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Abstract

In the vertex cover problem, the input is a graph G and an integer k, and the goal is to decide whether there is a set of vertices S of size at most k such that every edge of G is incident on at least one vertex in S. We study the vertex cover problem on graphs with maximum degree 4 and minimum degree at least 2, parameterized by r = k - n/3. We give an algorithm for this problem whose running time is $O^*(1.6253^r)$. As a corollary, we obtain an $O^*(1.2403^k)$ -time algorithm for vertex cover on graphs with maximum degree 4.

Keywords graph algorithms, parameterized complexity.

1 Introduction

For an undirected graph G, a vertex cover of G is a set of vertices S such that every edge of G is incident on at least one vertex in S. In the parameterized vertex cover problem, the input is a graph G and an integer k and the goal is to decide whether there is a vertex cover of G of size at most k. The parameterized vertex cover problem has been studied extensively. The first parameterized algorithm for vertex cover was given in [3]. Improved algorithms were given in [2,4,5,7,9–12,14]. The parameterized vertex cover problem was also studied on graphs with maximum degree 3 [5,6,8,13,15] and maximum degree 4 [1,5].

Consider the vertex problem on graphs with maximum degree Δ . It is easy to eliminate vertices with degree at most 1 from the graph. Thus, we can assume that the input graph has minimum degree at least 2. It is easy to show that the minimum size of a vertex cover in a graph with maximum degree Δ and minimum degree at least 2 is at least $\frac{2}{2+\Delta}n$, where *n* is the number of vertices (cf. [15]). Therefore, it is more natural to use the "above guarantee" parameter $r_{\Delta} = k - \frac{2}{2+\Delta}n$. For $\Delta = 3$, Xiao [15] gave an algorithm with $O^*(1.6651^{r_3})$ running time. An $O(c^{r_{\Delta}})$ -time algorithm for vertex cover can also give an algorithm for vertex cover parameterized by *k*. Suppose that there is an exponential algorithm for vertex cover on graphs with maximum degree Δ whose running time is $O^*(d^n)$. Let $\alpha = 1/(2/(2 + \Delta) + \log_c d)$. Given an instance (G, k) of vertex cover, if $n > \alpha k$, run the parameterized algorithm on (G, k) in $O^*(c^{r_{\Delta}}) = O^*(d^{\alpha k})$ time, and otherwise run the exponential algorithm on G in $O^*(d^n) = O^*(d^{\alpha k})$ time. For graphs with maximum degree 3, combining the $O^*(1.6651^{r_3})$ -time algorithm of Xiao [15] with the $O(1.0836^n)$ -time algorithm of Xiao and Nagamochi [16] gives an $O^*(1.1558^k)$ -time algorithm, which is faster than the previous algorithms given for this problem [5, 6, 8, 13].

In this paper we give an algorithm for vertex cover on graphs with maximum degree 4 and minimum degree at least 2 whose running time is $O^*(1.6253^{r_4})$. Combining

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our algorithm with the $O(1.1376^n)$ -time algorithm of Xiao and Nagamochi [17] gives an $O^*(1.2403^k)$ -time algorithm for vertex cover on graphs with maximum degree 4. This algorithm is faster than the $O^*(1.2637^k)$ -time algorithm of Agrawal et al. [1].

Our algorithm is similar to the algorithm of Xiao [15] for graphs with maximum degree 3. Unlike the algorithm of Xiao that uses one branching rule, our algorithm uses several branching rules. The branching rules of our algorithm are based on the rules of the algorithm of Chen et al. [5] with minor modifications in order obtain a faster algorithm. Since the rules of our algorithm are based on the rules of [5], we omit the proof of correctness of these rules.

2 Preliminaries

For a graph G and a set of vertices S, G - S is the graph obtained from G by removing the vertices of S and the edges incident on these vertices. For a set of vertices $S, N(S) = (\bigcup_{v \in S} N(v)) \setminus S$.

The merge operation on a set of vertices S in a graph G generates a graph G' by deleting the vertices of S and adding a new vertex v^* . The vertex v^* is adjacent to a vertex u in G' if and only if there is an edge in G between u and a vertex in S.

2.1 Reduction rules

In this section we describe several reduction rules that are used by our algorithm.

Rule (D0) and Rule (D1) below handle vertices with degree at most 1. While we assume that the input graph has minimum degree 2, graphs generated during the algorithm can contain vertices with degree at most 1 and these rules are used to eliminate such vertices.

(D0) If there is a vertex v with degree 0, delete v from G.

(D1) If there is a vertex v with degree 1, delete the unique neighbor of v from G and decrease k by 1.

A general crown in a graph G is a pair C, H of disjoint nonempty sets of vertices such that the vertices in C have degree 0 in G - H. A crown C, H is called good if G - C - H has minimum degree at least 2. A proper crown is a general crown C, H such that there is a matching between C and H of size |H|. An almost crown is a general crown C, H such that there such that |H| = |C| + 1 and $|N(S)| \ge |S| + 1$ for every $\emptyset \ne S \subseteq C$. We use the following reduction rules from [7].

(C1) If C, H is a proper crown, delete $C \cup H$ from G and decrease k by |H|.

(C2) If C, H is an almost crown and H is not an independent set, delete $C \cup H$ from G and decrease k by |H|.

(C3) If C, H is an almost crown and H is an independent set, merge the vertices in $C \cup H$ and decrease k by |H| - 1.

Lemma 1 (Xiao [15]). Let G be a graph with minimum degree at least 2. If C, H is a general crown and $|C| \ge |H| - 1$, it is possible in polynomial time to either find a good proper crown C', H', or to conclude that C, H is an almost crown.

3 The Algorithm

In this section we give an algorithm for vertex cover parameterized by $r = r_4$ on graphs with maximum degree at most 4 and minimum degree at least 2. We first describe an algorithm VCBASE₃ for solving the vertex cover problem on *connected* graphs with maximum degree at most 3 and minimum degree at least 2. The algorithm maintains the following invariants on the current graph: (1) The minimum degree is at least 2. (2) The maximum degree is at most 3. (3) Except for the initial instance, every connected component is not 3-regular.

Before describing algorithm VCBASE₃, we give a procedure BRANCH₃. The input to the procedure is an instance (G, k) of vertex cover and sets of vertices S_1, \ldots, S_t . The goal of the procedure is to performs branching on the instances $(G - S_1, k - |S_1|), \ldots,$ $(G - S_t, k - |S_t|)$, after applying Rule (D0) and Rule (D1) on each instance. However, it is possible that due to many applications of Rule (D0), the decrease in r in at least one branch will be too small. In this case, the procedure performs a suitable reduction rule, or make a recursive call. When procedure BRANCH₃ makes a recursive call, it marks one of the sets. Procedure BRANCH₃ performs the following steps.

- 1. For i = 1, ..., t, repeatedly apply Rule (D0) and Rule (D1) on the instance $(G S_i, k |S_i|)$. Let (G_i, k_i) be resulting instance. Let C_i be a set containing all the vertices that were deleted by applications of Rule (D0). Let H_i be a set containing S_i and all the vertices that were deleted by applications of Rule (D1) (note that $(G_i, k_i) = (G C_i H_i, k |H_i|)$).
- 2. If there is i such that S_i is not marked, $C_i \neq \emptyset$, $H_i \neq \emptyset$, and $|C_i| \ge |H_i| 1$:
 - (a) Apply Lemma 1 on the general crown C_i, H_i and either obtain a good proper crown C, H in G, or conclude that C_i, H_i is an almost crown in G.
 - (b) If C, H is a proper crown, apply Rule (C1) on (G, k) and C, H, and let (G', k') be the resulting instance. Return VCBASE₃(G', k').
 - (c) If H_i is not an independent set, apply Rule (C2) on (G, k) and C_i, H_i , and let (G', k') be the resulting instance. Return VCBASE₃(G', k').
 - (d) If $|N(H_i) \setminus C_i| \leq 2$, apply Rule (C3) on (G, k) and C_i, H_i , and let (G', k') be the resulting instance. Return VCBASE₃(G', k').
 - (e) Otherwise, call BRANCH₃($G, k, \{H_i, N(H_i)\}$), where H_i is marked.
- 3. Return VCBASE₃(G_1, k_1) $\lor \cdots \lor$ VCBASE₃(G_t, k_t).

Note that the graph G' in line 2b, 2c, or 2d has minimum degree at least 2 (as C, H and C_i, H_i are good crowns).

Lemma 2. If line 2b, 2c, or 2d of procedure BRANCH₃ is executed, the value of r does not increase when moving from the instance (G, k) to the instance (G', k').

Proof. Suppose first that line 2b is executed. Since G has minimum degree at least 2, the number of edges between C and H is at least 2|C|. Each vertex in H has degree at most 4 (we note that we used 4 and not 3 since we will need this lemma also for the algorithm on graphs with maximum degree 4), and therefore the number of edges between C and H is at most 4|H|. It follows that $2|C| \leq 4|H|$. By definition, k' = k - |H| and |V(G')| = |V(G)| - |C| - |H|. Therefore, the value of r decreases by $|H| - (|C| + |H|)/3 = (2|H| - |C|)/3 \geq 0$.

Now suppose that line 2c is executed. By definition, $k' = k - |H_i|$ and $|V(G')| = |V(G)| - (|C_i| + |H_i|) = |V(G)| - (2|H_i| - 1)$. Therefore, the value of r decreases by $|H_i| - (2|H_i| - 1)/3 = (|H_i| + 1)/3 \ge 1$. Finally, if line 2d is executed, $k' = k - (|H_i| - 1)$ and $|V(G')| = |V(G)| - (|C_i| + |H_i| - 1) = |V(G)| - 2(|H_i| - 1)$. Therefore, the value of r decreases by $|H_i| - 1 - 2(|H_i| - 1)/3 = (|H_i| - 1)/3 \ge \frac{1}{3}$.

Lemma 3. Suppose that line 3 of procedure BRANCH₃ is executed. For every i such that S_i is not marked, when moving from the instance (G, k) to the instance (G_i, k_i) , the value of r decreases by at least $(|H_i| + 2)/3$ if $|H_i| \ge 2$, decreases by $\frac{2}{3}$ if $|H_i| = 1$, and does not change if $|H_i| = 0$.

Proof. Suppose that $|H_i| \ge 2$. We have that $|C_i| \le |H_i| - 2$ otherwise line 3 of the algorithm would not be executed. By definition, in the instance (G_i, k_i) the value of k decreases by $|H_i|$ and the value of n decreases by $|H_i| + |C_i|$. Therefore, the value of r decreases by $|H_i| - (|H_i| + |C_i|)/3 = (2|H_i| - |C_i|)/3 \ge (|H_i| + 2)/3$.

If $|H_i| = 1$ then $|C_i| = 0$ otherwise line 3 of the algorithm would not be executed. Therefore, the value of r decreases by $1 - 1/3 = \frac{2}{3}$. Finally, if $|H_i| = 0$ then $|C_i| = 0$, and the value of r does not change.

We now analyze a call to BRANCH₃ in which S_1 is marked and line 3 is executed. By definition, $S_1 = H'$ and $S_2 = N(H')$, where C', H' is an almost crown that was obtained during the parent call to BRANCH₃. Note that since Rules (D0) and (D1) cannot be applied on G - C' - H', it follows that $H_1 = S_1 = H'$ and $C_1 = C'$. When moving from (G, k) to (G_1, k_1) , the value of k decreases by $|H_1| = |H'|$ and the value of n decreases by $|H_1| + |C_1| = 2|H'| - 1$. Therefore, the value of r decreases by $|H'| - (2|H'| - 1) = (|H'| + 1)/3 \ge 1$. By Lemma 3, when moving from (G, k) to (G_2, k_2) , the value of r decreases by at least $(|H_2| + 2)/3 \ge (|H'| + 4)/3 \ge 2 (|H_2| \ge |S_2| = |N(H')| = |C'| + |N(H') \setminus C'| \ge |C'| + 3 = |H'| + 2)$. Therefore, the branching vector in this case is at least (1, 2) and the branching number is at most 1.6181.

We now describe algorithm VCBASE₃. The algorithm applies the first applicable rule from the following rules.

- (1) If r < 0, return 'no'.
- (2) If $V(G) = \emptyset$ return 'yes'.

(3) If there is a vertex v with degree 2 whose two neighbors are not adjacent and $|N(N(v)) \setminus \{v\}| \leq 2$, apply Rule (C3) on (G,k) with $C = \{v\}$ and H = N(v), and let (G',k') be the resulting instance. Return BRANCH₃ $(G',k',\{\emptyset\})$.

Note that if Rule (3) is applied, when moving from (G, k) to (G', k'), the value of r decreases by $|H| - (|C| + |H|)/3 = \frac{1}{3}$.

(4) If there is a vertex v with degree 2 whose two neighbors are not adjacent, return BRANCH₃($G, k, \{N(v), N(N(v))\}$).

We now analyze the branching number of Rule (4). We can assume that the call to BRANCH₃ executes line 3 (if line 2b, 2c or 2d is executed then no branching is performed, and if line 2e is executed, we already showed that the branching number is at most 1.6181). By Lemma 3, when procedure BRANCH₃ processes the set $S_1 = N(v)$, the decrease in r is at least $(|H_1| + 2)/3 \ge (|S_1| + 2)/3 = \frac{4}{3}$. Additionally, when procedure BRANCH₃ processes the set $S_2 = N(N(v))$, the decrease in r is at least $(|H_2| + 2)/3 \ge (|S_2| + 2)/3 \ge 2$. Therefore, the branching vector is at least $(\frac{4}{3}, 2)$, and the branching number is at most 1.5248.

(5) If there is a vertex v with degree 2, return BRANCH₃(G, k, $\{N(v)\}$).

(6) If G is 3-regular, select an arbitrary vertex v and return BRANCH₃ $(G, k, \{\{v\}, N(v)\})$.

We do not analyze the branching number of Rule (6) since this rule is applied at most once and therefore does not affect the time complexity of the algorithm. The running time of algorithm VCBASE₃ is $O^*(1.6181^r)$. In order to handle non-connected graphs with maximum degree at most 3, we use an algorithm VC₃ that performs the following steps.

1. If G is connected return $VCBASE_3(G, k)$.

- 2. Let G' be a connected component of G.
- 3. For $k' = \lceil |V(G')|/3 \rceil, \dots, k \lceil (n |V(G')|)/3 \rceil$, if VCBASE₃(G', k') returns 'yes', return VC₃(G V(G'), k k').
- 4. Return 'no'.

Note that in line 3 we have $k' \leq k - (n - |V(G')|)/3$ and therefore $k' - |V(G')|/3 \leq k - n/3 = r$. Thus, the time complexity of all the calls to VCBASE₃ in line 3 is $O^*(1.6181^r)$. It follows that the running time of algorithm VC₃ is $O^*(1.6181^r)$.

We now describe an algorithm VCBASE for connected graphs with maximum degree 4 and minimum degree at least 2. The algorithm maintains the following invariants on the current graph: (1) The minimum degree is at least 2. (2) The maximum degree is 4. (3) Except for the initial instance, every connected component is not 4-regular. Let BRANCH be a procedure identical to BRANCH₃ except that the calls to VCBASE₃ are replaced with calls to VCBASE. Algorithm VCBASE uses Rules (1)–(5) above, where the calls to BRANCH₃ are replaced with calls to BRANCH, and the following rules.

(7) If the maximum degree of G is at most 3, return $VC_3(G,k)$.

(8) If G is 4-regular, select an arbitrary vertex v and return BRANCH $(G, k, \{\{v\}, N(v)\})$.

(9) If v is a vertex with degree 3 such that there is an edge between two neighbors u, w of v, return BRANCH $(G, k, \{N(v), N(x)\})$, where x is the third neighbor of v.

By Lemma 3, the branching vector of Rule (9) is at least $(\frac{5}{3}, \frac{5}{3})$ (since |N(v)| = 3 and $|N(x)| \ge 3$), and the branching number is at most 1.5158.

(10) If there is a vertex v with degree 3 and a vertex t such that $|N(t) \cap N(v)| \ge 2$, return BRANCH $(G, k, \{N(v), \{v, t\}\})$.

By Lemma 3, the branching vector of Rule (10) is at least $(\frac{5}{3}, \frac{4}{3})$, and the branching number is at most 1.5906.

For the following rules note that there is a connected component of G that contains at least one vertex with degree 3 and at least one vertex with degree 4. Therefore, there is a vertex v with degree 3 that is adjacent to a vertex with degree 4. Denote the neighbors of v by z, u, w, where deg(z) = 4. Note that since Rule (10) cannot be applied, the sets $N(z) \setminus \{v\}, N(u) \setminus \{v\}$, and $N(w) \setminus \{v\}$ are pairwise disjoint.

(11) If at least one vertex from u, w has degree 4, return BRANCH $(G, k, \{N(v), \{z\} \cup N(u) \cup N(w), N(z)\})$.

By Lemma 3, the branching vector of Rule (11) is at least $(\frac{5}{3}, 3, 2)$ (since |N(v)| = 3, $|\{z\} \cup N(u) \cup N(w)| \ge 7$, and |N(z)| = 4). We can improve the bound on the branching vector as follows. Consider the processing of $S_3 = N(z)$ in procedure BRANCH. We consider two cases. For the first case, assume that Rule (D1) is not applied in line 1 of procedure BRANCH. We claim that in this case Rule (D0) is applied only on z. Suppose conversely that the rule is applied on a vertex $x \ne z$. By definition, $N(x) \subseteq N(z)$. x cannot be u or v since the sets $N(z) \setminus \{v\}$, $N(u) \setminus \{v\}$, and $N(w) \setminus \{v\}$ are pairwise disjoint. Therefore, $N(x) \subseteq N(z) \setminus \{v\}$. The graph G has minimum degree 3, and therefore $N(x) = N(z) \setminus \{v\}$. It follows that Rule (10) can be applied on x, a contradiction. Thus, Rule (D0) is applied only on z. It follows that $H_3 = N(z)$ and $C_3 = \{z\}$. Therefore, the decrease in r is $4 - 5/3 = \frac{7}{3}$. If Rule (D1) is applied at least once, $|H_3| \ge |S_3| + 1 = 5$. By Lemma 3 the decrease in r is at least $(|H_3| + 2)/3 \ge \frac{7}{3}$. We obtained that the branching vector is at least $(\frac{5}{3}, 3, \frac{7}{3})$ and the branching number is at most 1.6253.

(12) Otherwise $(\deg(u) = \deg(w) = 3)$, return BRANCH $(G, k, \{\{z\}, N(z)\})$.

By Lemma 3, the branching vector of Rule (12) is at least $(\frac{2}{3}, 2)$ (since |N(z)| = 4). We can improve the bound on the branching vector as follows. Consider the processing of $S_1 = \{z\}$ in procedure BRANCH. Since G has minimum degree 3 and $|S_1| = 1$, we have that $G - S_1$ has minimum degree at least 2. Therefore, Rules (D0) and (D1) are not applied by procedure BRANCH, so $G_1 = G - \{z\}$ and $k_1 = k - 1$. In the instance (G_1, k_1) , v has degree 2 and its two neighbors u, w are not adjacent (since Rule (9) could not be applied on (G, k)). Moreover, $|N_{G_1}(\{u, w\}) \setminus \{v\}| = 4$ (since $N(u) \setminus \{v\}$ and $N(w) \setminus \{v\}$ are disjoint, and these sets do not contain z). Therefore, in the instance (G_1, k_1) , Rule (4) can be applied on v. We change the algorithm to force that in the instance (G_1, k_1) , Rule (3) will be applied on v (even though the requirement $|N(N(v)) \setminus \{v\}| \le 2$ is not satisfied). The application of Rule (3) on v decreases the value of r by $\frac{1}{3}$. Taking this decrease into account, the branching vector of Rule (12) is at least $(\frac{2}{3} + \frac{1}{3}, 2)$ and the branching number is at most 1.6181.

We now show that the algorithm maintains the three invariants defined above. Suppose that the current input graph G to VCBASE satisfies the invariants. We need to show that the graphs generated from G on which VCBASE is recursively called also satisfy the invariants. This is straightforward for all the rules of the algorithm except Rule (12). Consider the application of Rule (12) on G and the application of Rule (3) on $G_1 = G - \{z\}$, and let G_2 be the resulting graph. We have $\deg_{G_2}(v^*) = |N_{G_1}(\{u, w\}) \setminus \{v\}| = 4$. All the other vertices in G_2 have degrees between 2 and 4. Therefore, G_2 satisfies Invariant (1) and Invariant (2). At least one of the two vertices in $N(u) \setminus \{v\}$ has degree 3 in G (otherwise Rule (11) can be applied on u). This vertex also has degree 3 in G_2 (since $N(z) \setminus \{v\}$ and $N(u) \setminus \{v\}$, and $N(w) \setminus \{v\}$ are pairwise disjoint) and therefore G_2 satisfies Invariant (3).

We obtain that algorithm VCBASE solves the vertex cover problem on connected graphs with maximum degree 4 and minimum degree at least 2 in $O^*(1.6253^r)$ time. To handle non-connected graph, we use an algorithm VC which is analogous to algorithm VC₃.

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