Well-posedness of constrained minimization problems via saddle-points

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Dedicated to Professor Jean Saint Raymond on his sixtieth birthday, with my greatest admiration and esteem

Here and in the sequel, X is a Hausdorff topological space, J, Φ are two real-valued functions defined in X, and a, b are two numbers in $[-\infty, +\infty]$, with a < b.

If $a \in \mathbf{R}$ (resp. $b \in \mathbf{R}$), we denote by M_a (resp. M_b) the set of all global minima of the function $J + a\Phi$ (resp. $J + b\Phi$), while if $a = -\infty$ (resp. $b = +\infty$), M_a (resp. M_b) stands for the empty set. We adopt the conventions inf $\emptyset = +\infty$, sup $\emptyset = -\infty$.

We also set

$$\alpha := \max \left\{ \inf_{X} \Phi, \sup_{M_b} \Phi \right\} ,$$

$$\beta := \min \left\{ \sup_{X} \Phi, \inf_{M_a} \Phi \right\} .$$

Note that, by Proposition 1 below, one has $\alpha \leq \beta$.

A usual, given a function $f: X \to \mathbf{R}$ and a set $C \subseteq X$, we say that the problem of minimizing f over C is well-posed if the following two conditions hold:

- the restriction of f to C has a unique global minimum, say \hat{x} ;
- every sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} f(x_n) = \inf_C f$, converges to \hat{x} .

A set of the type $\{x \in X : f(x) \leq r\}$ is said to be a sub-level set of f. Clearly, when the sub-level sets of f are sequentially compact, the problem of minimizing f over a sequentially closed set C is well-posed if and only if $f_{|C|}$ has a unique global minimum.

The aim of the present paper is to establish the following result:

THEOREM 1. - Assume that $\alpha < \beta$ and that, for each $\lambda \in]a,b[$, the function $J + \lambda \Phi$ has sequentially compact sub-level sets and admits a unique global minimum in X.

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed.

Moreover, if we denote by \hat{x}_r the unique global minimum of $J_{|\Phi^{-1}(r)}$ $(r \in]\alpha, \beta[)$, the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $]\alpha, \beta[$.

Theorem 1 should be regarded as the definitive abstract result coming out from the saddle-point method developed in [4], [5], [6], [7], in specific settings.

The main tool used to prove Theorem 1 is provided by the following mini-max result:

THEOREM 2. - Let $I \subseteq \mathbf{R}$ be an interval and f a real-valued function defined in $X \times I$. Assume that there exists a number $\rho^* > \sup_I \inf_X f$, and a point $\hat{\lambda} \in I$ such that, for each $\rho \leq \rho^*$, the following conditions hold:

- (i) the set $\{\lambda \in I : f(x,\lambda) > \rho\}$ is connected for all $x \in X$;
- (ii) the set $\{x \in X : f(x,\lambda) \leq \rho\}$ is sequentially closed for all $\lambda \in I$ and sequentially compact for $\lambda = \hat{\lambda}$;
- (iii) for each compact interval $T \subseteq I$ for which $\sup_T \inf_X f < \rho$, there exists a continuous function $\varphi: T \to X$ such that $f(\varphi(\lambda), \lambda) < \rho$ for all $\lambda \in T$.

Then, one has

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) .$$

PROOF. We strictly follow the proof Theorem 2 of [3]. First, fix a non-decreasing sequence $\{I_n\}$ of compact sub-intervals of I, with $\hat{\lambda} \in I_1$, such that $\bigcup_{n \in \mathbb{N}} I_n = I$. Now, fix $n \in \mathbb{N}$. We claim that

$$\sup_{\lambda \in I_n} \inf_{x \in X} f(x, \lambda) = \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) . \tag{1}$$

Arguing by contradiction, suppose that

$$\sup_{\lambda \in I_n} \inf_{x \in X} f(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) .$$

Fix ρ satisfying

$$\sup_{\lambda \in I_n} \inf_{x \in X} f(x, \lambda) < \rho < \min \left\{ \rho^*, \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \right\} .$$

Set

$$S = \{(x, \lambda) \in X \times I_n : f(x, \lambda) < \rho\}$$

as well as, for each $\lambda \in I_n$,

$$S^{\lambda} = \{ x \in X : (x, \lambda) \in S \} .$$

Since $\sup_{I_n} \inf_X f < \rho$, one has $S^{\lambda} \neq \emptyset$ for all $\lambda \in I_n$. Let $I_n = [a_n, b_n]$. Put

$$A = \left\{ (x, \lambda) \in S : \lambda < b_n , \sup_{s \in]\lambda, b_n]} f(x, s) > \rho \right\}$$

and

$$B = \left\{ (x,\lambda) \in S : \lambda > a_n , \sup_{s \in [a_n,\lambda[} f(x,s) > \rho \right\} .$$

Observe that $S^{a_n} \times \{a_n\} \subseteq A$ and $S^{b_n} \times \{b_n\} \subseteq B$. Indeed, let $x_1 \in S^{a_n}$ and $x_2 \in S^{b_n}$. Since $\rho < \inf_X \sup_{I_n} f$, there are $t, s \in I_n$ such that $\min\{f(x_1, t), f(x_2, s)\} > \rho$. Since $\sup\{f(x_1, a_n), f(x_2, b_n)\} < \rho$, it follows that $t > a_n$ and $s < b_n$. Consequently, $(x_1, a_n) \in A$

and $(x_2, b_n) \in B$. Furthermore, observe that if $(x_0, \lambda_0) \in A$ and if $\mu \in]\lambda_0, b_n]$ is such that $f(x_0, \mu) > \rho$, then, in view of (ii), the set

$$(\{x \in X : f(x,\mu) > \rho\} \times [a_n,\mu]) \cap S$$

is sequentially open in S, contains (x_0, λ_0) and is contained in A. In other words, A is sequentially open in S. Analogously, it is seen that B is sequentially open in S. We now prove that $S = A \cup B$. Indeed, let $(x, \lambda) \in S \setminus A$. We have seen above that $S^{a_n} \times \{a_n\} \subseteq A$, and so $\lambda > a_n$. If $\lambda = b_n$, the fact that $(x, \lambda) \in B$ has been likewise proved above. Suppose $\lambda < b_n$. Thus, we have $\sup_{s \in [\lambda, b_n]} f(x, s) \leq \rho$. From this, it clearly follows that $\sup_{s \in [a_n, \lambda[} f(x, s) > \rho \text{ (note that } f(x, \lambda) < \rho), \text{ and so } (x, \lambda) \in B$. Furthermore, we have $A \cap B = \emptyset$. Indeed, if $(x_1, \lambda_1) \in A \cap B$, there would be $t_1, s_1 \in I_n$, with $t_1 < \lambda_1 < s_1$, such that $\min\{f(x_1, t_1), f(x_1, s_1)\} > \rho$. By (i), the set $\{s \in I : f(x_1, s) > \rho\}$ is an interval, and so we would have $f(x_1, \lambda_1) > \rho$, against the fact that $(x_1, \lambda_1) \in S$. Now, in view of (iii), consider a continuous function $\varphi : I_n \to X$ such that

$$f(\varphi(\lambda), \lambda) < \rho$$

for all $\lambda \in I_n$. Let $h: I_n \to X \times I_n$ be defined by setting

$$h(\lambda) = (\varphi(\lambda), \lambda)$$

for all $\lambda \in I_n$. Since h is continuous, the set $h(I_n)$ is sequentially connected ([2], Theorem 2.2). But, having in mind that $h(I_n) \subseteq S$ and that $h(I_n)$ meets both A and B (since $h(a_n) \in A$ and $h(b_n) \in B$), the properties of A, B proved above would imply that $h(I_n)$ is sequentially disconnected, a contradiction. So, (1) holds. Finally, let us prove the theorem. Again arguing by contradiction, suppose that

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) .$$

Choose r satisfying

$$\sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) < r < \min \left\{ \rho^*, \inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \right\} .$$

For each $n \in \mathbb{N}$, put

$$C_n = \left\{ x \in X : \sup_{\lambda \in I_n} f(x, \lambda) \le r \right\}.$$

Note that $C_n \neq \emptyset$. Indeed, otherwise, we would have

$$r \leq \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) = \sup_{\lambda \in I_n} \inf_{x \in X} f(x, \lambda) \leq \sup_{\lambda \in I} \inf_{x \in X} f(x, \lambda) .$$

Consequently, $\{C_n\}$ is a non-increasing sequence of non-empty sequentially closed subsets of the sequentially compact set $\{x \in X : f(x, \hat{\lambda}) \leq \rho^*\}$. Therefore, one has $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Let $x^* \in \bigcap_{n \in \mathbb{N}} C_n$. Then, one has

$$\sup_{\lambda \in I} f(x^*, \lambda) = \sup_{n \in \mathbb{N}} \sup_{\lambda \in I_n} f(x^*, \lambda) \le r$$

and so

$$\inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \le r ,$$

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a contradiction. The proof is complete.

We will also use the following proposition.

PROPOSITION 1 ([4], Proposition 1). - Let Y be a nonempty set, $f, g: Y \to \mathbf{R}$ two functions, and λ, μ two real numbers, with $\lambda < \mu$. Let \hat{y}_{λ} be a global minimum of the function $f + \lambda g$ and let \hat{y}_{μ} be a global minimum of the function $f + \mu g$.

Then, one has

$$g(\hat{y}_{\mu}) \leq g(\hat{y}_{\lambda})$$
.

If either \hat{y}_{λ} or \hat{y}_{μ} is strict and $\hat{y}_{\lambda} \neq \hat{y}_{\mu}$, then

$$g(\hat{y}_{\mu}) < g(\hat{y}_{\lambda})$$
.

Proof of Theorem 1. First, for each $\lambda \in]a,b[$, denote by \hat{y}_{λ} the unique global minimum in X of $J + \lambda \Phi$. Let us prove that the function $\lambda \to \hat{y}_{\lambda}$ is continuous in]a,b[. To this end, fix $\lambda^* \in]a,b[$. Let $\{\lambda_n\}$ be any sequence in]a,b[converging to λ^* and let $[c,d] \subset]a,b[$ be a compact interval containing $\{\lambda_n\}$. Fix $\rho > \sup_{n \in \mathbb{N}} \inf_{x \in X} (J(x) + \lambda_n \Phi(x))$. Clearly, we have

$$\bigcup_{\lambda \in [c,d]} \{ x \in X : J(x) + \lambda \Phi(x) \le \rho \} \subseteq$$

$$\subseteq \{x \in X : J(x) + c\Phi(x) \le \rho\} \cup \{x \in X : J(x) + d\Phi(x) \le \rho\}.$$

From this, due to the choice of ρ , we infer that the sequence $\{\hat{y}_{\lambda_n}\}$ is contained in the the set on the right-hand side which is clearly sequentially compact. Hence, there is a subsequence $\{\hat{y}_{\lambda_{n_k}}\}$ converging to some $y^* \in X$. Taking into account that the sequence $\{\Phi(\hat{y}_{\lambda_{n_k}})\}$ is bounded (by Proposition 1) and that the function $J + \lambda^*\Phi$ is sequentially lower semicontinuous, for each $x \in X$, we then have

$$\begin{split} J(y^*) + \lambda^* \Phi(y^*) & \leq \liminf_{k \to \infty} (J(\hat{y}_{\lambda_{n_k}}) + \lambda^* \Phi(\hat{y}_{\lambda_{n_k}})) = \\ & = \liminf_{k \to \infty} (J(\hat{y}_{\lambda_{n_k}}) + \lambda_{n_k} \Phi(\hat{y}_{\lambda_{n_k}}) + (\lambda^* - \lambda_{n_k}) \Phi(\hat{y}_{\lambda_{n_k}})) = \\ & = \liminf_{k \to \infty} (J(\hat{y}_{\lambda_{n_k}}) + \lambda_{n_k} \Phi(\hat{y}_{\lambda_{n_k}})) \leq \lim_{k \to \infty} (J(x) + \lambda_{n_k} \Phi(x)) = J(x) + \lambda^* \Phi(x) \;. \end{split}$$

Hence y^* is the global minimum of $J + \lambda^* \Phi$, that is $y^* = \hat{y}_{\lambda^*}$, which shows the continuity of $\lambda \to \hat{y}_{\lambda}$ at λ^* . Now, fix $r \in]\alpha, \beta[$ and consider the function $f: X \times \mathbf{R} \to \mathbf{R}$ defined by

$$f(x,\lambda) = J(x) + \lambda(\Phi(x) - r)$$

for all $(x, \lambda) \in X \times \mathbf{R}$. Clearly, the the restriction of the function f to $X \times]a, b[$ satisfies all the assumptions of Theorem 1. In particular, (iii) is satisfied taking $\varphi(\lambda) = \hat{y}_{\lambda}$. Consequently, we have

$$\sup_{\lambda \in]a,b[} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in]a,b[} (J(x) + \lambda(\Phi(x) - r)) . \tag{2}$$

Note that

$$\sup_{\lambda \in]a,b[} \inf_{x \in X} f(x,\lambda) \le \sup_{\lambda \in [a,b] \cap \mathbf{R}} \inf_{x \in X} f(x,\lambda) \le$$

$$\le \inf_{x \in X} \sup_{\lambda \in [a,b] \cap \mathbf{R}} f(x,\lambda) = \inf_{x \in X} \sup_{\lambda \in]a,b[} f(x,\lambda)$$

and so from (2) it follows

$$\sup_{\lambda \in [a,b] \cap \mathbf{R}} \inf_{x \in X} (J(x) + \lambda(\Phi(x) - r)) = \inf_{x \in X} \sup_{\lambda \in [a,b] \cap \mathbf{R}} (J(x) + \lambda(\Phi(x) - r)) . \tag{3}$$

Now, observe that the function $\inf_{x\in X} f(x,\cdot)$ is upper semicontinuous in $[a,b]\cap \mathbf{R}$ and that

$$\lim_{\lambda \to +\infty} \inf_{x \in X} f(x, \lambda) = -\infty$$

if $b = +\infty$ (since $r > \inf_X \Phi$), and

$$\lim_{\lambda \to -\infty} \inf_{x \in X} f(x, \lambda) = -\infty$$

if $a = -\infty$ (since $r < \sup_X \Phi$). From this, it clearly follows that there exists $\hat{\lambda}_r \in [a, b] \cap \mathbf{R}$ such that

$$\inf_{x \in X} f(x, \hat{\lambda}_r) = \sup_{\lambda \in [a,b] \cap \mathbf{R}} \inf_{x \in X} f(x, \hat{\lambda}_r) .$$

Since

$$\sup_{\lambda \in [a,b] \cap \mathbf{R}} f(x,\lambda) = \sup_{\lambda \in [a,b[} f(x,\lambda)$$

for all $x \in X$, the sub-level sets of the function $\sup_{\lambda \in [a,b] \cap \mathbf{R}} f(\cdot,\lambda)$ are sequentially compact. Hence, there exists $\hat{x}_r \in X$ such that

$$\sup_{\lambda \in [a,b] \cap \mathbf{R}} f(\hat{x}_r, \lambda) = \inf_{x \in X} \sup_{\lambda \in [a,b] \cap \mathbf{R}} f(x, \lambda) .$$

Then, thanks to (3), $(\hat{x}_r, \hat{\lambda}_r)$ is a saddle-point of f, that is

$$J(\hat{x}_r) + \hat{\lambda}_r(\Phi(\hat{x}_r) - r) = \inf_{x \in X} (J(x) + \hat{\lambda}_r(\Phi(x) - r)) = J(\hat{x}_r) + \sup_{\lambda \in [a,b] \cap \mathbf{R}} \lambda(\Phi(\hat{x}_r) - r) .$$
(4)

First of all, from (4) it follows that \hat{x}_r is a global minimum of $J + \hat{\lambda}_r \Phi$. We now show that $\Phi(\hat{x}_r) = r$. We distinguish four cases.

- $a = -\infty$ and $b = \infty$. In this case, the equality $\Phi(\hat{x}_r) = r$ follows from the fact that $\sup_{\lambda \in \mathbf{R}} \lambda(\Phi(\hat{x}_r) - r)$ is finite.

- $a > -\infty$ and $b = +\infty$. In this case, the finiteness of $\sup_{\lambda \in [a, +\infty[} \lambda(\Phi(\hat{x}_r) - r)$ implies that $\Phi(\hat{x}_r) \leq r$. But, if $\Phi(\hat{x}_r) < r$, from (4), we would infer that $\hat{\lambda}_r = a$ and so $\hat{x}_r \in M_a$. This would imply $\inf_{M_a} \Phi < r$, contrary to the choice of r.

- $a = -\infty$ and $b < +\infty$. In this case, the finiteness of $\sup_{\lambda \in]-\infty,b]} \lambda(\Phi(\hat{x}_r) - r)$ implies that $\Phi(\hat{x}_r) \geq r$. But, if $\Phi(\hat{x}_r) > r$, from (4) again, we would infer $\hat{\lambda}_r = b$, and so $\hat{x}_r \in M_b$. Therefore, $\sup_{M_b} \Phi > r$, contrary to the choice of r.

- $-\infty < a$ and $b < +\infty$. In this case, if $\Phi(\hat{x}_r) \neq r$, as we have just seen, we would have either $\inf_{M_a} \Phi < r$ or $\sup_{M_b} \Phi > r$, contrary to the choice of r.

Having proved that $\Phi(\hat{x}_r) = r$, we also get that $\hat{\lambda}_r \in]a, b[$. Indeed, if $\hat{\lambda}_r \in \{a, b\}$, we would have either $\hat{x}_r \in M_a$ or $\hat{x}_r \in M_b$ and so either $\inf_{M_a} \Phi \leq r$ or $\sup_{M_b} \Phi \geq r$, contrary to the choice of r. From (4) once again, we furthermore infer that any global minimum of $J_{|\Phi^{-1}(r)}$ (and \hat{x}_r is so) is a global minimum of $J + \hat{\lambda}_r \Phi$ in X. But, since $\hat{\lambda}_r \in]a, b[$, $J + \hat{\lambda}_r \Phi$ has exactly one global minimum in X which, therefore, coincides with \hat{x}_r . Since the sub-level sets of $J + \hat{\lambda}_r \Phi$ are sequentially compact, we then conclude that any minimizing sequence in X for $J + \hat{\lambda}_r \Phi$ converges to \hat{x}_r . But any minimizing sequence in $\Phi^{-1}(r)$ for J is a minimizing sequence for $J + \hat{\lambda}_r \Phi$, and so it converges to \hat{x}_r . Consequently, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed, as claimed.

Now, let us prove the other assertions made in thesis. By Proposition 1, it clearly follows that the function $\lambda \to \Phi(\hat{y}_{\lambda})$ is non-increasing in]a,b[and that its range is contained in $[\alpha,\beta]$. On the other hand, by the first assertion of the thesis, this range contains $]\alpha,\beta[$. Of course, from this it follows that the function $\lambda \to \Phi(\hat{y}_{\lambda})$ is continuous in]a,b[. Now, observe that the function $\lambda \to \inf_{x \in X} (J(x) + \lambda \Phi(x))$ is concave and hence continuous in]a,b[. This, in particular, implies that the function $\lambda \to J(\hat{y}_{\lambda})$ is continuous in]a,b[. Now, for each $r \in]\alpha,\beta[$, put

$$\Lambda_r = \{ \lambda \in]a, b[: \Phi(\hat{y}_\lambda) = r \} .$$

Let us prove that the multifunction $r \to \Lambda_r$ is upper semicontinuous in $]\alpha, \beta[$. Of course, it is enough to show that the restriction of the multifunction to any bounded open sub-interval of $]\alpha, \beta[$ is upper semicontinuous. So, let $s, t \in]\alpha, \beta[$, with s < t. Let $\mu, \nu \in]a, b[$ be such that $\Phi(\hat{y}_{\mu}) = t$, $\Phi(\hat{y}_{\nu}) = s$. By Proposition 1, we have

$$\bigcup_{r\in]s,t[}\Lambda_r\subseteq [\mu,\nu]\ .$$

Then, to show that the restriction of multifunction $r \to \Lambda_r$ to]s,t[is upper semicontinuous, it is enough to prove that its graph is closed in $]s,t[\times[\mu,\nu]]$ ([1], Theorem 7.1.16). But, this latter fact follows immediately from the continuity of the function $\lambda \to \Phi(\hat{y}_{\lambda})$. At this point, we observe that, for each $r \in]\alpha,\beta[$, the function $\lambda \to \hat{y}_{\lambda}$ is constant in Λ_r . Indeed, let $\lambda,\mu \in \Lambda_r$ with $\lambda \neq \mu$. If it was $\hat{y}_{\lambda} \neq \hat{y}_{\mu}$, by Proposition 1 it would follow

$$r = \Phi(\hat{y}_{\lambda}) \neq \Phi(\hat{y}_{\mu}) = r$$
,

an absurd. Hence, the function $r \to \hat{x}_r$, as composition of the upper semicontinuous multifunction $r \to \Lambda_r$ and the continuous function $\lambda \to \hat{y}_{\lambda}$, is continuous. Analogously, the continuity of the function $r \to J(\hat{x}_r)$ follows observing that it is the composition of $r \to \Lambda_r$ and the continuous function $\lambda \to J(\hat{y}_{\lambda})$. The proof is complete.

REMARK 1. - We want to point out that, under the assumptions of Theorem 1, we have actually proved that, for each $r \in]\alpha, \beta[$, there exists $\hat{\lambda}_r \in]a, b[$ such that the unique global minimum of $J + \hat{\lambda}_r \Phi$ belongs to $\Phi^{-1}(r)$.

When $a \ge 0$, we can obtain a conclusion dual to that of Theorem 1, under the same key assumption.

THEOREM 3. - Let $a \ge 0$. Assume that, for each $\lambda \in]a,b[$, the function $J+\lambda\Phi$ has sequentially compact sub-level sets and admits a unique global minimum in X.

Set

$$\gamma := \max \left\{ \inf_{X} J, \sup_{\hat{M}_{a}} J \right\} ,$$

$$\delta := \min \left\{ \sup_{X} J, \inf_{\hat{M}_{b}} J \right\} ,$$

where

$$\hat{M}_a = \begin{cases} M_a & \text{if } a > 0 \\ \emptyset & \text{if } a = 0 \end{cases},$$

$$\hat{M}_b = \begin{cases} M_b & \text{if } b < +\infty \\ \inf_X \Phi & \text{if } b = +\infty \end{cases}.$$

Assume that $\gamma < \delta$.

Then, for each $r \in]\gamma, \delta[$, the problem of minimizing Φ over $J^{-1}(r)$ is well-posed.

Moreover, if we denote by \tilde{x}_r the unique global minimum of $\Phi_{|J^{-1}(r)}$ $(r \in]\gamma, \delta[)$, the functions $r \to \tilde{x}_r$ and $r \to \Phi(\tilde{x}_r)$ are continuous in $]\gamma, \delta[$.

PROOF. Let $\mu \in]b^{-1}, a^{-1}[$. Then, since $\mu^{-1} \in]a, b[$ and

$$\Phi + \mu J = \mu (J + \mu^{-1} \Phi) ,$$

we clearly have that the function $J+\mu\Phi$ has sequentially compact sub-level sets and admits a unique global minimum. At this point, the conclusion follows applying Theorem 1 with the roles of J an Φ interchanged.

We now state the version of Theorem 1 obtained in the setting of a reflexive Banach space endowed with the weak topology.

THEOREM 4. - Let X be a sequentially weakly closed set in a reflexive real Banach space. Assume that $\alpha < \beta$ and that, for each $\lambda \in]a,b[$, the function $J + \lambda \Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in X.

Then, for each $r \in]\alpha, \beta[$, the problem of minimizing J over $\Phi^{-1}(r)$ is well-posed in the weak topology.

Moreover, if we denote by \hat{x}_r the unique global minimum of $J_{|\Phi^{-1}(r)}$ $(r \in]\alpha, \beta[)$, the functions $r \to \hat{x}_r$ and $r \to J(\hat{x}_r)$ are continuous in $]\alpha, \beta[$, the first one in the weak topology.

PROOF. Our assumptions clearly imply that, for each $\lambda \in]a,b[$, the sub-level sets of $J+\lambda\Phi$ are sequentially weakly compact, by the Eberlein- Smulyan theorem. Hence, considering X with the relative weak topology, we are allowed to apply Theorem 1, from which the conclusion directly follows.

Analogously, from Theorem 3 we get

THEOREM 5. - Let $a \ge 0$ and let X be a sequentially weakly closed set in a reflexive real Banach space. Assume that, for each $\lambda \in]a,b[$, the function $J+\lambda\Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in X. Assume also that $\gamma < \delta$, where γ, δ are defined as in Theorem 3.

Then, for each $r \in]\gamma, \delta[$, the problem of minimizing Φ over $J^{-1}(r)$ is well-posed in the weak topology.

Moreover, if we denote by \tilde{x}_r the unique global minimum of $\Phi_{|J^{-1}(r)}$ $(r \in]\gamma, \delta[)$, the functions $r \to \tilde{x}_r$ and $r \to \Phi(\tilde{x}_r)$ are continuous in $]\gamma, \delta[$, the first one in the weak topology.

Finally, it is worth noticing that Theorem 1 also offers the perspective of a novel way of seeing whether a given function possesses a global minimum. Let us formalize this using Remark 1.

THEOREM 6. - Assume that b > 0 and that, for each $\lambda \in]0, b[$, the function $J + \lambda \Phi$ has sequentially compact sub-level sets and admits a unique global minimum, say \hat{y}_{λ} . Assume also that

$$\lim_{\lambda \to 0^+} \Phi(\hat{y}_{\lambda}) < \sup_{X} \Phi . \tag{5}$$

Then, one has

$$\lim_{\lambda \to 0^+} \Phi(\hat{y}_{\lambda}) = \inf_{M} \Phi ,$$

where M is the set of all global minima of J in X.

PROOF. We already know that the function $\lambda \to \Phi(\hat{y}_{\lambda})$ is non-increasing in]a,b[and that its range is contained in $[\alpha,\beta]$. We claim that

$$\beta = \lim_{\lambda \to 0^+} \Phi(\hat{y}_{\lambda}) .$$

Assume the contrary. Let us apply Theorem 1, with a = 0 (so, $M_0 = M$), using the conclusion pointed out in Remark 1. Choose r satisfying

$$\lim_{\lambda \to 0^+} \Phi(\hat{y}_{\lambda}) < r < \beta .$$

Then, (since also $\alpha < r$) it would exist $\hat{\lambda}_r \in]0, b[$ such that $\Phi(\hat{y}_{\hat{\lambda}_r}) = r$, contrary to the choice of r. At this point, the conclusion follows directly from (5).

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