Maximal Independent Sets in Generalised Caterpillar Graphs

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Abstract

A caterpillar graph is a tree which on removal of all its pendant vertices leaves a chordless path. The chordless path is called the backbone of the graph. The edges from the backbone to the pendant vertices are called the hairs of the caterpillar graph. Ortiz and Villanueva (C.Ortiz and M.Villanueva, Discrete Applied Mathematics, 160(3): 259-266, 2012) describe an algorithm, linear in the size of the output, for finding a family of maximal independent sets in a caterpillar graph.

In this paper, we propose an algorithm, again linear in the output size, for a generalised caterpillar graph, where at each vertex of the backbone, there can be any number of hairs of length one and at most one hair of length two.

Keywords: Maximal Independent Set; MIS; Caterpillar Graphs; Generalised Caterpillar Graphs; Generating all MIS; Algorithm

1 Introduction

A caterpillar graph $C(P_k)$ (See Figure 1) is a tree which on removal of all its pendant vertices (vertices h_i and l_j in the figure) results in a chordless path $P_k = \{v_1, v_2, ..., v_k\}$ of k vertices. The path P_k is called the backbone of the caterpillar graph $C(P_k)$ and the edges from the backbone to the pendant vertices (edges (v_i, h_i) and (v_j, l_j) in Figure 1) are called its hairs. In $C(P_k)$ all hairs are of length one. Harary and Schwenk [3], introduced Caterpillar graphs by saying:

"Caterpillar is a tree which metamorphoses into a path when its cocoon of endpoints is removed".

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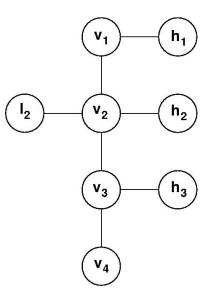


Figure 1: An example for a caterpillar graph

In chemical graph theory, caterpillar graphs are useful in studying topological properties of benzenoid hydrocarbons[2]. In fact, Basil and Sherif [2]observe that

"It is amazing that nearly all graphs that played an important role in what is now called "chemical graph theory" may be related to caterpillar trees."

Ortiz and Villanueva [4] describe an algorithm for enumerating a family of maximal independent sets in caterpillar graphs. The algorithm takes time linear in the size of the output, i.e., is linear in the sum of sizes of all maximal independent sets. They also propose $CC^2(P_k)$, a generalisation in which hairs have length exactly two.

In this paper, we consider a still more generalised version of caterpillar graphs (See Figure 2). In this generalised version, we allow a backbone vertex v_i to have up to one hair of length two and any number of hairs of length one; in particular v_i may have no hairs at all. If $P_k = \{v_1, ..., v_k\}$, a chordless path of length k is the backbone, then we denote the generalised caterpillar graph by $C^{1,2}(P_k)$.

A complete caterpillar graph $CC(P_k)$ is a (usual) caterpillar graph such that there is at least one hair at each of its backbone vertices. The graph of Figure 1 is not complete as vertex v_4 does not have any hair. The contraction graph G_k of a (usual) caterpillar graph $C(P_k)$ is the graph obtained by contracting, for each backbone vertex v_i of the $C(P_k)$, all the pendant vertices incident at v_i to a single vertex [4].

An independent set or a stable set is a set of vertices in a graph such that no

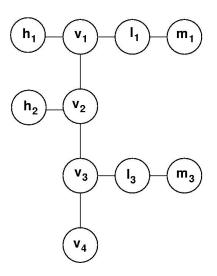


Figure 2: An example for a generalised caterpillar graph

two vertices in the set are adjacent. That is, it is a set I of vertices of a graph G such that if I contains two vertices, say a and b, then ab is not an edge of G. The size or cardinality of an independent set I is the number of vertices in the set I. An independent set I will be called a maximal independent set if every vertex v is either in I or is adjacent to a vertex in I.

Valiant [6] shows that the problem of counting the number of maximal independent sets is #P-complete for general graphs. Tsukiyama et al. [5] show that we can enumerate a family of maximal independent sets of a general connected graph in O(nmm(G)) time; here n is the number of vertices, m is the number of edges and m(G) is the number of maximal independent sets of a graph G. Ortiz and Villanueva [4] show that $m(C(P_k))$, the number of maximal independent sets of a (usual) caterpillar graph $C(P_k)$ is the same as $m(G_k)$, the number of maximal independent sets of its contraction graph, G_k . They give an algorithm to find a family of maximal independent sets of a caterpillar graph in time polynomial in the number of maximal independent sets. In this paper, we obtain a similar result for generalised caterpillar graphs.

Vertices of generalised caterpillar graph can be partitioned into stages. Vertex v_i together with vertices in hairs incident at v_i will be said to be at stage *i*. Formally, if we delete vertex v_i from $C^{1,2}(P_k)$, the graph may split into several components. The vertices in components which do not contain either vertex v_{i-1} or v_{i+1} will be at stage *i* (along with vertex v_i).

Let $x_1, x_2, ..., x_{m_i}$ be the pendant vertices in hairs of length one at stage i of $C^{1,2}(P_k)$. If any of these vertices is in a maximal independent set S, then v_i cannot be in the independent set S. Conversely, if v_i is not in S, then we have to put all these vertices in the maximal independent set S. Thus, any pendant vertex belonging to a hair of length one at stage i is contained in a maximal

independent set S, if and only if all pendant vertices at stage i are contained in S.

Hence, the number of maximal independent sets of $C^{1,2}(P_k)$ is independent of the number of hairs of length one at each stage (provided there is at least one such hair). Hence from now on, we assume that $C^{1,2}(P_k)$ has at most one hair of length one at each stage i (wherever there is at least one such hair) and we denote this hair by $v_i h_i$. We will denote the hair of length two at stage i by $v_i l_i m_i$, wherever it exists.

2 Structure of Maximal Independent Sets in generalised Caterpillar Graphs

If S is any maximal independent set of $C^{1,2}(P_k)$, let S_i be the subset of S containing only vertices at stage *i*. Clearly, $S = S_1 \cup S_2 \cup ..., S_k$. In a caterpillar graph, the only vertices from two different stages, adjacent to each other are the vertices v_i and v_{i+1} (or v_{i-1} and v_i).

In case, if a hair of length one is present at stage i (hair of length two may or may not be there), then either h_i or v_i (but not both) will be present in any maximal independent set. Hence, in this case, exactly one of h_i or v_i will be in the set S_i .

Similarly, if a hair of length two is present at stage i (hair of length one may or may not be present), then either m_i or l_i (but not both) will be present in any maximal independent set. Hence, in this case, exactly one of m_i or l_i will be present in S_i .

Also observe that as v_i and l_i are adjacent, both of them cannot be present in any independent set.

In general, either both hairs of length one and two, or only one or neither may be present at stage i. In all there are exactly three possibilities.

Case 1 $(v_i \in S_i)$: If $v_i \in S_i$, then if a hair of length two is present then $l_i \notin S_i$ and hence $m_i \in S_i$. Moreover, even if we have hair of length one, then as v_i and h_i are adjacent, $h_i \notin S_i$, thus

 $S_i = \{v_i, m_i\}$ if hair of length two is present $S_i = \{v_i\}$ otherwise

Case 2 $(l_i \in S_i)$: If $l_i \in S_i$, then $v_i \notin S_i$ and $m_i \notin S_i$. If a hair of length one is present at stage *i*, then as $v_i \notin S_i$, $h_i \in S_i$, thus, in this case $S_i = \{h_i, l_i\}$ if hair of length one is present $S_i = \{l_i\}$ otherwise

Case 3 $(v_i, l_i \notin S_i)$ If length one hair is present, then as $v_i \notin S_i$ we must have $h_i \in S_i$. If length two hair is present then as $l_i \notin S_i$, $m_i \in S_i$. Hence, in this case,

 $S_i = \{h_i, m_i\}$ if hairs of length one and two are both present

 $S_i = \{h_i\}$ if only hair of length one is present

 $S_i = \{m_i\}$ if only hair of length two is present

 $S_i = \Phi$ otherwise

Thus, in each of these cases, all vertices at stage i, except possibly v_i , will either be in the independent set S_i or will be adjacent to a vertex in S_i . If $v_i \in S_i$, the set S_i will be a maximal independent set of the subgraph at stage i.

Thus, we have the following lemma:

Lemma 1: $S = S_1 \cup S_2 \cup ... S_k$ is a Maximal Independent Set of generalised caterpillar graph $C^{1,2}(P_k)$ if and only if following conditions hold:

(1) for each v_i which is not adjacent to a vertex in S_i , either $v_{i-1} \in S_{i-1}$ or $v_{i+1} \in S_{i+1}$.

(2) both $v_{i-1} \in S_{i-1}$ and $v_i \in S_i$ should not simultaneously hold.

Proof: As each S_i is an independent set, and as each vertex of S_i except possibly v_i is either in S_i , or has a neighbour in S_i , the set S will be a maximal independent set iff either each $v_i \in S$ or each v_i has a neighbour in S. Vertex $v_i \in S$ iff $v_i \in S_i$.

If $v_i \notin S_i$, then v_i has a neighbour in S iff one of the following conditions hold

(1) either v_i has a neighbour in S_i , or

(2) $v_{i-1} \in S$ or

(3) $v_{i+1} \in S$.

For the set S to be independent, clearly the second condition must hold. The lemma thus follows. []

3 Finding Maximal Independent Sets in generalised Caterpillar Graphs

Let us assume that $S = S_1 \cup S_2 \cup ... S_k$ is a maximal independent set of $C^{1,2}(P_k)$. Then, for $i \leq k$, we classify the set S_i depending upon the "status" of vertex v_i .

Type 1: If $v_i \in S_i$, then we will say S_i is of Type 1.

Type 2: If $v_i \notin S_i$, but some neighbour of v_i is in S_i , then we will say S_i is of Type 2. In this case either $h_i \in S_i$ or $l_i \in S_i$.

Type 3: If no neighbour of v_i is in S_i , but $v_{i-1} \in S_{i-1}$, then we will say S_i is of Type 3.

Type 4: If $v_{i-1} \notin S_{i-1}$, $v_i \notin S_i$ and no neighbour of v_i is in S_i , then we will say that S_i is of Type 4. In this case, for the set S to be maximal, $v_{i+1} \in S_{i+1}$.

The last stage S_k cannot be of Type 4, as in that case neither $v_k \in S$ nor $v_{k-1} \in S$. And hence, as v_k does not have a neighbour in S_k , v_k will not have any neighbour in S (there is no vertex v_{k+1}), violating maximality. Further, as S_1 is the first independent set, S_1 cannot be of Type 3 (there is no vertex v_0).

We store the types of hairs present at stage i in an array T. The entry T[i] = 0 if there are no hairs at stage i

T[i] = 1 if there are only length one hairs at stage i

T[i] = 2 if there is only length two hair at stage i

T[i] = 3 if stage *i* has both types of hairs

Then the table below summarises the discussion above and gives the list of all S_i 's of each type.

T[i]	S_i of type 1	S_i of type 2	S_i of type 3	S_i of type 4
0	$\{v_i\}$	none	ϕ	ϕ
1	$\{v_i\}$	$\{h_i\}$	none	none
2	$\{v_i, m_i\}$	$\{l_i\}$	$\{m_i\}$	$\{m_i\}$
3	$\{v_i, m_i\}$	$\{h_i, l_i\}, \{h_i, m_i\}$	none	none

Table 1: The possible instances of S_i and their types depending on the hairs present

Theorem 1: $S = S_1 \cup S_2 \cup ... S_k$ is a maximal independent set of a generalised caterpillar graph $C^{1,2}(P_k)$ if and only if for $1 \le i \le k-1$, (S_i, S_{i+1}) is of one of the following forms, and S_k is not of the type 4.

- (1) (type 1, type 2)
- (2) (type 1, type 3)
- (3) (type 2, type x), where x is 1,2 or 4
- (4) (type 3, type x), where x is 1,2 or 4
- (5) (type 4, type 1)

Proof: Let us prove the 'only if' part first. Let us assume that S is a maximal independent set of $C^{1,2}(P_k)$. We have one of the following three cases, depending on the type of S_i .

(1) If S_i is of Type 1, then as $v_i \in S_i$, S_{i+1} cannot be of Type 4. As $v_i \in S_i$, $v_{i+1} \notin S_{i+1}$, hence S_{i+1} cannot be of Type 1. If v_{i+1} has a neighbour in S_{i+1} , then S_{i+1} will be of Type 2, otherwise of Type 3.

(2) If S_i is of Type 4, then as we saw earlier, $v_{i+1} \in S_{i+1}$, and hence S_{i+1} will be of Type 1.

(3) If S_i is of Type 2 or of Type 3, then as $v_i \notin S_i$, S_{i+1} cannot be of Type 3. If $v_{i+1} \in S_{i+1}$, then S_{i+1} will be of Type 1. If a neighbour of v_{i+1} is in S_{i+1} , then S_{i+1} will be of Type 2. If $v_{i+1} \notin S_{i+1}$, and if it does not have any neighbour in S_{i+1} , then S_{i+1} , then S_{i+1} will be of Type 4.

To prove the 'if' part, we need to prove that for $1 \leq i \leq k$, if (S_i, S_{i+1}) is in one of the forms listed (and S_k is not of Type 4), then S is a maximal independent set. As each vertex at stage *i*, except possibly v_i , is either in S_i or is adjacent to a vertex in S_i , we only need to show that

(a) Both $v_i \in S_i$ and $v_{i+1} \in S_{i+1}$ cannot simultaneously hold.

(b) If $v_i \notin S_i$, then v_i has a neighbour in S.

If $v_i \in S_i$, then S_i will be of Type 1, and if $v_{i+1} \in S_{i+1}$, then S_{i+1} will also be of Type 1. As we do not have the form (Type 1, Type 1), the first condition

holds.

If $v_i \notin S_i$, then S_i will not be of Type 1. If S_i is of Type 2, then v_i will have a neighbour in S_i , and hence in S. If S_i is of Type 3, then as the only permissible form for (S_{i-1}, S_i) is (Type 1,Type 3). Thus, S_{i-1} has to be of Type 1, and $v_{i-1} \in S_{i-1}$; or v_i will have a neighbour v_{i-1} in S_{i-1} , and hence in S.

Finally, if S_i is of Type 4, then as the only permissible form is (Type 4, Type 1). Thus, S_{i+1} has to be of Type 1 and $v_{i+1} \in S_{i+1}$ (for $i \neq k$) and so v_{i+1} will have a neighbour in S_{i+1} , or in S. Hence, S will be both independent and maximal. []

4 Finding Family of Maximal Independent Sets

From Theorem 1, if (S_i, S_{i+1}) is of one of the forms listed, then $S = S_1 \cup S_2 \cup ...S_k$ will be a maximal independent set. Thus, to find a family of all maximal independent sets, we need to find all such valid sequences. For this, we construct a directed k-level graph L_k such that any maximal independent set in the generalised caterpillar graph corresponds to a source-sink (source to sink) path in L_k .

A k-level graph $G = (V, E, \phi)$ with $k \leq n$ is a graph with an assignment of levels $\phi : V \to \{1, 2, ..., k\}$ that partitions the vertex set into k pairwise disjoint subsets, $V_1, V_2, ..., V_k$ such that $V = V_1 \cup V_2 \cup ... V_k$. Further, if (uv) is an edge in G, then u and v are not in the same level [1]. In our level graph L_k , the edges are only from level i to level i + 1.

We will denote the set of vertices of L_k by U. The k levels in L_k are numbered from 1 to k; level i in L_k corresponding to stage i in $C^{1,2}(P_k)$. Roughly speaking, at each level i, the vertices in L_k will correspond to one possible instance of S_i . The total number of vertices present at any level i of the level graph, will depend on the types of hairs present at stage i. We will see that the number of vertices at each level will be at most five.

We will use two labels "type" and "index" on vertices of L_k . Type of a (new) vertex will correspond to the type of corresponding S_i . Index will be one in all but one case. From Table 1, observe that, in all but one case, for any value of T[i], there is at most one instance of S_i . If T[i] = 3, then there are two instances of S_i . We use the label "index" to distinguish these cases.

In more detail, we add vertices at level *i* as follows. First, for each *i*, we first add a vertex p_i and set $type(p_i) = 1$ and $index(p_i) = 1$. This will correspond to the case when $v_i \in S_i$.

Depending upon type of hairs present at v_i , we add other vertices at level i as follows:

Only Length one hairs (T[i] == 1): Add a vertex s_i and set $type(s_i) = 2$ and $index(s_i) = 1$. This will correspond to the case when S_i is of type 2, and $S_i = \{h_i\}$.

Only Length two hairs (T[i] == 2): In this case, also, we first add a vertex s_i and set $type(s_i) = 2$ and $index(s_i) = 1$. This will correspond to the case when S_i is of type 2, and $S_i = \{l_i\}$. Further, we also

(a) for $i \ge 2$, we add a vertex t_i and set $type(t_i) = 3$ and $index(t_i) = 1$.

(b) and for $i \leq k-1$, we add a vertex u_i and set $type(u_i) = 4$ and $index(u_i) = 1$. These cases correspond to the case when S_i is of type 3 or of type 4. In these cases, $S_i = \{m_i\}$.

Both Length one and two hairs (T[i] == 3): Add two vertices q_i and r_i and set $type(q_i) = type(r_i) = 2$; $index(q_i) = 1$ and $index(r_i) = 2$.

This corresponds to the case when S_i is of type 2. Here S_i can be either $\{h_i, l_i\}$ or $\{h_i, m_i\}$, thus we need two vertices, one for each case. We also use the label "*index*" to distinguish these two cases.

No Hairs (T[i] == 0): In this case

(a) for $i \ge 2$, we add a vertex t_i and set $type(t_i) = 3$ and $index(t_i) = 1$.

(b) and for $i \leq k-1$, we add a vertex u_i and set $type(u_i) = 4$ and $index(u_i) = 1$.

These cases correspond to the case when S_i is of type 3 or of type 4. In these cases, $S_i = \Phi$.

This is summarised in Table 2 below:

vertex	type	index	Value of $T[i]$	set S_i
p_i	1	1	0, 1, 2 or 3	$v_i \in S_i$
s_i	2	1	1 or 2	$S_i = \{h_i\} \text{ or } S_i = \{l_i\}$
q_i	2	1	3	$S_i = \{h_i, l_i\}$
r_i	2	2	3	$S_i = \{h_i, m_i\}$
t_i	3	1	0 or 2	$S_i = \{m_i\} \text{ or } S_i = \Phi$
u_i	4	1	0 or 2	$S_i = \{m_i\} \text{ or } S_i = \Phi$

Table 2: The vertices at level i in L_k and the corresponding S_i

Next we add following edges. We add an edge from a vertex a in level i to a vertex b in the next level i + 1, if (type(a), type(b)) is one of the following (listed in Theorem 1):

(1,2), (1,3), (2,1), (2,2), (2,4), (3,1), (3,2), (3,4) or (4,1)

Formally, we add the following edges from a vertex a of level i to a vertex b of level i + 1, for $1 \le i \le k - 1$:

(a) For each vertex a of type 1, we add an edge (a, b) in L_k , if vertex b is either of type 2 or of type 3.

(b) For each vertex a of type 2 or type 3, we add an edge (a, b), if vertex b is of type 1, type 2, or of type 4 (i.e., b is not of type 3).

(c) For each vertex a of type 4 we add an edge (a, b), if vertex b is of type 1.

All vertices at Level 1 will be treated as sources and all vertices at level k as sinks, and hence any path from level 1 to level k in L_k will be a source-sink path.

As these edges correspond exactly to valid (S_i, S_{i+1}) pairs of Theorem 1, hence again from Theorem 1, it follows that any source-sink path in L_k will correspond to a valid set of sequence $S_1, S_2, ..., S_k$ and conversely. Thus, if we find all source-sink paths in L_k , we can find all the maximal independent sets in $C^{1,2}(P_k)$. This can be done by using a generalised depth first procedure (DFS), similar to the one used by [4].

We basically, put the start vertex in a stack and then for each neighbour of the start vertex (as the new start vertex), we call the procedure recursively. We stop when the start vertex for the current call is a sink vertex, and print the entire stack; and also remove this vertex from the stack (backtrack to the previous level).

The number of source-sink paths in L_k will be the number of maximal independent sets in $C^{1,2}(P_k)$. We can easily obtain maximal independent sets from the source-sink path. Let $w_1, w_2, ..., w_k$ be the vertices in a source sink path of L_k . Then we reconstruct the S_i corresponding to each P_i as follows (see Table 1):

If $type(w_i) == 1$, then if T[i] == 3 or T[i] == 2 then $S_i = \{v_i, m_i\}$ else $S_i = \{v_i\}$

If $type(w_i) == 3$ or $type(w_i) == 4$, then if T[i] == 2 then $S_i = \{m_i\}$ if T[i] == 0 then $S_i = \Phi$

If $type(w_i) == 2$, then if (T[i] == 1) then $S_i = \{h_i\}$ if (T[i] == 2) then $S_i = \{l_i\}$ if T[i] = 3if $index(w_i) == 1$ then $S_i = \{h_i, l_i\}$, else if $index(w_i) == 2$ then $S_i = \{h_i, m_i\}$

A theorem similar to that of Ortiz and Villanueva [4], also holds for the generalised caterpillar graph.

Theorem 2: We can enumerate all maximal independent sets of $C^{1,2}(P_k)$ in $O(km(C^{1,2}(P_k)))$ time, where $m(C^{1,2}(P_k))$ is the number of maximal independent sets of $C^{1,2}(P_k)$.

Proof: For each source-sink path P of L_k , generalised depth first procedure is called once for each vertex of P. Since any edge in L_k is between adjacent levels, the number of vertices in the path is k + 1. Hence there are only O(k)calls to the procedure. As each call takes O(1) time, except when we reached a sink, in which case it takes O(k) time. But as we reach sink only once for a path, the algorithm takes O(k) time, for each path found.

As there are $m(C^{1,2}(P_k))$ such paths, the algorithm takes $O(km(C^{1,2}(P_k)))$ time. []

As each maximal independent set has k vertices, the algorithm is essentially linear in the output size.

5 Example

Let us illustrate the algorithm for the graph given in Figure 2, we have the following vertices of type 1 in it:

Level 1: p_1 , $type(p_1) = 1$, $index(p_1) = 1$ Level 2: p_2 , $type(p_2) = 1$, $index(p_2) = 1$ Level 3: p_3 , $type(p_3) = 1$, $index(p_3) = 1$ Level 4: p_4 , $type(p_4) = 1$, $index(p_4) = 1$ We have the following vertices of type 2: Level 1: T[i] = 3. Two vertices q_1 and r_1 , $type(q_1) = 2$, $index(q_1) = 1$, $type(r_1) = 2, index(r_1) = 2$ Level 2: T[i] = 1. Vertex s_2 , $type(s_2) = 2$, $index(s_2) = 1$ Level 3: T[i] = 2. Vertex s_3 , $type(s_3) = 2$, $index(s_3) = 1$ Level 4: T[i] = 0. No vertex. We have the following vertices of type 3 in it: Level 1: T[i] = 3. None Level 2: T[i] = 1. None Level 3: T[i] = 2. Vertex t_3 , $type(t_3) = 3$, $index(t_3) = 1$ Level 4: T[i] = 0. Vertex t_4 , $type(t_4) = 3$, $index(t_4) = 1$ The following are vertices of type 4.

Level 1: T[i] = 3. None Level 2: T[i] = 1. None

 $L_{1} = \frac{1}{2} T[i] = 1.$ None

Level 3: T[i] = 2. Vertex u_3 , $type(u_3) = 4$, $index(u_3) = 1$

Level 4: T[i] = 0. None as this is the last level

The edges in the example are as shown in Figure 3.

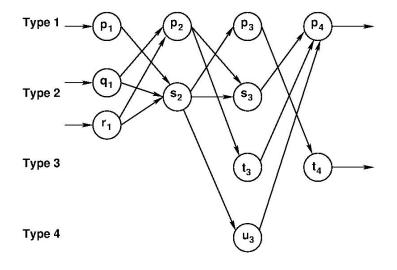


Figure 3: Construction of L_k from $C^{1,2}(P_k)$

Each source-sink path corresponds to a maximal independent set. For example, the path $p_1 - s_2 - u_3 - p_4$ in L_k corresponds to the maximal independent

set $\{v_1, m_1, h_2, m_3, v_4\}$ in $C^{1,2}(P_k)$. Similarly, the path $r_1 - s_2 - p_3 - t_4$ in L_k corresponds to the maximal independent set $\{h_1, m_1, h_2, v_3, m_3\}$ in $C^{1,2}(P_k)$.

Conclusions

We discuss the problem of finding a family of maximal independent sets in a generalised caterpillar graph. We show that this problem can also be reduced to the problem of finding all source-sink paths in a level graph. The proposed algorithm takes time linear in the output size (total number of vertices in all maximal independent sets).

Further, we believe, that this algorithm can be extended for another generalisation of caterpillar graph, where each vertex of backbone has bounded number of hairs of length more than one. It may also be possible to generalise the algorithm for some other generalisations of caterpillar graphs, possibly having a different set of hairs. We may have to identify the new set of possible S_i 's at each stage i and classifying them into possibly some other different "types".

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