# Well-posedness of constrained minimization problems via saddle-points 

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#### Abstract

Dedicated to Professor Jean Saint Raymond on his sixtieth birthday, with my greatest admiration and esteem


Here and in the sequel, $X$ is a Hausdorff topological space, $J, \Phi$ are two real-valued functions defined in $X$, and $a, b$ are two numbers in $[-\infty,+\infty$ ], with $a<b$.

If $a \in \mathbf{R}$ (resp. $b \in \mathbf{R}$ ), we denote by $M_{a}$ (resp. $M_{b}$ ) the set of all global minima of the function $J+a \Phi$ (resp. $J+b \Phi$ ), while if $a=-\infty$ (resp. $b=+\infty$ ), $M_{a}$ (resp. $M_{b}$ ) stands for the empty set. We adopt the conventions $\inf \emptyset=+\infty, \sup \emptyset=-\infty$.

We also set

$$
\begin{aligned}
& \alpha:=\max \left\{\inf _{X} \Phi, \sup _{M_{b}} \Phi\right\}, \\
& \beta:=\min \left\{\sup _{X} \Phi, \inf _{M_{a}} \Phi\right\} .
\end{aligned}
$$

Note that, by Proposition 1 below, one has $\alpha \leq \beta$.
A usual, given a function $f: X \rightarrow \mathbf{R}$ and a set $C \subseteq X$, we say that the problem of minimizing $f$ over $C$ is well-posed if the following two conditions hold:

- the restriction of $f$ to $C$ has a unique global minimum, say $\hat{x}$;
- every sequence $\left\{x_{n}\right\}$ in $C$ such that $\lim _{n \rightarrow \infty} f\left(x_{n}\right)=\inf _{C} f$, converges to $\hat{x}$.

A set of the type $\{x \in X: f(x) \leq r\}$ is said to be a sub-level set of $f$. Clearly, when the sub-level sets of $f$ are sequentially compact, the problem of minimizing $f$ over a sequentially closed set $C$ is well-posed if and only if $f_{\mid C}$ has a unique global minimum.

The aim of the present paper is to establish the following result:
THEOREM 1. - Assume that $\alpha<\beta$ and that, for each $\lambda \in] a, b[$, the function $J+\lambda \Phi$ has sequentially compact sub-level sets and admits a unique global minimum in $X$.

Then, for each $r \in] \alpha, \beta\left[\right.$, the problem of minimizing $J$ over $\Phi^{-1}(r)$ is well-posed.
Moreover, if we denote by $\hat{x}_{r}$ the unique global minimum of $J_{\mid \Phi^{-1}(r)}(r \in] \alpha, \beta[)$, the functions $r \rightarrow \hat{x}_{r}$ and $r \rightarrow J\left(\hat{x}_{r}\right)$ are continuous in $] \alpha, \beta[$.

Theorem 1 should be regarded as the definitive abstract result coming out from the saddle-point method developed in [4], [5], [6], [7], in specific settings.

The main tool used to prove Theorem 1 is provided by the following mini-max result:

THEOREM 2. - Let $I \subseteq \mathbf{R}$ be an interval and $f$ a real-valued function defined in $X \times I$. Assume that there exists a number $\rho^{*}>\sup _{I} \inf _{X} f$, and a point $\hat{\lambda} \in I$ such that, for each $\rho \leq \rho^{*}$, the following conditions hold:
(i) the set $\{\lambda \in I: f(x, \lambda)>\rho\}$ is connected for all $x \in X$;
(ii) the set $\{x \in X: f(x, \lambda) \leq \rho\}$ is sequentially closed for all $\lambda \in I$ and sequentially compact for $\lambda=\hat{\lambda}$;
(iii) for each compact interval $T \subseteq I$ for which $\sup _{T} \inf _{X} f<\rho$, there exists a continuous function $\varphi: T \rightarrow X$ such that $f(\varphi(\lambda), \lambda)<\rho$ for all $\lambda \in T$.

Then, one has

$$
\sup _{\lambda \in I} \inf _{x \in X} f(x, \lambda)=\inf _{x \in X} \sup _{\lambda \in I} f(x, \lambda) .
$$

PROOF. We strictly follow the proof Theorem 2 of [3]. First, fix a non-decreasing sequence $\left\{I_{n}\right\}$ of compact sub-intervals of $I$, with $\hat{\lambda} \in I_{1}$, such that $\cup_{n \in \mathbf{N}} I_{n}=I$. Now, fix $n \in \mathbf{N}$. We claim that

$$
\begin{equation*}
\sup _{\lambda \in I_{n}} \inf _{x \in X} f(x, \lambda)=\inf _{x \in X} \sup _{\lambda \in I_{n}} f(x, \lambda) . \tag{1}
\end{equation*}
$$

Arguing by contradiction, suppose that

$$
\sup _{\lambda \in I_{n}} \inf _{x \in X} f(x, \lambda)<\inf _{x \in X} \sup _{\lambda \in I_{n}} f(x, \lambda) .
$$

Fix $\rho$ satisfying

$$
\sup _{\lambda \in I_{n}} \inf _{x \in X} f(x, \lambda)<\rho<\min \left\{\rho^{*}, \inf _{x \in X} \sup _{\lambda \in I_{n}} f(x, \lambda)\right\}
$$

Set

$$
S=\left\{(x, \lambda) \in X \times I_{n}: f(x, \lambda)<\rho\right\}
$$

as well as, for each $\lambda \in I_{n}$,

$$
S^{\lambda}=\{x \in X:(x, \lambda) \in S\}
$$

Since $\sup _{I_{n}} \inf _{X} f<\rho$, one has $S^{\lambda} \neq \emptyset$ for all $\lambda \in I_{n}$. Let $I_{n}=\left[a_{n}, b_{n}\right]$. Put

$$
A=\left\{(x, \lambda) \in S: \lambda<b_{n}, \sup _{\left.s \in] \lambda, b_{n}\right]} f(x, s)>\rho\right\}
$$

and

$$
B=\left\{(x, \lambda) \in S: \lambda>a_{n}, \sup _{s \in\left[a_{n}, \lambda[ \right.} f(x, s)>\rho\right\}
$$

Observe that $S^{a_{n}} \times\left\{a_{n}\right\} \subseteq A$ and $S^{b_{n}} \times\left\{b_{n}\right\} \subseteq B$. Indeed, let $x_{1} \in S^{a_{n}}$ and $x_{2} \in S^{b_{n}}$. Since $\rho<\inf _{X} \sup _{I_{n}} f$, there are $t, s \in I_{n}$ such that $\min \left\{f\left(x_{1}, t\right), f\left(x_{2}, s\right)\right\}>\rho$. Since $\sup \left\{f\left(x_{1}, a_{n}\right), f\left(x_{2}, b_{n}\right)\right\}<\rho$, it follows that $t>a_{n}$ and $s<b_{n}$. Consequently, $\left(x_{1}, a_{n}\right) \in A$
and $\left(x_{2}, b_{n}\right) \in B$. Furthermore, observe that if $\left(x_{0}, \lambda_{0}\right) \in A$ and if $\left.\left.\mu \in\right] \lambda_{0}, b_{n}\right]$ is such that $f\left(x_{0}, \mu\right)>\rho$, then, in view of (ii), the set

$$
\left(\{x \in X: f(x, \mu)>\rho\} \times\left[a_{n}, \mu[) \cap S\right.\right.
$$

is sequentially open in $S$, contains $\left(x_{0}, \lambda_{0}\right)$ and is contained in $A$. In other words, $A$ is sequentially open in $S$. Analogously, it is seen that $B$ is sequentially open in $S$. We now prove that $S=A \cup B$. Indeed, let $(x, \lambda) \in S \backslash A$. We have seen above that $S^{a_{n}} \times\left\{a_{n}\right\} \subseteq A$, and so $\lambda>a_{n}$. If $\lambda=b_{n}$, the fact that $(x, \lambda) \in B$ has been likewise proved above. Suppose $\lambda<b_{n}$. Thus, we have $\sup _{\left.s \in] \lambda, b_{n}\right]} f(x, s) \leq \rho$. From this, it clearly follows that $\sup _{s \in\left[a_{n}, \lambda[ \right.} f(x, s)>\rho$ (note that $f(x, \lambda)<\rho$ ), and so $(x, \lambda) \in B$. Furthermore, we have $A \cap B=\emptyset$. Indeed, if $\left(x_{1}, \lambda_{1}\right) \in A \cap B$, there would be $t_{1}, s_{1} \in I_{n}$, with $t_{1}<\lambda_{1}<s_{1}$, such that $\min \left\{f\left(x_{1}, t_{1}\right), f\left(x_{1}, s_{1}\right)\right\}>\rho$. By $(i)$, the set $\left\{s \in I: f\left(x_{1}, s\right)>\rho\right\}$ is an interval, and so we would have $f\left(x_{1}, \lambda_{1}\right)>\rho$, against the fact that $\left(x_{1}, \lambda_{1}\right) \in S$. Now, in view of (iii), consider a continuous function $\varphi: I_{n} \rightarrow X$ such that

$$
f(\varphi(\lambda), \lambda)<\rho
$$

for all $\lambda \in I_{n}$. Let $h: I_{n} \rightarrow X \times I_{n}$ be defined by setting

$$
h(\lambda)=(\varphi(\lambda), \lambda)
$$

for all $\lambda \in I_{n}$. Since $h$ is continuous, the set $h\left(I_{n}\right)$ is sequentially connected ([2], Theorem 2.2). But, having in mind that $h\left(I_{n}\right) \subseteq S$ and that $h\left(I_{n}\right)$ meets both $A$ and $B$ (since $h\left(a_{n}\right) \in A$ and $h\left(b_{n}\right) \in B$ ), the properties of $A, B$ proved above would imply that $h\left(I_{n}\right)$ is sequentially disconnected, a contradiction. So, (1) holds. Finally, let us prove the theorem. Again arguing by contradiction, suppose that

$$
\sup _{\lambda \in I} \inf _{x \in X} f(x, \lambda)<\inf _{x \in X} \sup _{\lambda \in I} f(x, \lambda) .
$$

Choose $r$ satisfying

$$
\sup _{\lambda \in I} \inf _{x \in X} f(x, \lambda)<r<\min \left\{\rho^{*}, \inf _{x \in X} \sup _{\lambda \in I} f(x, \lambda)\right\} .
$$

For each $n \in \mathbf{N}$, put

$$
C_{n}=\left\{x \in X: \sup _{\lambda \in I_{n}} f(x, \lambda) \leq r\right\}
$$

Note that $C_{n} \neq \emptyset$. Indeed, otherwise, we would have

$$
r \leq \inf _{x \in X} \sup _{\lambda \in I_{n}} f(x, \lambda)=\sup _{\lambda \in I_{n}} \inf _{x \in X} f(x, \lambda) \leq \sup _{\lambda \in I} \inf _{x \in X} f(x, \lambda)
$$

Consequently, $\left\{C_{n}\right\}$ is a non-increasing sequence of non-empty sequentially closed subsets of the sequentially compact set $\left\{x \in X: f(x, \hat{\lambda}) \leq \rho^{*}\right\}$. Therefore, one has $\cap_{n \in \mathbf{N}} C_{n} \neq \emptyset$. Let $x^{*} \in \cap_{n \in \mathbf{N}} C_{n}$. Then, one has

$$
\sup _{\lambda \in I} f\left(x^{*}, \lambda\right)=\sup _{n \in \mathbf{N}} \sup _{\lambda \in I_{n}} f\left(x^{*}, \lambda\right) \leq r
$$

and so

$$
\inf _{x \in X} \sup _{\lambda \in I} f(x, \lambda) \leq r
$$

a contradiction. The proof is complete.
We will also use the following proposition.
PROPOSITION 1 ([4], Proposition 1). - Let $Y$ be a nonempty set, $f, g: Y \rightarrow \mathbf{R}$ two functions, and $\lambda, \mu$ two real numbers, with $\lambda<\mu$. Let $\hat{y}_{\lambda}$ be a global minimum of the function $f+\lambda g$ and let $\hat{y}_{\mu}$ be a global minimum of the function $f+\mu g$.

Then, one has

$$
g\left(\hat{y}_{\mu}\right) \leq g\left(\hat{y}_{\lambda}\right)
$$

If either $\hat{y}_{\lambda}$ or $\hat{y}_{\mu}$ is strict and $\hat{y}_{\lambda} \neq \hat{y}_{\mu}$, then

$$
g\left(\hat{y}_{\mu}\right)<g\left(\hat{y}_{\lambda}\right) .
$$

Proof of Theorem 1. First, for each $\lambda \in] a, b\left[\right.$, denote by $\hat{y}_{\lambda}$ the unique global minimum in $X$ of $J+\lambda \Phi$. Let us prove that the function $\lambda \rightarrow \hat{y}_{\lambda}$ is continuous in $] a, b[$. To this end, fix $\left.\lambda^{*} \in\right] a, b\left[\right.$. Let $\left\{\lambda_{n}\right\}$ be any sequence in $] a, b\left[\right.$ converging to $\lambda^{*}$ and let $[c, d] \subset] a, b\left[\right.$ be a compact interval containing $\left\{\lambda_{n}\right\}$. Fix $\rho>\sup _{n \in \mathbf{N}} \inf _{x \in X}\left(J(x)+\lambda_{n} \Phi(x)\right)$. Clearly, we have

$$
\begin{gathered}
\bigcup_{\lambda \in[c, d]}\{x \in X: J(x)+\lambda \Phi(x) \leq \rho\} \subseteq \\
\subseteq\{x \in X: J(x)+c \Phi(x) \leq \rho\} \cup\{x \in X: J(x)+d \Phi(x) \leq \rho\} .
\end{gathered}
$$

From this, due to the choice of $\rho$, we infer that the sequence $\left\{\hat{y}_{\lambda_{n}}\right\}$ is contained in the the set on the right-hand side which is clearly sequentially compact. Hence, there is a subsequence $\left\{\hat{y}_{\lambda_{n_{k}}}\right\}$ converging to some $y^{*} \in X$. Taking into account that the sequence $\left\{\Phi\left(\hat{y}_{\lambda_{n_{k}}}\right)\right\}$ is bounded (by Proposition 1) and that the function $J+\lambda^{*} \Phi$ is sequentially lower semicontinuous, for each $x \in X$, we then have

$$
\begin{gathered}
J\left(y^{*}\right)+\lambda^{*} \Phi\left(y^{*}\right) \leq \liminf _{k \rightarrow \infty}\left(J\left(\hat{y}_{\lambda_{n_{k}}}\right)+\lambda^{*} \Phi\left(\hat{y}_{\lambda_{n_{k}}}\right)\right)= \\
=\liminf _{k \rightarrow \infty}\left(J\left(\hat{y}_{\lambda_{n_{k}}}\right)+\lambda_{n_{k}} \Phi\left(\hat{y}_{\lambda_{n_{k}}}\right)+\left(\lambda^{*}-\lambda_{n_{k}}\right) \Phi\left(\hat{y}_{\lambda_{n_{k}}}\right)\right)= \\
=\liminf _{k \rightarrow \infty}\left(J\left(\hat{y}_{\lambda_{n_{k}}}\right)+\lambda_{n_{k}} \Phi\left(\hat{y}_{\lambda_{n_{k}}}\right)\right) \leq \lim _{k \rightarrow \infty}\left(J(x)+\lambda_{n_{k}} \Phi(x)\right)=J(x)+\lambda^{*} \Phi(x) .
\end{gathered}
$$

Hence $y^{*}$ is the global minimum of $J+\lambda^{*} \Phi$, that is $y^{*}=\hat{y}_{\lambda^{*}}$, which shows the continuity of $\lambda \rightarrow \hat{y}_{\lambda}$ at $\lambda^{*}$. Now, fix $\left.r \in\right] \alpha, \beta[$ and consider the function $f: X \times \mathbf{R} \rightarrow \mathbf{R}$ defined by

$$
f(x, \lambda)=J(x)+\lambda(\Phi(x)-r)
$$

for all $(x, \lambda) \in X \times \mathbf{R}$. Clearly, the the restriction of the function $f$ to $X \times] a, b[$ satisfies all the assumptions of Theorem 1. In particular, (iii) is satisfied taking $\varphi(\lambda)=\hat{y}_{\lambda}$. Consequently, we have

$$
\begin{equation*}
\sup _{\lambda \in] a, b[ } \inf _{x \in X}(J(x)+\lambda(\Phi(x)-r))=\inf _{x \in X} \sup _{\lambda \in] a, b[ }(J(x)+\lambda(\Phi(x)-r)) \tag{2}
\end{equation*}
$$

Note that

$$
\begin{aligned}
& \sup _{\lambda \in] a, b[ } \inf _{x \in X} f(x, \lambda) \leq \sup _{\lambda \in[a, b] \cap \mathbf{R}} \inf _{x \in X} f(x, \lambda) \leq \\
& \leq \inf _{x \in X} \sup _{\lambda \in[a, b] \cap \mathbf{R}} f(x, \lambda)=\inf _{x \in X} \sup _{\lambda \in] a, b[ } f(x, \lambda)
\end{aligned}
$$

and so from (2) it follows

$$
\begin{equation*}
\sup _{\lambda \in[a, b] \cap \mathbf{R}} \inf _{x \in X}(J(x)+\lambda(\Phi(x)-r))=\inf _{x \in X} \sup _{\lambda \in[a, b] \cap \mathbf{R}}(J(x)+\lambda(\Phi(x)-r)) . \tag{3}
\end{equation*}
$$

Now, observe that the function $\inf _{x \in X} f(x, \cdot)$ is upper semicontinuous in $[a, b] \cap \mathbf{R}$ and that

$$
\lim _{\lambda \rightarrow+\infty} \inf _{x \in X} f(x, \lambda)=-\infty
$$

if $b=+\infty\left(\right.$ since $\left.r>\inf _{X} \Phi\right)$, and

$$
\lim _{\lambda \rightarrow-\infty} \inf _{x \in X} f(x, \lambda)=-\infty
$$

if $a=-\infty\left(\right.$ since $\left.r<\sup _{X} \Phi\right)$. From this, it clearly follows that there exists $\hat{\lambda}_{r} \in[a, b] \cap \mathbf{R}$ such that

$$
\inf _{x \in X} f\left(x, \hat{\lambda}_{r}\right)=\sup _{\lambda \in[a, b] \cap \mathbf{R}} \inf _{x \in X} f\left(x, \hat{\lambda}_{r}\right)
$$

Since

$$
\sup _{\lambda \in[a, b] \cap \mathbf{R}} f(x, \lambda)=\sup _{\lambda \in] a, b[ } f(x, \lambda)
$$

for all $x \in X$, the sub-level sets of the function $\sup _{\lambda \in[a, b] \cap \mathbf{R}} f(\cdot, \lambda)$ are sequentially compact. Hence, there exists $\hat{x}_{r} \in X$ such that

$$
\sup _{\lambda \in[a, b] \cap \mathbf{R}} f\left(\hat{x}_{r}, \lambda\right)=\inf _{x \in X} \sup _{\lambda \in[a, b] \cap \mathbf{R}} f(x, \lambda) .
$$

Then, thanks to $(3),\left(\hat{x}_{r}, \hat{\lambda}_{r}\right)$ is a saddle-point of $f$, that is

$$
\begin{equation*}
J\left(\hat{x}_{r}\right)+\hat{\lambda}_{r}\left(\Phi\left(\hat{x}_{r}\right)-r\right)=\inf _{x \in X}\left(J(x)+\hat{\lambda}_{r}(\Phi(x)-r)\right)=J\left(\hat{x}_{r}\right)+\sup _{\lambda \in[a, b] \cap \mathbf{R}} \lambda\left(\Phi\left(\hat{x}_{r}\right)-r\right) \tag{4}
\end{equation*}
$$

First of all, from (4) it follows that $\hat{x}_{r}$ is a global minimum of $J+\hat{\lambda}_{r} \Phi$. We now show that $\Phi\left(\hat{x}_{r}\right)=r$. We distinguish four cases.

- $a=-\infty$ and $b=\infty$. In this case, the equality $\Phi\left(\hat{x}_{r}\right)=r$ follows from the fact that $\sup _{\lambda \in \mathbf{R}} \lambda\left(\Phi\left(\hat{x}_{r}\right)-r\right)$ is finite.
- $a>-\infty$ and $b=+\infty$. In this case, the finiteness of $\sup _{\lambda \in[a,+\infty[ } \lambda\left(\Phi\left(\hat{x}_{r}\right)-r\right)$ implies that $\Phi\left(\hat{x}_{r}\right) \leq r$. But, if $\Phi\left(\hat{x}_{r}\right)<r$, from (4), we would infer that $\hat{\lambda}_{r}=a$ and so $\hat{x}_{r} \in M_{a}$. This would imply $\inf _{M_{a}} \Phi<r$, contrary to the choice of $r$.
- $a=-\infty$ and $b<+\infty$. In this case, the finiteness of $\sup _{\lambda \in]-\infty, b]} \lambda\left(\Phi\left(\hat{x}_{r}\right)-r\right)$ implies that $\Phi\left(\hat{x}_{r}\right) \geq r$. But, if $\Phi\left(\hat{x}_{r}\right)>r$, from (4) again, we would infer $\hat{\lambda}_{r}=b$, and so $\hat{x}_{r} \in M_{b}$. Therefore, $\sup _{M_{b}} \Phi>r$, contrary to the choice of $r$.
- $-\infty<a$ and $b<+\infty$. In this case, if $\Phi\left(\hat{x}_{r}\right) \neq r$, as we have just seen, we would have either $\inf _{M_{a}} \Phi<r$ or $\sup _{M_{b}} \Phi>r$, contrary to the choice of $r$.

Having proved that $\Phi\left(\hat{x}_{r}\right)=r$, we also get that $\left.\hat{\lambda}_{r} \in\right] a, b\left[\right.$. Indeed, if $\hat{\lambda}_{r} \in\{a, b\}$, we would have either $\hat{x}_{r} \in M_{a}$ or $\hat{x}_{r} \in M_{b}$ and so either $\inf _{M_{a}} \Phi \leq r$ or $\sup _{M_{b}} \Phi \geq r$, contrary to the choice of $r$. From (4) once again, we furthermore infer that any global minimum of $J_{\mid \Phi^{-1}(r)}$ (and $\hat{x}_{r}$ is so) is a global minimum of $J+\hat{\lambda}_{r} \Phi$ in $X$. But, since $\left.\hat{\lambda}_{r} \in\right] a, b\left[, J+\hat{\lambda}_{r} \Phi\right.$ has exactly one global minimum in $X$ which, therefore, coincides with $\hat{x}_{r}$. Since the sub-level sets of $J+\hat{\lambda}_{r} \Phi$ are sequentially compact, we then conclude that any minimizing sequence in $X$ for $J+\hat{\lambda}_{r} \Phi$ converges to $\hat{x}_{r}$. But any minimizing sequence in $\Phi^{-1}(r)$ for $J$ is a minimizing sequence for $J+\hat{\lambda}_{r} \Phi$, and so it converges to $\hat{x}_{r}$. Consequently, the problem of minimizing $J$ over $\Phi^{-1}(r)$ is well-posed, as claimed.

Now, let us prove the other assertions made in thesis. By Proposition 1, it clearly follows that the function $\lambda \rightarrow \Phi\left(\hat{y}_{\lambda}\right)$ is non-increasing in $] a, b[$ and that its range is contained in $[\alpha, \beta]$. On the other hand, by the first assertion of the thesis, this range contains $] \alpha, \beta[$. Of course, from this it follows that the function $\lambda \rightarrow \Phi\left(\hat{y}_{\lambda}\right)$ is continuous in $] a, b[$. Now, observe that the function $\lambda \rightarrow \inf _{x \in X}(J(x)+\lambda \Phi(x))$ is concave and hence continuous in $] a, b\left[\right.$. This, in particular, implies that the function $\lambda \rightarrow J\left(\hat{y}_{\lambda}\right)$ is continuous in $] a, b[$. Now, for each $r \in] \alpha, \beta[$, put

$$
\Lambda_{r}=\{\lambda \in] a, b\left[: \Phi\left(\hat{y}_{\lambda}\right)=r\right\} .
$$

Let us prove that the multifunction $r \rightarrow \Lambda_{r}$ is upper semicontinuous in $] \alpha, \beta[$. Of course, it is enough to show that the restriction of the multifunction to any bounded open subinterval of $] \alpha, \beta[$ is upper semicontinuous. So, let $s, t \in] \alpha, \beta[$, with $s<t$. Let $\mu, \nu \in] a, b[$ be such that $\Phi\left(\hat{y}_{\mu}\right)=t, \Phi\left(\hat{y}_{\nu}\right)=s$. By Proposition 1, we have

$$
\bigcup_{r \in] s, t[ } \Lambda_{r} \subseteq[\mu, \nu]
$$

Then, to show that the restriction of multifunction $r \rightarrow \Lambda_{r}$ to $] s, t$ is upper semicontinuous, it is enough to prove that its graph is closed in $] s, t[\times[\mu, \nu]$ ([1], Theorem 7.1.16). But, this latter fact follows immediately from the continuity of the function $\lambda \rightarrow \Phi\left(\hat{y}_{\lambda}\right)$. At this point, we observe that, for each $r \in] \alpha, \beta\left[\right.$, the function $\lambda \rightarrow \hat{y}_{\lambda}$ is constant in $\Lambda_{r}$. Indeed, let $\lambda, \mu \in \Lambda_{r}$ with $\lambda \neq \mu$. If it was $\hat{y}_{\lambda} \neq \hat{y}_{\mu}$, by Proposition 1 it would follow

$$
r=\Phi\left(\hat{y}_{\lambda}\right) \neq \Phi\left(\hat{y}_{\mu}\right)=r,
$$

an absurd. Hence, the function $r \rightarrow \hat{x}_{r}$, as composition of the upper semicontinuous multifunction $r \rightarrow \Lambda_{r}$ and the continuous function $\lambda \rightarrow \hat{y}_{\lambda}$, is continuous. Analogously, the continuity of the function $r \rightarrow J\left(\hat{x}_{r}\right)$ follows observing that it is the composition of $r \rightarrow \Lambda_{r}$ and the continuous function $\lambda \rightarrow J\left(\hat{y}_{\lambda}\right)$. The proof is complete.

REMARK 1. - We want to point out that, under the assumptions of Theorem 1, we have actually proved that, for each $r \in] \alpha, \beta\left[\right.$, there exists $\left.\hat{\lambda}_{r} \in\right] a, b[$ such that the unique global minimum of $J+\hat{\lambda}_{r} \Phi$ belongs to $\Phi^{-1}(r)$.

When $a \geq 0$, we can obtain a conclusion dual to that of Theorem 1, under the same key assumption.

THEOREM 3. - Let $a \geq 0$. Assume that, for each $\lambda \in] a, b[$, the function $J+\lambda \Phi$ has sequentially compact sub-level sets and admits a unique global minimum in $X$.

Set

$$
\begin{aligned}
\gamma & :=\max \left\{\inf _{X} J, \sup _{\hat{M}_{a}} J\right\}, \\
\delta & :=\min \left\{\sup _{X} J, \inf _{\hat{M}_{b}} J\right\},
\end{aligned}
$$

where

$$
\begin{gathered}
\hat{M}_{a}= \begin{cases}M_{a} & \text { if } a>0 \\
\emptyset & \text { if } a=0,\end{cases} \\
\hat{M}_{b}= \begin{cases}M_{b} & \text { if } b<+\infty \\
\inf _{X} \Phi & \text { if } b=+\infty\end{cases}
\end{gathered}
$$

Assume that $\gamma<\delta$.
Then, for each $r \in] \gamma, \delta\left[\right.$, the problem of minimizing $\Phi$ over $J^{-1}(r)$ is well-posed.
Moreover, if we denote by $\tilde{x}_{r}$ the unique global minimum of $\Phi_{\mid J^{-1}(r)}(r \in] \gamma, \delta[)$, the functions $r \rightarrow \tilde{x}_{r}$ and $r \rightarrow \Phi\left(\tilde{x}_{r}\right)$ are continuous in $] \gamma, \delta[$.

PROOF. Let $\mu \in] b^{-1}, a^{-1}\left[\right.$. Then, since $\left.\mu^{-1} \in\right] a, b[$ and

$$
\Phi+\mu J=\mu\left(J+\mu^{-1} \Phi\right)
$$

we clearly have that the function $J+\mu \Phi$ has sequentially compact sub-level sets and admits a unique global minimum. At this point, the conclusion follows applying Theorem 1 with the roles of $J$ an $\Phi$ interchanged.

We now state the version of Theorem 1 obtained in the setting of a reflexive Banach space endowed with the weak topology.

THEOREM 4. - Let $X$ be a sequentially weakly closed set in a reflexive real Banach space. Assume that $\alpha<\beta$ and that, for each $\lambda \in] a, b[$, the function $J+\lambda \Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in $X$.

Then, for each $r \in] \alpha, \beta\left[\right.$, the problem of minimizing $J$ over $\Phi^{-1}(r)$ is well-posed in the weak topology.

Moreover, if we denote by $\hat{x}_{r}$ the unique global minimum of $J_{\mid \Phi^{-1}(r)}(r \in] \alpha, \beta[)$, the functions $r \rightarrow \hat{x}_{r}$ and $r \rightarrow J\left(\hat{x}_{r}\right)$ are continuous in $] \alpha, \beta[$, the first one in the weak topology.

PROOF. Our assumptions clearly imply that, for each $\lambda \in] a, b[$, the sub-level sets of $J+\lambda \Phi$ are sequentially weakly compact, by the Eberlein- Smulyan theorem. Hence, considering $X$ with the relative weak topology, we are allowed to apply Theorem 1, from which the conclusion directly follows.

Analogously, from Theorem 3 we get
THEOREM 5. - Let $a \geq 0$ and let $X$ be a sequentially weakly closed set in a reflexive real Banach space. Assume that, for each $\lambda \in] a, b[$, the function $J+\lambda \Phi$ is sequentially weakly lower semicontinuous, has bounded sub-level sets and has a unique global minimum in $X$. Assume also that $\gamma<\delta$, where $\gamma, \delta$ are defined as in Theorem 3.

Then, for each $r \in] \gamma, \delta\left[\right.$, the problem of minimizing $\Phi$ over $J^{-1}(r)$ is well-posed in the weak topology.

Moreover, if we denote by $\tilde{x}_{r}$ the unique global minimum of $\Phi_{\mid J^{-1}(r)}(r \in] \gamma, \delta[)$, the functions $r \rightarrow \tilde{x}_{r}$ and $r \rightarrow \Phi\left(\tilde{x}_{r}\right)$ are continuous in $] \gamma, \delta[$, the first one in the weak topology.

Finally, it is worth noticing that Theorem 1 also offers the perspective of a novel way of seeing whether a given function possesses a global minimum. Let us formalize this using Remark 1.

THEOREM 6. - Assume that $b>0$ and that, for each $\lambda \in] 0, b[$, the function $J+\lambda \Phi$ has sequentially compact sub-level sets and admits a unique global minimum, say $\hat{y}_{\lambda}$. Assume also that

$$
\begin{equation*}
\lim _{\lambda \rightarrow 0^{+}} \Phi\left(\hat{y}_{\lambda}\right)<\sup _{X} \Phi . \tag{5}
\end{equation*}
$$

Then, one has

$$
\lim _{\lambda \rightarrow 0^{+}} \Phi\left(\hat{y}_{\lambda}\right)=\inf _{M} \Phi
$$

where $M$ is the set of all global minima of $J$ in $X$.
PROOF. We already know that the function $\lambda \rightarrow \Phi\left(\hat{y}_{\lambda}\right)$ is non-increasing in $] a, b[$ and that its range is contained in $[\alpha, \beta]$. We claim that

$$
\beta=\lim _{\lambda \rightarrow 0^{+}} \Phi\left(\hat{y}_{\lambda}\right) .
$$

Assume the contrary. Let us apply Theorem 1, with $a=0$ (so, $M_{0}=M$ ), using the conclusion pointed out in Remark 1. Choose $r$ satisfying

$$
\lim _{\lambda \rightarrow 0^{+}} \Phi\left(\hat{y}_{\lambda}\right)<r<\beta .
$$

Then, (since also $\alpha<r$ ) it would exist $\left.\hat{\lambda}_{r} \in\right] 0, b\left[\right.$ such that $\Phi\left(\hat{y}_{\hat{\lambda}_{r}}\right)=r$, contrary to the choice of $r$. At this point, the conclusion follows directly from (5).

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