# A lower bound for online rectangle packing* 

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#### Abstract

We slightly improve the known lower bound on the asymptotic competitive ratio for online bin packing of rectangles. We present a complete proof for the new lower bound, whose value is above 1.91.


## 1 Introduction

Bin packing [24] is a well-studied combinatorial optimization problem. The goal is to partition items with rational sizes in $(0,1]$ into subsets of total sizes at most 1 , called bins. In the online version, items are presented one by one, such that every item is assigned irrevocably to a bin before the next item arrives. This classic variant is also called one-dimensional bin packing.

Rectangle packing is a generalization of bin packing where every item is an axis parallel oriented rectangle. Each rectangle $r_{i}$ has a height $0<h\left(r_{i}\right) \leq 1$ and a width $0<w\left(r_{i}\right) \leq 1$. The objective is to partition input rectangles into subsets, such that every subset can be packed into a bin, where a bin is a unit square. Packing should be done such that rectangles will not intersect, but their boundaries can touch each other and they can also touch the boundary of the bin. As rectangles are oriented, they cannot be rotated. In the online variant, rectangles are presented one by one, as in the one-dimensional version. There are two scenarios; the one where the specific packing of a rectangle is decided upon arrival (the position inside the bin), and the less strict one, where the algorithm keeps subsets of rectangles that can be packed into bins, but the exact packing can decided at termination. Typically, positive results are proved for the first version while negative results are proved for the second one, and thus, all results are valid for both versions.

For an algorithm $A$ for some bin packing problem, and an input $I$, the number of bins used by $A$ is denoted by $A(I)$. In particular, for an optimal offline algorithm $O P T$ that receives $I$ as a set, its cost is denoted by $\operatorname{OPT}(I)$, and this is the minimum number of bins required for packing $I$. The approximation ratio, or competitive ratio if $A$ is online, for input $I$ is $\frac{A(I)}{O P T(I)}$. The absolute approximation ratio or absolute competitive ratio is $\sup _{I}\left\{\frac{A(I)}{O P T(I)}\right\}$, and the asymptotic approximation ratio or asymptotic competitive ratio $R(A)$ (which is never larger than the absolute

[^0]one) is
$$
R(A)=\lim _{N \rightarrow \infty} \sup _{I}\left\{\left.\frac{A(I)}{O P T(I)} \right\rvert\, O P T(I) \geq N\right\}=\limsup _{N \rightarrow \infty} \max _{I}\left\{\left.\frac{A(I)}{O P T(I)} \right\rvert\, O P T(I)=N\right\}
$$

For one-dimensional online bin packing, it is known that the asymptotic competitive ratio is in [1.5427809, 1.57828956] [3, 4] (see also [6, 23, 26, 28, 25, 31]).

For rectangle packing, there is a number of articles where various algorithms are designed [11, 12, 13, 27, 16, 21]. Where the last work is the one of Han et al. [21, and the current best asymptotic competitive ratio is still above 2.5. The history of lower bounds is as follows. Galambos showed a lower bound of 1.6 on the asymptotic competitive ratio of any algorithm [19]. This was improved by Galambos and van Vliet to approximately 1.808 by applying the same idea multiple times [20]. By increasing the number of types of items in every part of the construction, an improved lower bound of approximately 1.851 was shown by van Vliet [29]. Finally, by applying an additional modification, a lower bound of 1.907 was claimed [8, 10]. For many years the lower bound of 1.907 was cited as an unpublished manuscript [10]. This result appears in the thesis of Blitz [8] that was not accessible for many years. That thesis [8] contains information that can assist in obtaining a proof, and can be seen as guidelines for obtaining it. A manuscript was published on arxiv with the details of an inferior result of approximately 1.859 [9] also appearing in [8] with a partial proof, where there are just nine types of items, while the value 1.907 was treated by many researchers as a conjecture.

The special case of rectangle packing, where all input items are squares was studied as well [11, 29, 8, 27, 15, 22, 1]. For this version there is also a large gap between the lower bound and upper bound on the asymptotic competitive ratio, where the lower bound is approximately 1.75 [1], while the upper bound is above 2.1 [22]. Another generalization of square packing is non-oriented packing of rectangles, where rectangles are still packed in an axis parallel manner, but they can be rotated by 90 degrees [17, 14]. This version is very different from the oriented one. For example, in the non-oriented version, given rectangles of heights 0.66 and widths of 0.34 , any bin can contain at most two such items, while the non-oriented version allows us to pack four such items into each bin.

As mentioned above, the previous lower bound on the asymptotic competitive ratio is known as 1.907 [8, 10], which was cited multiple times, but there is no full proof of this result. While the thesis of Blitz [8] has a number useful guidelines for the proof, including the input and properties of a certain linear program (LP) and its dual (see below), it does not contain a complete and precise proof, and only the proof of the lower bound 1.859 was recovered completely [9]. This last construction is based on the nine types of items appearing in the bottom three rows in Figure 1 In our work, we use the guidelines of Blitz that were provided for an intermediate result with 12 item types, which was a lower bound of approximately 1.905 [8]. We modify the input by replacing the first item with a potentially infinite sequence of items, so instead of 12 item types used, we have the last 11 types, and we use a large number of types instead of the first type. This approach allows us to provide a complete proof and show a slightly higher lower bound of 1.9100449 on the asymptotic competitive ratio of any online algorithm for the packing problem of rectangles into unit square bins. The construction of Blitz giving a lower bound of 1.907 consists of another row of items on
the top compared to Figure $\mathbb{1}$, that is, the small empty space in the top of the bin in Figure $\mathbb{1}$ also contains three items. However, replacing the first item of this construction leads to a result inferior to the one which we prove, since the first item type out of the 15 has very small height, and replacing it with a sequence of items only increases the bound by a very small amount. On the other hand, replacing the first item of the construction with nine item types does not increase the lower bound above 1.9.

We briefly discuss the relation between the proof methods. Here, we do not provide the details of proving the results of Blitz [8] using our method, since we show a better result. However, using our proof methods it is possible to recover all three lower bounds mentioned the thesis of Blitz [8] for rectangle packing, and many of the required properties are proved here. For comparison between the two methods (which are related), we describe the approach of [29, 8] for proving lower bounds on the competitive ratio for inputs of a specific form. Such an input has several types of items fixed in advance, where the input is of the form that at each time a large number of identical items arrive (those are items of some type), and then the input may be stopped (if the number of bins already used by the algorithm is relatively high) or it may continue (if not all item types were presented yet). Note that not all lower bounds for the asymptotic competitive ratio of bin packing problems have this structure, and inputs may have branching or clusters of items of close but slightly different sizes [5, 2, 1, though many results do have the form we discuss here [28, 18, 7, 6].

For the kind of inputs we described here, which will be used in our construction, it is possible to analyze packing patterns. A pattern is a multiset of items that can be packed into a bin. One can generate all such patterns for a given input or they can be analyzed without generating them. If the number of types is constant as we assume here, it is possible to write an LP whose variables are the numbers of patterns of every kind. The LP states the relation between numbers of items and numbers of bins with all possible patterns, i.e., numbers of items are counted as a function of the number of patterns containing such items (multiplied by the numbers of items in different patterns), and it is ensured that all items are indeed packed. Patterns are partitioned into subsets where every subset consists of patterns whose bins are first used after the arrival of one type of items. This is done since bins only count towards the cost of the algorithm starting the arrival time of the first items packed into them. Obviously, there are also constraints stating that the competitive ratio is not violated. The inputs are sufficiently large such that the absolute competitive ratio and the asymptotic one are equal. The cost of the algorithm is also based on numbers of suitable patterns, while the optimal cost is computed based on the input. There is work where this LP is solved [30, and work where the dual LP is analyzed too [29, 8, 9]. In the primal LP, there are two constraints for every item type. Thus, the dual LP has two variables for every item type. It is frequently the case that in an optimal solution to the dual LP the two sets of variables differ by just a multiplicative factor.

Here, we do not use a linear program, though a linear program corresponding to our approach would not contain a variable for every pattern, but just a variable for the number of bins opened by an online algorithm after the arrival of a type of items. Thus, we would have one variable for all patterns of one subset in the partition of patterns. We use weights for items, and these weights are strongly related to the values of variables in solutions to dual LP's. In our method, it is required
to find an upper bound on the total weight of any pattern in every subset of the partition, and this is also required in the method with LS's and their dual LP's described above. Thus, we can use the tables of data [8], both for total weights and for optimal solutions. As these tables were given without proof, we fill this gap and provide proofs. For our method it is not required to know the precise values (for maximum weights and costs of optimal solutions) but only upper bounds on these values. The method we apply was used in the past [7, 1]. We use it here as in [7] even though the packing problem here is different, since the method defined there can be used for many bin packing variants.

## 2 Lower bound

Our input is based on a modification of one of the inputs of Blitz [8], where the first item is replaced with a sequence of items. Note that this is not the input for which a lower bound of approximately 1.907 was claimed, but an inferior input for which the claimed lower bound was approximately 1.905 .

Let $N>0$ be a large integer divisible by $5^{k} \cdot 7224$. Let $k \geq 4$ be an integer, which is seen as a constant that is independent of $N$. Let $\delta>0$ and $\varepsilon>0$ be very small values, such that $\delta<\frac{1}{2^{3 k+50}}$ (where in particular, $2^{50} \delta<\frac{1}{20}$ ) and $\varepsilon<0.0001$. Let

$$
h_{1}=\frac{1}{43}+\varepsilon, \quad h_{2}=\frac{1}{7}+\varepsilon, \quad h_{3}=\frac{1}{3}+\varepsilon, \quad \text { and } \quad h_{4}=\frac{1}{2}+\varepsilon .
$$

Note that $h_{1}+h_{2}+h_{3}+h_{4}=\frac{1805}{1806}+4 \varepsilon<1$.
The input consists of $k+9$ item types. The first $k$ item types have heights of $h_{1}$, the next three item types have heights of $h_{2}$, the following three item types have heights of $h_{3}$, and the final three item types have heights of $h_{4}$. The input may stop after each one of the $k+9$ item types, and in case that some item type is presented, there are $N$ identical items of this type. For $i=1, \ldots, k$, the $i$ th item type out of the first $k$ types is denoted by type $\ell_{1 i}$. For $i=1,2, \ldots, k-2$, it is defined by its width

$$
w_{1 i}=\frac{1+\delta}{5^{k-i-1}},
$$

and therefore our addition to the original input [8] is replacing an item whose width is just below $\frac{1}{4}$ by items slightly wider than negative powers of 5 . We also let the widths of type $\ell_{1(k-1)}$ be $w_{1(k-1)}=\frac{1+2^{40} \delta}{4}$, and the width of type $\ell_{1 k}$ is defined as $w_{1 k}=\frac{1+2^{40} \delta}{2}$. Dimensions for all item types are also given in Table 1, and an illustration is given in figure 1. We let $h_{1 i}=h_{1}$ for $1 \leq i \leq k$, and $h_{j i}=h_{j}$ for $j=2,3,4$ and $i=0,1,2$.

Thus, for $1 \leq i \leq k-3$, we have $w_{1(i+1)}=5 \cdot w_{1 i}$, and we also have $w_{1 k}=2 \cdot w_{1(k-1)}$. For any integer $1 \leq t \leq k-2$, the total width of $t$ items, consisting of exactly one item of every type $\ell_{1 i}$ for any $1 \leq i \leq t$, is

$$
\sum_{i=1}^{t} w_{1 i}=(1+\delta) \sum_{i=1}^{t} \frac{1}{5^{k-i-1}}=(1+\delta) \frac{1}{5^{k-t-1}} \sum_{i=1}^{t} \frac{1}{5^{t-i}}=(1+\delta) \frac{1}{5^{k-t-1}} \sum_{j=0}^{t-1} \frac{1}{5^{j}}
$$

$$
=(1+\delta) \frac{1}{5^{k-t-1}} \frac{1-\frac{1}{5^{t}}}{1-\frac{1}{5}}=(1+\delta) \frac{1}{4 \cdot 5^{k-t-2}}\left(1-\frac{1}{5^{t}}\right)=(1+\delta)\left(\frac{1}{4 \cdot 5^{k-t-2}}-\frac{1}{4 \cdot 5^{k-2}}\right)<\frac{1-2^{42} \delta}{4 \cdot 5^{k-t-2}},
$$

by $2^{42}+1<2^{43}$ and since

$$
\frac{2^{43} \delta}{4 \cdot 5^{k-t-2}}<\frac{1+\delta}{4 \cdot 5^{k-2}}
$$

holds by $t \leq k-2$ and

$$
\frac{4 \cdot 5^{k-2}}{4 \cdot 5^{k-t-2}}=5^{t} \leq 5^{k-2} \quad \text { while } \quad \frac{1+\delta}{2^{43} \delta}>\frac{1}{2^{43} \delta}>2^{3 k+7}>8^{k} .
$$

In particular for $t=k-2$, the total width is below $\frac{1}{4}$. Thus, we also have

$$
\sum_{i=1}^{k-1} w_{1 i}<\frac{1-2^{42} \delta}{4}+\frac{1+2^{40} \delta}{4}<\frac{1}{2} \quad \text { and } \quad \sum_{i=1}^{k} w_{1 i}<\frac{1-2^{42} \delta}{4}+3 \cdot \frac{1+2^{40} \delta}{4}<1
$$

The next three types are denoted by $\ell_{20}, \ell_{21}, \ell_{22}$, and their widths are $w_{20}=\frac{1}{4}-2^{32} \delta>\frac{1}{5}$, $w_{21}=\frac{1}{4}+2^{30} \delta$, and $w_{22}=\frac{1}{2}+2^{31} \delta$, respectively. The following three types are denoted by $\ell_{30}, \ell_{31}$, $\ell_{32}$, and their widths are $w_{30}=\frac{1}{4}-2^{22} \delta>\frac{1}{5}, w_{31}=\frac{1}{4}+2^{20} \delta$, and $w_{22}=\frac{1}{2}+2^{21} \delta$, respectively. The last three types are denoted by $\ell_{40}, \ell_{41}, \ell_{42}$, and their widths are $w_{40}=\frac{1}{4}-2^{12} \delta>\frac{1}{5}, w_{41}=\frac{1}{4}+2^{10} \delta$, and $w_{42}=\frac{1}{2}+2^{11} \delta$, respectively. Note that $w_{j 0}+w_{j 1}+w_{j 2}<1$ and $w_{j 0}+w_{j 1}<\frac{1}{2}$ for $j=2,3,4$, but
$w_{20}+3 \cdot w_{1(k-1)} \geq 2 \cdot w_{20}+2 \cdot w_{1(k-1)} \geq 3 \cdot w_{20}+w_{1(k-1)}=3 \cdot\left(\frac{1}{4}-2^{32} \delta\right)+\frac{1+2^{40} \delta}{4}>1-2^{34} \delta+2^{38} \delta>1$ and

$$
2 \cdot w_{20}+w_{1 k}=2 \cdot w_{20}+2 \cdot w_{1(k-1)}>1 .
$$

In addition, we have

$$
w_{(j+1) 0}+3 w_{j 1} \geq 2 w_{(j+1) 0}+2 w_{j 1}=3 w_{(j+1) 0}+w_{j 1}=3\left(\frac{1}{4}-2^{52-10 j} \delta\right)+\left(\frac{1}{4}+2^{60-10 j} \delta\right)>1
$$

and

$$
2 w_{(j+1) 0}+w_{j 2}=2 w_{(j+1) 0}+2 w_{j 1}>1 \quad \text { for } \quad j=2,3
$$

It can be seen that $w_{20}<w_{30}<w_{40}$ while $w_{1(k-1)}>w_{21}>w_{31}>w_{41}$ and $w_{1 k)}>w_{22}>w_{32}>$ $w_{42}$.

We say that an item type is later than another type if it is presented later in the input. The weight $v_{j i}$ for an item of type $\ell_{j i}$ is given in Table 1. The weights were selected based on dual variables provided in [8].

We let $V_{j i}$ denote the maximum total weight of a bin containing items of the types consisting of type $\ell_{j i}$ and later types. Let $\Omega_{j i}$ be an upper bound on $\frac{O P T_{j i}}{N}$, where $O P T_{j i}$ is the cost of an optimal solution for the input up to type $\ell+j i$ items. Note that [8] contains tables with costs of


Figure 1: An illustration of the input in terms of one bin of an optimal solution for the entire input, if all item types are presented. Item types arrive ordered from top to bottom and from left to right.
optimal solutions and $V_{j i}$ values for the part of the input that is identical to ours, though it does not contain proofs of all the claimed values.

We use a theorem defined for inputs for bin packing problems, such that the inputs have the form explained in the introduction. The input consists of "batches" of identical items without branching. Substituting our notation, the theorem states that

$$
\frac{\sum_{i=1}^{k} v_{1 i}+\sum_{j=2}^{4} \sum_{i=0}^{2} v_{j i}}{Q},
$$

where

$$
Q=\left(\Omega_{11} \cdot V_{11}+\sum_{i=2}^{k}\left(\Omega_{1 i}-\Omega_{1(i-1)}\right) V_{1 i}\right)+\left(\left(\Omega_{20}-\Omega_{1 k}\right) V_{20}+\left(\Omega_{21}-\Omega_{20}\right) V_{21}+\left(\Omega_{22}-\Omega_{21}\right) V_{22}\right)
$$

$$
+\left(\sum_{j=3}^{4}\left(\left(\Omega_{j 0}-\Omega_{(j-1) 2}\right) V_{j 0}+\left(\Omega_{j 1}-\Omega_{j 0}\right) V_{j 1}+\left(\Omega_{j 2}-\Omega_{j 1}\right) V_{j 2}\right)\right)
$$

is a lower bound on the asymptotic competitive ratio (see [7]).
Note that one can use an upper bound on $V_{j i}$ rather than the actual value if all multipliers are positive, which will be the case here. This will hold as we will ensure that the sequence of upper bounds on $\frac{O P T_{j i}}{N}$ will be monotonically non-decreasing. In fact, many of the values that we use for $V_{i j}$ and $\Omega_{j i}$ are not just upper bounds, but they are the precise values, though we do not prove this property and do not use it. In order to apply the formula, one has to show that all optimal solutions are of order of growth $\Theta(N)$ (since we are interested in a lower bound on the asymptotic competitive ratio), which will be shown later.

We have

$$
\sum_{i=1}^{k} v_{1 i}+\sum_{j=2}^{4} \sum_{i=0}^{2} v_{j i}=68.25-\frac{1}{4 \cdot 5^{k-1}}
$$

since $\sum_{i=k-1}^{k} v_{1 i}+\sum_{j=2}^{4} \sum_{i=0}^{2} v_{j i}=67$ and

$$
\sum_{i=1}^{k-2} v_{1 i}=\sum_{i=1}^{k-2} \frac{1}{5^{k-i-2}}=\sum_{j=0}^{k-3} \frac{1}{5^{j}}=\frac{1-\frac{1}{5^{k-2}}}{4 / 5}=1.25-\frac{1}{4 \cdot 5^{k-1}},
$$

and

$$
Q \leq \frac{1}{168} \cdot\left(42 \cdot\left(5-\frac{1}{5^{k-3}}\right) / 5^{k-3}+\sum_{i=2}^{k-2} 42 \cdot\left(5-\frac{1}{5^{k-i-2}}\right) \cdot\left(\frac{1}{5^{k-i-2}}-\frac{1}{5^{k-(i-1)-2}}\right)\right.
$$

$+1 \cdot 126+2 \cdot 112+6 \cdot 96+6 \cdot 72+12 \cdot 68+14 \cdot 48+14 \cdot 42+28 \cdot 36+21 \cdot 24+21 \cdot 18+42 \cdot 12)$.
Since

$$
\begin{aligned}
& \sum_{i=2}^{k-2}\left(5-\frac{1}{5^{k-i-2}}\right) \cdot\left(\frac{1}{5^{k-i-2}}-\frac{1}{5^{k-(i-1)-2}}\right)=\sum_{i=2}^{k-2}\left(5-\frac{1}{5^{k-i-2}}\right) \cdot \frac{4}{5^{k-i-1}} \\
& \quad=4 \cdot \sum_{i=2}^{k-2} \frac{1}{5^{k-i-2}}-0.8 \cdot \sum_{i=2}^{k-2} \frac{1}{25^{k-i-2}}=4 \cdot \sum_{u=0}^{k-4} \frac{1}{5^{u}}-0.8 \cdot \sum_{u=0}^{k-4} \frac{1}{25^{u}} \\
& =4 \cdot\left(\frac{1-(1 / 5)^{k-3}}{0.8}\right)-0.8 \cdot\left(\frac{1-(1 / 25)^{k-3}}{0.96}\right)=\frac{25}{6}-\frac{1}{5^{k-4}}+\frac{1}{6 \cdot 5^{2 k-7}}
\end{aligned}
$$

we have,

$$
Q \leq \frac{1}{168} \cdot\left(42 / 5^{k-4}-42 / 5^{2 k-6}+42\left(\frac{25}{6}-\frac{1}{5^{k-4}}+\frac{1}{6 \cdot 5^{2 k-7}}\right)+5828\right)=\frac{6003-\frac{7}{5^{2 k-6}}}{168}
$$

We get

$$
r \geq \frac{68.25-\frac{1}{4.5^{k-1}}}{Q} \geq \frac{68.25-\frac{1}{4.5^{k-1}}}{\frac{6003}{168}-\frac{1}{24 \cdot 5^{2 k-6}}} .
$$

Letting $k$ grow to infinity, we get $\frac{11466}{6003}=\frac{1274}{667} \approx 1.9100449$.

| Item type <br> $\ell_{j i}$ | width <br> $w_{j i}$ | height <br> $h_{j i}$ | weight <br> $v_{j i}$ | upper bound <br> on $V_{j i}$ | upper bound on <br> $168 \cdot O P T_{j i} / N \quad\left(\Omega_{j i}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $j=1,1 \leq i \leq k-2$ | $\frac{1+\delta}{5^{k-i-1}}$ | $\frac{1}{43}+\varepsilon$ | $\frac{1}{5^{k-i-2}}$ | $42\left(5-\frac{1}{\left.5^{k-i-2}\right)}\right.$ | $\frac{1}{5^{k-i-2}}$ |
| $j=1, i=k-1$ | $\frac{1+2^{40} \delta}{4}$ | $\frac{1}{43}+\varepsilon$ | 1 | 126 | 2 |
| $j=1, i=k$ | $\frac{1+2^{40} \delta}{2}$ | $\frac{1}{43}+\varepsilon$ | 2 | 112 | 4 |
| $j=2, i=0$ | $\frac{1}{4}-2^{32} \delta$ | $\frac{1}{7}+\varepsilon$ | 4 | 96 | 10 |
| $j=2, i=1$ | $\frac{1}{4}+2^{30} \delta$ | $\frac{1}{7}+\varepsilon$ | 4 | 72 | 16 |
| $j=2, i=2$ | $\frac{1}{2}+2^{31} \delta$ | $\frac{1}{7}+\varepsilon$ | 8 | 68 | 28 |
| $j=3, i=0$ | $\frac{1}{4}-2^{22} \delta$ | $\frac{1}{3}+\varepsilon$ | 6 | 48 | 42 |
| $j=3, i=1$ | $\frac{1}{4}+2^{20} \delta$ | $\frac{1}{3}+\varepsilon$ | 6 | 42 | 56 |
| $j=3, i=2$ | $\frac{1}{2}+2^{21} \delta$ | $\frac{1}{3}+\varepsilon$ | 12 | 36 | 84 |
| $j=4, i=0$ | $\frac{1}{4}-2^{12} \delta$ | $\frac{1}{2}+\varepsilon$ | 6 | 24 | 105 |
| $j=4, i=1$ | $\frac{1}{4}+2^{10} \delta$ | $\frac{1}{2}+\varepsilon$ | 6 | 18 | 126 |
| $j=4, i=2$ | $\frac{1}{2}+2^{11} \delta$ | $\frac{1}{2}+\varepsilon$ | 12 | 12 | 168 |

Table 1: A summary of the input and all values required for the proof of the lower bound. The first four columns contain definitions, and the contents of the remaining two columns are proved in the text.

Lemma 2.1 For every valid pair $j, i$, we have $O P T_{j i} \leq \Omega_{j i}$, where $\Omega_{j i}$ is stated in Table $\mathbb{1}$.
Proof. Let $j=1$. Let $1 \leq i \leq k$, and consider a subset of items consisting of one item of every type $\ell_{1 a}$ for $1 \leq a \leq i$. For $1 \leq i \leq k-2$, the set has total width below $\frac{1}{4.5^{k-i-2}}$, and therefore one can pack them into a rectangle of height $\frac{1}{43}+\varepsilon$ and width $\frac{1}{4.5^{k-i-2}}$. A bin can be split into 42 rows of height $\frac{1}{42}$ and $4 \cdot 5^{k-i-2}$ columns of width $\frac{1}{4 \cdot 5^{k-i-2}}$, resulting in $42 \cdot 4 \cdot 5^{k-i-2}$ such rectangles. Thus,

$$
O P T_{j i} \leq \frac{N}{42 \cdot 4 \cdot 5^{k-i-2}}=\frac{N}{168} \cdot \frac{1}{5^{k-i-2}},
$$

and we let $\Omega_{1 i}=\frac{1}{168} \cdot \frac{1}{5^{k-i-2}}$. For $i=k-1$, the total width is below $\frac{1}{2}$, so the columns will be of width $\frac{1}{2}$, and $\Omega_{1 k}=\frac{1}{84}$. For $i=k$, the total width is below 1 , so the columns will be of width 1 (that is, there are no columns), and $\Omega_{1 k}=\frac{1}{42}$.

For $j=2,3,4$, an item of type $\ell_{j 0}$ can be packed into a rectangle of width $\frac{1}{4}$ and the corresponding height ( $\frac{1}{7}$ for $j=2, \frac{1}{3}$ for $j=3$, and $\frac{1}{2}$ for $j=1$ ), two items, one of type $\ell_{j 0}$ and one of type $\ell_{j 1}$ have total width below $\frac{1}{2}$, and they can be packed into a rectangle of width $\frac{1}{2}$ and the
corresponding height. For three items, one of each type out of $\ell_{j 0}, \ell_{j 1}$, and $\ell_{j 2}$, the total width is below 1 , they can be packed into a rectangle of width 1 and the corresponding height.

Bounding $O P T_{20}$ is done as follows. Create $\frac{N}{24}$ bins with six rows of height $\frac{1}{7}+\varepsilon$ and six rows of height $\frac{1}{43}+\varepsilon$. This is possible since $\varepsilon=0.0001$. Every row of height $\frac{1}{7}+\varepsilon$ is split into four columns of width $\frac{1}{4}$. This allows us to pack all items of type $\ell_{20}$, as there are 24 areas in every bin that can contain an item of type $\ell_{20}$ each. Additionally, we can pack one item of each type $\ell_{1 i}$ (where $1 \leq i \leq k$ ) into every row of height $\frac{1}{43}+\varepsilon$, allowing to pack six items of every such type into every bin, and leaving $\frac{3 N}{4}$ items of each such type unpacked. These items are packed into $\frac{3 N}{4 \cdot 42}$ additional bins, each having 42 rows of height $\frac{1}{43}+\varepsilon$. Thus, we let $\Omega_{20}=\frac{10}{168}$.

Bounding $O P T_{21}$ is done similarly, but pairs of a type $\ell_{20}$ item and a type $\ell_{21}$ item are packed into areas of width $\frac{1}{2}$, so they occupy $\frac{N}{12}$ bins and $\frac{N}{2}$ items of each type $\ell_{1 i}$ remain unpacked and they require $\frac{N}{2 \cdot 42}$ bins, leading to the definition $\Omega_{21}=\frac{16}{168}$. For $O P T_{22}$, triples of a type $\ell_{20}$ item, a type $\ell_{21}$ item, and a type $\ell_{22}$ item are packed into areas of width 1 , so they occupy $\frac{N}{6}$ bins, and all items of types $\ell_{1 i}$ are packed into the rows of height $\frac{1}{43}+\varepsilon$ in the same bins. This leads to the definition $\Omega_{22}=\frac{28}{168}$.

Bounding $O P T_{30}$ is done as follows. Create $\frac{N}{8}$ bins with two rows of height $\frac{1}{3}+\varepsilon$, two rows of height $\frac{1}{7}+\varepsilon$ and two rows of height $\frac{1}{43}+\varepsilon$. This is possible since $\varepsilon=0.0001$. Every row of height $\frac{1}{3}+\varepsilon$ is split into four columns of width $\frac{1}{4}$. This allows us to pack all items of type $\ell_{30}$, as every bin has eight areas where such an item can be packed. Additionally, we can pack one item of each type $\ell_{1 i}$ into every row of height $\frac{1}{43}+\varepsilon$, and we can pack one item of each type $\ell_{2 i}$ into every row of height $\frac{1}{7}+\varepsilon$, leaving $\frac{3 N}{4}$ items of each such type unpacked. These items are packed into $\frac{3 N}{4 \cdot 6}$ additional bins, each having six rows of height $\frac{1}{7}+\varepsilon$ and six rows of height $\frac{1}{43}+\varepsilon$. Thus, we let $\Omega_{30}=\frac{42}{168}$.

Bounding $O P T_{31}$ is done similarly, but pairs of a type $\ell_{30}$ item and a type $\ell_{31}$ item are packed into areas of width $\frac{1}{2}$, so they occupy $\frac{N}{4}$ bins and $\frac{N}{2}$ items of each type $\ell_{j i}$ for $j=1,2$ remain unpacked and they require $\frac{N}{12}$ bins, leading to the definition $\Omega_{31}=\frac{56}{168}$. For $O P T_{32}$, triples of a type $\ell_{30}$ item, a type $\ell_{31}$ item, and a type $\ell_{32}$ item are packed into areas of width 1 , so they occupy $\frac{N}{2}$ bins, and all items of types $\ell_{1 i}$ and $\ell_{2 t}$ are packed into the rows of height $\frac{1}{43}+\varepsilon$ in the same bins. This leads to the definition $\Omega_{32}=\frac{84}{168}$.

Bounding $O P T_{40}$ is done as follows. Create $\frac{N}{4}$ bins with a row of every height out of $\frac{1}{2}+\varepsilon$, $\frac{1}{3}+\varepsilon, \frac{1}{7}+\varepsilon$, and $\frac{1}{42}+\varepsilon$. This is possible since $\varepsilon=0.0001$. Every row of height $\frac{1}{2}+\varepsilon$ is split into four columns of width $\frac{1}{4}$. This allows us to pack all items of type $\ell_{40}$. Additionally, we can pack one item of each type $\ell_{j i}$ for any $j \in\{1,2,3\}$ and any $i$ into the other rows, leaving $\frac{3 N}{4}$ items of each such type unpacked. These items are packed into $\frac{3 N}{4 \cdot 2}$ additional bins, each having two rows of every height excluding $\frac{1}{2}+\varepsilon$. Thus, we let $\Omega_{40}=\frac{105}{168}$.

Bounding $O P T_{41}$ is done similarly, but pairs of a type $\ell_{40}$ item and a type $\ell_{41}$ item are packed into areas of width $\frac{1}{2}$, so they occupy $\frac{N}{2}$ bins and $\frac{N}{2}$ items of each type $\ell_{j i}$ for $j=1,2,3$ remain unpacked and they require $\frac{N}{4}$ bins, leading to the definition $\Omega_{31}=\frac{126}{168}$. For $O P T_{32}$, triples of a type $\ell_{30}$ item, a type $\ell_{31}$ item, and a type $\ell_{32}$ item are packed into areas of width 1 , so they occupy
$N$ bins, and all items of other types are packed into the rows of the other three heights. This leads to the definition $\Omega_{32}=1$.

We are left with the task of bounding $V_{i j}$. The bounds will be proved using a sequence of lemmas, where the first one is general an it is used in several proofs.

Lemma 2.2 Let $b_{w}$ and $b_{h}$ be positive integers. Consider an item size such that the width is in $\left(\frac{1}{b_{w}+1}, 1\right]$, and the height is in $\left(\frac{1}{b_{h}+1}, 1\right]$. Consider a bin that contains $f$ items of this type (and possibly other items). Then, $f \leq b_{w} \cdot b_{h}$.

Proof. Consider the bin and draw $b_{h}$ horizontal lines. Considering also the bottom and top of the bin, the distances between any two consecutive lines will be $\frac{1}{b_{h}+1}$. Since the height of the items is above $\frac{1}{b_{h}+1}$, every item contains a part of at least one line in its interior (such a line is not the bottom or top). Since the width of every item is above $\frac{1}{b_{w}+1}$, and items cannot overlap (except for their boundary), there can be at most $b_{w}$ items containing a part of a line. For every item, associate it with a line that it contains a part of it (if there is more than one such line, choose one arbitrarily). As there are $b_{h}$ lines with at most $b_{w}$ items each, there are at most $b_{w} \cdot b_{h}$ items of this type.

The last lemma shows in particular that all optimal solutions have order of growth $\Omega(N)$, as the first $N$ items have sides larger than $\frac{1}{5^{k-2}}$ and $\frac{1}{43}$, respectively, so the cost of any solution is at least $\frac{N}{42 \cdot\left(5^{k-2}-1\right)}$. An upper bound of $O(N)$ on the cost of an optimal solution for every input follows from the total number of items which is $(k+9) N$.

We use the concept of dominance as in [8]. For an item type $\ell_{j i}$ and an item of type $\ell_{j^{\prime} i^{\prime}}$, if there are integers $c_{w}$ and $c_{h}$ such that $w_{j i} \geq c_{w} \cdot w_{j^{\prime} i^{\prime}}$ and $h_{j i} \geq c_{h} \cdot h_{j^{\prime} i^{\prime}}$, while $v_{j i} \leq c_{w} \cdot c_{h} \cdot v_{j^{\prime} i^{\prime}}$, we say that type $\ell_{j^{\prime} i^{\prime}}\left(c_{w}, c_{h}\right)$-dominates (or simply dominates) type $\ell_{j i}$ in the sense that in the calculation of the maximum weight of any feasible bin, items of type $\ell_{j i}$ do not need to be considered, as every such item can be replaced with $c_{w} \cdot c_{h}$ items of type $\ell_{j^{\prime} i^{\prime}}$, without decreasing the total weight. Note that the dominance relation is transitive. The value $c_{w} \cdot c_{h}$ is called the factor of dominance.

Lemma 2.3 1. For every $j \in\{2,3,4\}$ and every $i=0,1$, type $\ell_{j i}$ dominates $\ell_{j(i+1)}$.
2. For $i=1,2, \ldots, k-1$, type $\ell_{1 i}$ dominates $\ell_{1(i+1)}$.
3. Item type $\ell_{1(k-2)}$ dominates item type $\ell_{20}$.
4. Item type $\ell_{20}$ dominates item type $\ell_{30}$.
5. Item type $\ell_{30}$ dominates item type $\ell_{40}$.

## Proof.

1. For every $j \in\{2,3,4\}$, the type $\ell_{j 0}$ dominates $\ell_{j 1}$ since the width of the former type is smaller, and their heights and weights are equal. Type $\ell_{j 1}$ dominates type $\ell_{j 2}$ since the width of the former is twice as small, their heights are equal, and the weight ratio satisfies $v_{j 2} / v_{j 1}=2$.
2. For $1 \leq i \leq k-3$, type $\ell_{1 i}$ dominates type $\ell_{1(i+1)}$ as their heights are equal, and $\frac{w_{1(i+1)}}{w_{1 i}}=$ $\frac{v_{1(i+1)}}{v_{1 i}}=5$. Type $\ell_{1(k-2)}$ dominated type $\ell_{1(k-1)}$ since their heights are equal, $w_{1(k-2)}<$ $w_{1(k-1)}$ and $v_{1(k-2)}=v_{1(k-1)}$. Type $\ell_{1(k-1)}$ dominated type $\ell_{1 k}$ since their heights are equal, $2 \cdot w_{1(k-1)}=w_{1(k-1)}$ and $2 \cdot v_{1(k-1)}=v_{1 k}$.
3. The height of item type $\ell_{1(k-2)}$ is $\frac{1}{43}+\varepsilon$, and the height of item type $\ell_{20}$ is $\frac{1}{7}+\varepsilon$. We have $6\left(\frac{1}{43}+\varepsilon\right)<\frac{1}{7}+\varepsilon$ as $\varepsilon<0.0001$. The width of item type $\ell_{1(k-2)}$ is $\frac{1+\delta}{5}$, and the width of item type $\ell_{20}$ is $\frac{1}{4}-2^{32} \delta$. We have $\frac{1+\delta}{5}<\frac{1}{4}-2^{32} \delta$ as $2^{64} \delta<1$. As the weight of six items of type $\ell_{1(k-2)}$ is 6 , while the weight of one item of type $\ell_{20}$ is 4 , the domination holds.
4. Item type $\ell_{20}$ has height $\frac{1}{7}+\varepsilon$ while item type $\ell_{30}$ has height $\frac{1}{3}+\varepsilon$, and we have $2\left(\frac{1}{7}+\varepsilon\right)<\frac{1}{3}+\varepsilon$, as $\varepsilon<0.0001$. Item type $\ell_{20}$ has smaller width than item type $\ell_{30}$. The weight of two items of type $\ell_{20}$ is 8 while the weight of one type $\ell_{30}$ item is 6 . Thus, the domination holds.
5. Item type $\ell_{30}$ has both smaller height and smaller width than an item of type $\ell_{40}$ and they have the same weights. Thus, the domination holds.

Lemma 2.4 In the following cases it is sufficient to consider bins containing only items of type $\ell_{j i}$ for the computation of $V_{j i}$.

1. The case $j=4$ and $i=0,1,2$.
2. The cases $j=2,3$ and $i=0$.
3. The case $j=1$ and $i \leq k-2$.

In the cases $j=1$ and $i=k-1, k$, it is sufficient to consider only $\ell_{1 i}$ and $\ell_{20}$. In the cases $j=2$ and $i=1,2$ it is sufficient to consider only $\ell_{j i}$ and $\ell_{30}$. In the cases $j=3$ and $i=1,2$ it is sufficient to consider only $\ell_{j i}$ and $\ell_{40}$

Proof. The three cases where one item type can be considered follow by transitivity of domination, since every such type dominates every later type.

In the other cases the mentioned two item types are sufficient as for every later type (later than $\ell_{j i}$, at least one of these two mentioned types dominates the later one.

Corollary 2.5 All $V_{j i}$ values in the table for the next cases are correct.
The case $j=4$ and $i=0,1,2$, the cases $j=2,3$ and $i=0$, and the case $j=1$ and $i \leq k-2$.
Proof. For these values we consider one type of item.
In the cases where $i=0$ and $j \geq 2$, the width of an item is in $\left(\frac{1}{5}, \frac{1}{4}\right]$, so $b_{w}=4$, and the heights for $j=2,3,4$ are in $\left(\frac{1}{b_{h}+1}, \frac{1}{b_{h}}\right]$ for $b_{h}=6,2,1$, respectively. Thus, taking the item weights into account we let $V_{20}=4 \cdot 24=96, V_{30}=6 \cdot 8=48$, and $V_{40}=6 \cdot 4=24$.

As $w_{41}>\frac{1}{4}$, we let $V_{41}=6 \cdot 3=18$, and as $w_{42}>\frac{1}{2}$, we let $V_{42}=12$.
Consider $j=1$, for which the height is above $\frac{1}{43}$. For $1 \leq i \leq k-2$, we let $V_{1 i}=\frac{1}{5^{k-i-2}} \cdot 42$. $\left(5^{k-i-1}-1\right)$ since the width of these items is above $\frac{1}{5^{k-i-1}}$ and the height is above $\frac{1}{43}$.

Lemma 2.6 We have $V_{31}=42$ and $V_{32}=36$.
Proof. To prove the first bound, we consider types $\ell_{31}$ and $\ell_{40}$. Since for these two types the widths are above $\frac{1}{5}$ and the heights are above $\frac{1}{3}$, no bin can contain more than eight such items. Moreover, for type $\ell_{40}$ the width is above $\frac{1}{2}$, so no bin can contain more than four such items. If the bin has at most seven items, we are done, as the weight of any item of one of these types is 6 and therefore we assume that there are eight such items.

Recall that we use the word intersecting with the meaning that the intersection is in the interior and not on the boundary. Draw two horizontal lines with distances of $\frac{1}{3}$ between consecutive lines including the top and bottom. Due to item heights, every item intersects at least one line (contains a part of a line in its interior) and we associate it with such a line. If it intersects both lines, we associate it with one of them.

As all widths are above $\frac{1}{5}$, every line can intersect at most four items, and as there are eight items, each associated with one of the lines, we find that every line intersects exactly four items, and no item intersects two lines. For a given line (one of the two), let $y_{31} \geq$ and $y_{40} \geq$ be the (integer) numbers of items of types $\ell_{31}$ and $\ell_{40}$ associated with this line, where $\ell_{31}+\ell_{40}=4$. We have

$$
1 \geq y_{31}\left(\frac{1}{4}+2^{20} \delta\right)+y_{40}\left(\frac{1}{4}-2^{12} \delta\right)=\left(y_{31}+y_{40}\right) / 4+\delta\left(2^{20} y_{31}-2^{12} y_{40}\right)=1+\delta\left(2^{20} y_{31}-2^{12} y_{40}\right)
$$

which implies $y_{40} \geq 2^{8} y_{31}$. The only solution is $y_{40}=4$ and $y_{31}=0$. However, this proves that the bin has eight type $\ell_{40}$ items, a contradiction.

To prove the second bound, note that a bin can contain at most two items of type $\ell_{32}$, as their heights are above $\frac{1}{3}$ and their widths are above $\frac{1}{2}$. Since a bin contains at most four items of type $\ell_{40}$, if there is at most one item of type $\ell_{32}$, we are done, as $w_{32}=12$. Assume that there are two such items. Drawing horizontal lines as before, every $\ell_{32}$ type item overlaps at least one such line, and as $w_{32}>\frac{1}{2}$, each item overlaps exactly one of the lines. Since $\left(\frac{1}{2}+2^{21} \delta\right)+2\left(\frac{1}{4}-2^{12} \delta\right)>1$, each line overlaps at most one item of type $\ell_{40}$. Since every type $\ell_{40}$ item overlaps at least one line, there are at most two such items, and the total weight is at most $2 \cdot 12+2 \cdot 6=36$.

Lemma 2.7 We have $V_{21}=72$ and $V_{22}=68$.
Proof. To prove the first bound, we consider types $\ell_{21}$ and $\ell_{30}$.
Here we draw six horizontal lines with distances of $\frac{1}{7}$ between consecutive lines including the top and bottom. Due to item heights, every type $\ell_{21}$ item intersects a line and we associate it with one such line. Every type $\ell_{30}$ item intersects at least two lines (as otherwise its height is at most $\frac{2}{7}$ ) and we associate it with exactly two such lines.

As all widths are above $\frac{1}{5}$, every line intersects at most four items, and it has at most four items associated with it. For a given line (one of the six), let $y_{21}$ and $y_{30}$ be the (integer) numbers of items of types $\ell_{21}$ and $\ell_{30}$ associated with it, where $\ell_{31}+\ell_{40} \leq 4$, as widths are larger than $\frac{1}{5}$. We have

$$
1 \geq y_{21}\left(\frac{1}{4}+2^{30} \delta\right)+y_{30}\left(\frac{1}{4}-2^{22} \delta\right)=\left(y_{21}+y_{30}\right) / 4+\delta\left(2^{30} y_{31}-2^{22} y_{40}\right)
$$

which implies that either $y_{21}+y_{30} \leq 3$, or that $y_{30}=4$ and $y_{21}=0$. The second option holds since in the case $y_{21}+y_{30} \leq 3$ we get $y_{30} \geq 2^{8} y_{21}$ similarly to the proof of the previous lemma.

For every item associated with one line, we assign its weight to the line, and for items associated with two lines, we assign half of the weight to each such line, so the weight is split equally between its two associated lines. Thus, an item of type $\ell_{21}$ assigns a weight of 4 to its line, and an item of type $\ell_{30}$ assigns a weight of 3 to each of its lines.

Consider a specific lines again. If $y_{21}+y_{30} \leq 3$, the line is assigned at most a weight of 12 . In the case $y_{30}=4$ and $y_{21}=0$, it is also assigned a weight of 12 (as the share of weight for every item is 3 ). As there are six lines, the total weight is at most $6 \cdot 12=72$.

To prove the second bound, we consider types $\ell_{22}$ and $\ell_{30}$. We draw lines and associate items as above. Every line can intersect at most one item of type $\ell_{22}$ as the width of such an item is above $\frac{1}{2}$. If a line does not have such an item associated with it, it can have at most four $\ell_{30}$ items associated with it. Otherwise, since $w_{22}+2 \cdot w_{30}>1$, it can have at most one type $\ell_{30}$ item associated with it. Let $x_{0}$ be the number of lines without an $\ell_{22}$ item associated with them and let $x_{1}=6-x_{0}$ be the number of lines having an $\ell_{22}$ item associated with them. As every $\ell_{30}$ item is associated with two lines, the number of $\ell_{30}$ items is at most

$$
\left\lfloor\frac{1}{2} \cdot\left(4 x_{0}+x_{1}\right)\right\rfloor=\left\lfloor\frac{1}{2} \cdot\left(3 x_{0}+6\right)\right\rfloor=\left\lfloor\frac{3 \cdot x_{0}}{2}\right\rfloor+3
$$

The number of $\ell_{30}$ items is also at most 8 , as their heights are above $\frac{1}{3}$, and their widths are above $\frac{1}{5}$.

The number of $\ell_{22}$ items is $x_{1}$. Using the weights of items ( 8 for type $\ell_{22}$ and 6 for type $\ell_{30}$ ), the total weight is at most

$$
6 \cdot\left(3+\left\lfloor\frac{3 x_{0}}{2}\right\rfloor\right)+8 x_{1}=18+6\left\lfloor\frac{3 x_{0}}{2}\right\rfloor+8\left(6-x_{0}\right)=66+6\left\lfloor\frac{3 x_{0}}{2}\right\rfloor-8 x_{0} \leq 66+x_{0}
$$

so for $x_{0} \leq 2$ the total weight does not exceed 68 . The total weight is also at most $6 \cdot 8+8 x_{1}$, using the property that there are $x_{1}$ items of type $\ell_{22}$ and at most eight items of type $\ell_{30}$, so for $x_{1} \leq 2$, the weight does not exceed 64 . The only remaining case is $x_{0}=x_{1}=3$. In this case we have $\left\lfloor\frac{3 x_{0}}{2}\right\rfloor=4$, and

$$
66+6\left\lfloor\frac{3 x_{0}}{2}\right\rfloor-8 x_{0}=66+6 \cdot 4-8 \cdot 3=66
$$

Lemma 2.8 We have $V_{1(k-1)}=126$ and $V_{1 k}=112$.
Proof. We will consider type $V_{20}$ for all bounds, and type $\ell_{1(k-1)}$ or $\ell_{1 k}$ for the two bounds. We will use the property that the width of type $\ell_{20}$ is above $\frac{1}{5}$.
$\frac{1}{4}-2^{32} \delta>\frac{1}{4}-\frac{1}{2^{30}}>0.24$, so any horizontal line can intersect the interior of at most four such items.

Recall that

$$
w_{20}+3 \cdot w_{1(k-1)} \geq 2 \cdot w_{20}+2 \cdot w_{1(k-1)} \geq 3 \cdot w_{20}+w_{1(k-1)}>1
$$

and

$$
2 \cdot w_{20}+w_{1 k}=2 \cdot w_{20}+2 \cdot w_{1(k-1)}>1
$$

We also use the property $2 w_{1 k}=4 w_{1(k-1)}>1$.
Here, we draw 42 lines of distances $\frac{1}{43}$ between consecutive lines, including the top and bottom. Since $h_{20}=\frac{1}{7}+\varepsilon$, every item of type $\ell_{20}$ contains parts of at least six lines (as an item with at most five lines in its interior has height at most $\frac{7}{43}$ ), and we associate it with exactly six lines. Any item of the other type contains a part of at least one line, and we associate it with one such line.

The calculation of $V_{1(k-1)}$ is as follows. A line can have at most three $\ell_{1(k-1)}$ items associated with it due to the width of this type that is larger than $\frac{1}{4}$. The maximum number of $\ell_{20}$ items for a line with $3,2,1$, and 0 such items can have at most the following numbers of $\ell_{20}$ items associated with it (respectively): $0,1,2$, and 4 . Let $x_{i}$ be the number of lines for which the number of $\ell_{1(k-1)}$ items associated with them is $i$, where $x_{0}+x_{1}+x_{2}+x_{3}=42$.

As there are six lines associated with every $\ell_{20}$ item, we have at most $\left\lfloor\frac{1}{6}\left(x_{2}+2 x_{1}+4 x_{0}\right)\right\rfloor$ items of type $\ell_{20}$. The weight of an $\ell_{1(k-1)}$ item is 1 , and the weight of a type $\ell_{20}$ item is 4 (so the share of every line associated with it is $\frac{2}{3}$ ). Thus the total weight is at most

$$
4\left\lfloor\frac{x_{2}+2 x_{1}+4 x_{0}}{6}\right\rfloor+\left(3 x_{3}+2 x_{2}+x_{1}\right) \leq 3\left(x_{0}+x_{1}+x_{2}+x_{3}\right)=3 \cdot 42=126 .
$$

The calculation of $V_{1 k}$ is as follows. A line can have at most one $\ell_{1 k}$ item associated with it, as its width is above $\frac{1}{2}$. The maximum number of $\ell_{20}$ items associated with a line with one $\ell_{1 k}$ item is one, and it there are no $\ell_{1 k}$ items, there can be at most four $\ell_{20}$ items associated with the line, as their widths are above $\frac{1}{5}$. Let $x_{0}$ and $x_{1}=42-x_{0}$ be the numbers of lines with no $\ell_{1 k}$ items associated with them, and with one associated $\ell_{1 k}$ item, respectively. As there are six lines associated with every $\ell_{20}$ item, we have at most $\left\lfloor\frac{1}{6}\left(4 x_{0}+x_{1}\right)\right\rfloor$ such items. The weight of an $\ell_{1 k}$ item is 2 . Thus the total weight is at most

$$
4\left\lfloor\frac{4 x_{0}+x_{1}}{6}\right\rfloor+2 x_{1} \leq \frac{8}{3}\left(x_{0}+x_{1}\right)=\frac{8}{3} \cdot 42=112 .
$$

We conclude with the following theorem.
Theorem 2.9 The asymptotic competitive ratio of any online algorithm for rectangle packing is at least $\frac{1274}{667} \approx 1.9100449$.

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