A solution to an open problem on lower against number in graphs

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Abstract

In [1] the problem of finding a sharp lower bound on lower against number of a general graph is mentioned as an open question. We solve the problem by establishing a tight lower bound on lower against number of a general graph in terms of order and maximum degree.

Keywords: Lower against number, maximal negative function.

1 Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use [2] for terminology and notations which are not defined here. The open neighborhood of a vertex v is denoted by $N(v)$, and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$. A graph G is called r-regular if $deg(v) = r$ for every $v \in V(G)$, and nearly r-regular if $deg(v) \in \{r-1, r\}$ for every $v \in V(G)$.
Let $S \subseteq V$. For a real-valued function $f : V \to R$ we define $f(S) = \sum_{v \in S} f(v)$. Also, $f(V)$ is the weight of f. A function $f : V \to \{-1, 1\}$ is called negative if $f(N[v]) \leq 1$, for every $v \in V(G)$. The maximum of values of $f(V(G))$, taken over all negative functions f, is called the against number $\beta_N(G)$. The author in [1] exhibited a real-world application of it to social networks. (This concept was introduced by Zelinka [3] as signed 2-independence number).
A negative function f of a graph G is maximal if there exist no negative function g such that $g \neq f$ and $g(v) \geq f(v)$ for every $v \in V(G)$. The

minimum of values of $f(V(G))$, taken over all maximal negative functions f, is called the lower against number and is denoted by $\beta_N^*(G)$.
In [1], Wang proved the following lower bounds on $\beta_N^*(G)$ for regular and nearly regular graphs.

Theorem 1.1. *Let G is an r-regular graph of order n. Then, $\beta_N^*(G) \geq (r+2-r^2)n/(r+2+r^2)$ for r even, and $\beta_N^*(G) \geq (1-r)n/(1+r)$ for r odd. This bound is best possible.*

Theorem 1.2. *For any nearly r-regular graph G of order n, $\beta_N^*(G) \geq (1-r)n/(1+r)$. Furthermore, this bound is sharp.*

Also, the author posed the following question as an open problem: *What is a sharp lower bound on $\beta_N^*(G)$ for a general graph G?*
Recently, Zhao in [4] proved that if G is a graph of order n with minimum degree δ and maximum degree Δ , then

$$\beta_N^*(G) \geq (\delta + 2 + \delta\Delta - 2\Delta^2)n)/(\delta + 2 - \delta\Delta + 2\Delta^2)$$

for δ even, and

$$\beta_N^*(G) \geq (\delta + 1 - \Delta + \delta\Delta - 2\Delta^2)n/(\delta + 1 + \Delta - \delta\Delta + 2\Delta^2)$$

for δ odd. Moreover he showed that these bounds are sharp.
In this paper, in answer to the question, we give a sharp lower bound on the lower against number of a general graph just in terms of order and maximum degree that is tighter that ones in [4]. Also, we conclude Theorem 1.1 and Theorem 1.2 as immediate results of our main theorem.

2 A lower bound on $\beta_N^*(G)$

We need the following lemma.

Lemma 2.1. *[1] A negative function f of a graph G is maximal if and only if for every $v \in V(G)$ with $f(v) = -1$, there exists at least one vertex $u \in N[v]$ such that $f(N[v]) = 0$ or 1.*

We are now in a position to present the main result of this paper.

Theorem 2.2. *Let G be a graph of order n with maximun degree Δ. Then*

$$\beta_N^*(G) \geq \begin{cases} \left(\dfrac{1-\Delta}{1+\Delta}\right)n & \Delta \geq \delta + 1 \quad or \quad \delta = \Delta \equiv 1 \ (mod\ 2) \\ \left(\dfrac{\Delta + 2 - \Delta^2}{\Delta + 2 + \Delta^2}\right)n & otherwise. \end{cases}$$

and these bounds are sharp.

Proof. If $\delta = \Delta \equiv 0$ $(mod\ 2)$, then desired result follows by Theorem 1.1. Hence in what follows we may assume that $\Delta \geq \delta+1$ or $\delta = \Delta \equiv 1$ $(mod\ 2)$. Let f be a maximal negative function of G with weight $f(V(G)) = \beta^*_N(G)$ and $M = \{v \in V | f(v) = -1\}$ and $P = \{v \in V | f(v) = 1\}$. Also, $m = |M|$ and $p = |P|$. For notational convenience, we set $l = \lfloor \frac{\Delta}{2} \rfloor + 1$ and $k = \lfloor \frac{\delta}{2} \rfloor + 1$. We define $A_i = \{v \in M || N(v) \cap P| = i\}$ and $a_i = |A_i|$, for all $0 \leq i \leq l$. Let $v \in M$. Since f is a negative function, then v has at most l neighbors in P. Therefore, P is the disjoint union, for $0 \leq i \leq l$, of the sets A_i. Now we get

$$n = p + m = p + \sum_{i=0}^{l} a_i. \tag{1}$$

On the other hand, if $[M, P]$ is the set of edges having one end point in M and the other in P, then

$$|[M, P]| = \sum_{i=1}^{l} i a_i \leq p\Delta. \tag{2}$$

Case 1. If $A_0 = \phi$. By inequalities (1) and (2), we have

$$n = p + \sum_{i=1}^{l} a_i \leq p + \sum_{i=1}^{l} i a_i \leq p + p\Delta.$$

Therefore, $p = (n + \beta^*_N(G))/2 \geq \dfrac{n}{1+\Delta}$, which implies the desired lower bound.

Case 2. If $A_0 \neq \phi$. Let $v \in A_0$. Obviously, $f(N[v]) \leq -2$. Now Lemma 1 implies that there exists a vertex $u \in N[v]$ such that $f(N[u]) = 0$ or 1. This shows that the set $Q = \{v \in N(A_0) | f(N[v]) = 0 \text{ or } 1\}$ is nonempty. Let $v \in \cup_{i=0}^{k-1} A_i$. Then

$$\begin{aligned} f(N[v]) &= |N[v] \cap P| - |N[v] \cap (V \setminus P)| = 2|N[v] \cap P| - |N[v]| \\ &\leq 2(k-1) - deg(v) - 1 \leq -1. \end{aligned}$$

Therefore v does not belong to Q. Hence, $Q \subseteq \cup_{i=k}^{l} A_i$. Suppose that $u \in Q \cap A_i$, for $k \leq i \leq l$. We claim that $|N(u) \cap A_o| \leq i - 1$. Suppose to the contrary that $|N(u) \cap A_o| \geq i$. Then

$$\begin{aligned} 0 \text{ or } 1 = f(N[u]) &= -1 + |N(u) \cap P| - |N(u) \cap M| \\ &\leq -1 + i - |N(u) \cap A_0| \leq -1 \end{aligned}$$

a contradiction. Thus $Q \cap A_i$ has at most $(i-1)|Q \cap A_i|$ neighbors in A_0. Since f is a maximal negative function, for every vertex $v \in A_0$ there exists a vertex $u \in Q$ such that $u \neq v$, which implies $u \in Q \cap A_i$, for some $k \leq i \leq l$. Hence $A_0 \subseteq \cup_{i=k}^{l} N(Q \cap A_i)$. Now we deduce that

$$a_0 = |A_0| \leq \sum_{i=k}^{l} |N(Q \cap A_i) \cap A_0| \leq \sum_{i=k}^{l} |Q \cap A_i|(i-1) \leq \sum_{i=k}^{l} (i-1)a_i.$$

By (1), we have

$$n = p + a_0 + \sum_{i=1}^{l} a_i \leq p + \sum_{i=k}^{l} (i-1)a_i + \sum_{i=1}^{k-1} a_i + \sum_{i=k}^{l} a_i.$$

Thus

$$n \leq p + \sum_{i=1}^{l} i a_i \leq p + p\Delta.$$

Therefore $p = (n + \beta_N^*(G))/2 \geq \dfrac{n}{\Delta + 1}$, as desired.
Since Theorem 1.2 (also Theorem 1.1) is a special case of this theorem, we see that this lower bound is sharp. $\square$

Comparing Theorem 2.2 with its corresponding result in [4] we can see that the lower bounds in Theorem 2.2 are tighter that their corresponding ones in [4]. Moreover, Theorem 1.1 and Theorem 1.2 are immediate results of Theorem 2.2 when $\delta = \Delta = r$.

References

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