PLANAR GRAPHS WITHOUT 4-CYCLES AND CLOSE TRIANGLES ARE $(2,0,0)$-COLORABLE

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Abstract. For a set of nonnegative integers $c_1, \dots, c_k$, a $(c_1, c_2, \dots, c_k)$-coloring of a graph G is a partition of $V(G)$ into $V_1, \dots, V_k$ such that for every i, $1 \le i \le k, G[V_i]$ has maximum degree at most c_i. We prove that all planar graphs without 4-cycles and no less than two edges between triangles are $(2,0,0)$-colorable.

1. Introduction

The coloring of planar graphs has a long history. The well-known Four Color Theorem, proved by Appel and Haken (see [1]-[2]) in the 1970s, states that all planar graphs are 4-colorable. Determining whether an arbitrary planar graph is 3-colorable is NP-complete; much attention has been given to proving sufficient conditions under which planar graphs are 3-colorable. The classic example is the theorem by Grötzch [9] showing that planar graphs without 3-cycles are 3-colorable.

Recently, the study of the coloring of planar graphs with 3 colors has taken a very interesting turn. Steinberg [17] in 1976 famously conjectured that planar graphs without 4-cycles and 5-cycles are 3-colorable. Erdős asked for the constant D such that planar graphs excluding cycles of lengths from 4 to D are 3-colorable. Borodin, Glebov, Raspaud, and Salavatipour [4] showed that $D \le 7$. After being open for almost 40 years, in a very recent paper [6], the Steinberg Conjecture was disproved by a counterexample. This surprising result suggests that the property of planar graphs being 3-colorable may be more rare than was previously thought, and spurs the search for more classes of planar graphs that are 3-colorable.

One interesting restriction that gives rise to classes of 3-colorable planar graphs involves forbidding triangles that are close together. This idea is illustrated in the famous conjecture by Havel.

Conjecture 1.1 (Havel, 1969)**.** *There is a constant C (perhaps as small as 4) such that any planar graph whose triangles are at distance at least C from each other is 3-colorable.*

This conjecture was resolved by Dvořák, Král' and Thomas [8] by showing the truth for any planar graph G with $d_\Delta(G) > 10^{100}$, where $d_\Delta(G)$ is the length of the shortest path between the vertices of any two 3-cycles. Clearly more work is needed to understand the constant C, but in the meantime, there have been advances that combine the hypotheses of the Steinberg and the Havel conjectures. For example, Borodin and Glebov [3] showed that any planar graph G without 5-cycles and satisfying $d_\triangle(G) \ge 2$ is 3-colorable.

With the recent counterexample to Steinberg's conjecture showing that it may be more difficult to find 3-colorable planar graphs than originally thought, it becomes more interesting to investigate

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"nearly" 3-colorable planar graphs. A graph G is $(c_1, c_2, \dots, c_k)$-colorable if $V(G)$ can be partitioned into k nonempty subsets $V_1, V_2, \dots, V_k$ such that the maximum degree of $G[V_i]$ is at most c_i. In other words, there exists a k-coloring such that for each color i, each vertex colored with i has at most c_i neighbors of the same color. Clearly, a graph is properly 3-colorable if and only if it is $(0,0,0)$-colorable. In [7], it is shown that every planar graph is $(2,2,2)$-colorable.

There are many results in this area; we refer interested readers to [16]. As an illustration, the following is a list of results known for 5-cycle-free planar graphs.

Theorem 1.2. *Let G be a planar graph without 5-cycles.*

- *If G also has no 4-cycles, then it is $(2,0,0)$- and $(1,1,0)$-colorable (*[5, 21, 12]*).*
- *If G has no intersecting triangles, then it is $(2,0,0)$- and $(1,1,0)$-colorable (*[14, 15]*).*
- *If G has no K_4^-, then it is $(1,1,1)$- and $(1,1,0)$-colorable. (*[13, 20]*).*

In [18], Wang and Xu proved that planar graphs without 4-cycles are $(1,1,1)$-colorable (in fact, $(1,1,1)$-choosable), and constructed a non-3-colorable planar graph that has no 4-cycles (and $d_\Delta = 1$). (See Figure 1.) Furthermore, although the result by Borodin and Glebov [3] (and other likewise results) forbids 5-cycles and not 4-cycles, their proof involves showing that there are no internal 4-cycles in a minimal counterexample.

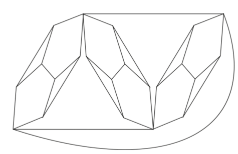

Figure 1: A non-$(0,0,0)$-colorable planar graph without 4-cycles.

This motivates the study of the 3-colorability of planar graphs without 4-cycles (but perhaps with 5-cycles) and satisfying $d_\triangle(G) \geq 2$. We conjecture that the following is true.

Conjecture 1.3. *If G is a planar graph without 4-cycles such that $d_\triangle(G) \geq 2$, then G is 3-colorable.*

In this paper, we prove a relaxation of Conjecture 1.3. Let $\mathcal{G}$ be the set of planar graphs with $d_\triangle(G) \geq 2$ and no 4-cycles.

Theorem 1.4. *If $G \in \mathcal{G}$, then G is $(2,0,0)$-colorable.*

To prove Theorem 1.4, we use the idea of superextendable colorings introduced by Xu in [20].

Definition A $(2,0,0)$-coloring ϕ of a subgraph H of G *superextends* to G if there exists a $(2,0,0)$-coloring ϕ_G of G that extends ϕ with the property that $\phi(v) \neq \phi(u)$ whenever $v \in H$ and $u \in G \cap N(v) - H$, where $N(v)$ is the set of neighbors of v. We say that a subgraph $H \subseteq G$ is *superextendable* to G if every $(2,0,0)$-coloring ϕ_H of H superextends to G. When we wish to specify G, we will say (G, H) is superextendable.

We need the following definition.

Definition A 6-cycle is *bad* if alternating vertices along the 6-cycle are matched to the vertices of a triangle. The triangle is called an *interior triangle* of a bad 6-cycle. (See Figure 2.) Otherwise, a 6-cycle is *good.*

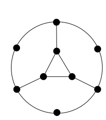

Figure 2: A bad 6-cycle.

Our approach is to prove the following stengthening of Theorem 1.4.

Theorem 1.5. *For each $G \in \mathcal{G}$, every triangle, 5-cycle and good 6-cycle in G is superextendable.*

Observe that the restriction to good 6-cycles is necessary. Otherwise, the graph in Figure 2 is a counterexample: precolor the vertices of degree 3 on the 6-cycle with color 1.

Assuming Theorem 1.5 holds, it is easy to verify that Theorem 1.4 also holds. If G has no triangles, then G is $(2,0,0)$-colorable (in fact, $(0,0,0)$-colorable by Grötzch's theorem). Otherwise, fix a $(2,0,0)$-coloring ϕ of some triangle; by Theorem 1.5, the coloring can be superextended to G, which is a $(2,0,0)$-coloring of G.

In Section 2, we highlight the advantage of proving Theorem 1.5, and we present some preliminary observations about a minimum counterexample to the theorem. The proof uses the discharging method, so Sections 3 and 4 contain, respectively, the reducible configurations and the discharging arguments.

2. Preliminaries and Definitions

The advantage of proving the stronger theorem involving superextendable colorings was noted by Xu in [20]. Let a cycle C in a plane graph G be a *separating* cycle if the deletion of C results in a disconnected graph. Let $int(C)$ denote the *interior* of C, and similarly $ext(C)$ the *exterior*, when the vertices of C are deleted. If a proper coloring of a separating cycle C can be extended to $int(C)$ and $ext(C)$ individually, then the union of the two colorings is a proper coloring. However, this property would not hold for $(2,0,0)$-colorings of the two subgraphs; a vertex of C precolored with color 1 may have two neighbors of color 1 in both $int(C)$ and $ext(C)$, so the union of the two colorings would contain a vertex of color 1 with four neighbors of color 1. The superextendable property allows us to combine colorings of $int(C)$ and $ext(C)$ into a $(2,0,0)$-coloring of the entire graph.

In order to illustrate this more clearly, we must introduce some notation that will be used for the remainder of the paper. Our proof of Theorem 1.5 is by contradiction, so we will let (G,C) for $G \in \mathcal{G}$ be a counterexample to the theorem of minimum order. That is, some fixed precoloring ϕ of a cycle C of length 3 or 5 or a good cycle of length 6 in G cannot be superextended to G, and G is the smallest graph with this property. Let $V(C)$ denote the vertices of the cycle, and let $|V(C)| = r$.

We first observe that C cannot be a separating cycle, analogous to Lemma 1 in [20]. Otherwise, ϕ can be superextended individually to $int(C)$ and $ext(C)$ by the minimality of G, and then the union of these two colorings would be a superextension of ϕ to G, a contradiction. Hence we may assume that G is drawn with C as the exterior face.

Lemma 2.1. *G does not contain separating triangles, 5-cycles or good 6-cycles.*

Proof. Suppose otherwise that G contains a separating cycle C', where the length of C' is 3 or 5, or C' is a good 6-cycle. Let G_1 be the subgraph of G induced by C' together with $ext(C')$, and G_2 the subgraph of G induced by C' together with $int(C')$. Note that C is contained in G_1. By the minimality of G, ϕ superextends to a $(2,0,0)$-coloring ϕ_{G_1} of G_1. Now ϕ_{G_1} restricted to C' is a $(2,0,0)$ precoloring of C', and again applying the minimality of G, it superextends to a $(2,0,0)$-coloring of G_2. The union of these two colorings is a superextension of ϕ to G, a contradiction. $\square$

The lack of separating short cycles provides additional information about the structure of C. The next lemma follows [20].

Lemma 2.2. *The cycle C is chordless, and for nonadjacent $x, y \in C$, $N(x) \cap N(y) \subseteq V(C)$.*

Proof. The conclusion is trivial if $r = 3$; suppose that $r = 5$ or $r = 6$.

If C has a chord and $r = 5$, then the chord separates C into a 3-cycle and a 4-cycle, contradicting $G \in \mathcal{G}$. If $r = 6$, then the chord would create either a 4-cycle and a 5-cycle, again contradicting $G \in \mathcal{G}$, or a 3-cycle and a 5-cycle. In the second case, since $V(C) \neq V(G)$, one of these cycles would have to be a separating cycle, contradicting Lemma 2.1.

Now consider nonadjacent x, y in $V(C)$. Suppose that there exists $v \in N(x) \cap N(y)$ where $v \notin V(C)$. If $r = 5$, then C together with xvy forms a 4-cycle and a 5-cycle, which is impossible. If $r = 6$, then C together with xvy forms either a 6-cycle and a 4-cycle, which is impossible, or two 5-cycles. Neither of the 5-cycles can be separating, but then $V(G) = C \cup \{v\}$, and it is easy to verify that such a graph is not a counterexample. Therefore x and y can have no such neighbor. $\square$

Now we introduce some definitions we use in the rest of the paper. In a $(2, 0, 0)$-coloring of G, a vertex v is 1-*saturated* if it is colored with 1 and has two neighbors of color 1; otherwise, it is called *nicely colored* (i.e., it is colored with 2 or 3, or it is colored with 1 but not 1-saturated). If v is nicely colored, then a neighbor of v can be (re)colored with 1. If v has at most three colored neighbors, then *nicely recoloring* v means v is either recolored with color 2 or 3 (if one of those colors is available), or v has at most one neighbor with color 1, and v remains color 1.

Lemma 2.3. *Suppose $\phi_{G'}$ is a superextension of ϕ to $G' \subset G$, and $v \in V(G')$. If v is nicely recolored, then the extension remains a superextension.*

Proof. Since the color of v is not changed to 1, v can be color 1 only if $\phi_{G'}(v) = 1$, in which case v must not have a neighbor of color 1 on C. $\square$

Suppose that x is a vertex of a face f. A neighbor v of x is an *outer neighbor* (with respect to f) if v is not on f. A k^+-vertex (or k^--vertex) in G is a vertex of degree at least (or at most) k. A $(k_1, k_2, \dots, k_t)$-*cycle* is a t-cycle whose vertices' degrees are $k_1, \dots, k_t$, respectively. A vertex is *triangular* if it lies on a 3-face, otherwise it is called *nontriangular*.

Let f be a 3-face in $int(C)$. If $v \notin V(f)$ is adjacent to a 3-vertex x on f, then f is called a *pendant face to* v, and x and v are pendant neighbors to each other.

A 3-vertex v on a 5-face in $int(C)$ is called *special* if its two neighbors on the face are 4^--vertices. If x is a 5^+-vertex with a special neighbor v, we call the pendant 5-face containing v a *pendant special 5-face* to x. (See Figure 3.) A 3-vertex in $int(C)$ is *potentially special* if two of its neighbors are 4^--vertices in $int(C)$. Note that a special vertex is also potentially special.

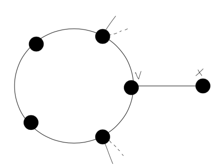

Figure 3: Special vertex v and a pendant special 5-face to x.

3. Reducible Configurations

Lemma 3.1. *There are no 2^--vertices in $int(C)$.*

Proof. Let v be a 2^--vertex in $int(C)$. Then ϕ superextends to $G - v$ by the minimality of G. Now v can be properly colored, a contradiction. $\square$

The following lemma is a foundational one for our paper, and similar lemmas appear in other related results (see for example [14]). We include the proof for completeness.

Lemma 3.2. *Every 3-vertex in $int(C)$ is adjacent to at least one 5^+-vertex or a vertex on C.*

Proof. Let $v \in int(C)$ be a 3-vertex not adjacent to a vertex of C. By the minimality of G, ϕ superextends to $G - v$. Since this coloring cannot extend to G, the colors $1, 2, 3$ must appear in $N(v)$, and the vertex $u \in N(v)$ colored with 1 is neither nicely colored, nor can it be recolored with 2 or 3. Hence u is adjacent to vertices colored with 2 and 3 and to two vertices of color 1, thus $d(u) \geq 5$. □

Lemma 3.3. *Suppose ϕ has been superextended to some subgraph of G, and let $v \in int(C)$ be a vertex that has exactly two colored neighbors, v_1 and v_2, both in $int(C)$. Then v can be recolored with color 1, unless one of the following holds:*

(1) v_1 and v_2 are adjacent and $d(v_1) + d(v_2) \geq 9$, or

(2) v_1 and v_2 are nonadjacent and one is a 5^+-vertex.

Proof. Suppose that v cannot be recolored with 1. Then if we recolor v with 1, some neighbor of v (say v_1) that is colored with 1 will have three neighbors of color 1. Note that color 2 and 3 must appear in $N(v_1)$, for otherwise we recolor v_1 with the absent color, so the degree of v_1 is at least 5. Assume further that $v_1v_2 \in E(G)$. If $d(v_1) + d(v_2) \leq 8$, then $d(v_1) = 5$ and $d(v_2) = 3$. As there are at most two different colors in $N(v_2)$, we can properly recolor v_2 (if its color is 1) or remove the color of v_2 and then properly recolor v_1 and v_2 in order(if its color is 2 or 3). Now in both cases, we can recolor v with 1. □

Observation 3.4. *Let v be a special or potentially special vertex with $N(v)$ in $int(C)$. By Lemma 3.2, v must be adjacent to a 5^+-vertex. By Lemma 3.3, in any precoloring of G in which the 5^+-neighbor v has not been colored, v can be recolored with 1.*

Lemma 3.5. *Let v be a vertex in $int(C)$. If v is adjacent to m special vertices and t pendant $(3, 3, 5^-)$- or $(3, 4, 4)$-faces, then $m + t \leq d(v) - 2$.*

Proof. Suppose to the contrary, that there exists some vertex $v \in int(C)$ with $m + t > d(v) - 2$. Consider $G' = G \backslash \{v\}$. We know that $G' \in \mathcal{G}$, so ϕ superextends to a coloring $\phi_{G'}$ of G'. By Lemma 3.3, the $m + t$ pendant or special vertices in $N(v)$ can be recolored with color 1, leaving at most one vertex in $N(v)$ with a different color. Now either color 2 or 3 is available to properly color v, superextending ϕ to G, a contradiction. □

Lemma 3.6. *Let $(v_1, \dots, v_6)$ be an internal 6-cycle with chord v_1v_3. Let $d(v_1) = d(v_3) = 3$, and v_4 be adjacent to k internal pendant $(3, 3, 5^-)$-, $(3, 4, 4)$-faces and special vertices. If $d(v_2) \leq 5$ and $f = v_1v_2v_3$ is not the interior triangle of any bad 6-cycle, then $d(v_5) \geq 5$, and $k \leq d(v_4) - 3$.*

Proof. Suppose the statement is not true. Consider the graph G' formed by deleting v_1 and v_3 and identifying v_4 and v_6 into a single vertex X.

We claim that $G' \in \mathcal{G}$. First of all, we do not create chords in C of G since both v_4 and v_6 are in $int(C)$. The only path of length 3 from v_6 to v_4 in G goes through v_1 and v_3, else there would be a separating 5-cycle in G or a triangle at vertex v_4 or v_6. Hence no new triangles are created by identifying v_4 and v_6. Further, since v_1 and v_3 are triangular in G, neither v_4 nor v_6 can be triangular. Thus $d_{\triangle}(G') \geq 2$. It remains to show that G' has no 4-cycles; such a 4-cycle could be created by a path of length 4 in G from v_4 to v_6. For such a path to exist, there must be a 4-cycle containing v_5, which does not exist, or a separating 6-cycle in G. Since G does not contain good separating 6-cycles and $f = v_1v_2v_3$ is not the interior triangle of any bad 6-cycle, G must contain a 6-cycle containing v_4, v_5, v_6 with a triangle inside, pendant to v_5. Let x be the pendant neighbor to v_5. But since there are no separating triangles or 5-cycles in G, v_5 and the two triangular neighbors of x are all degree 3, contradicting Lemma 3.2. Hence there is no such bad 6-cycle, and $G' \in \mathcal{G}$.

By the minimality of G, we know that there exists a superextension $\phi_{G'}$ of ϕ to G'. We claim we can extend $\phi_{G'}$ to a coloring ϕ_G that superextends ϕ to G, a contradiction. Let $\phi_G(x) = \phi_{G'}(x)$ for $x \in V(G') \backslash \{X\}$ and $\phi_G(v_4) = \phi_G(v_6) = \phi_{G'}(X) = \alpha$. Recolor v_2 so that it is nicely colored. It remains to color v_1 and v_3, and to verify that v_5 is adjacent to at most two neighbors of color 1 when it is colored with 1.

If $\alpha = 2$ (or symmetrically 3), then we can properly color v_3. Since v_1 has three nicely colored neighbors, it can be colored, completing the superextension to G. Hence we may assume $\alpha = 1$.

Suppose that either $\phi_G(v_5) \neq 1$, or $\phi_G(v_5) = 1$ and v_4 and v_6 are the only neighbors of v_5 that are colored with 1. At least one of $\{v_4, v_6\}$ is nicely colored with 1; assume by symmetry that v_4 is nicely colored. Properly color v_1, and now there is a color available for v_3, again a contradiction. If $d(v_5) < 5$, then v_5 can be recolored in this way, hence $d(v_5) \geq 5$.

Now suppose that $\phi_G(v_5) = 1$ and v_5 has a third neighbor colored with 1. (Since X was adjacent to v_5 in G', v_5 cannot have more than three neighbors with color 1.) Observe that v_3 is a pendant neighbor of v_4. Remove the color of v_4, and by Lemma 3.3, recolor with color 1 the $k-1$ other neighbors of v_4 that are special vertices or pendant neighbors on $(3,3,5^-)$- or $(3,4,4)$-faces. Since $d(v_4) \leq k+2$, v_4 has at most one neighbor colored from $\{2,3\}$. Hence v_4 can be properly colored. If v_2 is not colored with 1, then we can color v_3 with 1 and properly color v_1. If v_2 is colored with 1, then v_3 and v_1 can be consecutively colored properly. Therefore $d(v_4) > k+2$. □

Lemma 3.7. *Let $v \in int(C)$ be a 5-vertex with $N(v) = \{v_i : 1 \leq i \leq 5\} \subseteq int(C)$. Let f_i be the face containing $v_i v v_{i+1}$ for $i = 1, 2$. If v_1 and v_3 are both 3-vertices that are on internal $(3,3,5^-)$- or $(3,4,4)$-faces and both f_1 and f_2 are 5-faces in int(C), then $d(v_2) \geq 4$.*

Proof. Suppose otherwise, that $d(v_2) = 3$. Then we discuss the following two cases.

Case 1: One of v_1 and v_3 (say v_1, by symmetry) is not on the interior triangle of a bad 6-cycle. Let G' be the graph formed by identifying v_2 and v_5 in $G - v$ into vertex X. First of all, we do not create chords of C, for otherwise, the chord must be incident with X, conradicting v_2 and v_5 in $\int(C)$. Note that no new triangles can be created, else there would be a separating 5-cycle in G, contradicting Lemma 2.1. Since $d_\triangle(G) \geq 2$, v_2 and its neighbors are all nontriangular, hence $d_\triangle(G') \geq 2$. Also, G' contains no 4-cyces, else there would be a 4-path between v_2 and v_5 which implies a separating good 6-cycle in G, contradicting Lemma 2.1. Therefore, $G' \in \mathcal{G}$. By the minimality of G, we know that there exists a superextension $\phi_{G'}$ of ϕ to G'. We show that $\phi_{G'}$ can be extended to a coloring ϕ_G of G. Let $\phi_G(v_2) = \phi_G(v_5) = \phi_{G'}(X)$, and let $\phi_G(x) = \phi_{G'}(x)$ for $x \in V(G) - \{v, v_2, v_5\}$. It remains to color v to arrive at a contradiction. Recolor v_1 and v_3 with 1 by Lemma 3.3. If $\phi_{G'}(X) = 1$, then v can be properly colored. If $\phi_{G'}(X) = 2$ or 3 (say 2), then v_4 must be colored 3, else we can color v properly. In this case, we can properly recolor v_1 and v_3, and then color v with 1.

Case 2: Both v_1 and v_3 are on the interior $(3,3,3)$-faces of bad 6-cycles. Let $N(v_i) = \{v, u_i, w_i\}$ for $i = 1, 2, 3$ such that w_1 and w_2 are on f_1 and u_2 and u_3 are on f_2. Let $G' = G - \{v_3, u_3, w_3\}$. By the minimality of G, there exists a superextension $\phi_{G'}$ of ϕ to G'. We first claim that v is 1-saturated. For otherwise, color v_3 with 1, and then either $\phi_{G'}(u_2) \neq 1$ and we color u_3 with 1 and then w_3 properly, or $\phi_{G'}(u_2) = 1$ and we color w_3, u_3 properly in order, a contradiction. Similarly, u_2 is also 1-saturated. Furthermore, neither v nor u_2 can be recolored, so their neighborhoods must have color set $\{1,1,2,3\}$. Further, v_1 must be colored with 1, or we could recolor it by Lemma 3.3. We claim that w_2 must also be colored with 1. For otherwise, we can recolor w_1 with 1 and then recolor v_1 properly, a contradiction. But since all neighbors of v_2 are colored with 1, v_2 can be recolored with a different color so that v can be nicely recolored, a contradiction again. □

Lemma 3.8. *Given a $(3,3,5^-)$-face f in int(C), the pendant neighbors of the 3-vertices on f either are in $V(C)$ or have degree at least 5.*

Proof. Consider a 3-face $f = xyz$ in $int(C)$, where $d(z) \leq 5$ and $d(y) = d(x) = 3$ (see Figure 4). Assume to the contrary that the outer neighbor y' of y has degree at most 4, but $y' \notin V(C)$. Consider $G' = G \backslash \{x, y\}$. Because $G' \in \mathcal{G}$, we know that there exists a superextension $\phi_{G'}$ of ϕ to

G'. Since z and y' have degree at most 3 in G', they can be nicely recolored. But now we extend ϕ_G to G by properly coloring x and then coloring y, a contradiction. $\square$

Figure 4: $(3,5^-,5)$-face and $(3,3,6)$-face used in Lemmas 3.8, 3.10 and 3.11

Lemma 3.9. *Let $v \in int(C)$ be a 4-vertex with $N(v) = \{v_i | 1 \le i \le 4\}$ in the clockwise order. If $f = v_1vv_2$ is an internal $(3,3,4)$-face, then neither v_3 nor v_4 can be a 3-vertex in $int(C)$.*

Proof. Without loss of generality, let v_3 be a 3-vertex in $int(C)$. Let G' be the graph formed by identifying v_2 and v_4 into X in $G - v$. First of all, we do not create chords in C of G, for otherwise, the chord must be at X, thus there is a 3-path connecting two vertices on C, which will create a separating 5 or good 6-cycle, a contradiction to Lemma 2.1. Note that no new triangles can be created, else there would be a separating 5-cycle in G, contradicting Lemma 2.1. Since $d_\triangle(G) \ge 2$, both v_2 and v_4 are nontriangular in $G - v$, hence $d_\triangle(G') \ge 2$. None of v_2, v, v_4 is on a bad 6-cycle, thus G' contains no 4-cycles, else there would be a separating good 6-cycle in G, contradicting Lemma 2.1. Therefore, $G' \in \mathcal{G}$. By the minimality of G, we know that there exists a superextension $\phi_{G'}$ of ϕ to G'. We show that $\phi_{G'}$ can be extended to a coloring ϕ_G of G. Let $\phi_G(v_2) = \phi_G(v_4) = \phi_{G'}(X)$, and let $\phi_G(x) = \phi_{G'}(x)$ for $x \in V(G) - \{v, v_2, v_4\}$. We claim this coloring extends to v, a contradiction.

If $\phi_{G'}(X) = 2$ (or 3), then we properly recolor v_1 and v_3 and color v with 1. If $\phi_{G'}(X) = 1$, then $N(v)$ has color set $\{1,1,2,3\}$, else we can color v with the missing color. By symmetry, let v_1 be colored with 2 and v_3 be colored with 3. We can recolor v_1 with either 3 if the outer neighbor of v_1 is colored with 1, or 1 otherwise. In either case, v can be colored with 2, a contradiction. $\square$

Lemma 3.10. *Suppose that $f = xyz$ is a $(3,5^-,5)$-face in $int(C)$, with $d(x) \le 5$, $d(y) = 3$, and $d(z) = 5$. Let the outer neighbors of z be z_1, z_2, z_3 in clockwise order so that x and z_1 are on the same face. Let y' be the outer neighbor of y(See Figure 4).*

(1) At most one of $\{z_1, z_2, y\}$ (and symmetrically $\{z_1, z_3, y\}$) is potentially special (and hence at most one is special).

(2) If z_2 and z_3 are potentially special, then either $y' \in V(C)$, or $d(z_1) \ge 5$.

Proof. Consider the graph G' formed by identifying vertices x and z_3 into vertex X, and deleting the vertex z. Note all 3-cycles in G' were 3-cycles in G, else there would be a separating 5-cycle in G, contradicting Lemma 2.1. Also, since z was incident to a 3-face, z_3 cannot be triangular, and hence $d_\triangle \ge 2$ is maintained in G'. We also claim that G' does not contain any 4-cycles. Any such 4-cycle would correspond to a path of length 4 in G between x and z_3, and such a path would imply a separating 6-cycle in G; such a 6-cycle must be bad. But since x is triangular, the 6-cycle could not have another interior triangle, a contradiction. Hence $G' \in \mathcal{G}$, and by the minimality of G, ϕ superextends to a $(2,0,0)$ coloring $\phi_{G'}$ of G'. We show that $\phi_{G'}$ can be extended to a coloring ϕ_G of G when the hypotheses fail. Let $\phi_G(x) = \phi_G(z_3) = \phi_{G'}(X) = \alpha$, and let $\phi_G(v) = \phi_{G'}(v)$ for all other $v \in V(G) - z$. It remains to color z to arrive at a contradiction.

(1) Assume first that z_1 and z_2 are both potentially special. Properly recolor y, and properly recolor z_1 and z_2. If $\alpha = 2$ (or symmetrically 3), then z can be colored with 1, unless z_1, z_2, and y are all colored with 1, in which case z can be colored with 3; even if $z_3 \in V(C)$, this would be a superextension of ϕ to G, a contradiction. If $\alpha = 1$, then z_1 and z_2 can be recolored with 1 by

Lemma 3.3, and z can be properly colored. Hence at most one of $\{z_1, z_2\}$ is potentially special. Repeating the proof with z_3 in place of z_2 shows that at most one of $\{z_1, z_3\}$ is potentially special.

Now suppose one of $\{z_1, z_2\}$ is potentially special (assume z_1 by symmetry), and y is, as well. Properly recolor y and z_1. If $\alpha = 2$ and z_2 is colored with 2 or 3, then z can be colored with 1. Otherwise, y and z_1 can be recolored with 1, and either 2 or 3 is available for z.

(2) Now assume that z_2 and z_3 are potentially special, $y' \notin V(C)$, and $d(z_1) \leq 4$.

If $\alpha = 1$, then recolor z_2 with 1. Now z can be colored with 2 or 3 unless $\phi_{G'}(z_1) \neq \phi_{G'}(y)$ and neither is color 1. If z_2, z_3, or x are not nicely colored with 1, then properly recolor them, and color 1 is now available for z. If they are all nicely colored with 1, then y can be recolored 1, unless $\phi_{G'}(y') = 1$, in which case y can be recolored with $\phi_{G'}(z_1)$. In either case, $\phi_{G'}(y)$ becomes available for z, a contradiction.

If $\alpha = 2$ (or symmetrically 3), consider the color on z_1. If $\phi_{G'}(z_1) \neq 1$, then properly recolor z_2 and y, and now color 1 is available for z. If $\phi_{G'}(z_1) = 1$, then recolor z_2 with 1 by Lemma 3.3. So z can be colored with 3, unless y is given color 3. In this case, consider z_1. Since $d(z_1) \leq 4$, the vertex z_1 can be nicely recolored, and we can color z with 1, a contradiction. □

Lemma 3.11. *Let f be a $(3, 3, 6)$-face in $int(C)$ with vertices x, y, z such that $d(z) = 6$. Then either a neighbor of z is in $V(C)$, or z has at most two potentially special neighbors.*

Proof. Suppose that no neighbors of z are in $V(C)$. Let z_1, z_2, z_3, and z_4 be the outer neighbors of z, labeled as in Figure 4. Let H_1 be the graph formed by identifying x, z_2, and z_4 in $G - \{z, y\}$ into a single vertex X_1, and H_2 be graph formed by identifying y, z_1, and z_3 in $G - \{z, x\}$ into a single vertex X_2. Let $S_1 = \{x, z_2, z_4\}$ and let $S_2 = \{y, z_1, z_3\}$. Assume by symmetry that the number of potentially special vertices in S_1 is at most the number of potentially special vertices in S_2. This implies that we will consider H_1 for this proof, but a similar argument would hold for H_2 if S_2 had more potentially special vertices.

Note that all 3-cycles in H_1 were 3-cycles in G, else there would be a separating 5-cycle in G, contradicting Lemma 2.1. Also, since z was incident to a 3-face, $d_\triangle \geq 2$ is maintained in H_1. We also claim that H_1 does not contain any 4-cycles. Any 4-cycle in H_1 would correspond to the contraction of the edges between two vertices in $\{x, z_2, z_4\}$, and that would imply a separating 6-cycle in G; such a 6-cycle must be good, since the outer neighbors of z cannot be triangular, but no such separating cycle exists. Thus $H_1 \in \mathcal{G}$, and by the minimality of G, we know that ϕ superextends to a $(2, 0, 0)$-coloring ϕ_{H_1} of H_1.

We claim that ϕ_{H_1} extends to a $(2, 0, 0)$-coloring ϕ_G of G that superextends ϕ, a contradiction. Let $\phi_G(v) = \phi_{H_1}(v)$ for $v \in H_1 \backslash \{X_1\}$, and $\phi_G(x) = \phi_G(z_2) = \phi_G(z_4) = \phi_{H_1}(X_1) = \alpha$. It remains to assign colors to y and z. Let y' be the outer neighbor of y. If S_2 contains at most one potentially special vertex, then by the minimality of S_1, the result holds. Hence we may assume S_2 contains at least two potentially special vertices, and by symmetry, we may assume z_1 is potentially special.

Suppose first that $\alpha = 1$. Recolor z_1 with 1 by Lemma 3.3. If $\phi_{H_1}(z_3) \neq \phi_{H_1}(y')$, then y can be colored with $\phi_{H_1}(z_3)$, leaving a color available for z. If $\phi_{H_1}(z_3) = \phi_{H_1}(y')$, then y can be colored with 1 and a color is left for z, unless $\phi_{H_1}(z_3) = \phi_{H_1}(y') = 1$ and y' is improperly colored (and cannot be nicely recolored). This implies that y is not potentially special, and hence z_3 is potentially special. Thus z_3 can be recolored with 1, and y and z can be colored with 2 and 3.

Otherwise, by symmetry we may assume that $\alpha = 2$. Properly color y, and then color 1 is available for z unless either all of S_2 receives color 1, or some vertex in S_2 is not nicely colored with 1 and cannot be nicely recolored. In the former case, color 3 is available for z. In the latter case, some neighbor of z in S_2 is not potentially special. But then the other two vertices in S_2 must be potentially special, and they can be recolored with 1. This leaves color 3 available for z. □

4. Discharging Procedure

We are now ready to present a discharging procedure that will complete the proof of the theorem. Let each vertex $v \in V(G)$ have an initial charge of $\mu(v) = 2d(v) - 6$, and each face $f \neq C$ in our fixed plane drawing of G have an initial charge of $\mu(f) = d(f) - 6$. Recall that the length of C is r; let $\mu(C) = r + 6$. By Euler's Formula, $\sum_{x \in V \cup F} \mu(x) = 0$.

Let $\mu^*(x)$ be the charge of $x \in V \cup F$ after the discharge procedure. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$ and $\mu^*(C) > 0$.

Let a t-face with exactly one vertex in C be an F'_t-face, and a t-face with two or more vertices in C be an F''_t-face for $t \in \{3, 5\}$. Note that by Lemma 2.2, no 3-face contains three vertices of C and no 5-face contains four consecutive vertices of C. Observe also that since $d_\Delta \geq 2$, a vertex can be incident to at most one 3-face.

We call a 5-vertex v *good* if it contains three consecutive neighbors that are neither special vertices on 5-faces nor on internal pendant $(3,3,5^-)$- or $(3,4,4)$-faces of v, furthermore, they are the nontriangular neighbors when v is on a 3-face. Otherwise, it is *bad*. Extending this, a 4^+-vertex in $int(C)$ is good if it is a nontriangular 4-vertex, a good 5-vertex, or a 6^+-vertex. We call a 5-face in $int(C)$ *rich* if it has one good 4^+-vertex and two or more other 5^+-vertices.

Below are the discharging rules:

(R1) If v is a 4-vertex and f is an incident face in $int(C)$, then v:

- (a) gives 2 to f when f is a $(3,3,4)$-face, and $\frac{5}{4}$ to f when f is any other triangular face.
- (b) gives $\frac{1}{2}$ to f when f is a 5-face and v is nontriangular, and gives $\frac{1}{4}$ to f when f is a 5-face and v is a triangular vertex with no incident $(3,3,4)$-face.

(R2) If $v \in int(C)$ is a d-vertex with $d \geq 5$, then v:

- (a) gives $\frac{3}{8}$ to each incident 5-face in $int(C)$ with exactly two 5^+-vertices that are consecutive, gives $\frac{1}{3}$ to each incident 5-face in $int(C)$ that is not rich and has at least three 5-vertices, and gives $\frac{1}{2}$ to each other incident 5-face, unless v is a bad 5-vertex and the 5-face is rich, in which case v gives $\frac{1}{4}$. In addition, v gives $\frac{1}{4}$ to each of its pendant special 5-faces in $int(C)$.
- (b) gives $1, \frac{5}{8}, \frac{1}{2}$ to pendant $(3,3,3)$-, $(3,3,5)$-faces, and $(3,4^-,4)$-faces in $int(C)$, respectively.
- (c) gives $\frac{7}{4}, \frac{3}{2}, 1$ to incident $(3,4^-,5)$-, $(3,5,5)$- and other incident 3-faces in $int(C)$, respectively (when $d = 5$).
- (d) gives $3, 2, 1$ to incident $(3,3,d)$-, $(3,4^+,d)$-, and $(4^+,4^+,d)$-faces in $int(C)$, respectively (when $d > 5$).

(R3) The initial charge of $r + 6$ on C is distributed as follows:

- (a) C gets $2d(v) - 6$ from each vertex $v \in C$, 1 from each 7^+-face.
- (b) C gives 3 to each 3-face in $F'_3 \cup F''_3$, 1 to each 5-face in $F'_5 \cup F''_5$, 1 to pendant $(3,3,5^-)$- and $(3,4,4)$-faces in $int(C)$, and $\frac{1}{4}$ to its pendant special 5-faces.

Lemma 4.1. *The face C has a positive final charge.*

Proof. Let t_3, t_5 be the number of pendant 3-faces and pendant special 5-faces at C, respectively. Assume that C gets a from 7^+-faces. Let $E(C, V(G) - C)$ be the set of edges between C and $V(G) - C$ and let $e(C, V(G) - C)$ be its size. Then by (R3),

$$\mu^*(C) = r + 6 + \sum_{v \in C}(2d(v) - 6) - 3(|F'_3 \cup F''_3|) - |F'_5 \cup F''_5| - t_3 - \frac{t_5}{4} + a$$

$$= r + 6 + 2\sum_{v \in C}(d(v) - 2) - 2r - 3(|F'_3 \cup F''_3|) - |F'_5 \cup F''_5| - t_3 - \frac{t_5}{4} + a$$

$$= 6 - r + 2e(C, V(G) - C) - 3(|F'_3 \cup F''_3|) - |F'_5 \cup F''_5| - t_3 - \frac{t_5}{4} + a. \tag{1}$$

We aim to balance the charge of 2 on each $e \in E(C, V(G)-C)$ with the charge distributed to the incident and pendant faces; we can view this as sharing a charge of 2 for each $e \in E(C, V(G)-C)$ with the faces.

(a) If e is on a 3-face $f \in F_3' \cup F_3''$, then e can give $\frac{3}{2}$ to f, $\frac{1}{4}$ to a potential pendant 5-face, and $\frac{1}{4}$ to a potential incident 5-face.

(b) If e is adjacent to a pendant 3-face, then it can give 1 to the 3-face and $\frac{1}{2}$ to each potential incident 5-face.

(c) If e is neither on a 3-face $f \in F_3' \cup F_3''$ nor adjacent to any 3-face, then it can give $\frac{3}{4}$ to each potential incident 5-face and $\frac{1}{4}$ to a potential pendant 5-face. In this case, e would have a surplus of at least $\frac{1}{4}$.

Observe first that pendant faces are collectively allocated $t_3 + \frac{t_5}{4}$ from $E(C, V(G) - C)$. Since each face in $F_3' \cup F_3''$ contains two edges in $E(C, V(G) - C)$, it is allocated a charge of 3. Each face in $F_5' \cup F_5''$ contains two edges in $E(C, V(G) - C)$, and it is allocated at least $\frac{1}{2} \cdot 2 = 1$, unless it shares an edge with a 3-face in $F_3' \cup F_3''$. In that case, it is not adjacent to pendant 3-face, so it gains $\frac{1}{4} + \frac{3}{4} = 1$. This implies that

$$2e(C, V(G) - C) - 3(|F_3' \cup F_3''|) - |F_5' \cup F_5''| - t_3 - \frac{t_5}{4} \geq 0.$$

Hence from (1), $\mu^*(C) > 0$ if C is a 3- or 5-cycle. When $r = 6$, $\mu^*(C) \geq 0$, with equality only if $a = 0$ and

$$2e(C, V(G) - C) - 3(|F_3' \cup F_3''|) - |F_5' \cup F_5''| - t_3 - \frac{t_5}{4} = 0.$$

This implies that each edge must be as in (a) or (b), and it is either the common edge of two 5-faces and adjacent to a pendant 3-face or the common edge of a 3-face and a 5-face and adjacent to a pendant 5-face. Note that edges on 3-faces in F_3'' cannot be adjacent to pendant 5-faces, so $F_3'' = \emptyset$.

Let $C = u_1u_2u_3u_4u_5u_6$. Suppose that u_1 is on a 3-face in F_3'. Then u_i must be a 2-vertex for $2 \leq i \leq 6$; otherwise u_1 and u_i must be on the same 5-face and thus $d_\Delta \geq 2$ implies u_i cannot be on a 3-face or adjacent to a pendant 3-face in $int(C)$. But in this case there is a 7^+-face, contradicting $a = 0$. Therefore $F_3' \cup F_3'' = \emptyset$.

Now if u_1, u_2 are 3^+-vertices and in the same 5-face $u_1u_2v_2vv_1$, then both v_1, v_2 are in pendant triangles, and v must be in both, contradicting $d_\Delta \geq 2$. Hence C contains no consecutive 3^+-vertices. Let $d(u_1) \geq 3$ and $d(u_i) = 2$ for $2 \leq i \leq s-1$ and $d(u_s) \geq 3$. Note that u_1, u_s are in the same 5-face. Then u_1, u_s have the same pendant 3-face. This implies that $u_s = u_3$. Again, there is no 7^+-faces, so one of u_4, u_5, u_6 must be 3^+-vertex. Therefore, by the above argument, it must be $d(u_5) \geq 3$ and $d(u_4) = d(u_6) = 2$. But then C is a bad 6-cycle, contrary to our assumption that C is good. □

Lemma 4.2. *Each face other than C has nonnegative final charge.*

Proof. Observe first that (R3a) is the only rule applied to 7^+-faces; therefore all such faces have a nonnegative final charge.

Suppose next that f is a face with $d(f) = 3$; the initial charge on f is -3.

If $f \in F_3'$, then f gets 3 from the vertex in $V(C)$ incident to f by (R3). If $f \in F_3''$, then f gets $\frac{3}{2}$ from the two 3^+-vertices in $V(C)$ incident to f, again by (R3). In either case, $\mu^*(f) \geq -3 + 3 = 0$.

Next, suppose $f \in int(C)$.

- If f is a $(3,3,3)$-face, then by Lemma 3.8, its outer neighbors either have degree at least 5 or lie on C. Hence by (R2b) and (R3), f gets 1 from each outer neighbor, and $\mu^*(f) \geq 0$.
- If f is a $(3,3,4)$-face, then f gets 2 from the incident 4-vertex by (R1a). Lemma 3.8 again guarantees that the outer neighbors of the 3-vertices on f either have degree at least 5 or

lie on C, and hence f gets at least $\frac{1}{2}$ from the outer neighbors of its 3-vertices by either (R2b) or (R3b). Hence $\mu^*(f) \geq -3 + \left(2 + \frac{1}{2} \cdot 2\right) = 0$.

- If f is a $(3,3,5)$-face, then by (R2c) and (R2b), it gets $\frac{7}{4}$ from the 5-vertex and at least $\frac{5}{8} \cdot 2$ from the two outer neighbors, so $\mu^*(f) \geq -3 + \left(\frac{7}{4} + \frac{5}{8} \cdot 2\right) = 0$.
- If f is a $(3,4,5)$-face, then f gets $\frac{5}{4}$ from its 4-vertex by (R1a) and $\frac{7}{4}$ from its 5-vertex by (R2c), hence $\mu^*(f) \geq 0$.
- If f is a $(3,4,4)$-face, then f gets $\frac{5}{4}$ from each incident 4-vertex by (R1a) and at least $\frac{1}{2}$ from the pendant vertex , and $\mu^*(f) = 0$.
- If f is a $(3,5,5)$-face or a $(3,3,6^+)$-face, then (R2c) or (R2d), respectively, imply that f receives a charge of 3 from its incident vertices.
- If f is a $(3,4^+,6^+)$-face, then f receives 2 from the 6^+-vertex by (R2d) and at least $\frac{5}{4}$ from the 4^+-vertex by (R1a), (R2c) or (R2d), and again, the final charge on f is nonnegative.
- If f is a $(4^+,4^+,4^+)$-face, then by (R1) and (R2), f gets at least 1 from each incident vertex.

Therefore the final charge on all 3-faces is nonnegative.

Assume now that $d(f) = 5$, so the initial charge on f is -1. If f is an F_5'- or F_5''-face, then by (R3b), f gets 1 from the incident vertices on C. Hence we let f be a 5-face in $int(C)$.

Suppose f contains at least three 5^+-vertices. If f is rich, then by (R2a), f receives $\frac{1}{2}$ from the good 5^+-vertex and at least $\frac{1}{4}$ from each of the other two (or more) 5^+-vertices, and $\mu^*(f) \geq -1 + \frac{1}{2} + \frac{1}{4} \cdot 2 = 0$. If f is not rich, then f receives $\frac{1}{3}$ from each and $\mu^*(f) \geq -1 + \frac{1}{3} \cdot 3 = 0$. Suppose f contains exactly two non-consecutive 5^+-vertices; then by (R2), f gets $\frac{1}{2}$ from each, and $\mu^*(f) \geq 0$. Similarly, since f receives $\frac{1}{2}$ from each nontriangular 4-vertex, $\mu^*(f) \geq 0$ when f has at least such 4-vertices, or one such 4-vertex and one 5^+-vertex. Hence we may assume that f contains at most two 5^+- and nontriangular 4-vertices, and when it has exactly two, they are consecutive 5^+-vertices on f.

Let $f = v_1v_2v_3v_4v_5$. By $d_\Delta(G) \geq 2$ and Lemma 3.9, if v_i is 4-vertex on f on an internal $(3,3,4)$-face in $int(C)$ (that is, v_i is a 4-vertex that does not give $\frac{1}{4}$ to f), then either v_{i-1} or v_{i+1} is a nontriangular 4^+-vertex. Hence when f has no 5^+- or nontriangular 4-vertices, f also contains no 4-vertex on an internal $(3,3,4)$-face. Since any 3-vertex on f must be a special vertex, $\mu^*(f) \geq -1 + \frac{1}{4} \cdot 5 = \frac{5}{4} > 0$ by (R1b) and (R2a). When v_1 is the only 5^+- or nontriangular 4-vertex on f, neither v_3 nor v_4 is a 4-vertex on an internal $(3,3,4)$-face, and f gets $\frac{1}{4}$ through each of v_3, v_4 and $\frac{1}{2}$ from v_1. Thus $\mu^*(f) \geq -1 + \frac{1}{2} + \frac{1}{4} \cdot 2 = 0$. When v_1 and v_2 are the only 5^+- or nontriangular 4-vertices on f, they must be 5^+-vertices. Further, v_4 cannot be a 4-vertex on an internal $(3,3,4)$-face, and by (R1b) and (R2a), $\mu^*(f) \geq -1 + \frac{3}{8} \cdot 2 + \frac{1}{4} = 0$. □

Clearly, each vertex on C has final charge 0, since all its (positive or negative) charges are given to C. Now we consider the vertices in $int(C)$. By Lemma 3.1, if $v \in int(C)$, then $d(v) \geq 3$. If $d(v) = 3$, then the initial charge on v is $2d(v) - 6 = 0$, and v does not distribute charge during discharging. Hence we consider $d(v) \geq 4$.

Suppose $d(v) = 4$. Vertex v distributes charge according to rule (R1). If v is triangular, then it gives 2 if it is incident with a $(3,3,4)$-face in $int(C)$, and at most $\frac{5}{4} + \frac{1}{4} \cdot 3 = 2$ otherwise. If v is nontriangular, then it gives at most $\frac{1}{2} \cdot 4 = 2$. Hence the charge of all 4-vertices after the discharge procedure is at least 0.

Lemma 4.3. *Triangular 5^+-vertices in $int(C)$ have nonnegative final charge.*

Proof. Let v be a triangular d-vertex in $int(C)$ with $d \geq 5$. Let f_0 be the 3-face incident with v.

If $f_0 \cap C \neq \emptyset$, then v does not send charge to f_0. Since v gives at most $\frac{1}{2}$ to its other incident faces and $\frac{1}{4}$ to each of the special pendant 5-faces, $\mu^*(v) \geq (2d-6) - \frac{1}{2} \cdot (d-1) - \frac{1}{4} \cdot d > 0$. Now assume $f_0 \in int(C)$.

Let $d = 5$.

- f_0 is a $(3, 4^-, 5)$-face. By Lemmas 3.8 and 3.10, one of the following must hold: (1) v is adjacent to at most one special vertex, (2) v is adjacent to two special vertices and incident with a face that contains a vertex in C, or (3) v is adjacent to two special vertices and a nontriangular 5^+-vertex, which implies v has at most two incident 5-faces with non-consecutive 5^+-vertices. In all cases, (R2) governs the distribution of charge: in case (1), $\mu^*(v) \geq 4 - \frac{7}{4} - \frac{1}{2} \cdot 4 - \frac{1}{4} = 0$; in case (2), $\mu^*(v) \geq 4 - \frac{7}{4} - \frac{1}{2} \cdot 3 - \frac{1}{4} \cdot 2 > 0$; and in case (3), if v is a bad 5-vertex, then $\mu^*(v) \geq 4 - \frac{7}{4} - \frac{1}{2} \cdot 2 - \frac{3}{8} \cdot 2 - \frac{1}{4} \cdot 2 = 0$; note that v cannot be a good 5-vertex, for otherwise, f_0 must be a $(3, 3, 5)$-face and the two 3-vertices are special, but Lemma 3.8 tells that they must be adjacent to a pendant 5^+-vertices, a contradiction.
- f_0 is a $(3, 5, 5)$-face. By Lemma 3.10, v is adjacent to at most two special vertices, and by (R2), $\mu^*(v) \geq 4 - \frac{3}{2} - \frac{1}{2} \cdot 4 - \frac{1}{4} \cdot 2 = 0$.
- f_0 is any other 3-face. By (R2c), v gives 1 to f_0. Lemma 3.5 implies that v has a maximum of three adjacent special vertices, so by (R2), $\mu^*(v) \geq 4 - 1 - \frac{1}{2} \cdot 4 - \frac{1}{4} \cdot 3 > 0$.

Let $d = 6$. First assume that f_0 is a $(3^+, 4^+, 6)$-face. By Lemma 3.5 and (R2), v gives $\frac{1}{4}$ to up to four adjacent special vertices. By (R2a), v gives at most $\frac{1}{2}$ to each incident 5-face, and by (R2d), v gives at most 2 to the incident 3-face. Thus $\mu^*(v) \geq 6 - (2 + \frac{1}{4} \cdot 5 + \frac{1}{2} \cdot 5) > 0$. So we may assume that f_0 is a $(3, 3, 6)$-face. By Lemma 3.11, v is adjacent to no more than two special vertices or one of its neighbors is in $V(C)$ (this implies that v has at most three incident 5-faces in $int(C)$). By (R2), in the former case, $\mu^*(v) \geq 6 - 3 - \frac{1}{2} \cdot 5 - \frac{1}{4} \cdot 2 = 0$, and in the latter case, $\mu^*(v) \geq 6 - 3 - \frac{1}{2} \cdot 3 - \frac{1}{4} \cdot 4 > 0$.

Finally, let $d \geq 7$. Then v is incident with at most $d-1$ faces of length 5, and by Lemma 3.5, at most $d-2$ special vertices. By (R2), $\mu^*(v) \geq 2d-6-3-\frac{1}{2} \cdot (d-1) - \frac{1}{4} \cdot (d-2) = \frac{1}{4}(5d-32) > 0$. □

In the rest of the paper, whenever mentioned, $(3, 3, 5^-)$-faces, $(3, 4, 4)$-faces, and 5-faces are in $int(C)$.

Lemma 4.4. *Nontriangular 6^+-vertices in $int(C)$ have nonnegative final charge.*

Proof. Let $v \in int(C)$ be a nontriangular 6^+-vertex and let t be the number of pendant $(3, 3, 5^-)$ or $(3, 4, 4)$-faces of v. By Lemma 3.5, v has at most $(d-t-2)$ pendant special 5-faces. By (R2), v gives at most 1 to each pendant 3-face, at most $\frac{1}{2}$ to each incident 5-face, and $\frac{1}{4}$ to each pendant special 5-face. So if $t \leq d-4$, then $\mu^*(v) \geq 2d-6-t-\frac{1}{2}d-\frac{1}{4}(d-t-2) = \frac{1}{4}(5d-22-3t) \geq \frac{1}{4}(2d-10) > 0$. If $t = d-2$, then v has no pendant special 5-faces, and $d_{\triangle}(G) \geq 2$ implies that v has at most 4 incident 5-faces. So $\mu^*(v) \geq 2d - 6 - (d-2) - \frac{1}{2}(4) = d - 3 \geq 0$. Since $t \leq d-2$ by Lemma 3.5, it remains only to check $t = d-3$.

If $d \geq 7$, then $\mu^*(v) \geq (2d-6) - (d-3) - \frac{1}{2}d - \frac{1}{4}(d-t-2) = \frac{1}{2}d - \frac{13}{4} > 0$. If $d = 6$ and at least one of the three pendant 3-faces of v is not a $(3, 3, 3)$-face, then v gives at most $\frac{5}{8}$ to this 3-face by (R2), so $\mu^*(v) \geq 6 - 1 \cdot 2 - \frac{5}{8} - \frac{1}{2} \cdot 6 - \frac{1}{4} = \frac{1}{8} > 0$. So we may assume that the 6-vertex v is adjacent to exactly three pendant $(3, 3, 3)$-faces. If v has at most five incident 5-faces, then $\mu^*(v) \geq 6 - 1 \cdot 3 - \frac{1}{2} \cdot 5 - \frac{1}{4} = \frac{1}{4} > 0$. If v has six incident 5-faces, then v is adjacent to no pendant special 5-faces since by Lemma 3.8, the outer neighbors of the 3-vertices on a $(3, 3, 3)$-face either are in V(C) or have degree at least 5. So $\mu^*(v) \geq 6 - 1 \cdot 3 - \frac{1}{2} \cdot 6 = 0$. □

Lemma 4.5. *Nontriangular 5-vertices in $int(C)$ have nonnegative final charge.*

Proof. Let $v \in int(C)$ be a nontriangular 5-vertex that is adjacent to t pendant $(3, 3, 5^-)$- or $(3, 4, 4)$-faces. Let $N(v) = \{v_i : 1 \leq i \leq 5\}$. Let f_i be the incident face of v containing v_i, v, v_{i+1} for $1 \leq i \leq 5$ (index modulo 5). By Lemma 3.5, v has at most $d - t - 2$ pendant special 5-faces. By (R2), v gives at most 1 to each pendant 3-face, at most $\frac{5}{8}$ to each pendant 3-face that is not a

$(3,3,3)$-face, at most $\frac{1}{2}$ to each incident 5-face, and $\frac{1}{4}$ to each pendant special 5-face. If $t \le 1$, then v has at most two pendant special 5-faces, and $\mu^*(v) \ge 4 - 1 - \frac{1}{2} \cdot 5 - 2 \cdot \frac{1}{4} = 0$. Since $t \le 3$ by Lemma 3.5, we may assume that $t = 2$ or $t = 3$.

Case 1: $t = 2$.

First we verify **Claim A:** If v_i is on a $(3,3,3)$-face, both f_i and f_{i-1} are 5-faces.

For otherwise, suppose v_1 is on a $(3,3,3)$-face and f_1 is not a 5-face. Since an interior triangle of a bad 6-face must share edges with three 5-faces, the $(3,3,3)$-face at v_1 is not the interior triangle of a bad 6-cycle. If f_5 is not a 5-face, then $\mu^*(v) \ge 4 - 2 \cdot 1 + \frac{1}{2} \cdot 3 + \frac{1}{4} > 0$. If f_5 is a 5-face, then it shares an edge with a $(3,3,3)$-face, thus by Lemma 3.6, since v has two pendant $(3,3,5^-)$- or $(3,4,4)$-faces, it has no pendant special 5-face. By (R2), $\mu^*(v) \ge 4 - 2 \cdot 1 + \frac{1}{2} \cdot 4 = 0$. Hence Claim A is established.

Without loss of generality, we may assume that either v_1 and v_2 or v_1 and v_3 are on $(3,3,5^-)$- or $(3,4,4)$-faces. First let v_1 and v_2 be on $(3,3,5^-)$- or $(3,4,4)$-faces. Note that f_1 is a 6^+-face since $d_\triangle(G) \ge 2$. By Claim A, neither v_1 nor v_2 is on a $(3,3,3)$-face. Hence by (R2), $\mu^*(v) \ge 4 - \frac{5}{8} \cdot 2 - \frac{1}{2} \cdot 4 - \frac{1}{4} = \frac{1}{2} > 0$.

Now let v_1, v_3 be on $(3,3,5^-)$- or $(3,4,4)$-faces. Note that v is a bad 5-vertex. We first suppose that one of f_1, f_2 (say f_1) is not a 5-face. By Claim A, v_1 is not on a $(3,3,3)$-face. So by (R2), $\mu^*(v) \ge 4 - 1 - \frac{5}{8} - \frac{1}{2} \cdot 4 - \frac{1}{4} = \frac{1}{8} > 0$.

Now assume that both f_1 and f_2 are 5-faces. By Lemma 3.7, $d(v_2) \ge 4$. If v has no pendant $(3,3,3)$-faces, then $\mu^*(v) \ge 4 - \frac{5}{8} \cdot 2 - \frac{1}{2} \cdot 5 - \frac{1}{4} = 0$. Hence we may assume that v_1 is on a $(3,3,3)$-face. By Claim A, f_5 is a 5-face. By Lemma 3.8, the pendant neighbors (in particular, on f_1 and f_5) of the $(3,3,3)$-faces are in $V(C)$ or have degree at least 5.

First consider the case that v_3 is also on a $(3,3,3)$-face. By Claim A, f_3 is also a 5-face, and the pendant neighbors (in particular, on f_2 and f_3) of the $(3,3,3)$-faces are in $V(C)$ or have degree at least 5. Note that v_2 cannot be triangular; further, three of its consecutive neighbors are 5^+-vertices. Hence v_2 is a good 4^+-vertex and both f_1 and f_3 are rich 5-faces. By (R2a), v gives $\frac{1}{4}$ to each of f_1 and f_2. Note that both v_4 and v_5 are next to a 5^+-neighbor respectively on f_3 and f_5, so they are not pendant special 3-vertices of v, and thus v has no pendent special 5-faces. So by (R2), $\mu^*(v) \ge 4 - 1 \cdot 2 - \frac{1}{4} \cdot 2 - \frac{1}{2} \cdot 3 = 0$.

Finally, assume that v_3 is not on a $(3,3,3)$-face. Suppose v_3 is on a $(3,4^-,4)$-face. If $d(v_2) \ne 5$, then v_2 is a good 4^+-vertex and f_1 is rich, so by (R2), $\mu^*(v) \ge 4 - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{2} \cdot 4 - \frac{1}{4} = 0$; if $d(v_2) = 5$, then $\mu^*(v) \ge 4 - 1 - \frac{1}{2} - \frac{1}{3} - \frac{3}{8} - \frac{1}{2} \cdot 3 - \frac{1}{4} = \frac{1}{24} > 0$. So we assume that v_3 is on a $(3,3,5)$-face. Then by Lemma 3.8, f_2 contains at least three 5^+- or good 4^+-vertices. So v_2 is a good 4^+-vertex and both f_1 and f_2 are rich 5-faces. By (R2a), v gives $\frac{1}{4}$ to f_1 and f_2. Then $\mu^*(v) \ge 4 - 1 - \frac{5}{8} - \frac{1}{4} \cdot 2 - \frac{1}{2} \cdot 3 - \frac{1}{4} = \frac{1}{8} > 0$.

Case 2: $t = 3$.

By Lemma 3.5, v has no pendant special 5-faces. Without loss of generality, we may assume that either v_1, v_2, v_3 or v_1, v_2, v_4 are on pendant $(3,3,5^-)$- or $(3,4,4)$-faces.

Assume first that v_1, v_2, v_3 are on pendant $(3,3,5^-)$- or $(3,4,4)$-faces. Since $d_\triangle(G) \ge 2$, neither f_1 nor f_2 is a 5-face; hence v has at most three incident 5-faces and is not on a bad 6-cycle. If v has at most two incident 5-faces, then $\mu^*(v) \ge 4 - 1 \cdot 3 - \frac{1}{2} \cdot 2 = 0$. Thus we suppose that f_3 and f_5 are 5-faces (with an incident 3-face). By Lemma 3.6, one of the neighbors of v_1 on its triangular face must have degree at least 4; the same is true for v_3. Hence neither of the pendant triangles at v_1 and v_3 are $(3,3,3)$-faces. By (R2), $\mu^*(v) \ge 4 - 1 - \frac{5}{8} \cdot 2 - \frac{1}{2} \cdot 3 = \frac{1}{4} > 0$.

Now assume that v_1, v_2, v_4 are on pendant $(3,3,5^-)$ or $(3,4,4)$-faces. Since $d_\triangle(G) \ge 2$, f_1 is not a 5-face. Then the 3-face at v_2 is not an interior triangle of a bad 6-cycle. By Lemma 3.6, either f_2 is a 5-face and v_2 is not on a $(3,3,3)$-face or f_2 is not a 5-face. Hence v gives at most

$\max\{1, \frac{5}{8} + \frac{1}{2}\} = \frac{9}{8}$ to f_2 and the pendant 3-face at v_2 by (R2). Similarly, v gives at most $\frac{9}{8}$ to f_5 and the pendant 3-face at v_1. If v_4 is not on a $(3,3,3)$-face, then $\mu^*(v) \geq 4 - \frac{9}{8} \cdot 2 - \frac{1}{2} \cdot 2 - \frac{5}{8} = \frac{1}{8} > 0$.

Hence we may assume that v_4 is on a $(3,3,3)$-face. If neither f_2 nor f_5 is a 5-face, then $\mu^*(v) \geq 4 - 1 \cdot 3 - \frac{1}{2} \cdot 2 = 0$, so we may also assume (by symmetry) that f_5 is a 5-face. If one of f_3, f_4 is not a 5-face, then $\mu^*(v) \geq 4 - 1 - \frac{9}{8} \cdot 2 - \frac{1}{2} = \frac{1}{4} > 0$, so we may assume that both f_3 and f_4 are 5-faces. Since f_1 is not a 5-face, the 3-face at v_1 is not the interior 3-face of a bad 6-cycle, so by Lemma 3.6, it cannot be a $(3,3,3)$-face. Now that both f_4, f_5 are 5-faces, by Lemma 3.7, $d(v_5) \geq 4$. By Lemma 3.8, the pendant neighbors of the $(3,3,3)$-face at v_4 are in $V(C)$ or have degree at least 5, hence f_4 has at least three 4^+-vertices. Since $d_\triangle(G) \geq 2$, v_4 cannot be a triangular 4-vertex. If $d(v_5) \neq 5$, then v_4 is a good 4^+-vertex and f_4 is a rich 5-face; by (R2), $\mu^*(v) \geq 4 - \frac{9}{8} \cdot 2 - 1 - \frac{1}{4} - \frac{1}{2} = 0$. So let $d(v_5) = 5$. If v_1 is on a $(3, 4^-, 4)$-face, then $\mu^*(v) \geq 4 - 1 - \frac{1}{2} - \frac{9}{8} - \frac{1}{2} \cdot 2 - \frac{1}{3} = \frac{1}{24} > 0$. If v_1 is on a $(3,3,5)$-face, then the 5-vertex must be on f_5 by Lemma 3.6. So by (R2), v gives $\frac{1}{3}$ to each of f_4 and f_5. Then $\mu^*(v) \geq 4 - 1 - \frac{5}{8} - \frac{1}{3} \cdot 2 - \frac{9}{8} - \frac{1}{2} = \frac{1}{12} > 0$. $\square$

Therefore all vertices have nonnegative charge after the discharge procedure.

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