

# Dual Characterizations of Set Containments with Strict Convex Inequalities

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## Abstract

Characterizations of the containment of a convex set either in an arbitrary convex set or in the complement of a finite union of convex sets (i.e. the set, described by reverse-convex inequalities) are given. These characterizations provide ways of verifying the containments either by comparing their corresponding dual cones or by checking the consistency of suitable associated systems. The convex sets considered in this paper are the solution sets of an arbitrary number of convex inequalities, which can be either weak or strict inequalities. Particular cases of dual characterizations of set containments have played key roles in solving large scale knowledge-based data classification problems where they are used to describe the containments as inequality constraints in optimization problems. The idea of evenly convex set (intersection of open half spaces), which was introduced by W. Fenchel in 1952, is used to derive the dual conditions, characterizing the set containments.

**Key words:** Set containment, convex functions, semi-infinite systems, existence theorems, dual cones, conjugacy.

## 1 Introduction

Consider the sets

$$F := \{x \in \mathbb{R}^n \mid f_t(x) < 0, \forall t \in S; f_t(x) \leq 0, \forall t \in W\} \quad (1.1)$$

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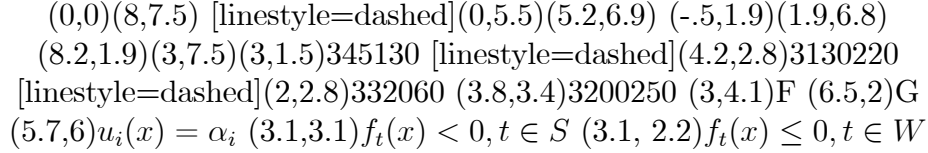
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and

$$G = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I; h_j(x) \geq 0, \forall j \in J\},$$

where  $S \cap W = \emptyset$ ,  $S \cup W \neq \emptyset$ ,  $I \cap J = \emptyset$ ,  $I \cup J \neq \emptyset$ , and all the functions,  $\{f_t, t \in S \cup W\}$ ,  $\{g_i, i \in I\}$ , and  $\{h_j, j \in J\}$ , are convex functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ . The set containment problem that is studied in this paper, consists of deciding whether  $F \subset G$  or not. Dual characterizations of such set containments have played a key role in solving large scale knowledge-based data classification problems where they are used to describe the containments as inequality constraints in optimization problems (see e.g., [2, 10, 11] and [8]). For instance, the incorporation of prior knowledge in the form of a polyhedral knowledge set in the construction of a linear classifier is modelled as the set containment  $F \subset G$  [2], where  $F$  is a given polyhedral convex set and  $G$  is a given closed halfspace. The dual characterizations of the set containment were obtained using the classical nonhomogeneous Farkas Lemma [9].

More recently, various extensions of the containment problem to more general situations have been obtained in [8] and [11], by means of mathematical programming theory and conjugacy theory, respectively, where  $S = \emptyset$  (i.e. without strict inequalities). In this paper we establish dual characterizations by allowing the systems defining  $F$  and  $G$  to contain strict inequalities, as depicted in Figure 1 below. Such kind of systems also arise naturally in the characterization of the stable containment; i.e., establishing conditions which guarantee that the inclusion is preserved under sufficiently small perturbations of the systems representing  $F$  and  $G$ .



**Figure 1:** Containment of the evenly convex set  $F = \{x \mid f_t(x) < 0, \forall t \in S; f_t(x) \leq 0, \forall t \in W\}$  in the polyhedral set  $G = \{x \mid u_i(x) \leq \alpha_i, \forall i \in I, u_i(x) < \alpha_i, \forall i \in J\}$ , where  $f_t : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function and  $u_i \in \mathbb{R}^n$  and  $\alpha_i \in \mathbb{R}$ .

The main basic tool in our approach in deriving the dual characterizations is the association of two dual cones in  $\mathbb{R}^{n+1}$ , say  $K$  and  $M$ , such that  $F \subset G$  if and only if  $M \subset K$ . Since  $M \subset K$  can be interpreted as a dual condition, the verification of the containment reduces to the effective calculus of the corresponding dual cones. In the case where  $F$  is the intersection of a family of open convex sets,  $\{x \in \mathbb{R}^n \mid f_t(x) < 0\}$ ,  $t \in S$ , with a family of closed convex sets,  $\{x \in \mathbb{R}^n \mid f_t(x) \leq 0\}$ ,  $t \in W$ ,  $F$  turns out to be an evenly convex set (i.e., the intersection of open halfspaces, see [1]), represented by means of a convex inequality system. The dual cones of closed convex sets were introduced in [3] in order to characterize large classes of closed

convex sets from a geometric point of view. The dual cones of evenly convex sets that are introduced for the first time in the present paper, play a central role in describing the dual conditions.

The paper is organized as follows. Section 2 contains the necessary notation and some basic results on convex as well as evenly convex sets to be used later. Section 3 develops calculus rules for the dual cone of a closed convex set. Section 4 considers stable containment of closed convex sets. Section 5 defines dual cones for evenly convex sets and develops calculus rules which are similar to those obtained in Section 3. These cones provide dual characterizations of containment for convex sets which are represented by means of strict constraints. Finally, Section 6 presents general existence theorems for several classes of convex systems which contain strict inequalities.

## 2 Preliminaries: Evenly Convex Sets

All the vectors in  $\mathbb{R}^n$  will be interpreted as column vectors. The inner product of two vectors  $u$  and  $x$  will be denoted by either  $u'x$  or  $u(x)$ , and the Euclidean distance between  $u$  and  $x$  will be denoted by  $d(u, x) = \|u - x\|$ . Given a set  $X \subseteq \mathbb{R}^n$ , we shall denote by  $\text{int } X$ ,  $\text{bd } X$ ,  $\text{cl } X$ ,  $\text{co } X$ , and  $\text{coneco } X$  the interior, the boundary, the closure, the convex hull and the convex cone generated by  $X$  respectively. By  $\mathbb{R}_+$  and  $\mathbb{R}_{++}$  we denote the sets of nonnegative and positive real numbers, respectively, so that  $\mathbb{R}_+ X := \{\lambda x \mid \lambda \geq 0, x \in X\}$  and  $\mathbb{R}_{++} X := \{\lambda x \mid \lambda > 0, x \in X\}$  are cones in  $\mathbb{R}^n$ , with the null vector  $0_n \in \mathbb{R}_+ X$ . The smallest convex cone containing  $X \cup \{0_n\}$  is  $\text{coneco } X = \mathbb{R}_+ \text{co } X$ .

Fenchel [1] defined the class of **evenly convex sets** as the intersections of open halfspaces. The set  $C$  is evenly convex if and only if for all  $x \notin C$  there exists a hyperplane  $H$  such that  $x \in H$  and  $H \cap C = \emptyset$ . The **evenly convex hull** of  $X$  [1], denoted by  $\text{eco } X$ , is the smallest evenly convex set which contains  $X$  (i.e., it is the intersection of all the open halfspaces which contain  $X$ ). It is known that  $\text{eco } X$  is obtained by eliminating from  $\text{clco } X$  those exposed faces which do not contain points of  $X$  (Proposition 2.1 in [4]). From the definition, given  $\bar{x} \in \mathbb{R}^n$ ,  $\bar{x} \notin \text{eco } X$  if and only if there exists  $z \in \mathbb{R}^n$  such that  $z'(x - \bar{x}) < 0$  for all  $x \in X$ .

The following existence theorem for linear inequality systems containing strict inequalities will be used later.

**Proposition 2.1** (*Theorem 3.1 [4]*) *Let  $S$  be non-empty. The system  $\{a'_t x < b_t, t \in S; a'_t x \leq b_t, t \in W\}$  is consistent if and only if*

$$0_{n+1} \notin \text{eco} \left[ \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S \right\} + \mathbb{R}_+ \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W \right\}; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right].$$

The **support function** of  $X$  is  $\sigma_X(u) = \sup_{x \in X} u(x)$  and the **indicator function**

of  $X$  is defined by

$$\delta_X(x) = \begin{cases} 0, & x \in X, \\ +\infty, & x \notin X. \end{cases}$$

Given a proper convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , the **conjugate function** of  $f$  is  $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$f^*(u) = \sup_{x \in \text{dom } f} \{u(x) - f(x)\},$$

where  $\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$  is the **domain** of  $f$ . The **epigraph** of  $f$  is defined by

$$\text{epi } f = \left\{ \begin{pmatrix} x \\ \gamma \end{pmatrix} \in \mathbb{R}^{n+1} \mid f(x) \leq \gamma, x \in \text{dom } f \right\}.$$

Many dual conditions are formulated in terms of  $\text{epi } f_t^*$ , where  $f_t$  defines a constraint. So, it is important to note that  $\text{epi } f_t^*$  can be expressed in terms of  $f_t$  by exploiting the information at an arbitrary point  $\bar{x} \in \text{dom } f_t$ . In fact, according to Proposition 2.1 in [8],

$$\text{epi } f_t^* = \bigcup_{\varepsilon \in \mathbb{R}_+} \left\{ \begin{pmatrix} v \\ \varepsilon + v'\bar{x} - f_t(\bar{x}) \end{pmatrix} \mid v \in \partial_\varepsilon f_t(\bar{x}) \right\}, \quad (2.1)$$

where  $\partial_\varepsilon f_t(\bar{x})$  is the  $\varepsilon$ -**subdifferential** of  $f_t$  at  $\bar{x}$ , i.e.,

$$\partial_\varepsilon f_t(\bar{x}) = \{v \in \mathbb{R}^n \mid f_t(x) \geq f_t(\bar{x}) + v'(x - \bar{x}) - \varepsilon, \forall x \in \text{dom } f_t\}.$$

Recall that the subdifferential of  $f_t$  at  $\bar{x}$  is  $\partial f_t(\bar{x}) = \bigcap_{\varepsilon \geq 0} \partial_\varepsilon f_t(\bar{x})$ .

The following result is fundamental for the characterization of containments of closed convex sets.

**Proposition 2.2** (*Lemma 3.1 [8]*) *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex lower semicontinuous (lsc) function, and let  $F = \{x \in \mathbb{R}^n \mid f(x) \leq 0\}$ . Then the following statements hold:*

$$(i) \ F \neq \emptyset \text{ if and only if } \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \text{cl}(\mathbb{R}_+ \text{epi } f^*).$$

$$(ii) \text{ If } F \neq \emptyset, \text{ then } \text{epi } \sigma_F = \text{cl}(\mathbb{R}_+ \text{epi } f^*).$$

### 3 Containments of Closed Convex Sets

Consider the sets

$$F = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0, \forall t \in W\}, \quad (3.1)$$

and

$$G = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I; h_j(x) \geq 0, \forall j \in J\},$$

where  $W \neq \emptyset$ ,  $I \cap J = \emptyset$ ,  $I \cup J \neq \emptyset$ , and all the functions,  $\{f_t, t \in W\}$ ,  $\{g_i, i \in I\}$ , and  $\{h_j, j \in J\}$ , are convex functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ .

Mangasarian [11] presented dual characterizations of the set containment  $F \subset G$  in the following cases:

**Case 1:**  $|W| < \infty$ ,  $|I| < \infty$ ,  $J = \emptyset$ , and all the involved functions are affine (i.e.,  $F$  and  $G$  are given polyhedral convex sets).

**Case 2:**  $|W| < \infty$ ,  $I = \emptyset$ ,  $|J| < \infty$ ,  $\{f_t, t \in W\}$  are affine functions and  $\{h_j, j \in J\}$  are quadratic convex functions (i.e.,  $F$  is a polyhedral convex set and  $G$  is a reverse-convex quadratic set).

**Case 3:**  $|W| < \infty$ ,  $I = \emptyset$ ,  $|J| < \infty$ , and  $\{f_t, t \in W\}$  and  $\{h_j, j \in J\}$  are differentiable convex functions (so that  $F$  is a closed convex set and  $G$  is a closed reverse-convex set, both sets defined by means of ordinary systems).

The recent paper [8] established dual characterizations of the containment problem in the following cases:

**Case 4:**  $W$  is arbitrary,  $|I| < \infty$ ,  $J = \emptyset$ ,  $\{f_t, t \in W\}$  are convex (affine) functions, and  $\{g_i, i \in I\}$  are affine functions (i.e.,  $F$  is the solution set of a convex (linear) semi-infinite system and  $G$  is a polyhedral convex set).

**Case 5:**  $W$  is arbitrary,  $I = \emptyset$ ,  $|J| < \infty$ , and  $\{f_t, t \in W\}$  and  $\{h_j, j \in J\}$  are convex functions (i.e.,  $F$  is as in Case 4 and  $G$  is a reverse-convex set described by means of reverse convex inequalities).

We assume that  $G$  is represented in a similar way when  $J = \emptyset$ . In relation to the reverse-convex set  $G$  in the Case 5, let us observe that we can express

$$G = \{x \in \mathbb{R}^n \mid h_j(x) \geq 0, \forall j \in J\} = \mathbb{R}^n \setminus \bigcup_{j \in J} G_j,$$

where  $G_j := \{x \in \mathbb{R}^n \mid h_j(x) < 0\}$  for all  $j \in J$ . Obviously,  $F \subset G$  if and only if  $F \cap G_j = \emptyset$  for all  $j \in J$ , so that the basic problem is to determine the existence of solution of a system similar to (1.1):

$$\{f_t(x) \leq 0, t \in W; f_t(x) < 0, t \in S; h_j(x) < 0\}.$$

Consequently, existence theorems for convex systems possibly containing strict inequalities play a double role in our approach. In fact, they provide tests for (1.1) to be consistent (otherwise the containment problem is trivial) and they provide dual characterizations of  $F \subset G$  when  $G$  is a reverse-convex set.

We begin by developing calculus rules for the dual cone of a closed convex set, revisiting Case 4 as an immediate application. We define the **weak dual cone** of the nonempty closed convex set  $F \subset \mathbb{R}^n$  as

$$K^{\leq} := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1} \mid a'x \leq b, \forall x \in F \right\} = \text{epi } \sigma_F. \quad (3.2)$$

Obviously,  $\text{coneco} \left\{ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \subset K^\leq$  and the equality holds if and only if  $F = \mathbb{R}^n$ . It is known that  $F$  is bounded if and only if  $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{int } K^\leq$  (see, e.g. Theorem 9.3 in [5]). The standard hyperplane separation arguments yield

$$F = \left\{ x \in \mathbb{R}^n \mid a'x \leq b, \forall \begin{pmatrix} a \\ b \end{pmatrix} \in K^\leq \right\}. \quad (3.3)$$

Observe the symmetry of (3.2) and (3.3): the index set of the linear system in one of the formulae is the solution set in the other one, and vice versa. Consequently, if  $G \neq \emptyset$  is another closed convex set with associated weak dual cone  $M^\leq$ , we have

$$F \subset G \Leftrightarrow M^\leq \subset K^\leq, \quad (3.4)$$

i.e., the containment of closed convex sets is actually reduced to checking the consistency of the given representation of  $F$  and, if  $F \neq \emptyset$ , the calculus of the respective weak dual cones. Then the dual characterization of the containment is the right hand side inclusion in (3.4).

For Case 4, the following existence theorem allows to check the nonemptiness of  $F$ . Such result can be seen as a convex counterpart of the existence theorem of Zhu [13] for linear systems in infinite dimensional spaces (see Lemma 4.1 in [5] for a semi-infinite version).

**Proposition 3.1** *Let  $F = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0, \forall t \in W\}$ , where  $f_t : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is proper, convex and lsc for all  $t \in T$ . Then  $F \neq \emptyset$  if and only if*

$$\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin \text{cl coneco} \left[ \bigcup_{t \in W} \text{epi } f_t^* \right]. \quad (3.5)$$

**Proof.** For each  $t \in W$ , we consider the function  $h_t := \gamma_t f_t$ , where

$$\gamma_t := \begin{cases} 1, & f_t(0_n) \leq 0, \\ f_t(0_n)^{-1}, & f_t(0_n) > 0. \end{cases}$$

Since the function  $h := \sup_{t \in W} h_t$  is proper, convex and lsc, according to Theorem 2.4.4 in [7], we have

$$\text{cl}(\mathbb{R}_+ \text{epi } h^*) = \text{cl coneco} \left[ \bigcup_{t \in W} \text{epi } h_t^* \right] = \text{cl coneco} \left[ \bigcup_{t \in W} \text{epi } f_t^* \right],$$

with  $F = \{x \in \mathbb{R}^n \mid h(x) \leq 0\}$ . The conclusion follows from Proposition 2.2(i).  $\square$

The nonhomogeneous Farkas Lemma for linear semi-infinite systems (Corollary 3.1.2 in [5], Corollary 3.3 in [8]) establishes that, if  $F = \{x \in \mathbb{R}^n \mid a'_t x \leq b_t, \forall t \in W\}$ , then

$$K^\leq = \text{cl coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}. \quad (3.6)$$

**Proposition 3.2** *Let  $F = \bigcap_{i \in I} F_i \neq \emptyset$ , where  $F_i$  is a closed convex set with weak duality cone  $K_i^\leq$ ,  $i \in I$ . Then*

$$K^\leq = \text{cl co} \left[ \bigcup_{i \in I} K_i^\leq \right].$$

**Proof.** Since  $F_i = \left\{ x \in \mathbb{R}^n \mid a'x \leq b, \quad \forall \begin{pmatrix} a \\ b \end{pmatrix} \in K_i^\leq \right\}$  for all  $i \in I$ ,

$$F = \left\{ x \in \mathbb{R}^n \mid a'x \leq b, \quad \forall \begin{pmatrix} a \\ b \end{pmatrix} \in \bigcup_{i \in I} K_i^\leq \right\}.$$

Then, by the nonhomogeneous Farkas Lemma and since  $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K_i^\leq$  for all  $i \in I$ , we have

$$K^\leq = \text{cl coneco} \left[ \left( \bigcup_{i \in I} K_i^\leq \right) \cup \left\{ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \right] = \text{cl co} [\bigcup_{i \in I} K_i^\leq]. \quad \square$$

**Proposition 3.3** *If  $F = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0, \forall t \in W\} \neq \emptyset$  and  $f_t : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is proper, convex and lsc for each  $t \in W$ , then the weak duality cone of  $F$  is*

$$K^\leq = \text{cl coneco} \left[ \bigcup_{t \in T} \text{epi } f_t^* \right].$$

**Proof.**  $F = \bigcap_{t \in W} F_t \neq \emptyset$ , with  $F_t := \{x \in \mathbb{R}^n \mid f_t(x) \leq 0\}$  for all  $t \in W$ . Then, by Propositions 3.2 and 2.2(ii),

$$\begin{aligned} K^\leq &= \text{cl co} \left[ \bigcup_{t \in W} K_t^\leq \right] = \text{cl co} \left[ \bigcup_{t \in W} \text{cl}(\mathbb{R}_+ \text{epi } f_t^*) \right] \\ &= \text{cl co} \left[ \bigcup_{t \in W} (\mathbb{R}_+ \text{epi } f_t^*) \right] = \text{cl coneco} \left[ \bigcup_{t \in W} \text{epi } f_t^* \right]. \quad \square \end{aligned}$$

Observe that  $K^\leq$  is the same cone which yields the consistency test (3.5). Theorem 2.4.4 in [7] provides an alternative proof of Proposition 3.3 (see Theorem 3.2 in [8]).

## 4 Stable Set Containments

In this section we see how the inclusion  $F \subset G$ , where  $F$  and  $G$  are represented by means of linear inequality systems, is preserved under sufficiently small perturbations of the data. To formulate the problem, let  $F$  and  $G$  be the solution sets of the systems

$$\xi = \{a'_t x \leq b_t, \quad t \in W\} \quad (4.1)$$

and

$$\eta = \{c'_i x \leq d_i, \ i \in I\}. \quad (4.2)$$

We say that the containment  $F \subset G$  is **stable** if it holds under arbitrary perturbations of the coefficients of  $\xi$  and  $\eta$ , provided that these perturbations are sufficiently small. In order to define the size of a perturbation, consider the set,  $\Theta_\xi$ , of all linear systems with the same number of unknowns and constraints as  $\xi$ . So the elements of  $\Theta_\xi$  are of the form

$$\xi_1 = \{(a_t^1)'x \leq b_t^1, \ t \in W\},$$

with  $a^1 : W \rightarrow \mathbb{R}^n$  and  $b^1 : W \rightarrow \mathbb{R}$ . The size of the perturbation which yields  $\xi_1$  from the **nominal system**  $\xi$  is defined as

$$\rho(\xi_1, \xi) := \sup_{t \in W} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_\infty.$$

It is easy to see that  $\rho$  defines a pseudometric on  $\Theta_\xi$  (observe that it is possible that  $\rho(\xi_1, \xi) = +\infty$ ). Similarly, the nominal system defining  $G, \eta$ , provides perturbed systems in a space of parameters,  $\Theta_\eta$ , and the size of the perturbation is also measured by means of the pseudometric of the uniform convergence. We denote by  $F_1$  and  $G_1$  the solution sets of  $\xi_1$  and  $\eta_1$ .

Precisely, the containment  $F \subset G$  is **stable**, if there exists a scalar  $\delta > 0$  such that  $F_1 \subset G_1$  if  $\rho(\xi_1, \xi) < \delta$  and  $\rho(\eta_1, \eta) < \delta$ .

We shall prove that the stable containment is basically the containment of a closed convex set in an open convex set, a particular case of containment of evenly convex sets.

Recall that  $\eta$  satisfies the **strongly Slater (SS) condition** if there exists  $\bar{x} \in \mathbb{R}^n$  and  $\varepsilon > 0$  such that  $c'_i \bar{x} \leq d_i - \varepsilon$  for all  $i \in I$ , i.e., if the system  $\{c'_i x + x_{n+1} \leq d_i, \ i \in I; -x_{n+1} < 0\}$ , is consistent, i.e. (by Proposition 2.1)

$$0_{n+2} \notin \text{eco} \left[ \begin{pmatrix} 0_n \\ -1 \\ 0 \end{pmatrix} + \mathbb{R}_+ \left\{ \begin{pmatrix} c_i \\ 1 \\ d_i \end{pmatrix}, \ i \in I \right\}; \begin{pmatrix} 0_{n+1} \\ 1 \end{pmatrix} \right].$$

**Proposition 4.1** *Let  $F \neq \emptyset$  and  $G$  be the solution sets of the linear system (4.1) and (4.2), respectively. Then the following statements hold:*

- (i) *If  $F \subset \text{int } G$ ,  $G$  is compact and  $\eta$  satisfies the SS condition, then  $F \subset G$  is stable.*
- (ii) *If  $F \subset G$  is stable and either  $\{a_t, t \in T\}$  or  $\{c_i, i \in I\}$  is bounded (e.g., one of the two systems is ordinary), then  $F \subset \text{int } G$ .*

**Proof.** (i) Assume that  $F \subset \text{int } G$ ,  $G$  is compact and  $\eta$  satisfies the SS condition.

Let  $\varepsilon := d(F, \text{bd } G) > 0$  ( $F$  is compact),  $U := \{x \in \mathbb{R}^n \mid d(x, F) < \frac{\varepsilon}{2}\}$ , and  $V := \mathbb{R}^n \setminus \text{cl } U$ . Obviously,  $U$  and  $V$  are disjoint open sets such that  $F \subset U$  and  $\text{bd } G \subset V$ , respectively.



Since  $F$  is bounded, the feasible set mapping associating to each  $\xi_1 \in \Theta_\xi$  its corresponding solution set mapping  $F_1$  is Berge upper semicontinuous (Corollary 6.2.1 in [5]). Hence, there exists  $\delta_0 > 0$  such that  $F_1 \subset U$  if  $\rho(\xi_1, \xi) < \delta_0$ .

The assumptions on  $G$  and  $\eta$  entail two consequences:

- (a) There exists  $\delta_1 > 0$  such that  $G_1 \cap U \neq \emptyset$  if  $\rho(\eta_1, \eta) < \delta_1$  (the SS condition of  $\eta$  is equivalent to the Berge lower semicontinuity of the feasible set mapping associating to each  $\eta_1 \in \Theta_\eta$  its solution set  $G_1$ , see e.g. Theorem 6.1 in [5]).
- (b) There exists  $\delta_2 > 0$  such that  $\text{bd } G_1 \subset V$  if  $\rho(\eta_1, \eta) < \delta_2$  (since  $G$  is a convex body and  $\eta$  satisfies the SS condition, the set valued mapping associating to each  $\eta_1 \in \Theta_\eta$  the boundary of its solution set,  $\text{bd } G_1$ , is Berge upper semicontinuous, according to Corollary 5.3 in [6]).

Let  $\delta = \min\{\delta_0, \delta_1, \delta_2\} > 0$  and let  $\xi_1 \in \Theta_\xi$  and  $\eta_1 \in \Theta_\eta$  such that  $\rho(\xi_1, \xi) < \delta$  and  $\rho(\eta_1, \eta) < \delta$ .

If  $U \not\subset G_1$ , we take  $x^1 \in U \setminus G_1$  and  $x^2 \in U \cap G_1$  (from (a)), and  $[x^1, x^2]$  must contain a point  $x^3 \in [x^1, x^2] \subset U$  such that  $x^3 \in \text{bd } G_1$ . Then  $x^3 \in U \cap (\text{bd } G_1) \subset U \cap V$ , by (b), contradicting  $U \cap V = \emptyset$ .

Therefore we have  $F_1 \subset U \subset G_1$ .

(ii) Now we assume that  $F \subset G$  but  $F \not\subset \text{int } G$ . Let  $\bar{x} \in F \setminus (\text{int } G)$ .

We shall prove that the inclusion  $F \subset G$  is unstable provided that one of the two sets of left-hand-side vectors is bounded.

First we assume that  $\{a_t, t \in T\}$  is bounded.

Since  $\text{int } G$  is evenly convex and  $\bar{x} \notin \text{int } G$ , there exists  $z \in \mathbb{R}^n$  such that  $z'(x - \bar{x}) < 0$  for all  $x \in \text{int } G$ . Thus,

$$z'(x - \bar{x}) \leq 0 \text{ for all } x \in G. \quad (4.3)$$

Given  $\gamma > 0$ , we consider the system  $\xi_\gamma = \{a_t'x \leq b_t + \gamma a_t'z, t \in W\} \in \Theta_\xi$ . Since the feasible set of  $\xi_\gamma$  is  $F_\gamma = F + \gamma z$ , we have  $\bar{x} + \gamma z \in F_\gamma$ . On the other hand,

$$z'[(\bar{x} + \gamma z) - \bar{x}] = \gamma \|z\|^2 > 0,$$

so that  $\bar{x} + \gamma z \notin G$  according to (4.3). Hence  $F_\gamma \not\subset G$ , with

$$\lim_{\gamma \searrow 0} \rho(\xi_\gamma, \xi) = \lim_{\gamma \searrow 0} \gamma \|z\| \sup_{t \in W} \|a_t\| = 0.$$

Now we assume that  $\{c_i, i \in I\}$  is bounded.

By the separation theorem (if  $\bar{x} \notin G$ ) and the supporting hyperplane theorem (if  $\bar{x} \in \text{bd } G$ ), there exists  $z \in \mathbb{R}^n$ ,  $z \neq 0_n$ , such that

$$z'(x - \bar{x}) \leq 0 \text{ for all } x \in G. \quad (4.4)$$

Given  $\gamma > 0$ , we consider the system  $\eta_\gamma = \{c'_i x \leq d_i - \gamma c'_i z, i \in I\}$ . Now we have  $G_\gamma = G - \gamma z$ . If  $\bar{x} \in G_\gamma$ , then  $\bar{x} + \gamma z \in G$  and (4.4) entails the following contradiction:

$$0 \geq z'[(\bar{x} + \gamma z) - \bar{x}] = \gamma \|z\|^2 > 0.$$

Since  $\bar{x} \in F \setminus G_\gamma$ , we have  $F \not\subset G_\gamma$ , with

$$\lim_{\gamma \searrow 0} \rho(\eta_\gamma, \eta) = \lim_{\gamma \searrow 0} \gamma \|z\| \sup_{i \in I} \|c_i\| = 0.$$

This completes the proof.  $\square$

In particular, if  $\eta$  is a minimal representation of a full dimensional polytope  $G$ , then

$$F \subset G \text{ is stable} \Leftrightarrow F \subset \text{int } G,$$

and the characterization of stable containment between closed convex sets is equivalent to the characterization of the containment of a closed convex set in an open convex set (the kind of problem we shall consider in the next section). This statement is not necessarily true for polyhedral sets (consider  $n = 2$ ,  $G = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$  and  $F = \{x \in \mathbb{R}^2 \mid x_2 \leq -1\}$ ).

## 5 The Containments of Evenly Convex Sets

We define the **strict dual cone** of a nonempty evenly convex set  $F \subset \mathbb{R}^n$  as

$$K^< := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1} \mid a'x < b, \forall x \in F \right\}. \quad (5.1)$$

Obviously,  $\{0_n\} \times \mathbb{R}_{++} \subset K^<$  and the equality holds if and only if  $F = \mathbb{R}^n$ . Since  $0_{n+1} \notin K^<$ ,  $K^<$  cannot be closed. In particular, if  $F$  is closed, we have  $K^<$  strictly contained in  $K^\leq$  as far as  $0_{n+1} \in K^\leq \setminus K^<$  (the supporting halfspaces for  $F$  also define elements of  $K^\leq \setminus K^<$ , if  $F \neq \mathbb{R}^n$ ).

The symmetric expression of (5.1) is now a straightforward consequence of the characterization of the evenly convex sets by means of the strong separation property from external points:

$$F = \left\{ x \in \mathbb{R}^n \mid a'x < b, \forall \begin{pmatrix} a \\ b \end{pmatrix} \in K^< \right\}. \quad (5.2)$$

As for closed convex sets, if  $M^<$  denotes the strict dual cone of a second evenly convex set  $G$ ,

$$F \subset G \Leftrightarrow M^< \subset K^<. \quad (5.3)$$

(5.3) reduces again the containment problem with evenly convex sets to checking the consistency of the given representations of  $F$  and  $G$  (by means of Proposition

2.1 if the given representation is linear and by means of the existence theorems in Section 6 otherwise) and, if  $F \neq \emptyset \neq G$ , the comparison of the respective strict dual cones. Their calculus is the main objective of this section. In particular cases  $K^<$  can be calculated through the weak dual cone of  $\text{cl} F$ ,  $\bar{K}^\leq$ . Obviously,  $\bar{K}^\leq$  is a closed convex cone and  $\bar{K}^\leq = K^\leq$  if  $F$  is closed.

From now on in this section we assume that  $F$  is a nonempty evenly convex set with associated strict dual cone  $K^<$ . It follows from the definitions that  $\text{cl} K^< = \bar{K}^\leq$ .

Thus the relative interiors of  $\bar{K}^\leq$  and  $K^<$  coincide and  $K^< = \text{int } \bar{K}^\leq$  if  $K^<$  is open.

Observe also that  $K^<$  provides useful information on  $F$ , for instance, the recession cone of  $F$  is

$$0^+ F = \left\{ y \in \mathbb{R}^n \mid a'y \leq 0, \forall \begin{pmatrix} a \\ b \end{pmatrix} \in K^< \right\}$$

(from (5.2)), and  $F$  is bounded if and only if  $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{int } K^<$ .

**Example 5.1** If  $F = \mathbb{R}_+^2$ , then  $\bar{K}^\leq = K^\leq = \mathbb{R}_-^2 \times \mathbb{R}_+$ , where  $\mathbb{R}_- := -\mathbb{R}_+$ , and  $K^< = \mathbb{R}_-^2 \times \mathbb{R}_{++}$ . Observe that  $K^<$  is neither closed nor open, with  $\text{cl } K^< = \bar{K}^\leq$ .

The next three results show the existence of a topological duality between the evenly convex sets and their respective strict dual cones. These relationships will allow us to calculate  $K^<$  from  $\bar{K}^\leq$  when  $F$  is either open or compact.

**Proposition 5.1** The cone  $K^< \cup \{0_{n+1}\}$  is closed if and only if the set  $F$  is open. In such a case,  $K^< = \bar{K}^\leq \setminus \{0_{n+1}\}$ .

**Proof.** Assume that  $F$  is not open. Let  $\bar{x} \in (\text{bd } F) \cap F$  and let  $a \neq 0_n$  and  $b \in \mathbb{R}$  such that  $a'x \leq b$  for all  $x \in F$  and  $a'\bar{x} = b$  ( $a'x = b$  is a supporting hyperplane for  $F$  at  $\bar{x}$ ). Then, clearly,  $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{cl}(K^< \cup \{0_{n+1}\})$ . On the other hand,  $a'\bar{x} = b$ , with  $\bar{x} \in F$ , entails  $\begin{pmatrix} a \\ b \end{pmatrix} \notin K^<$ . Since  $a \neq 0_n$ ,  $\begin{pmatrix} a \\ b \end{pmatrix} \notin K^< \cup \{0_{n+1}\}$  and we conclude that this set is not closed.

Now we assume that  $K^< \cup \{0_{n+1}\}$  is not closed. Let

$$\left\{ \begin{pmatrix} a_r \\ b_r \end{pmatrix} \right\} \subset K^< \cup \{0_{n+1}\}$$

such that

$$\lim_r \begin{pmatrix} a_r \\ b_r \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \notin K^< \cup \{0_{n+1}\}.$$

Let  $\bar{x} \in F$  such that  $a'\bar{x} \geq b$ . Since  $a'_r \bar{x} < b_r$  for all  $r \in \mathbb{N}$ , we have, for  $r \rightarrow \infty$ ,  $a'\bar{x} \leq b$ , i.e.,  $a'\bar{x} = b$ . We shall prove that  $\bar{x} \in (\text{bd } F) \cap F$ , so that  $F$  cannot be open.

We have  $a \neq 0_n$  (otherwise  $\begin{pmatrix} a \\ b \end{pmatrix} = 0_{n+1} \in K^< \cup \{0_{n+1}\}$ ), and  $a'x = b$  defines a hyperplane containing  $\bar{x}$ . Since  $\begin{pmatrix} a_r \\ b_r \end{pmatrix} \in K^< \cup \{0_{n+1}\}$  for all  $r \in \mathbb{N}$ , either we have  $a'_r x = b_r$  for all  $x \in F$  (if  $\begin{pmatrix} a_r \\ b_r \end{pmatrix} = 0_{n+1}$ ) or  $a'_r x < b_r$  for all  $x \in F$ , otherwise. Taking limits,  $a'x \leq b$  for all  $x \in F$  and so  $a'x = b$  is a supporting hyperplane for  $F$  at  $\bar{x}$ . Hence  $\bar{x} \in (\text{bd } F) \cap F$ .

If  $K^< \cup \{0_{n+1}\}$  is closed, then  $K^< \cup \{0_{n+1}\} = \bar{K}^{\leq}$  and, since  $0_{n+1} \notin K^<$ ,  $K^< = \bar{K}^{\leq} \setminus \{0_{n+1}\}$ .  $\square$

From Proposition 5.1 we get a particular version of Farkas Lemma for linear strict inequalities.

**Corollary 5.1** *If  $|S| < \infty$ ,  $a'x < b$  is a consequence of the consistent system  $\{a'_t x < b_t, t \in S\}$  if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \left[ \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \right] \setminus \{0_{n+1}\}.$$

**Proof.** The assumptions guarantee that  $F := \{x \in \mathbb{R}^n \mid a'_t x < b_t, t \in S\}$  is an open subset of  $\mathbb{R}^n$ . The weak dual cone of  $\text{cl } F$  is, by the nonhomogeneous Farkas Lemma, the polyhedral convex cone

$$\bar{K}^{\leq} = \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\},$$

and the conclusion follows from Proposition 5.1.  $\square$

**Proposition 5.2** *If the cone  $K^<$  is relatively open, then the set  $F$  is closed.*

**Proof.** Assume that  $F$  is not closed. Let  $\{x_r\} \subset F$  such that  $\lim_r x_r = \bar{x} \notin F$ . Since  $F$  is an evenly convex set, there exists  $a \neq 0_n$  and  $b \in \mathbb{R}$  such that  $a'x < b$  for all  $x \in F$  and  $a'\bar{x} = b$ . Obviously,  $\begin{pmatrix} a \\ b \end{pmatrix} \in K^<$ .

Take an arbitrary small  $\varepsilon > 0$ . Since  $\lim_r a'x_r = a'\bar{x} = b > b - \varepsilon$ , there exists  $m \in \mathbb{N}$  such that  $a'x_m > b - \varepsilon$ , with  $x_m \in F$ . Then  $\begin{pmatrix} a \\ b - \varepsilon \end{pmatrix} \notin K^<$ , with  $\left\| \begin{pmatrix} a \\ b - \varepsilon \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \varepsilon$  and  $\begin{pmatrix} a \\ b - \varepsilon \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} - \varepsilon \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{aff } K^<$ . Hence  $\begin{pmatrix} a \\ b \end{pmatrix}$  does not belong to the relative interior of  $K^<$ .  $\square$

Example 5.1 shows that the converse of Proposition 5.2 is not true. Next we show that the compactness of  $F$  guarantees the openness (not only relative) of  $K^<$ .

**Proposition 5.3** *If  $F$  is compact, then  $K^<$  is open. In such a case,  $K^< = \text{int } \bar{K}^{\leq}$ .*

**Proof.** Assume that  $F$  is compact. Then its support function  $\sigma_F$  is continuous on  $\mathbb{R}^n$ .

Let  $\begin{pmatrix} a \\ b \end{pmatrix} \in K^<$ . Since  $\sigma_F(a) = \max_{x \in F} a'x < b$ , there exists  $\varepsilon > 0$  such that  $\sigma_F(a) < b - \varepsilon$ . Let  $\delta > 0$  such that  $\delta < \frac{\varepsilon}{2}$  and

$$|\sigma_F(c) - \sigma_F(a)| < \frac{\varepsilon}{2} \quad \text{if } \|c - a\| < \delta.$$

We shall prove that  $K^<$  contains the open ball, in  $\mathbb{R}^{n+1}$ , centered at  $\begin{pmatrix} a \\ b \end{pmatrix}$  with radius  $\delta$ . In fact, if  $\left\| \begin{pmatrix} c \\ d \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right\| < \delta$ , then

$$\sigma_F(c) < \sigma_F(a) + \frac{\varepsilon}{2} < b - \frac{\varepsilon}{2} < d.$$

Thus  $\max_{x \in F} c'x < d$ , i.e.,  $\begin{pmatrix} c \\ d \end{pmatrix} \in K^<$ .  $\square$

Proposition 5.3 yields another version of Farkas Lemma for linear strict inequalities.

**Corollary 5.2** *If the solution set of  $\{a'_t x < b_t, t \in S\}$  is compact, then  $a'x < b$  is a consequence of that system if and only if*

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \text{int coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

**Proof.** Since  $F := \{x \in \mathbb{R}^n \mid a'_t x < b_t, \forall t \in S\}$  is compact, by Proposition 5.3,

$$\begin{aligned} K^< &= \text{int cl coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \\ &= \text{int coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}. \end{aligned} \quad \square$$

**Example 5.2** *It is easy to see that the unit closed disk  $F := \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$  is the solution set of the system of strict inequalities*

$$\{(\cos t)x_1 + (\sin t)x_2 < s, (t, s) \in [0, 2\pi] \times ]1, +\infty[ \}.$$

Since  $F = \{x \in \mathbb{R}^2 \mid (\cos t)x_1 + (\sin t)x_2 \leq 1, t \in [0, 2\pi]\}$ , we have

$$\begin{aligned} \bar{K}^{\leq} &= \text{coneco} \left\{ \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}, t \in [0, 2\pi]; \begin{pmatrix} 0_2 \\ 1 \end{pmatrix} \right\} \\ &= \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1, x_3 \geq 0\}. \end{aligned}$$

Thus,

$$K^< = \text{int } \bar{K}^{\leq} = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1, x_3 > 0\}.$$

The next result can be interpreted as the general Farkas Lemma for systems of strict inequalities.

**Proposition 5.4** *If  $F = \{x \in \mathbb{R}^n \mid a'_t x < b_t, \forall t \in S\}$ , then*

$$K^< = \text{eco } \mathbb{R}_{++} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

**Proof.** We have to prove that  $K^< = \text{eco } X$ , where  $X$  is the cone

$$X := \mathbb{R}_{++} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

Assume that  $\begin{pmatrix} a \\ b \end{pmatrix} \notin \text{eco } X$ . Then either  $\begin{pmatrix} a \\ b \end{pmatrix} \notin \text{cl co } X$  or  $\begin{pmatrix} a \\ b \end{pmatrix}$  belongs to a certain exposed face of  $\text{cl co } X$  which does not contain points of  $X$ , otherwise. In both cases, there exists a hyperplane containing  $\begin{pmatrix} a \\ b \end{pmatrix}$  and  $0_{n+1}$  which does not contain points of  $X$ .

Let  $\begin{pmatrix} c \\ d \end{pmatrix} \neq 0_{n+1}$  such that

$$\begin{pmatrix} c \\ d \end{pmatrix}' \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix}' \begin{pmatrix} v \\ w \end{pmatrix} < 0 \quad \text{for all} \quad \begin{pmatrix} v \\ w \end{pmatrix} \in X. \quad (5.4)$$

From (5.4), since  $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in X$ , we get  $d < 0$ . Let  $\bar{x} := |d|^{-1}c$ . Multiplying by  $|d|^{-1}$  each expression of (5.4), we obtain  $a'\bar{x} = b$  and  $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} v \\ w \end{pmatrix} < 0$  for all  $\begin{pmatrix} v \\ w \end{pmatrix} \in X$ . In particular, since  $\begin{pmatrix} a_t \\ b_t \end{pmatrix} \in X$  if  $t \in S$ , we get  $a'_t \bar{x} < b_t$  for all  $t \in S$ . Therefore  $\bar{x} \in F$  and  $a'\bar{x} = b$ , and this entails  $\begin{pmatrix} a \\ b \end{pmatrix} \notin K^<$ .

Now we assume  $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{eco } X$ . Since

$$\text{eco } X \subset \text{cl co } X \subset \text{cl cone co } \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} = \bar{K}^{\leq},$$

we have  $a'x \leq b$  for all  $x \in F$ . We claim that  $\begin{pmatrix} a \\ b \end{pmatrix} \in K^<$ . Indeed, if there exists  $\bar{x} \in F$  such that  $a'\bar{x} = b$  then

$$\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \left[ \lambda \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right] = \lambda(a'_t \bar{x} - b_t) < 0, \quad \forall \lambda > 0, \forall t \in S, \quad (5.5)$$

and

$$\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \left[ \lambda \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right] = -\lambda < 0, \quad \forall \lambda > 0. \quad (5.6)$$

It follows from (5.5) and (5.6) that  $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} v \\ w \end{pmatrix} < 0$  for all  $\begin{pmatrix} v \\ w \end{pmatrix} \in X$  and hence,

$$\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \left[ \begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right] < 0 \quad \text{for all } \begin{pmatrix} v \\ w \end{pmatrix} \in X \quad (5.7)$$

as  $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} a \\ b \end{pmatrix} = 0$ . Recalling the last characterization of  $\text{eco } X$  in the preliminaries, (5.7) entails  $\begin{pmatrix} a \\ b \end{pmatrix} \notin \text{eco } X$ . This contradiction proves that  $\begin{pmatrix} a \\ b \end{pmatrix} \in K^<$ .  $\square$

Observe that  $\{a'_t x < b_t, t \in S\}$  is consistent if and only if

$$\{(\lambda a_t)'x < \lambda b_t, (\lambda, t) \in \mathbb{R}_{++} \times S; (\mu 0_n)'x < \mu, \mu \in \mathbb{R}_{++}\}$$

is consistent if and only if (by Proposition 2.1)

$$\begin{aligned} 0_{n+1} &\notin \text{eco} \left[ \mathbb{R}_{++} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right] \\ &= \text{eco } \mathbb{R}_{++} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Thus  $K^<$  (defined as in Proposition 5.4) characterizes the consistency of  $\{a'_t x < b_t, t \in S\}$  by  $0_{n+1} \notin K^<$ , in the same way as  $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \notin K^\leq$  (defined in (3.6)) characterizes the consistency of  $\{a'_t x \leq b_t, t \in W\}$ .

The next result is the counterpart of Proposition 3.2 for strict dual cones.

**Proposition 5.5** *Let  $F = \bigcap_{i \in I} F_i \neq \emptyset$ , where  $F_i$  is an evenly convex set with strict dual cone  $K_i^<$ ,  $i \in I$ . Then*

$$K^< = \text{eco} \left[ \bigcup_{i \in I} K_i^< \right].$$

**Proof.** By (5.2),  $F_i = \left\{ x \in \mathbb{R}^n \mid a'x < b, \forall \begin{pmatrix} a \\ b \end{pmatrix} \in K_i^< \right\}$  for all  $i \in I$ . Then

$$F = \left\{ x \in \mathbb{R}^n \mid a'x < b, \forall \begin{pmatrix} a \\ b \end{pmatrix} \in \bigcup_{i \in I} K_i^< \right\},$$

and, recalling Proposition 5.4 and that  $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K_i^<$  for all  $i \in I$ , we have

$$K^< = \text{eco } \mathbb{R}_{++} \left[ \bigcup_{i \in I} K_i^< \right] = \text{eco } \left[ \bigcup_{i \in I} K_i^< \right].$$

□

Now we can calculate the strict dual cone of certain evenly convex sets.

**Corollary 5.3** *Let  $F = \{x \in \mathbb{R}^n \mid f_t(x) < 0, \forall t \in S\} \neq \emptyset$ , with  $f_t : \mathbb{R}^n \rightarrow \mathbb{R}$  convex for all  $t \in S$ . Then the strict dual cone of  $F$  is*

$$K^< = \text{eco} \left[ \left\{ \bigcup_{t \in S} \text{cl}(\mathbb{R}_+ \text{epi } f_t^*) \right\} \setminus \{0_{n+1}\} \right].$$

*In the particular case of  $f_t(x) = a'_t x - b_t$  for all  $t \in S$  (i.e.,  $F$  is the solution set of a system of linear strict inequalities), it holds*

$$K^< = \text{eco} \left[ \left( \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \right) \setminus \{0_{n+1}\} \right].$$

**Proof.** Given  $t \in S$ , we denote  $F_t := \{x \in \mathbb{R}^n \mid f_t(x) < 0\}$ , which is a nonempty open convex set. The weak dual cone of

$$\text{cl } F_t = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0\}$$

is, according to Proposition 2.2,

$$\bar{K}_t^< = \text{cl}(\mathbb{R}_+ \text{epi } f_t^*).$$

Then, recalling Proposition 5.1, the strict dual cone of  $F_t$  is

$$K_t^< = [\text{cl}(\mathbb{R}_+ \text{epi } f_t^*)] \setminus \{0_{n+1}\},$$

and, by Proposition 5.5, we have

$$K^< = \text{eco} \left[ \left\{ \bigcup_{t \in S} \text{cl}(\mathbb{R}_+ \text{epi } f_t^*) \right\} \setminus \{0_{n+1}\} \right].$$

If  $f_t(x) = a'_t x - b_t$ , then  $f_t^* = b_t + \delta_{\{a_t\}}$  and

$$\text{cl}(\mathbb{R}_+ \text{epi } f_t^*) = \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\},$$

so that

$$\begin{aligned} K^< &= \text{eco} \left[ \left( \bigcup_{t \in S} \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \right) \setminus \{0_{n+1}\} \right] \\ &= \text{eco} \left[ \left( \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in S; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \right) \setminus \{0_{n+1}\} \right]. \end{aligned}$$

□



**Example 5.1 (revisited)**  $F = \{x \in \mathbb{R}^2 \mid -x_1 < \lambda, \forall \lambda \in \mathbb{R}_{++}; -x_2 < \mu, \forall \mu \in \mathbb{R}_{++}\}$ . Since

$$\text{coneco} \left\{ \begin{pmatrix} -1 \\ 0 \\ \lambda \end{pmatrix}, \lambda > 0; \begin{pmatrix} 0 \\ -1 \\ \mu \end{pmatrix}, \mu > 0; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \setminus \{0_3\} = \mathbb{R}_-^2 \times \mathbb{R}_{++},$$

and this set is evenly convex,  $K^< = \mathbb{R}_-^2 \times \mathbb{R}_{++}$ .

**Corollary 5.4** *Let  $F = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0, \forall t \in W\} \neq \emptyset$ , with  $f_t : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  proper convex lsc for all  $t \in W$ . Let  $K_t^<$  be the strict dual cone of  $F_t := \{x \in \mathbb{R}^n \mid f_t(x) \leq 0\}, t \in W$ . Then*

$$K^< = \text{eco} \left[ \bigcup_{t \in W} K_t^< \right],$$

where

$$K_t^< = \begin{cases} \text{int}(\mathbb{R}_+ \text{epi } f_t^*), & \text{if } F_t \text{ is bounded,} \\ \left( \text{coneco} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \right) \setminus \mathbb{R}_+ \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\}, & \text{if } f_t(x) = a_t'x - b_t. \end{cases}$$

**Proof.** It is a straightforward consequence of Propositions 5.3 (if  $F_t$  is bounded) and 5.5, and the evenly convex property of  $K_t^<$  (if  $f_t$  is an affine function).  $\square$

**Example 5.2 (revisited)**  $F = \{x \in \mathbb{R}^2 \mid (\cos t)x_2 + (\sin t)x_1 \leq 1, \forall t \in [0, 2\pi]\}$ .

By Corollary 5.4, recalling that  $\text{eco } X = X$  if  $X$  is open and convex,

$$\begin{aligned} K^< &= \text{eco} \left\{ \bigcup_{t \in [0, 2\pi]} \left[ \left( \text{coneco} \left\{ \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right) \setminus \mathbb{R}_+ \left\{ \begin{pmatrix} \cos t \\ \sin t \\ 1 \end{pmatrix} \right\} \right] \right\} \\ &= \text{eco} \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1, x_3 > 0\} \\ &= \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < 1, x_3 > 0\}. \end{aligned}$$

If  $F = \{x \in \mathbb{R}^n \mid f_t(x) < 0, \forall t \in S; f_t(x) \leq 0, \forall t \in W\} \neq \emptyset$ , its strict dual cone is  $\text{eco}(M^< \cup N^<)$ , where  $M^<$  and  $N^<$  are the strict dual cones of

$$\{x \in \mathbb{R}^n \mid f_t(x) < 0, \forall t \in S\}$$

and

$$\{x \in \mathbb{R}^n \mid f_t(x) \leq 0, \forall t \in W\},$$

respectively.  $M^<$  and  $N^<$  can be calculated by means of Corollaries 5.3 and 5.4.

## 6 General Existence Theorems and Applications to Set Containments

This section provides three existence theorems for convex systems of the form

$$\xi = \{f_t(x) < 0, t \in S; f_t(x) \leq 0, t \in W\},$$

with  $S \neq \emptyset$  (otherwise Proposition 3.1 applies). All the proposed consistency tests are expressed (or can be expressed by means of (2.1)) in terms of the information on  $\{f_t, t \in S \cup W\}$ , and the proofs will be derived from the existence theorem for linear systems (Proposition 2.1). The first result replaces linearity by sublinearity in Proposition 2.1.

**Proposition 6.1** *Let  $f_t(x) = g_t(x) - b_t$ , with  $g_t : \mathbb{R}^n \rightarrow \mathbb{R}$  sublinear and  $b_t \in \mathbb{R}$  for all  $t \in S \cup W$ . Then  $\xi$  is consistent if and only if*

$$0_{n+1} \notin \text{eco} \left[ \left( \bigcup_{t \in S} \partial g_t(0_n) \times \{b_t\} \right) + \mathbb{R}_+ \left( \bigcup_{t \in W} \partial g_t(0_n) \times \{b_t\} \right); \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right]. \quad (6.1)$$

**Proof.** Since  $g_t$  is sublinear and continuous, it can be expressed as

$$g_t(x) = \max_{v \in \partial g_t(0_n)} v'x \text{ for all } x \in \mathbb{R}^n. \quad (6.2)$$

Consequently,  $\xi$  has the same solution set as the linear system

$$\eta := \left\{ \begin{array}{l} a'_{tv}x < b_t, \ a_{tv} \in \partial g_t(0_n), \ t \in S \\ a'_{tv}x \leq b_t, \ a_{tv} \in \partial g_t(0_n), \ t \in W \end{array} \right\}.$$

Applying Proposition 2.1, we conclude that  $\eta$  is consistent if and only if condition (6.1) holds.  $\square$

It is easy to extend Proposition 6.1 to the case that each function  $f_t$  can be expressed as  $f_t(x) = g_t(x - x_t) - b_t$ , with  $g_t : \mathbb{R}^n \rightarrow \mathbb{R}$  sublinear,  $x_t \in \mathbb{R}^n$  and  $b_t \in \mathbb{R}$  (replace  $x$  with  $x - x_t$  and  $0_n$  with  $x_t$  in  $\eta$ ). Typical examples of such functions are  $f_t(x) = \sqrt{(x - x_t)'A_t(x - x_t)} - b_t$  where  $A_t$  is a positive definite symmetric matrix  $x_t \in \mathbb{R}^n$  and  $b_t > 0$ , so that the solution set of  $\xi$  is the intersection of (open and closed) ellipsoids.

Next, we relax in another way the assumption on  $\{f_t, t \in W\}$  in Proposition 6.1.

**Proposition 6.2** *Let  $f_t : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  proper convex lsc for all  $t \in W$  and let  $f_t(x) = g_t(x) - b_t$ , with  $g_t : \mathbb{R}^n \rightarrow \mathbb{R}$  sublinear for all  $t \in S$ . Then  $\xi$  is consistent if and only if*

$$0_{n+1} \notin \text{eco} \left[ \left( \bigcup_{t \in S} \partial g_t(0_n) \times \{b_t\} \right) + \mathbb{R}_+ \left\{ \bigcup_{t \in W} \text{epi } f_t^* \right\}; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right]. \quad (6.3)$$

**Proof.** Given  $t \in S$ , by (6.2), the solution set of  $\{f_t(x) < 0\}$  is the same as the solution set of

$$\{a'_{tv}x < b_t, a_{tv} \in \partial g_t(0_n)\}.$$

On the other hand, given  $t \in W$ , we can write

$$f_t(x) = f_t^{**}(x) = \sup_{x^* \in \text{dom } f_t^*} [x^*(x) - f_t^*(x^*)] \quad \text{for all } x \in \mathbb{R}^n.$$

Thus  $f_t(x) \leq 0$  if and only if  $x^*(x) - f_t^*(x^*) \leq 0$  for all  $x^* \in \text{dom } f_t^*$  if and only if

$$x^*(x) \leq f_t^*(x^*) + \gamma \quad \text{for all } (x^*, \gamma) \in \text{dom } f_t^* \times \mathbb{R}_+.$$

Then  $\xi$  is consistent if and only  $\eta$  is consistent, where

$$\eta := \left\{ \begin{array}{l} a'_{tv}x < b_t, a_{tv} \in \partial g_t(0_n), t \in S \\ x^*(x) \leq f_t^*(x^*) + \gamma, (x^*, \gamma) \in \text{dom } f_t^* \times \mathbb{R}_+, t \in W \end{array} \right\}.$$

Applying again Proposition 2.1, we conclude that  $\eta$  is consistent if and only if (6.3) holds.  $\square$

The sublinearity assumption in Proposition 6.2 can be relaxed by requiring that each function  $f_t$ ,  $t \in S$ , is the maximum of a family of affine functions (compare with (6.2)).

**Proposition 6.3** *Let  $f_t : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  proper convex lsc for all  $t \in W$ . We also assume that, for each  $t \in S$ , there exists a compact set  $C_t \subset \mathbb{R}^{n+1}$  such that  $f_t(x) = \max_{(a,b) \in C_t} (ax - b)$  for all  $x \in \mathbb{R}^n$ . Then  $\xi$  is consistent if and only if*

$$0_{n+1} \notin \text{eco} \left[ \left( \bigcup_{t \in S} C_t \right) + \mathbb{R}_+ \left\{ \bigcup_{t \in W} \text{epi } f_t^* \right\}; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right]. \quad (6.4)$$

**Proof.** Given  $t \in S$ ,  $f_t(x) < 0$  if and only if

$$\begin{pmatrix} a \\ b \end{pmatrix}' \begin{pmatrix} x \\ -1 \end{pmatrix} < 0 \quad \text{for all } \begin{pmatrix} a \\ b \end{pmatrix} \in C_t.$$

Reasoning as in Proposition 6.2,  $\xi$  is consistent if and only if  $\eta$  is consistent, with

$$\eta := \left\{ \begin{array}{l} a'x < b, \begin{pmatrix} a \\ b \end{pmatrix} \in C_t, t \in S \\ x^*(x) \leq f_t^*(x^*) + \gamma, (x^*, \gamma) \in \text{dom } f_t^* \times \mathbb{R}_+, t \in W \end{array} \right\},$$

and  $\eta$  turns out to be consistent if and only if (6.4) holds.  $\square$

In order to summarize the consequences of the previous existence theorems for the containment problem, let us denote by  $\mathcal{A}$  the set of affine functions on  $\mathbb{R}^n$ , by

$\mathcal{S}$  the family of differences between sublinear (possibly composed with translations) and constant functions, by  $\mathcal{M}$  the family of functions which can be expressed as  $\max_{(a,b) \in C} (ax - b)$  for a certain compact set  $C \subset \mathbb{R}^{n+1}$ , and by  $\mathcal{C}$  the class of proper convex lsc functions.

Given  $F = \{x \in \mathbb{R}^n \mid f_t(x) < 0, \forall t \in S; f_t(x) \leq 0, \forall t \in W\}$ ,  $F \neq \emptyset$  is characterized in the following cases:

- $S = \emptyset$  and  $\{f_t, t \in W\} \subset \mathcal{C}$ .
- $S \neq \emptyset$  and  $\{f_t, t \in S \cup W\}$  is contained in either  $\mathcal{A}$  or  $\mathcal{S}$ .
- $S \neq \emptyset, \{f_t, t \in W\} \subset \mathcal{C}$ , and  $\{f_t, t \in S\}$  is contained in either  $\mathcal{S}$  or  $\mathcal{M}$ .

Concerning the containment of  $F \neq \emptyset$  in the reverse-convex set  $\mathbb{R}^n \setminus G_j$ , with  $G_j = \{x \in \mathbb{R}^n \mid h_j(x) < 0\}$ , it is characterized in the following cases:

- $\{f_t, t \in S \cup W; h_j\}$  is contained in either  $\mathcal{A}$  or  $\mathcal{S}$ .
- $\{f_t, t \in W\} \subset \mathcal{C}$  and  $\{f_t, t \in S; h_j\}$  is contained in either  $\mathcal{S}$  or  $\mathcal{M}$ .

Obviously, the containment of  $F \neq \emptyset$  in the reverse-convex set  $\mathbb{R}^n \setminus \bigcup_{j \in J} G_j$ , with  $G_j = \{x \in \mathbb{R}^n \mid h_j(x) < 0\}, j \in J$ , is characterized if  $F \subset \mathbb{R}^n \setminus G_j$  is characterized for all  $j \in J$ .

Finally, observe that the proofs of Propositions 6.1–6.3 are based upon the explicit construction of a linear representation of  $F$ , that is,  $\eta$ . From this representation it is possible to obtain the strict dual cone of  $F \neq \emptyset$  just applying Corollaries 5.3 and 5.4.

**Example 6.1** Let  $F = \{x \in \mathbb{R}^n \mid g_t(x) < b_t, \forall t \in S; g_t(x) \leq b_t, \forall t \in W\} \neq \emptyset$ , with  $g_t, t \in S \cup W$  as in Proposition 6.1. Then the strict dual cone of  $F$  is  $K^< = \text{eco}(M^< \cup N^<)$ , where

$$M^< = \text{eco} \left[ \text{coneco} \left\{ \bigcup_{t \in S} \partial g_t(0_n) \times \{b_t\}; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \setminus \{0_{n+1}\} \right]$$

and

$$N^< = \text{eco} \left[ \underbrace{\bigcup_{\substack{\begin{pmatrix} a \\ b \end{pmatrix} \in \bigcup_{t \in W} \partial g_t(0_n) \times \{b_t\}}} \left\{ \text{coneco} \left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\} \setminus \mathbb{R}_+ \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\} \right\} \right]}.$$

Similar expressions can be given for the strict dual cone of the solutions set of  $\xi$ , under the assumptions of Proposition 6.2 and 6.3, provided that  $\xi$  is consistent.

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