.
\end{aligned} \tag{34}$$

From (32), (33) and (34), we conclude that

$$\begin{aligned}
\frac{f(S \cup \{s\} \cup \{t\}) - f(S \cup \{t\})}{f(S \cup \{s\}) - f(S)} &\leq \frac{1 - \pi_{n_2} \prod_{\{m:(m,n_2) \in S\}} q_{n_2 m} q_{n_2 m_2}}{1 - \pi_{n_2} \prod_{\{m:(m,n_2) \in S\}} q_{n_2 m}} \\
&\leq \frac{1 - \pi_{n_2} q_{n_2 m_2}}{1 - \pi_{n_2}}.
\end{aligned} \tag{35}$$

From (35), we obtain that the elemental curvature $\alpha = \max_{n \in N, m \in M} \frac{1 - \pi_n q_{nm}}{1 - \pi_n}$. Both π_n and q_{nm} are probabilities, which indicates that $\alpha \geq 1$.

5 Conclusions

In this paper, we have described how the greedy algorithm achieves a $\frac{1}{1+\mu}$ approximation for a monotone non-decreasing function with a general matroid constraint, where $\mu = \alpha$, if $0 < \alpha \leq 1$; $\mu = \frac{\alpha^K (1-\alpha^K)}{K(1-\alpha)}$, if $\alpha > 1$. The parameter α represents the "elemental curvature" of f, and K is the cardinality of the largest maximal independent sets. We also show that a greedy algorithm can find a solution with a $1 - (\frac{\alpha + \cdots + \alpha^{k-1}}{1 + \alpha + \cdots + \alpha^{k-1}})^k$ approximation for a uniform matroid, where k is the cardinality of the solution. Demonstrative examples are presented showing the usefulness of these approximation bounds: 1) the approximation bound for maximizing monotone non-decreasing functions using a greedy algorithm is expressed in a unified framework; 2) a tighter approximation can be obtained when the bound of α is known.

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