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CALCULUS OF TANGENT SETS AND DERIVATIVES OF SET VALUED MAPS UNDER METRIC SUBREGULARITY CONDITIONS

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Abstract: In this paper we intend to give some calculus rules for tangent sets in the sense of Bouligand and Ursescu, as well as for corresponding derivatives of set-valued maps. Both first and second order objects are envisaged and the assumptions we impose in order to get the calculus are in terms of metric subregularity of the assembly of the initial data. This approach is different from those used in alternative recent papers in literature and allows us to avoid compactness conditions. A special attention is paid for the case of perturbation set-valued maps which appear naturally in optimization problems.

Keywords: Bouligand tangent sets · Ursescu tangent sets · metric regularity · set-valued derivatives · perturbation maps

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1 Introduction

The importance of the tangent sets in the study of various mathematical problems, including optimization, viability theory, and control theory is well known: Chapter 4 in the comprehensive monograph [2] clearly emphasizes and illustrates this idea which is the basic layer of many papers in literature. On the other hand, the most useful theoretical constructions of tangent sets are those called in [2] the contingent and the adjacent tangent sets which we call here Bouligand and respectively Ursescu tangent sets after the names of the mathematicians who firstly introduced these concepts.

Once one has a concept of tangent set to an arbitrary set in a normed vector space, then one can construct a corresponding derivative for set-valued maps, and after that the question of calculus rules for these objects arises naturally. Several recent papers in literature are devoted to the study of such calculus for Bouligand and Ursescu first and second order tangent sets: we cite here [13], [12], [11] and the references therein. The starting point of this paper is the remark that, in general, the quoted papers employ quite strong conditions on the initial data in order to get the desired calculus. One speaks here about conditions concerning several types of generalized compactness for the graphs of underlying set-valued maps and such requirements are known to be quite strong in infinite dimensional spaces. Notice as well that the second-order conditions used in [11] are also of the same nature. It is true that there is a difference between the conditions used in [13] (working on finite dimensional vector spaces for first order objects) and [11] (working on general Banach spaces for second order objects) as the authors of the later paper emphasize, but, however, the techniques and the arguments are not so different. Another remark is that in all these papers the approach inherits the main line of arguments and techniques from [10, Chapter 3].

In contrast, we prefer here to follow the way open in [2, Chapter 4] allowing, as we shall see, to use metric (sub)regularity assumptions which are more adequate for infinite dimensional setting. The role of metric regularity in the regularity theory of constraint systems is nowadays well known (see the monographs [16], [19], [14], [3] for the main results in these topics and for detailed discussions) and this fact give credit to our method. Other differences with respect to the previous papers in literature are as follows: we impose conditions on the assembly of the initial data rather than separate condition for each object and the calculus rules we obtain refer, in general, not to equalities but to those inclusions which are shown to be the right ones to use in getting necessary optimality conditions in some vector optimization problems.

The paper is organized as follows. The second section introduces the main notations, concepts and auxiliary results needed in the rest of the paper. The third section concentrates on some general results concerning the calculus of the first and second order tangent sets under metric subregularity assumptions. Then a calculus rule for the derivatives of the sum between set-valued maps is given and the proof is based on the reduction of this situation to the general case of tangent sets previously obtained. An application of this calculus rule to vector optimization problems and an example underlying the differences with respect to the requirements used in the other papers in literature are given. The fourth section employs a similar technique to the case of generalized perturbation maps and several inclusion concerning the coderivatives of these maps are presented. After every important result of the paper we present sufficient conditions in terms of Fréchet normal cones to ensure the fulfillment of the metric subregularity condition imposed in hypotheses. The paper ends with a short section which displays the main conclusion of this work.

2 Preliminaries and tools

In the sequel, we suppose that the involved spaces are Banach, unless otherwise stated. In this setting, $B(x,r)$ and $D(x,r)$ denote the open and the closed ball with center x and radius r, respectively. If $x \in X$ and $A \subset X$, one defines the distance from x to A as $d(x,A) := \inf\{\|x-a\| \mid a \in A\}$. As usual, we use the convention $d(x,\emptyset) = \infty$. For a non-empty set $A \subset X$ we put $\operatorname{cl} A$ and $\operatorname{int} A$ for its topological closure and interior, respectively. On a product of normed vector spaces we consider the sum norm and the corresponding topology. Usually, the zero element of X is denoted by 0_X.

Let $F : X \rightrightarrows Y$ be a multifunction. The domain and the graph of F are denoted respectively by $\operatorname{Dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F := \{(x,y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$ then $F(A) := \bigcup_{x\in A} F(x)$. The inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$.

One says that F is open at linear rate $c > 0$ around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ and a positive number $\varepsilon > 0$ such that, for every $(x,y) \in \operatorname{Gr} F \cap (U \times V)$ and every $\rho \in (0, \varepsilon)$,

$$B(y, \rho c) \subset F(B(x, \rho)).$$

It is well known that this property is equivalent to the metric regularity property of F around $(\overline{x}, \overline{y})$ which requires to exist $a > 0$ and two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that for every $u \in U$ and every $v \in V$ to have

$$d(u, F^{-1}(v)) \leq ad(v, F(u)).$$

Recall as well that F is said to have the Aubin property at a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if there exist $L > 0$, $r > 0$ such that for all $x', x'' \in D(\overline{x}, r)$ we have

$$F(x') \cap D(\overline{y}, r) \subset F(x'') + L \left\| x' - x'' \right\| D(0, 1).$$

Denote by X^* the topological dual of X. As announced, we shall use in the main sections of the paper the Fréchet normal cones in order to write sufficient conditions for the fulfillment of the metric regularity assumptions we use in our results. Here are the main facts which allow us to do this.

Definition 2.1 *Let X be a normed vector space, S be a non-empty subset of X and let $x \in S$. The Fréchet normal cone to S at x is*

$$\widehat{N}(S, x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u - x)}{\|u - x\|} \leq 0 \right\}. \tag{2.1}$$

Definition 2.2 *Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative at $(\overline{x}, \overline{y})$ is the set-valued map $\widehat{D}^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by*

$$\widehat{D}^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

If $F = f$ is a single-valued function, then we write $\widehat{D}^* f(\overline{x})$ for $\widehat{D}^* F(\overline{x}, \overline{y})$. If f is Fréchet differentiable at $\overline{x}$, then $\widehat{D}^* f(\overline{x})(y^*) = \{\nabla^* f(\overline{x})(y^*)\}$ for every $y^* \in Y^*$, where ∇ denotes the Fréchet differential, while $\nabla^* f(\overline{x})$ stands for the $[\nabla f(\overline{x})]^*$ (the adjoint operator of $\nabla f(\overline{x})$).

We recall a well-known openness result for set-valued maps (see, e.g., [15, Theorem 5.6] or [14, Theorem 4.1]).

Theorem 2.3 *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a set-valued map with closed graph and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then the following assertions are equivalent:*

(i) There exist $r > 0, s > 0$ and $c > 0$ such that for every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$ and every $y^* \in Y^*, x^* \in \widehat{D}^* F(x, y)(y^*)$,

$$c \left\| y^* \right\| \leq \left\| x^* \right\|. \tag{2.2}$$

(ii) There exist $\alpha > 0, \beta > 0, c > 0$ and $\varepsilon > 0$ such that for every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, \alpha) \times B(\overline{y}, \beta)]$, every $a \in (0, c)$ and every $\rho \in (0, \varepsilon]$,

$$B(y, \rho a) \subset F(B(x, \rho)).$$

From an inspection of the proof of this basic result, one observes that the implication $(i) \Rightarrow (ii)$ remains true even outside the class of Asplund spaces (see, for more details, [7]):

- If $\operatorname{Gr} F$ is convex if is enough that X, Y to be Banach spaces.
- If one replace the Fréchet constructions with some other generalized differentiation objects having reasonable good behavior on appropriate classes of spaces.

We look in this paper to the case of the restriction of a single-valued map $f : X \longrightarrow Y$ to a nonempty closed set $M \subset X$. We consider then the set-valued map $F_{f,M} : X \rightrightarrows Y$ given by

$$F_{f,M}(x) = \begin{cases} \{f(x)\}, x \in M \\ \emptyset, x \notin M. \end{cases}$$

In fact, one can write $F_{f,M} = f + \Delta_M$, where Δ_M is the indicator mapping of M,

$$\Delta_M : X \rightrightarrows Y, \Delta_M(x) = \begin{cases} \{0\} \subset Y, x \in M \\ \emptyset, x \notin M. \end{cases}.$$

It is well known (and easy to see) that $\widehat{D}^*\Delta_M(\overline{x}, 0)(y^*) = \widehat{N}(M, \overline{x})$ for every $\overline{x} \in M$ and $y^* \in Y^*$.

Of course, $\operatorname{Gr} F_{f,M} = \operatorname{Gr} f \cap (M \times Y)$ and in order to write condition (2.2) for $F_{f,M}$ we need some calculus rules.

One such calculus is proved in [6, Theorem 4.3]. Let $f : X \to Y$ be calm at $\overline{x}$, i.e. there exist $l > 0$ and a neighborhood U of $\overline{x}$ such that

$$\|f(x) - f(\overline{x})\| \leq l \|x - \overline{x}\| \text{ for every } x \in U.$$

and $G : X \rightrightarrows Y$ be an arbitrary set-valued map such that $(\overline{x}, \overline{y}) \in \operatorname{Gr} G$. Then

$$\hat{D}^*(G - f)(\overline{x}, \overline{y} - f(\overline{x}))(y^*) \subset \bigcap_{x^* \in \hat{D}^* f(\overline{x})(y^*)} [\hat{D}^* G(\overline{x}, \overline{y})(y^*) - x^*] \text{ for all } y^* \in Y^*, \tag{2.3}$$

provided that $\hat{D}^* f(\overline{x})(y^*) \neq \emptyset$. Furthermore, inclusion (2.3) holds as equality if f is Fréchet differentiable at $\overline{x}$. Summing up all the facts already presented, for every $\overline{x} \in M$, if f is Fréchet differentiable at $\overline{x}$,

$$\hat{D}^* F_{f,M}(\overline{x}, f(\overline{x}))(y^*) = \widehat{N}(M, \overline{x}) + \nabla^* f(\overline{x})(y^*), \forall y^* \in Y^*.$$

This formula and Theorem 2.3 allow us to give the following consequence.

Corollary 2.4 *Let X, Y be Asplund spaces, $f : X \longrightarrow Y$ be a Fréchet differentiable function, $M \subset X$ be a closed set and $\overline{x} \in M$. Suppose that the following assumption is satisfied: there exist $c > 0$, $r > 0$ such that for every $x \in M \cap B(\overline{x}, r)$ and every $y^* \in Y^*$, $x^* \in \widehat{N}(M, x) + \nabla^* f(x)(y^*)$,*

$$c \|y^*\| \leq \|x^*\|.$$

Then $F_{f,M}$ is metrically regular around $(\overline{x}, f(\overline{x}))$, that is, there exist $s > 0$, $\mu > 0$ s.t. for every $u \in B(\overline{x}, s) \cap M$ and $v \in B(f(\overline{x}), s)$

$$d(u, f^{-1}(v) \cap M) \leq \mu \|v - f(u)\|.$$

The property displayed in the above conclusion is met in literature under the name of metric regularity of f around $(\overline{x}, f(\overline{x}))$ with respect to M, a terminology we use here as well.

However, we use as well in this paper a weaker condition (see, e.g., [1], [20]): one says that a set-valued map $F : X \rightrightarrows Y$ is metrically subregular at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if there exist $a > 0$ and $U \in \mathcal{V}(\overline{x})$, such that for every $u \in U$

$$d(u, F^{-1}(\overline{y})) \leq ad(\overline{y}, F(u)).$$

With the above notation, we say that the function f is metrically subregular at $(\overline{x}, f(\overline{x}))$ with respect to M $(\overline{x} \in M)$ if $F_{f,M}$ is metrically subregular at $(\overline{x}, f(\overline{x}))$.

More precisely, f is metrically subregular at $(\overline{x}, f(\overline{x}))$ with respect to M $(\overline{x} \in M)$ if there exist $s > 0$, $\mu > 0$ s.t. for every $u \in B(\overline{x}, s) \cap M$

$$d(u, f^{-1}(f(\overline{x})) \cap M) \leq \mu \left\| f(\overline{x}) - f(u) \right\|.$$

Of course, metric regularity around a point implies metric subregularity at that point. For several aspects on the links and the differences between these notions the reader is refered to [14, p. 178] and [3, Section 3H]. For sufficient conditions of subregularity in terms of coderivative one can consult the recent paper [20], but in order to avoid too many technicalities, we use in the sequel the specialization in various particular cases of the condition in Corollary 2.4 which implies metric regularity and whence metric subregularity of the underlying functions.

We introduce some other main definitions we use in the sequel. The objects are standard, but they are used under different names in literature (see, for instance, [2]).

Definition 2.5 *Let D be a nonempty subset of X and $\overline{x} \in X$.*

(i) The first order Bouligand tangent cone to D at $\overline{x}$ is the set

$$T_B(D, \overline{x}) = \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, \forall n \in \mathbb{N}, \overline{x} + t_n u_n \in D\}$$

where $(t_n) \downarrow 0$ means $(t_n) \subset (0, \infty)$ and $(t_n) \to 0$.

(ii) If $x_1 \in X$ the second order Bouligand tangent set to D at $(\overline{x}, x_1)$ is the set

$$T_B^2(D, \overline{x}, x_1) = \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, \forall n \in \mathbb{N}, \overline{x} + t_n x_1 + t_n^2 u_n \in D\}.$$

(iii) The first order Ursescu tangent cone to D at $\overline{x}$ is the set

$$T_U(D, \overline{x}) = \{u \in X \mid \forall (t_n) \downarrow 0, \exists (u_n) \to u, \forall n \in \mathbb{N}, \overline{x} + t_n u_n \in D\}.$$

(iv) If $x_1 \in X$ the second order Ursescu tangent set to D at $(\overline{x}, x_1)$ is the set

$$T_U^2(D, \overline{x}, x_1) = \{u \in X \mid \forall (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \overline{x} + t_n x_1 + t_n^2 u_n \in D\}.$$

Remark 2.6 *For both types of tangent sets ($* \in \{B, U\}$) one has that $T_*^2(D, \overline{x}, x_1) = \emptyset$ if $x_1 \notin T_*(D, \overline{x})$ and, in general, $T_*^2(D, \overline{x}, 0) = T_*(D, \overline{x})$. In the same notation, the sets $T_*(D, \overline{x})$ are closed cones (not necessarily convex) but, in general, $T_*^2(D, \overline{x}, x_1)$ are not cones. Moreover, $T_*(D, \overline{x}) = T_*(\operatorname{cl} D, \overline{x})$ and $T_*^2(D, \overline{x}, x_1) = T_*^2(\operatorname{cl} D, \overline{x}, x_1)$. If $\overline{x} \in \operatorname{int} A$, then, $T_*(D, \overline{x}) = T_*(D \cap A, \overline{x})$. It is also clear that $T_B(D, \overline{x}) \neq \emptyset$ if and only if $\overline{x} \in \operatorname{cl} D$ and, obviously, $T_U(D, \overline{x}) \subset T_B(D, \overline{x})$.*

The next well-known properties (see [2, Tables 4.1 and 4.2]) are important in the sequel.

Proposition 2.7 *Let $D \subset X, E \subset Y$ be closed sets, $x \in D, y \in E, x_1 \in X, y_1 \in Y$ and $f : D \to Y$ be a Fréchet differentiable map.*

(i) Then

$$T_U(D,x) \times T_B(E,y) \subset T_B(D \times E,(x,y)) \subset T_B(D,x) \times T_B(E,y),$$
$$T_U(D,x) \times T_U(E,y) = T_U(D \times E,(x,y))$$

and

$$T_U^2(D,x,x_1) \times T_B^2(E,y,y_1) \subset T_B^2(D \times E,(x,y),(x_1,y_1)) \subset T_B^2(D,x,x_1) \times T_B^2(E,y,y_1).$$

(ii) If $x \in D \cap f^{-1}(E)$ then

$$T_B(D \cap f^{-1}(E),x) \subset T_B(D,x) \cap \nabla f(x)^{-1}(T_B(E,f(x)))$$

and

$$T_U(D \cap f^{-1}(E),x) \subset T_U(D,x) \cap \nabla f(x)^{-1}(T_U(E,f(x))).$$

(iii) If $M \subset X, \overline{x} \in M, x_1 \in X$ and $A : X \to Y$ is a bounded linear operator, then

$$\operatorname{cl} A(T_B(M,\overline{x})) \subset T_B(A(M),A(\overline{x})),$$

and

$$\operatorname{cl} A(T_B^2(M,\overline{x},x_1)) \subset T_B^2(A(M),A(\overline{x}),A(x_1)).$$

3 General calculus of tangent sets and derivatives

Let us now give a calculus result for the above sets. The main line follows the method implemented in [2, Chapter 4]. We formulate now the conditions which ensure the converse inclusions in Proposition 2.7 (ii). These conditions are the more general ones met in the literature.

Theorem 3.1 *Let X,Y be Banach spaces, $D \subset X, E \subset Y$ be closed sets and $f : X \to Y$ be a continuously Fréchet differentiable map and $\overline{x} \in D \cap f^{-1}(E)$. Suppose that $g : X \times Y \longrightarrow Y$, $g(x,y) := f(x) - y$ is metrically subregular at $(\overline{x}, f(\overline{x}), 0)$ with respect to $D \times E$.*
Then

$$T_B(D,\overline{x}) \cap \nabla f(\overline{x})^{-1}(T_U(E,f(\overline{x}))) \subset T_B(D \cap f^{-1}(E),\overline{x})$$
$$T_U(D,\overline{x}) \cap \nabla f(\overline{x})^{-1}(T_B(E,f(\overline{x}))) \subset T_B(D \cap f^{-1}(E),\overline{x})$$
$$T_U(D,\overline{x}) \cap \nabla f(\overline{x})^{-1}(T_U(E,f(\overline{x}))) = T_U(D \cap f^{-1}(E),\overline{x}).$$

If moreover, f is twice continuously differentiable, then for every $x_1 \in X$:

$$T_B^2(D,\overline{x},x_1) \cap \nabla f(\overline{x})^{-1}(T_U^2(E,f(\overline{x}),\nabla f(\overline{x})(x_1)) - 2^{-1}\nabla^2 f(\overline{x})(x_1,x_1)) \subset T_B^2(D \cap f^{-1}(E),\overline{x},x_1),$$
$$T_U^2(D,\overline{x},x_1) \cap \nabla f(\overline{x})^{-1}(T_B^2(E,f(\overline{x}),\nabla f(\overline{x})(x_1)) - 2^{-1}\nabla^2 f(\overline{x})(x_1,x_1)) \subset T_B^2(D \cap f^{-1}(E),\overline{x},x_1),$$
$$T_U^2(D,\overline{x},x_1) \cap \nabla f(\overline{x})^{-1}(T_U^2(E,f(\overline{x}),\nabla f(\overline{x})(x_1)) - 2^{-1}\nabla^2 f(\overline{x})(x_1,x_1)) \subset T_U^2(D \cap f^{-1}(E),\overline{x},x_1).$$

Proof. According to the metric subregularity assumption, there exists $s > 0$, $\mu > 0$ s.t. for every $(x, y) \in [B(\overline{x}, s) \times B(f(\overline{x}), s)] \cap (D \times E)$

$$d((x, y), g^{-1}(0) \cap (D \times E)) \leq \mu \|f(x) + y\| . \tag{3.1}$$

Take $u \in T_B(D, \overline{x}) \cap \nabla f(\overline{x})^{-1}(T_U(E, f(\overline{x})))$, i.e. $u \in T_B(D, \overline{x})$ and $\nabla f(\overline{x})(u) \in T_U(E, f(\overline{x}))$. Then there exist $(t_n) \downarrow 0, (u_n) \to u$, $v_n \to \nabla f(x)(u)$ with $\overline{x} + t_n u_n \in D$ and $f(\overline{x}) + t_n v_n \in E$ for all n large enough. Then one can apply (3.1) for every pair $(u, v) = (\overline{x} + t_n u_n, f(\overline{x}) + t_n v_n)$. Then for every n large enough, there exists $(p_n, q_n) \in D \times E$ with $f(p_n) = q_n$ and

$$\|(\overline{x} + t_n u_n, f(\overline{x}) + t_n v_n) - (p_n, q_n)\| < \mu \|f(\overline{x} + t_n u_n) - f(\overline{x}) - t_n v_n\| + t_n^2.$$

Then for every n as above, $p_n \in D \cap f^{-1}(E)$ and

$$\|\overline{x} + t_n u_n - p_n\| < \mu \|f(\overline{x} + t_n u_n) - f(\overline{x}) - t_n v_n\| + t_n^2$$

whence

$$\left\|t_n^{-1}(p_n - \overline{x}) - u_n\right\| < \mu \left\|t_n^{-1}[f(\overline{x} + t_n u_n) - f(\overline{x})] - v_n\right\| + t_n.$$

Since $t_n^{-1}[f(\overline{x} + t_n u_n) - f(\overline{x})] \stackrel{n\to\infty}{\longrightarrow} \nabla f(\overline{x})(u)$, we infer that $u_n' := t_n^{-1}(p_n - \overline{x}) \to u$ which allows us to conclude the proof of the first inclusion of the theorem. Now, the other two first-order relations are similar. Notice that for the equality in the third relation one take into account Proposition 2.7 (ii).

The proof of the second-order relations is similar. Nevertheless, we illustrate it with the first inclusion. Take $u \in T_B^2(D, \overline{x}, x_1) \cap \nabla f(\overline{x})^{-1}(T_U^2(E, f(\overline{x}), \nabla f(\overline{x})(x_1)) - 2^{-1}\nabla^2 f(\overline{x})(x_1, x_1))$, i.e. $u \in T_B^2(D, \overline{x}, x_1)$ and $\nabla f(\overline{x})(u) + 2^{-1}\nabla^2 f(\overline{x})(x_1, x_1) \in T_U^2(E, f(\overline{x}), \nabla f(\overline{x})(x_1))$. Then there exist $(t_n) \downarrow 0, (u_n) \to u$ s.t. $\overline{x} + t_n x_1 + t_n^2 u_n \in D$ and $v_n \to \nabla f(\overline{x})(u) + 2^{-1}\nabla^2 f(\overline{x})(x_1, x_1)$ with $f(\overline{x}) + t_n \nabla f(\overline{x})(x_1) + t_n^2 v_n \in E$ for all n large enough. Again, one applies (3.1) for the pairs $(\overline{x} + t_n x_1 + t_n^2 u_n, f(\overline{x}) + t_n \nabla f(\overline{x})(x_1) + t_n^2 v_n)$. For all n large enough, there exists $(p_n, q_n) \in D \times E$ with $f(p_n) = q_n$ and

$$\begin{aligned}&\left\|(\overline{x} + t_n x_1 + t_n^2 u_n, f(\overline{x}) + t_n \nabla f(\overline{x})(x_1) + t_n^2 v_n) - (p_n, q_n)\right\| \\ &< \mu \left\|f(\overline{x} + t_n x_1 + t_n^2 u_n) - f(\overline{x}) - t_n \nabla f(\overline{x})(x_1) - t_n^2 v_n\right\| + t_n^3.\end{aligned}$$

Then $p_n \in D \cap f^{-1}(E)$ and

$$\begin{aligned}&\left\|t_n^{-2}(p_n - \overline{x} - t_n x_1) - u_n\right\| \\ &< \mu \left\|t_n^{-2}[f(\overline{x} + t_n x_1 + t_n^2 u_n) - f(\overline{x}) - t_n \nabla f(\overline{x})(x_1)] - v_n\right\| + t_n.\end{aligned}$$

Since $t_n^{-2}[f(\overline{x} + t_n x_1 + t_n^2 u_n) - f(\overline{x}) - t_n \nabla f(\overline{x})(x_1)] \to \nabla f(\overline{x})(u) + 2^{-1}\nabla^2 f(\overline{x})(x_1, x_1)$ we get that $u_n' := t_n^{-2}(p_n - \overline{x} - t_n x_1) \to u$ and $\overline{x} + t_n x_1 + t_n^2 u_n' = p_n \in D \cap f^{-1}(E)$.

Then the proof concludes here. □

Remark 3.2 *Following Corollary 2.4, a sufficient condition for the metric (sub)regularity assumption in Theorem 3.1 could be written down (on Asplund spaces) as follows: there exist $c > 0$, $r > 0$ such that for every $(x, y) \in (D \times E) \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$ and every $y^* \in Y^*$, $(u^*, v^*) \in \widehat{N}(D, x) \times \widehat{N}(E, y) + (\nabla^* f(x)(y^*), -y^*)$*

$$c \|y^*\| \leq \|(u^*, v^*)\| .$$

Definition 3.3 *Let $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. The first order Bouligand derivative of F at $(\overline{x},\overline{y})$ is the set valued map $D_B F(\overline{x},\overline{y})$ from X into Y defined by*

$$\operatorname{Gr} D_B F(\overline{x},\overline{y}) = T_B(\operatorname{Gr} F, (\overline{x},\overline{y})),$$

and if $(x_1,y_1) \in X \times Y$, the second order Bouligand derivative of F at $(\overline{x},\overline{y})$ with respect to (x_1,y_1) is the set valued map $D_B^2 F((\overline{x},\overline{y}),(x_1,y_1))$ from X into Y defined by

$$\operatorname{Gr} D_B^2 F((\overline{x},\overline{y}),(x_1,y_1)) = T_B^2(\operatorname{Gr} F, (\overline{x},\overline{y}),(x_1,y_1)).$$

Now the first and second order Ursescu derivative has similar definition.

The first part of the following definition was introduced by J.-P. Penot [17].

Definition 3.4 *Let $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. (i) The Dini lower derivative of F at $(\overline{x},\overline{y})$ is the multifunction $D_D F(\overline{x},\overline{y})$ from X into Y given, for every $u \in X$, by*

$$\begin{aligned} D_D F(\overline{x},\overline{y})(u) = \{ v \in Y \mid \forall (t_n) \downarrow 0, \forall (u_n) \to u, \exists (v_n) \to v, \exists n_0 \in \mathbb{N}, \\ \forall n \geq n_0, \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n) \}. \end{aligned}$$

(ii) If $(x_1,y_1) \in X \times Y$, the second order Dini lower derivative of F at $(\overline{x},\overline{y})$ with respect to (x_1,y_1) is the multifunction $D_D^2 F(\overline{x},\overline{y})(x_1,y_1)$ from X into Y given, for every $u \in X$, by

$$\begin{aligned} D_D^2 F((\overline{x},\overline{y}),(x_1,y_1))(u) = \{ v \in Y \mid \forall (t_n) \downarrow 0, \forall (u_n) \to u, \exists (v_n) \to v, \exists n_0 \in \mathbb{N}, \\ \forall n \geq n_0, \overline{y} + t_n y_1 + t_n^2 v_n \in F(\overline{x} + t_n x_1 + t_n^2 u_n) \}. \end{aligned}$$

Remark 3.5 *Obviously, the next inclusions are true for all $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, $(x_1,y_1) \in X \times Y$ and $u \in X$:*

$$\begin{aligned} D_D F(\overline{x},\overline{y})(u) \subset D_U F(\overline{x},\overline{y})(u) \subset D_B F(\overline{x},\overline{y})(u), \\ D_D^2 F((\overline{x},\overline{y}),(x_1,y_1))(u) \subset D_U^2 F((\overline{x},\overline{y}),(x_1,y_1))(u) \subset D_B^2 F((\overline{x},\overline{y}),(x_1,y_1))(u) \end{aligned}$$

One says that a set A is derivable at a point $\overline{x} \in A$ if $T_B(A,\overline{x}) = T_U(A,\overline{x})$ (see [2]). Similarly, one says that a set-valued map that F is derivable (terminology of [2]) or proto-differentiable (terminology of [18]) at $\overline{x}$ relative to $\overline{y} \in F(\overline{x})$ if its graph is derivable at $(\overline{x},\overline{y})$, i.e. $D_U F(\overline{x},\overline{y}) = D_B F(\overline{x},\overline{y})$. One says that F is semi-differentiable at $\overline{x}$ relative to $\overline{y} \in F(\overline{x})$ if $D_D F(\overline{x},\overline{y}) = D_B F(\overline{x},\overline{y})$. It is clear that semi-differentiability implies proto-differentiability. The next result is easy to prove.

Proposition 3.6 *Suppose that F has the Aubin property around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and $u \in X$. Then:*

$$\begin{aligned} D_D F(\overline{x},\overline{y})(u) = D_U F(\overline{x},\overline{y})(u) = \{ v \in Y \mid \forall (t_n) \downarrow 0, \exists (v_n) \to v, \forall n \in \mathbb{N}, \\ \overline{y} + t_n v_n \in F(\overline{x} + t_n u) \}. \end{aligned}$$

Hence, if the set-valued map $F : X \rightrightarrows Y$ has the Aubin property around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ and it is proto-differentiable at $\overline{x}$ relative to $\overline{y}$, then it is semi-differentiable at $\overline{x}$ relative to $\overline{y}$.

However, as shown in [4], semi-differentiability is a quite strong assumption which ensures some simple results concerning the derivative calculus of the sum of set-valued maps. More explicitly, if $F_1, F_2 : X \rightrightarrows Y$ are set-valued maps, for the sum $F_1 + F_2 : X \rightrightarrows Y$ given by

$$\begin{aligned} (F_1 + F_2)(x) &= F_1(x) + F_2(x) \\ &= \{ y \in Y \mid \exists y_1 \in F_1(x), \exists y_2 \in F_2(x), y = y_1 + y_2 \}, \end{aligned}$$

one can easily prove (see [4]) that for $(\overline{x}, y_1) \in \operatorname{Gr} F_1$, $(\overline{x}, y_2) \in \operatorname{Gr} F_2$ if either F_1 is semi-differentiable at $\overline{x}$ relative to $\overline{y}_1$ or F_2 is semi-differentiable at $\overline{x}$ relative to $\overline{y}_2$, then, for every $u \in X$,

$$D_B F_1(\overline{x}, \overline{y}_1)(u) + D_B F_2(\overline{x}, y_2)(u) \subset D_B(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2)(u)$$

and

$$D_U F_1(\overline{x}, \overline{y}_1)(u) + D_U F_2(\overline{x}, \overline{y}_2)(u) \subset D_U(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2)(u).$$

Despite the fact that it is a quite heavy assumption, this concept of semi-differentiability is employed (in conjunction with some compactness requirements) as the basic ingredient in several recent papers concerning the calculus rules for both first and second order derivatives: see, for instance [11], [12] and the references therein. We prefer here to avoid such assumptions and to work with the far more natural hypotesis of proto-differentiability.

Before passing to the main results, we give some similar definitions for second-order objects: one says that A is second-order derivable at $\overline{x}$ in the direction x_1 if $T_B^2(A, \overline{x}, x_1) = T_U^2(A, \overline{x}, x_1)$, and one says that F is second-order proto-differentiable at $\overline{x}$ relative to $\overline{y} \in F(\overline{x})$ in the direction (x_1, y_1) if $D_B^2 F((\overline{x}, \overline{y}), (x_1, y_1)) = D_U^2 F((\overline{x}, \overline{y}), (x_1, y_1))$.

Let $F : X \rightrightarrows Y$ be a set-valued map and $f : X \to X$ be a single-valued map. Then we denote by $F \circ f$ the set-valued map from X to Y given by $(F \circ f)(x) = F(f(x))$ for every $x \in X$. We are ready to present the first main result of the paper. The proof is based on a transformation method which allows us to reduce the calculus of the involved objects to the pattern of Theorem 3.1.

Theorem 3.7 *Let $F_1, F_2 : X \rightrightarrows Y$ be set-valued maps with closed graph, $f : X \to X$ be a continuously differentiable single-valued map and $(\overline{x}, \overline{y}_1) \in \operatorname{Gr} F_1$, $(\overline{x}, \overline{y}_2) \in \operatorname{Gr}(F_2 \circ f)$. Suppose that the function $g : (X \times Y)^2 \longrightarrow X$, $g(\alpha, \beta, \gamma, \delta) = f(\alpha) - \gamma$ is metrically subregular at $(\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2, 0_X)$ with respect to $\operatorname{Gr} F_1 \times \operatorname{Gr} F_2$.*

(i) If either F_1 is proto-differentiable at $\overline{x}$ relative to $\overline{y}_1$ or F_2 is proto-differentiable at $f(\overline{x})$ relative to $\overline{y}_2$, then, for every $u \in X$,

$$D_B F_1(\overline{x}, \overline{y}_1)(u) + D_B F_2(f(\overline{x}), \overline{y}_2)(\nabla f(\overline{x})(u)) \subset D_B(F_1 + F_2 \circ f)(\overline{x}, \overline{y}_1 + \overline{y}_2)(u).$$

(ii) Let $x \in X$ and $y_1, y_2 \in Y$. If f is linear and either F_1 is second-order proto-differentiable at $\overline{x}$ relative to $\overline{y}_1$ in the direction (x, y_1) or F_2 is second-order proto-differentiable at $\overline{x}$ relative to $\overline{y}_2$ in the direction $(f(x), y_2)$, then, for every $u \in X$,

$$\begin{aligned}&D_B^2 F_1((\overline{x}, \overline{y}_1), (x, y_1))(u) + D_B^2 F_2((f(\overline{x}), \overline{y}_2), (f(x), y_2))(\nabla f(\overline{x})(u)) \\ &\subset D_B^2(F_1 + F_2 \circ f)((\overline{x}, \overline{y}_1 + \overline{y}_2), (f(x), y_1 + y_2))(u).\end{aligned}$$

Proof. (i) Let us consider the following auxiliary functions:

$$\varphi : (X \times Y)^2 \to X \times Y, \quad \varphi(\alpha, \beta, \gamma, \delta) = (\alpha, \beta + \delta)$$

and

$$\psi : (X \times Y)^2 \to X, \quad \psi(\alpha, \beta, \gamma, \delta) = f(\alpha) - \gamma.$$

It is not difficult to see that $\mathrm{Gr}(F_1+F_2\circ f)=\varphi\left((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0_X)\right)$ because

$$\begin{aligned}\varphi\left((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0_X)\right)&=\varphi\left((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\{(x,y,z,t)\in(X\times Y)^2\mid f(x)=z\}\right)\\&=\varphi(\{(x,y,f(x),t)\mid y\in F_1(x),t\in F_2(f(x))\})\\&=\{(x,y+t)\mid x\in X,y+t\in(F_1+F_2\circ f)(x)\}\\&=\mathrm{Gr}(F_1+F_2\circ f).\end{aligned}$$

The linearity of φ and Proposition 2.7 (iii), ensures:

$$\begin{aligned}\mathrm{Gr}\,D_B(F_1+F_2\circ f)(\overline{x},\overline{y}_1+\overline{y}_2)&=T_B(\mathrm{Gr}(F_1+F_2\circ f),(\overline{x},\overline{y}_1+\overline{y}_2))\\&=T_B(\varphi\left((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0_X)\right),\varphi(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2))\\&\supset\mathrm{cl}\left(\varphi(T_B((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0_X),(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2)))\right).\end{aligned}$$

Our hypotheses allow us to apply Theorem 3.1 for $D:=\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2$, $E:=\{0_X\}$ and $f:=\psi$. Because of the very particular form of E, one observes that the assumption on g ensures that $h:(X\times Y\times X\times Y)\times X\longrightarrow X$, $h(\alpha,\beta,\gamma,\delta,\varepsilon)=f(\alpha)-\gamma-\varepsilon$ is submetrically regular at $(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2,0_X,0_X)$ with respect to $(\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\times\{0_X\}$, so we can indeed specialize Theorem 3.1 to the case described above. Taking into account the equality $T_U(\{0_X\},0_X)=T_B(\{0_X\},0_X)=\{0_X\}$, we successively have

$$\begin{aligned}T_B((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0_X),(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2))&\supset T_B(\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2,(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2))\\&\quad\cap\nabla\psi(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2)^{-1}(T_U(\{0_X\},0_X))\\&=T_B(\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2,(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2))\\&\quad\cap\nabla\psi(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2)^{-1}(0_X).\end{aligned}$$

From Proposition 2.7 (i) and the proto-differentiability assumption, we get the following chain of inclusions:

$$\begin{aligned}&\mathrm{Gr}\,D_B(F_1+F_2\circ f)(\overline{x},\overline{y}_1+\overline{y}_2)\\&\supset\varphi(T_B(\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2,(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2))\cap\nabla\psi(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2)^{-1}(0_X))\\&=\varphi\left(T_B(\mathrm{Gr}\,F_1,(\overline{x},\overline{y}_1))\times T_B(\mathrm{Gr}\,F_2,(f(\overline{x}),\overline{y}_2))\cap\nabla\psi(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2)^{-1}(0_X)\right)\\&=\varphi(\{(u,v,w,p)\in(X\times Y)^2\mid(u,v)\in\mathrm{Gr}\,D_BF_1(\overline{x},\overline{y}_1),\\&\qquad(w,p)\in\mathrm{Gr}\,D_BF_2(f(\overline{x}),\overline{y}_2),\nabla f(\overline{x})(u)=w\})\\&=\{(u,v+p)\mid(u,v)\in\mathrm{Gr}\,D_BF_1(\overline{x},\overline{y}_1),(w,p)\in\mathrm{Gr}\,D_BF_2(f(\overline{x}),\overline{y}_2),\nabla f(\overline{x})(u)=w\}\\&=\mathrm{Gr}(D_BF_1(\overline{x},\overline{y}_1)(\cdot)+D_BF_2(f(\overline{x}),\overline{y}_2)(\nabla f(\overline{x})(\cdot))).\end{aligned}$$

The proof of the first-order calculus rule is complete.

(ii) For the second part,

$$\begin{aligned}&\mathrm{Gr}\,D_B^2(F_1+F_2\circ f)((\overline{x},\overline{y}_1+\overline{y}_2),(x,y_1+y_2))\\&=T_B^2(\mathrm{Gr}(F_1+F_2\circ f),(\overline{x},\overline{y}_1+\overline{y}_2),(x,y_1+y_2))\\&=T_B^2(\varphi\left((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0)\right),\varphi(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2),\varphi(x,y_1,f(x),y_2))\\&\supset\varphi\left(T_B^2((\mathrm{Gr}\,F_1\times\mathrm{Gr}\,F_2)\cap\psi^{-1}(0),(\overline{x},\overline{y}_1,f(\overline{x}),\overline{y}_2),(x,y_1,f(x),y_2))\right).\end{aligned}$$

Now, we apply Theorem 3.1 for the same data as before. Taking into account the equality $T_U^2(\{0_X\}, 0_X, 0_X) = T_B^2(\{0_X\}, 0_X, 0_X)) = \{0_X\}$ and because $\nabla f(\overline{x})(x) = f(x)$ (since f is linear), we successively have

$$\begin{aligned}
&T_B^2(\operatorname{Gr} F_1 \times \operatorname{Gr} F_2) \cap \psi^{-1}(0_X), (\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2), (x, y_1, f(x), y_2)) \\
&\supset T_B^2(\operatorname{Gr} F_1 \times \operatorname{Gr} F_2, (\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2), (x, y_1, f(x), y_2)) \\
&\cap \nabla\psi(\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2)^{-1}(0_X).
\end{aligned}$$

Therefore, from the second-order proto-differentiability condition,

$$\begin{aligned}
&\operatorname{Gr} D_B^2(F_1 + F_2 \circ f)((\overline{x}, \overline{y}_1 + \overline{y}_2), (x, y_1 + y_2)) \\
&\supset \varphi\left(T_B^2(\operatorname{Gr} F_1 \times \operatorname{Gr} F_2, (\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2), (x, y_1, f(x), y_2)) \cap \nabla\psi(\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2)^{-1}(0_X)\right) \\
&= \varphi\left(T_B^2(\operatorname{Gr} F_1, (\overline{x}, \overline{y}_1), (x, y_1)) \times T_B^2(\operatorname{Gr} F_2, (f(\overline{x}), \overline{y}_2), (f(x), y_2)\right) \cap \nabla\psi(\overline{x}, \overline{y}_1, f(\overline{x}), \overline{y}_2)^{-1}(0_X)) \\
&= \varphi(\{(u, v, w, p) \in (X \times Y)^2 \mid (u, v) \in \operatorname{Gr} D_B^2 F_1((\overline{x}, \overline{y}_1), (x, y_1)), \\
&\qquad\qquad (w, p) \in \operatorname{Gr} D_B^2 F_2((f(\overline{x}), \overline{y}_2)(f(x), y_2)), \nabla f(\overline{x})(u) = w\} \\
&= \{(u, v + p) \mid (u, v) \in \operatorname{Gr} D_B^2 F_1((\overline{x}, \overline{y}_1), (x, y_1)), \\
&\qquad\qquad (w, p) \in \operatorname{Gr} D_B^2 F_2((f(\overline{x}), \overline{y}_2), (f(x), y_2)), \nabla f(\overline{x})(u) = w\} \\
&= \operatorname{Gr}(D_B^2 F_1((\overline{x}, \overline{y}_1), (x, y_1))(\cdot) + D_B^2 F_2((f(\overline{x}), \overline{y}_2), (f(x), y_2))(\nabla f(\overline{x})(\cdot))).
\end{aligned}$$

Consequently, the final formula is proved. □

Remark 3.8 *Use again Corollary 2.4 in order to get a sufficient condition for the metric (sub)regularity assumption in Theorem 3.7 (on Asplund spaces): there exist $c > 0$, $r > 0$ such that for every $(x_1, y_1, x_2, y_2) \in (\operatorname{Gr} F_1 \times \operatorname{Gr} F_2) \cap [B(\overline{x}, r) \times B(\overline{y}, r) \times B(f(\overline{x}), r) \times B(\overline{y}, r)]$ and every $x^* \in X^*$, $(u_1^*, v_1^*, u_2^*, v_2^*) \in \widehat{N}(\operatorname{Gr} F_1, (x_1, y_1)) \times \widehat{N}(\operatorname{Gr} F_2, (x_2, y_2)) + (\nabla^* f(x_1)(x^*), 0_{Y^*}, -x^*, 0_{Y^*})$*

$$c\,\|x^*\| \leq \|(u_1^*, v_1^*, u_2^*, v_2^*)\|.$$

As one can see, we do not obtain equalities in Theorem 3.7, but we do not use any semi-differentiability or other compactness-like assumptions (compare with [12, Proposition 2]). Nevertheless, we would like to emphasize that the inclusions in Theorem 3.7 are exactly those needed in order to deduce optimality conditions for some special types of vector optimization problems in terms of the initial data. For instance, take $F_1, F_2 : X \rightrightarrows Y$ set-valued maps and $C \subset Y$ a closed convex cone with nonempty interior which, as usual, gives a partial order relation on Y. Consider the general vector optimization problem

$$(P) \qquad\qquad \min(F_1 + F_2) \text{ s.t. } x \in X.$$

A point $(\overline{x}, \overline{y}) \in \operatorname{Gr}(F_1 + F_2)$ is called a weak Pareto solution of (P) if

$$((F_1 + F_2)(X) - \overline{y}) \cap (-\operatorname{int} C) = \emptyset.$$

For several motivations and comments on the minimizing the sum (or difference) of two (single or set-valued) mappings we refer to the recent works [8], [9]. Theorem 3.7 allows us to formulate a necessary optimality condition for (P).

Theorem 3.9 *Let $F_1, F_2 : X \rightrightarrows Y$, be set-valued maps with closed graph, $(\overline{x}, \overline{y}_1) \in \operatorname{Gr} F_1$, $(\overline{x}, \overline{y}_2) \in \operatorname{Gr} F_2$ and let $(\overline{x}, \overline{y}_1 + \overline{y}_2)$ be a weak Pareto solution of (P). Suppose that the function $g : (X \times Y)^2 \to X$, $g(\alpha, \beta, \gamma, \delta) = \alpha - \gamma$ is metrically subregular at $(\overline{x}, \overline{y}_1, \overline{x}, \overline{y}_2, 0_X)$ with respect to $\operatorname{Gr} F_1 \times \operatorname{Gr} F_2$.*

(i) If either F_1 is proto-differentiable at $\overline{x}$ relative to $\overline{y}_1$ or F_2 is proto-differentiable at $\overline{x}$ relative to $\overline{y}_2$ then, for every $u \in X$,

$$[D_B F_1(\overline{x}, \overline{y}_1)(u) + D_B F_2(\overline{x}, \overline{y}_2)(u)] \cap (-\operatorname{int} C) = \emptyset.$$

(ii) Let $x \in X$ and $y_1, y_2 \in Y$ with $y_1 + y_2 \in -C$. If either F_1 is second-order proto-differentiable at $\overline{x}$ relative to $\overline{y}_1$ in the direction (x, y_1) or F_2 is second-order proto-differentiable at $\overline{x}$ relative to $\overline{y}_2$ in the direction (x, y_2) then for every $u \in X$,

$$\left[D_B^2 F_1((\overline{x}, \overline{y}_1)(x, y_1))(u) + D_B^2 F_2((\overline{x}, \overline{y}_2), (x, y_2))(u)\right] \cap (-\operatorname{int} C) = \emptyset.$$

Proof. Since $(\overline{x}, \overline{y}_1 + \overline{y}_2)$ is a weak Pareto solution of (P), then, following [10, Chapter 3] or [4, Lemma 3.4],

$$T_B((F_1 + F_2)(X), \overline{y}_1 + \overline{y}_2) \cap (-\operatorname{int} C) = \emptyset$$

and for any $z \in -C$

$$T_B^2((F_1 + F_2)(X), \overline{y}_1 + \overline{y}_2, z) \cap (-\operatorname{int} C) = \emptyset.$$

But it is easy to see that for any $u \in X$,

$$D_B(F_1 + F_2)(\overline{x}, \overline{y}_1 + \overline{y}_2)(u) \subset T_B((F_1 + F_2)(X), \overline{y}_1 + \overline{y}_2)$$

and for any $u \in X$, $(x_1, y_1, y_2) \in X \times Y \times Y$,

$$D_B^2(F_1 + F_2)((\overline{x}, \overline{y}_1 + \overline{y}_2), (x_1, y_1 + y_2))(u) \subset T_B^2((F_1 + F_2)(X), \overline{y}_1 + \overline{y}_2, y_1 + y_2).$$

Summing up these facts and applying as well Theorem 3.7, with the identity map instead of f, one gets the conclusions. $\square$

Moreover, we would like to notice that the nature of our conditions of metric subregularity in Theorem 3.7 are very different from the semi-differentiability conditions imposed in [12, Proposition 2], [11] and [13, Proposition 2.1]. The next example puts the accent on these differences.

Example 3.10 Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$F(x) = \begin{cases} \emptyset, x = \frac{1}{n}, n \in \mathbb{N} \setminus \{0\} \\ x, \text{ otherwise.} \end{cases}$$

where $\mathbb{R}$ and $\mathbb{N}$ denotes the sets of real numbers and natural numbers, respectively. It is easy to observe that $D_B F(0,0)(u) = D_U F(0,0)(u) = \{u\}$ for any $u \in \mathbb{R}$ but $D_D F(0,0)(0) = \emptyset$, whence F is proto-differentiable but not semi-differentiable at $0_{\mathbb{R}}$ relative to $0_{\mathbb{R}}$. Consider now $F_1 = F_2 := F$. The metric subregularity condition in our Theorem 3.7 is nevertheless fulfilled. Moreover, even the metric regularity around the reference point holds. To see this, take $s > 0$, $p \in (-s, s)$, $\alpha, \gamma \in (-s, s)$. It is enough to show that there is $\mu > 0$ (independent of the previous data) s.t.

$$\inf\{d((\alpha, \alpha, \gamma, \gamma), (a, a, c, c)) \mid a - c = p\} \leq \mu \left|p - (\alpha - \gamma)\right|$$

i.e.

$$2 \inf\{|\alpha - a| + |(\gamma + p) - a| \mid a \in \mathbb{R}\} \leq \mu \left|(\gamma + p) - \alpha\right|$$

and this is clearly true for any $\mu \geq 2$.

4 Calculus of derivatives of perturbation maps

The same technique as in the above section is applied now in order to get calculus for derivatives of generalized perturbation map. Let X, Y, Z be Banach spaces, $F, K : X \times Y \rightrightarrows Z$ be set-valued maps. Define (see [13], [12]) $G : X \times Z \rightrightarrows Y$ the set-valued map given by

$$G(x,z) = \{y \in Y \mid z \in F(x,y) + K(x,y)\}.$$

We report now a result concerning a calculus rule for the Bouligand derivatives of G. Observe first that for $(x,z,y) \in \operatorname{Gr} G$, there exist $q_z \in F(x,y)$ and $t_z \in K(x,y)$ s.t. $z = q_z + t_z$. We use this notation in the sequel.

Theorem 4.1 *In the above notations, suppose that the graphs of F and K are closed, $(\overline{x},\overline{z},\overline{y}) \in \operatorname{Gr} G$ and the function $i : (X \times Y \times Z)^2 \to X \times Y$, $i(x,y,z,u,v,t) = (x-u, y-v)$ is metrically subregular at $(\overline{x},\overline{y},q_{\overline{z}},\overline{x},\overline{y},t_{\overline{z}},0_X,0_Y)$ with respect to $\operatorname{Gr} F \times \operatorname{Gr} K$.*

(i) Then, for every $(u,w) \in X \times Z$,

$$\{v \in Y \mid w \in D_B F(\overline{x},\overline{y},q_{\overline{z}})(u,v) + D_B K(\overline{x},\overline{y},t_{\overline{z}})(u,v)\} \subset D_B G(\overline{x},\overline{z},\overline{y})(u,w),$$

(ii) Let $(u,v,t) \in X \times Y \times Z$ with $t = t_1 + t_2$ $(t_1,t_2 \in Z)$ and suppose that either F is second-order proto-differentiable at $(\overline{x},\overline{y})$ relative to $q_{\overline{z}}$ in the direction (u,v,t_1) or K is second-order proto-differentiable at $(\overline{x},\overline{y})$ relative to $t_{\overline{z}}$ in the direction (u,v,t_2). Then for any $(\alpha,\beta) \in X \times Y$ one has

$$\begin{aligned}&\gamma_1 \in D_B^2 F((\overline{x},\overline{y},q_{\overline{z}}),(u,v,t_1))(\alpha,\beta), \gamma_2 \in D_B^2 F((\overline{x},\overline{y},q_{\overline{z}}),(u,v,t_1))(\alpha,\beta)\\ &\quad \Rightarrow \beta \in \operatorname{Gr} D_B^2 G(\overline{x},\overline{z},\overline{y})(u,t,v)(\alpha,\gamma_1+\gamma_2).\end{aligned}$$

Proof. (i) Take

$$\varphi : (X \times Y \times Z)^2 \to X \times Z \times Y, \quad \varphi(x,y,z,u,v,t) = (x, z+t, y)$$

and

$$\psi : (X \times Y \times Z)^2 \to X \times Y, \quad \psi(x,y,z,u,v,t) = (x-u, y-v).$$

Then

$$\begin{aligned}&\varphi((\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X,0_Y))\\ &= \varphi(\{(x,y,z,u,v,t) \mid (x,y,z) \in \operatorname{Gr} F, (u,v,t) \in \operatorname{Gr} K, x = u, y = v\})\\ &= \varphi(\{(x,y,z,x,y,t) \mid (x,y,z) \in \operatorname{Gr} F, (x,y,t) \in \operatorname{Gr} K\})\\ &= \{(x,z+t,y) \mid z \in F(x,y), t \in K(x,y)\}\\ &= \{(x,w,y) \mid w \in F(x,y) + K(x,y)\} = \operatorname{Gr} G.\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{Gr} D_B G(\overline{x},\overline{z},\overline{y}) &= T_B(\operatorname{Gr} G, (\overline{x},\overline{z},\overline{y}))\\ &= T_B(\varphi((\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X,0_Y)), (\overline{x}, q_{\overline{z}} + t_{\overline{z}}, \overline{y}))\\ &= T_B(\varphi((\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X,0_Y)), \varphi(\overline{x},\overline{y},q_{\overline{z}},\overline{x},\overline{y},t_{\overline{z}}))\\ &\supset \operatorname{cl}\left(\varphi(T_B((\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X,0_Y), (\overline{x},\overline{y},q_{\overline{z}},\overline{x},\overline{y},t_{\overline{z}})))\right).\end{aligned}$$

Using the metric subregularity assumption, the function $j : (X \times Y \times Z)^2 \times (X \times Y) \to X \times Y$, $j(x, y, z, u, v, t, p, q) = (x-u-p, y-v-q)$ is metrically subregular at $(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}}, 0_X, 0_Y, 0_X, 0_Y)$ with respect to $(\operatorname{Gr} F \times \operatorname{Gr} K) \times \{(0_X, 0_Y)\}$ whence applying Theorem 3.1 one has:

$$\begin{aligned}
&T_B((\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X, 0_Y), (\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})) \\
&\supset T_B(\operatorname{Gr} F \times \operatorname{Gr} K, (\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})) \cap \nabla\psi(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})^{-1}(0_X, 0_Y)).
\end{aligned}$$

We employ now the proto-differentiability assumption and, accordingly, one has

$$\begin{aligned}
\operatorname{Gr} D_B G(\overline{x}, \overline{z}, \overline{y}) &\supset \varphi((T_B(\operatorname{Gr} F, (\overline{x}, \overline{y}, q_{\overline{z}})) \times T_B(\operatorname{Gr} K, (x, y, t_{\overline{z}}))) \cap \nabla\psi(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})^{-1}(0_X, 0_Y)) \\
&= \varphi(\{(u, v, p, r, s, j) \mid p \in D_B F(\overline{x}, \overline{y}, q_{\overline{z}})(u, v), j \in D_B K(\overline{x}, \overline{y}, t_{\overline{z}})(r, s), u = r, v = s\}) \\
&= \{(u, p + j, v) \mid p \in D_B F(\overline{x}, \overline{y}, q_{\overline{z}})(u, v), j \in D_B K(\overline{x}, \overline{y}, t_{\overline{z}})(u, v)\}.
\end{aligned}$$

We conclude that

$$\{v \in Y \mid w \in D_B F(\overline{x}, \overline{y}, q_{\overline{z}})(u, v) + D_B K(\overline{x}, \overline{y}, t_{\overline{z}})(u, v)\} \subset D_B G(\overline{x}, \overline{z}, \overline{y})(u, w),$$

hence the first conclusion.

(ii) For the second part, we firstly write

$$\begin{aligned}
&\operatorname{Gr} D_B^2 G(\overline{x}, \overline{z}, \overline{y})(u, t, v) = \\
&= T_B^2(\operatorname{Gr} G, (\overline{x}, \overline{z}, \overline{y})(u, t, v)) \\
&= T_B^2(\varphi((\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X, 0_Y)), \varphi(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}}), \varphi(u, v, t_1, u, v, t_2)) \\
&\supset \varphi\left(T_B^2(\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X, 0_Y), (\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_1, u, v, t_2))\right).
\end{aligned}$$

By similar arguments,

$$\begin{aligned}
&T_B^2(\operatorname{Gr} F \times \operatorname{Gr} K) \cap \psi^{-1}(0_X, 0_Y), (\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_1, u, v, t_2)) \\
&\supset T_B^2(\operatorname{Gr} F \times \operatorname{Gr} K, (\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_1, u, v, t_2)) \\
&\cap \nabla\psi(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})^{-1}(0_X, 0_Y).
\end{aligned}$$

The second-order proto-differentiability condition ensures

$$\begin{aligned}
&\operatorname{Gr} D_B^2 G(\overline{x}, \overline{z}, \overline{y})(u, t, v) \\
&\supset \varphi\left(T_B^2(\operatorname{Gr} F \times \operatorname{Gr} K, (\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_1, u, v, t_2)) \cap \nabla\psi(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})^{-1}(0_X, 0_Y)\right) \\
&= \varphi\left(T_B^2(\operatorname{Gr} F, (\overline{x}, \overline{y}, q_{\overline{z}}), (u, v, t_1)) \times T_B^2(\operatorname{Gr} K, (\overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_2)\right) \\
&\qquad \cap \nabla\psi(\overline{x}, \overline{y}, q_{\overline{z}}, \overline{x}, \overline{y}, t_{\overline{z}})^{-1}(0_X, 0_Y)) \\
&= \varphi(\{(a, b, c, d, e, f) \in (X \times Y \times Z)^2 \mid (a, b, c) \in \operatorname{Gr} D_B^2 F((\overline{x}, \overline{y}, q_{\overline{z}}), (u, v, t_1)), \\
&\qquad (d, e, f) \in \operatorname{Gr} D_B^2 K((\overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_2)), a = d, b = e\} \\
&= \{(a, c + f, b) \mid (a, b, c) \in \operatorname{Gr} D_B^2 F((\overline{x}, \overline{y}, q_{\overline{z}}), (u, v, t_1)), \\
&\qquad (a, b, f) \in \operatorname{Gr} D_B^2 K((\overline{x}, \overline{y}, t_{\overline{z}}), (u, v, t_2))\}.
\end{aligned}$$

Consequently, the second part of the conclusion easily follows. □

Remark 4.2 *On the basis of Corollary 2.4, a sufficient condition for the metric (sub)regularity assumption in Theorem 3.7 (on Asplund spaces) reads as follows: there exist* $c > 0$, $r > 0$ *such that for every* $(x_1, y_1, z_1, x_2, y_2, z_2) \in (\operatorname{Gr} F \times \operatorname{Gr} K) \cap [B(\overline{x}, r) \times B(\overline{y}, r) \times B(q_{\overline{z}}, r) \times B(\overline{x}, r) \times B(\overline{y}, r) \times B(t_{\overline{z}}, r)]$ *and every* $(x^*, y^*) \in X^* \times Y^*$, $(u_1^*, v_1^*, w_1^*, u_2^*, v_2^*, w_2^*) \in \widehat{N}(\operatorname{Gr} F, (x_1, y_1, z_1)) \times \widehat{N}(\operatorname{Gr} K, (x_2, y_2, z_2)) + (x^*, y^*, 0_{Z^*}, -x^*, -y^*, 0_{Z^*})$

$$c \|(x^*, y^*)\| \leq \|(u_1^*, v_1^*, w_1^*, u_2^*, v_2^*, w_2^*)\|.$$

We consider now a generalization of the perturbation map from the Rockafellar's paper [18] (see [18, Theorem 5.4]). Let, as above, X, Y, Z be general Banach spaces, $f : X \times Y \to Z$ be a continuously differentiable single-valued map, $D \subset Y, E \subset Z$ be closed subsets and define the set-valued $F : X \rightrightarrows Y$ defined by:

$$F(x) = \{y \in D \mid f(x, y) \in E\}.$$

The calculus of the derivatives of this model set-valued map can also be captured by the approach we use in this paper.

Theorem 4.3 *In the above notations, suppose that* D, E *are closed,* $(\overline{x}, \overline{y}) \in X \times Y$ *with* $\overline{x} \in D, f(\overline{x}, \overline{y}) \in E$ *and* $g : X \times Y \times Z \longrightarrow Z$, $g(x, y, z) = f(x, y) - z$ *is metrically subregular at* $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}), 0_Z)$ *with respect to* $(X \times D) \times E$.

(i) If either D *is derivable at* $\overline{y}$ *or* E *is derivable at* $f(\overline{x}, \overline{y})$*, then for every* $u \in X$,

$$D_B F(\overline{x}, \overline{y})(u) = \{v \in T_B(D, \overline{y}) \mid \nabla f(\overline{x}, \overline{y})(u, v) \in T_B(E, f(\overline{x}, \overline{y}))\}.$$

(ii) Suppose that f *is twice continuously differentiable. If either* D *is second-order derivable at* $\overline{y}$ *or* E *is second-order derivable at* $f(\overline{x}, \overline{y})$*, then for every* $x_1 \in X$ *and* $y_1 \in Y$

$$\begin{aligned}
&\{v \in T_B^2(D, \overline{y}, y_1) \mid \nabla f(\overline{x}, \overline{y})(u, v) \in T_B^2(E, f(\overline{x}, \overline{y}), \nabla f(\overline{x}, \overline{y})(x_1, y_1)) \\
&\quad - 2^{-1} \nabla^2 f(\overline{x}, \overline{y})((x_1 y_1), (x_1, y_1))\} \\
&\quad = D_B^2 F(\overline{x}, \overline{y})(x_1, y_1)
\end{aligned}$$

Proof. (i) Obviously, $\operatorname{Gr} F = (X \times D) \cap f^{-1}(E)$. Then, using Proposition 2.7 (ii),

$$\begin{aligned}
T_B(\operatorname{Gr} F, (\overline{x}, \overline{y})) &= T_B((X \times D) \cap f^{-1}(E), (\overline{x}, \overline{y})) \\
&\subset T_B(X \times D, (\overline{x}, \overline{y})) \cap \nabla f(\overline{x}, \overline{y})^{-1}(T_B(E, f(\overline{x}, \overline{y}))),
\end{aligned}$$

Clearly, $T_B(X \times D, (\overline{x}, \overline{y})) = X \times T_B(D, \overline{y})$ (see Proposition 2.7 (i)), so

$$T_B(\operatorname{Gr} F, (\overline{x}, \overline{y})) \subset (X \times T_B(D, \overline{y})) \cap \nabla f(\overline{x}, \overline{y})^{-1}(T_B(E, f(\overline{x}, \overline{y}))).$$

In order to prove the converse inclusion observe that in our hypotheses one can apply Theorem 3.1 for $X \times D, E$ and f. Therefore, using as well the derivability assumption on the sets,

$$T_B((X \times D) \cap f^{-1}(E), (x, y)) \supset (X \times T_B(D, y)) \cap \nabla f(x, y)^{-1}(T_B(E, f(x, y))).$$

The conclusion follows.

(ii) This time, the second part of Theorem 3.1 yields the conclusion:

$$T_B^2(X \times D, (\overline{x}, \overline{y}), (x_1, y_1)) \cap \nabla f(\overline{x}, \overline{y})^{-1}(T_B^2(E, f(\overline{x}, \overline{y}), \nabla f(\overline{x}, \overline{y})(x_1, y_1))$$
$$- 2^{-1}\nabla^2 f(\overline{x}, \overline{y})((x_1 y_1), (x_1, y_1)))$$
$$\subset T_B^2((X \times D) \cap f^{-1}(E), (\overline{x}, \overline{y}), (x_1, y_1)).$$

The fact that $T_B^2(X \times D, (\overline{x}, \overline{y}), (x_1, y_1)) = X \times T_B^2(D, \overline{y}, y_1)$ ends the proof of this part. The other inclusion is a simple application of second-order Taylor formula. □

Remark 4.4 *As we have proceeded before, we give now (on Asplund spaces) the condition for the metric (sub)regularity, according to Corollary 2.4: there exist $c > 0$, $r > 0$ such that for every $(x, y, z) \in [X \times D \times E] \cap [B(\overline{x}, r) \times B(\overline{y}, r) \times B(f(\overline{x}, \overline{y}), r)]$ and every $z^* \in Z^*$, $(u^*, v^*, w^*) \in X \times \widehat{N}(D, y) \times \widehat{N}(E, z) + (\nabla^* f(x, y)(z^*), -z^*)$*

$$c \|z^*\| \leq \|(u^*, v^*, w^*)\|.$$

5 Concluding remarks

This paper applies a method based on metric subregularity assumptions in order to compute the first and second order derivatives of several set-valued maps. This approach is natural in infinite dimensional setting since allows us to eliminate any compactness requirement. Moreover, we have described, on Asplund spaces, sufficient conditions for our hypotheses in terms of Fréchet normal cones and we have stressed that similar conditions could be imposed as well on other appropriate classes of Banach spaces and corresponding well-behaved normals. In the case of set-valued mappings with closed convex graphs these limitation can be dropped. We have mainly concentrated on the sum of set-valued mappings and on perturbation maps. Nevertheless, we would like to emphasize that this method could be applied in further investigations for other models as well and the inclusion obtained in such a way could be useful in the study of other vector optimization programs, different from the one considered here.

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