History-dependent mixed variational problems in contact mechanics

Mircea Sofonea, Andaluzia Matei

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Abstract We consider a new class of mixed variational problems arising in Contact Mechanics. The problems are formulated on the unbounded interval of time $[0, +\infty)$ and involve history-dependent operators. For such problems we prove existence, uniqueness and continuous dependence results. The proofs are based on results on generalized saddle point problems and various estimates, combined with a fixed point argument. Then, we apply the abstract results in the study of a mathematical model which describes the frictionless contact between a viscoplastic body and an obstacle, the so-called foundation. The process is quasistatic and the contact is modelled with normal compliance and unilateral constraint, in such a way that the stiffness coefficient depends on the history of the penetration. We prove the unique weak solvability of the contact problem, as well as the continuous dependence of its weak solution with respect to the viscoplastic constitutive function, the applied forces, the contact conditions and the initial data.

Keywords History-dependent operator · Mixed variational problem · Lagrange multiplier · Viscoplastic material · Frictionless contact · Normal compliance · History-dependent stiffness coefficient · Unilateral constraint · Variational formulation · Weak solution

1 Introduction

Mixed variational problems involving Lagrange multipliers are used both in analysis and mechanics, in the study of minimization problems. They provide a useful framework in

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which a large number of problems involving unilateral constraints can be cast and can be solved numerically. Their study is based on arguments on duality, the saddle points theory and fixed point. The literature in the field is extensive, see for instance [5,7,10,11,16] and the references therein. There, existence and uniqueness results in the study of stationary variational problems with Lagrange multipliers can be found, together with various applications in Solid Mechanics. A recent existence result in the study of evolutionary problems with Lagrange multipliers was obtained in [20]. The analysis of various mixed variational problems associated to contact models can be found in [12–15,21–23] and, more recently, in [2,4,24], for instance.

In this paper we deal with a new class of mixed variational problems involving Lagrange multipliers, which arise in the study of various quasistatic contact problems with elastic, viscoelastic and viscoplastic bodies. The trait of novelty consists in the fact that the problems are evolutionary, are defined on an unbounded interval of time and involve history-dependent operators. The statement of the problem is as follows. Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two real Hilbert spaces. We denote by $\mathbb{R}_+$ the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$, and we use the notation $C(\mathbb{R}_+; X)$ and $C(\mathbb{R}_+; Y)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X and Y, respectively. Also, we consider two operators $A : X \to X$ and $\mathcal{S} : C(\mathbb{R}_+; X) \times C(\mathbb{R}_+; Y) \to C(\mathbb{R}_+; X)$, a bilinear form $b : X \times Y \to \mathbb{R}$, two functions $f, h : \mathbb{R}_+ \to X$ and a set $\Lambda \subset Y$. With these data we introduce the following problem.

Problem 1 Find the functions $u : \mathbb{R}_+ \to X$ and $\lambda : \mathbb{R}_+ \to \Lambda$ such that

$$(Au(t), v)_X + (\mathcal{S}(u,\lambda)(t), v)_X + b(v, \lambda(t)) = (f(t), v)_X \quad \forall\, v \in X, \tag{1.1}$$

$$b(u(t), \mu - \lambda(t)) \le b(h(t), \mu - \lambda(t)) \quad \forall\, \mu \in \Lambda, \tag{1.2}$$

for all $t \in \mathbb{R}_+$.

Our aim in this paper is threefold. The first one is to study the unique solvability of Problem 1. To this end, we use a result related to a generalized saddle point problem proved in [21], combined with a fixed point result obtained in [27] and show that, under appropriate conditions, Problem 1 has a unique solution (u, λ) such that $u \in C(\mathbb{R}_+; X)$ and $\lambda \in C(\mathbb{R}_+; Y)$. The second aim is to study the behavior of the solution of Problem 1 with respect to a perturbation of the data. To this end we use monotonicity properties and arguments of convergence in the spaces $C(\mathbb{R}_+; X)$ and $C(\mathbb{R}_+; Y)$ which allow us to prove a convergence result. Finally, our third aim is to show how our abstract results can be used in the analysis of mathematical models in Contact Mechanics. To this end we consider a quasistatic process of contact between a viscoplastic body and a deformable foundation. The contact is with normal compliance and finite penetration, in such a way that the stiffness coefficient depends on the history of the penetration. Considering such kind of model leads to a new and interesting mathematical problem, governed by two history-dependent operators. We provide the analysis of this problem, which includes its unique weak solvability and the continuous dependence of the solution with respect to the data. The proofs are based on the abstract results obtained in the study of Problem 1. In this way we fully exemplify the cross fertilization between the models and applications, in one hand, and the nonlinear functional analysis, on the other hand.

The rest of the paper is structured as follows. In Sect. 2 we introduce the mixed variational problem, list the assumptions on the data, then we state and prove our main abstract existence and uniqueness result, Theorems 2.1. In Sect. 3 we introduce a perturbation of the problem, then we state and prove a convergence result, Theorem 3.1. Then, in Sect. 4, we describe our mathematical model of contact, list the assumptions on the data and derive its variational formulation. In Sect. 5 we prove the unique weak solvability of the model. The proof is based on the abstract result provided by Theorem 2.1. Finally, in Sect. 6 we prove the continuous dependence of the weak solution of the contact problem with respect to the data. The proof is based on the abstract result provided by Theorem 3.1.

We end this introduction with some notation and preliminaries. First, everywhere in this paper we denote by $\mathbb{N}$ the set of positive integers. Given a normed space U and a subset $K \subset U$ we use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that, if U is a Banach space, then $C(\mathbb{R}_+; U)$ can be organized in a canonical way as a Fréchet space, i.e. a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Details can be found in [6] and [19], for instance. Here we restrict ourselves to recall that the convergence of a sequence $(u_k)_k$ to the element u, in the space $C(\mathbb{R}_+; U)$, can be described as follows:

$$\begin{cases} u_k \to u \quad \text{in } C(\mathbb{R}_+; U) \ \text{ as } \ k \to \infty \ \text{ if and only if} \\ \max\limits_{r \in [0,n]} \|u_k(r) - u(r)\|_U \to 0 \ \text{ as } k \to \infty, \quad \text{for all } \ n \in \mathbb{N}. \end{cases} \tag{1.3}$$

We also recall the following fixed point result.

Theorem 1.1 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $\mathcal{L} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator. Assume that there exists $m \in \mathbb{N}$ with the following property: for each $n \in \mathbb{N}$ there exist two constants $c_n \geq 0$ and $k_n \in [0, 1)$ such that*

$$\|\mathcal{L}u(t) - \mathcal{L}v(t)\|_X^m \leq c_n \int_0^t \|u(s) - v(s)\|_X^m \, ds + k_n \|u(t) - v(t)\|_X^m \tag{1.4}$$

for all $u, \ v \in C(\mathbb{R}_+; X)$ and for any $t \in [0, n]$. Then the operator $\mathcal{L}$ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

Note that in (1.4) and below the notation $\mathcal{L}\eta(t)$ represents the value of the function $\mathcal{L}\eta$ at the point t, i.e. $\mathcal{L}\eta(t) = (\mathcal{L}\eta)(t)$. The proof of Theorem 1.1 can be found in [27]. We shall use this fixed point result twice, in Sects. 2 and 5 of the paper.

2 An abstract existence and uniqueness result

In this section we prove the unique solvability of Problem 1. To this end we assume that the data satisfy the following condition.

$$\begin{cases} \text{(a) There exists } m_A > 0 \text{ such that} \\ \quad (Au - Av, \ u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall\, u, v \in X. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \quad \|Au - Av\|_X \leq L_A \, \|u - v\|_X \quad \forall\, u, v \in X. \end{cases} \tag{2.1}$$

$$\begin{cases} \text{For each } n \in \mathbb{N} \text{ there exist } d_n \geq 0 \text{ and } r_n \geq 0 \text{ such that} \\ \quad \|\mathcal{S}(u_1, \lambda_1)(t) - \mathcal{S}(u_2, \lambda_2)(t)\|_X \\ \quad \leq d_n(\|u_1(t) - u_2(t)\|_X + \|\lambda_1(s) - \lambda_2(s)\|_Y) \\ \qquad + r_n \displaystyle\int_0^t (\|u_1(s) - u_2(s)\|_X + \|\lambda_1(s) - \lambda_2(s)\|_Y) \, ds \\ \qquad \forall\, u_1, \ u_2 \in C(\mathbb{R}_+; X), \ \forall\, \lambda_1, \lambda_2 \in C(\mathbb{R}_+; Y), \ \forall\, t \in [0, n]. \end{cases} \tag{2.2}$$

$$\begin{cases} b : X \times Y \to \mathbb{R} \text{ is a bilinear form such that} \\ \text{(a) There exists } M_b > 0 \text{ such that} \\ \quad |b(v, \mu)| \le M_b \|v\|_X \|\mu\|_Y \quad \forall\, v \in X,\ \mu \in Y. \\ \text{(b) There exists } \alpha > 0 \text{ such that} \\ \quad \inf_{\mu \in Y, \mu \ne 0_Y} \ \sup_{v \in X, v \ne 0_X} \dfrac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \ge \alpha. \end{cases} \tag{2.3}$$

$$f \in C(\mathbb{R}_+; X), \quad h \in C(\mathbb{R}_+; X). \tag{2.4}$$

$$\Lambda \text{ is a closed convex unbounded subset of } Y \text{ that contains } 0_Y. \tag{2.5}$$

Our main result in this section is the following.

Theorem 2.1 *Assume* (2.1)–(2.5). *There exists* $d_0 > 0$ *which depends only on* A *and* b *such that, if* $d_n < d_0$ *for all positive integers* n, *then Problem* 1 *has a unique solution* (u, λ). *Moreover, the solution satisfies* $u \in C(\mathbb{R}_+; X)$ *and* $\lambda \in C(\mathbb{R}_+; \Lambda)$.

The proof of Theorem 2.1 will be carried out in several steps. To this end, we assume in what follows that (2.1)–(2.5) hold. The first step is given by the following result.

Lemma 2.2 *Given* $g,\ k \in X$, *there exists a unique pair* $(u, \lambda) \in X \times \Lambda$ *such that*

$$(Au, v)_X + b(v, \lambda) = (g,\ v)_X \quad \forall\, v \in X, \tag{2.6}$$

$$b(u, \mu - \lambda) \le b(k, \mu - \lambda) \quad \forall\, \mu \in \Lambda. \tag{2.7}$$

In addition, if $(u_1,\ \lambda_1)$ *and* $(u_2,\ \lambda_2)$ *are the solutions of the problem* (2.6)–(2.7) *corresponding to the data* $g_1,\ k_1 \in X$ *and* $g_2,\ k_2 \in X$, *respectively, then there exists* c_0 *which depends only on* A *and* b *such that*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le c_0(\|g_1 - g_2\|_X + \|k_1 - k_2\|_X). \tag{2.8}$$

Proof The existence and uniqueness part of the lemma corresponds to Theorem 5.2 in [21] and, for this reason, we skip its proof. The estimate (2.8) shows the Lischitz continuous dependence of the solution with respect to the data and corresponds to Theorem 5.8 in [21]. Nevertheless, since the size of the constant c_0 in this estimate will play an important role in what follows, for the convenience of the reader, we present the proof of (2.8). Thus, consider $g_i, k_i \in X$ and denote by (u_i, λ_i), the solution of the problem (2.6)–(2.7), corresponding to the data $g_i,\ k_i \in X$, for each $i = 1,\ 2$. Then, using (2.6) it follows that

$$(Au_1 - Au_2, v)_X + b(v, \lambda_1 - \lambda_2) = (g_1 - g_2, v)_X \quad \forall\, v \in X \tag{2.9}$$

and, using (2.1)(b) we find that

$$b(v, \lambda_1 - \lambda_2) \le \|g_1 - g_2\|_X \|v\|_X + L_A \|u_1 - u_2\|_X \|v\|_X \quad \forall\, v \in X.$$

We now use (2.3)(b) and the previous inequality to obtain that

$$\alpha\, \|\lambda_1 - \lambda_2\|_Y \le \|g_1 - g_2\|_X + L_A \|u_1 - u_2\|_X. \tag{2.10}$$

On the other hand, (2.7) yields

$$b(u_1 - u_2, \lambda_2 - \lambda_1) \le b(k_1 - k_2, \lambda_2 - \lambda_1)$$

and, therefore, using condition (2.3)(a) we find that

$$b(u_1 - u_2, \lambda_2 - \lambda_1) \le M_b\, \|k_1 - k_2\|_X \|\lambda_1 - \lambda_2\|_Y. \tag{2.11}$$

We now take $v = u_1 - u_2$ in (2.9) and use (2.11) in the resulting inequality to deduce that

$$(Au_1 - Au_2, u_1 - u_2)_X \le (g_1 - g_2, u_1 - u_2)_X + M_b \,\|k_1 - k_2\|_X \|\lambda_1 - \lambda_2\|_Y.$$

Therefore, using the assumption (2.1)(a) it follows that

$$m_A \|u_1 - u_2\|_X^2 \le \|g_1 - g_2\|_X \|u_1 - u_2\|_X + M_b \|k_1 - k_2\|_X \|\lambda_1 - \lambda_2\|_Y. \tag{2.12}$$

Inequalities (2.12) and (2.10) imply that

$$m_A \|u_1 - u_2\|_X^2 \le \frac{\|g_1 - g_2\|_X^2}{2c_1} + \frac{c_1 \|u_1 - u_2\|_X^2}{2} + \frac{M_b^2 \|k_1 - k_2\|_X^2}{2c_2} + \frac{c_2 \|\lambda_1 - \lambda_2\|_Y^2}{2}, \tag{2.13}$$

$$\|\lambda_1 - \lambda_2\|_Y^2 \le \frac{2}{\alpha^2}\big(\|g_1 - g_2\|_X^2 + L_A^2 \|u_1 - u_2\|_X^2\big) \tag{2.14}$$

respectively, where c_1, c_2 are arbitrary positive constants. We now choose c_1 and c_2 such that

$$m_A - \frac{c_1}{2} - \frac{c_2 L_A^2}{\alpha^2} > 0.$$

Therefore, from (2.13) and (2.14) we deduce that there exists $c_3 > 0$, which depends only on A and b, such that

$$\|u_1 - u_2\|_X^2 \le c_3\big(\|g_1 - g_2\|_X^2 + \|k_1 - k_2\|_X^2\big). \tag{2.15}$$

Finally, combining (2.14) and (2.15) we obtain (2.8) with c_0 depending only on A and b, which concludes the proof. □

The next step is given by the following result.

Lemma 2.3 *Given $\eta \in C(\mathbb{R}_+; X)$, there exists a unique couple of functions $(u_\eta, \lambda_\eta) \in C(\mathbb{R}_+; X) \times C(\mathbb{R}_+; \Lambda)$ such that*

$$(Au_\eta(t), v)_X + (\eta(t), v)_X + b(v, \lambda_\eta(t)) = (f(t),\ v)_X \quad \forall\, v \in X, \tag{2.16}$$

$$b(u_\eta(t), \mu - \lambda_\eta(t)) \le b(h(t), \mu - \lambda_\eta(t)) \quad \forall\, \mu \in \Lambda, \tag{2.17}$$

for all $t \in \mathbb{R}_+$. In addition, given $\eta_1,\ \eta_2 \in C(\mathbb{R}_+; X)$ and denoting by $u_{\eta_1},\ u_{\eta_2}$ the corresponding couples of functions which verify (2.16)–(2.17) *at each $t \in \mathbb{R}_+$, then*

$$\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_X + \|\lambda_{\eta_1}(t) - \lambda_{\eta_2}(t)\|_Y \le c_0 \|\eta_1(t) - \eta_2(t)\|_X \tag{2.18}$$

for all $t \in \mathbb{R}_+$.

Proof Let $t \in \mathbb{R}_+$ be fixed. We use Lemma 2.2 with $g = f(t) - \eta(t)$ and $k = h(t)$ to obtain the existence of a unique couple $(u_\eta(t), \lambda_\eta(t)) \in X \times \Lambda$ which satisfies (2.16)–(2.17). Next, we consider $t_1, t_2 \in \mathbb{R}_+$ and denote $\eta(t_i) = \eta_i$, $u(t_i) = u_i$, $\lambda(t_i) = \lambda_i$, $f(t_i) = f_i$, $h(t_i) = h_i$ and $g_i = f_i - \eta_i$. Then, using inequality (2.8), it follows that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le c_0\,(\|f_1 - f_2\|_X + \|\eta_1 - \eta_2\|_X + \|h_1 - h_2\|_X).$$

Therefore, since $f,\ \eta,\ h \in C(\mathbb{R}_+; X)$ we conclude that $u_\eta \in C(\mathbb{R}_+; X)$ and $\lambda_\eta \in C(\mathbb{R}_+; \Lambda)$. Finally, the estimate (2.18) is obtained by using arguments similar to those used above, based on inequality (2.8). □

We now use Lemma 2.3 to introduce the operator $\mathcal{L} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ defined by equality

$$\mathcal{L}\eta = \mathcal{S}(u_\eta, \lambda_\eta) \quad \forall\, \eta \in C(\mathbb{R}_+; X). \tag{2.19}$$

We have the following fixed-point result.

Lemma 2.4 *There exists $d_0 > 0$ which depends only on A and b such that, if $d_n < d_0$ for all positive integers n, then $\mathcal{L}$ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.*

Proof Let $\eta_1, \eta_2 \in C(\mathbb{R}_+; X)$, $n \in \mathbb{N}$ and let $t \in [0, n]$. Then, using (2.19) and (2.2) we have

$$\begin{aligned}\|\mathcal{L}\eta_1(t) - \mathcal{L}\eta_2(t)\|_X &\le d_n(\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_X + \|\lambda_{\eta_1}(t) - \lambda_{\eta_2}(t)\|_Y)\\ &+ r_n \int_0^t (\|u_{\eta_1}(s) - u_{\eta_2}(s)\|_X + \|\lambda_{\eta_1}(s) - \lambda_{\eta_2}(s)\|_Y)\, ds.\end{aligned}$$

Therefore, inequality (2.18) yields

$$\|\mathcal{L}\eta_1(t) - \mathcal{L}\eta_2(t)\|_X \le c_0 d_n \|\eta_1(t) - \eta_2(t)\|_X + c_0 r_n \int_0^t \|\eta_1(s) - \eta_2(s)\|_X\, ds. \tag{2.20}$$

Let $d_0 = \frac{1}{c_0}$ and note that Lemma 2.2 shows that d_0 depends only on A and b. Assume now that $d_n < d_0$ for all $n \in \mathbb{N}$. Then inequality (2.20) and Theorem 1.1 show that $\mathcal{L}$ has a unique fixed point, which concludes the proof. □

We now have all the ingredients to prove Theorem 2.1.

Proof Let $d_0 > 0$ be defined as above and recall that d_0 depends only on A and b. Assume that $d_n < d_0$ for all positive integers n, and denote by η^* the unique fixed point of the operator $\mathcal{L}$ provided in Lemma 2.4. Then, using (2.16), (2.17) and definition (2.19) of the operator $\mathcal{L}$ it is easy to see that $(u_{\eta^*}, \lambda_{\eta^*})$ is a solution of Problem 1 and, moreover, it has the regularity $(u_{\eta^*}, \lambda_{\eta^*}) \in C(\mathbb{R}_+; X) \times C(\mathbb{R}_+; \Lambda)$. This concludes the existence part of the theorem. The uniqueness part follows from the uniqueness of the fixed point of the operator $\mathcal{L}$, guaranteed by Lemma 2.4. □

3 A convergence result

We now turn to the dependence of the solution with respect to the data. To this end, everywhere in this section we assume that (2.1)–(2.5) hold and we denote by (u, λ) the solution of Problem 1 provided by Theorem 2.1. Moreover, we assume that for each $\rho > 0$ the operator $\mathcal{S}_\rho : C(\mathbb{R}_+; X) \times C(\mathbb{R}_+, Y) \to C(\mathbb{R}_+; X)$, and the functions f_ρ, $h_\rho : \mathbb{R}_+ \to X$ are given, and represent perturbations of the data $\mathcal{S}$, f and h, respectively. With these data, for each $\rho > 0$, we consider the following problem.

Problem 2 Find the functions $u_\rho : \mathbb{R}_+ \to X$ and $\lambda_\rho : \mathbb{R}_+ \to \Lambda$ such that

$$(Au_\rho(t), v)_X + (\mathcal{S}_\rho(u_\rho, \lambda_\rho)(t), v)_X + b(v, \lambda_\rho(t)) = (f_\rho(t),\, v)_X \quad \forall\, v \in X, \tag{3.1}$$

$$b(u_\rho(t), \mu - \lambda_\rho(t)) \le b(h_\rho(t), \mu - \lambda_\rho(t)) \quad \forall\, \mu \in \Lambda, \tag{3.2}$$

for all $t \in \mathbb{R}_+$.

We assume that, for each $\rho > 0$, the following conditions hold.

$$\left\{\begin{array}{l} \text{For each } n \in \mathbb{N} \text{ there exist } d_{\rho n} \geq 0 \text{ and } r_{\rho n} \geq 0 \text{ such that} \\ \quad \|\mathcal{S}_\rho(u_1, \lambda_1)(t) - \mathcal{S}_\rho(u_2, \lambda_2)(t)\|_X \\ \quad \leq d_{\rho n}(\|u_1(t) - u_2(t)\|_X + \|\lambda_1(t) - \lambda_2(t)\|_Y) \\ \qquad + r_{\rho n} \displaystyle\int_0^t (\|u_1(s) - u_2(s)\|_X + \|\lambda_1(s) - \lambda_2(s)\|_Y)\, ds \\ \qquad \forall\, u_1,\ u_2 \in C(\mathbb{R}_+; X),\ \forall\, \lambda_1, \lambda_2 \in C(\mathbb{R}_+; Y),\ \forall\, t \in [0, n]. \end{array}\right. \tag{3.3}$$

$$f_\rho \in C(\mathbb{R}_+, X), \quad h_\rho \in C(\mathbb{R}_+, X). \tag{3.4}$$

Under these assumptions, if $d_{\rho n} < d_0$ for all $n \in \mathbb{N}$, Theorem 2.1 guarantees the existence of a unique solution (u_ρ, λ_ρ) to Problem 2 such that $u_\rho \in C(\mathbb{R}_+; X)$ and $\lambda_\rho \in C(\mathbb{R}_+; \Lambda)$. Our interest lies in the behavior of the solution of the perturbed problem as ρ tends to zero. To this end we consider the following additional assumptions, in which d_0 represents the constant in Theorem 2.1.

$$\left\{\begin{array}{l} \text{For each } n \in \mathbb{N} \text{ there exist } H_n : \mathbb{R}_+ \to \mathbb{R}_+, \\ J_n : C(\mathbb{R}_+; X) \times C(\mathbb{R}_+; Y) \to \mathbb{R}_+ \text{ and } R_n \geq 0 \text{ such that} \\ \text{(a) } \|\mathcal{S}_\rho(u, \lambda)(t) - \mathcal{S}(u, \lambda)(t))\|_X \leq H_n(\rho) J_n(u, \lambda) \\ \qquad \forall\, (u, \lambda) \in C(\mathbb{R}_+; X \times \Lambda),\ \forall\, t \in [0, n],\ \forall\, \rho > 0. \\ \text{(b) } r_{\rho n} \leq R_n \quad \forall\, \rho > 0. \\ \text{(c) } \lim\limits_{\rho \to 0} H_n(\rho) = 0. \end{array}\right. \tag{3.5}$$

$$\text{There exists } \widetilde{d}_0 \text{ such that } \quad d_{\rho n} \leq \widetilde{d}_0 < d_0 \quad \forall\, n \in \mathbb{N},\ \forall\, \rho > 0. \tag{3.6}$$

$$f_\rho \to f, \quad h_\rho \to h \quad \text{in} \quad C(\mathbb{R}_+; X) \quad \text{as} \quad \rho \to 0. \tag{3.7}$$

We have the following convergence result.

Theorem 3.1 *Assume* (3.5)–(3.7). *Then the solution* (u_ρ, λ_ρ) *of Problem* 2 *converges to the solution* (u, λ) *of Problem* 1 *i.e.*

$$u_\rho \to u \quad \text{in} \quad C(\mathbb{R}_+; X), \quad \lambda_\rho \to \lambda \quad \text{in} \quad C(\mathbb{R}_+; Y) \quad \text{as } \rho \to 0. \tag{3.8}$$

Proof Let $\rho > 0$, $n \in \mathbb{N}$ and let $t \in [0, n]$. We note that the system (3.1)–(3.2) represents a system of the form (2.6)–(2.7) in which

$$g = f_\rho(t) - \mathcal{S}_\rho(u_\rho, \lambda_\rho)(t) \quad \text{and} \quad h = h_\rho(t).$$

Also, the system (1.1)–(1.2) is a system of the form (2.6)–(2.7) in which

$$g = f(t) - \mathcal{S}(u, \lambda)(t) \quad \text{and} \quad h = h(t).$$

Therefore, using the estimate (2.8) yields

$$\begin{aligned} &\|u_\rho(t) - u(t)\|_X + \|\lambda_\rho(t) - \lambda(t)\|_Y \\ &\quad \leq c_0(\|f_\rho(t) - f(t)\|_X + \|\mathcal{S}_\rho(u_\rho, \lambda_\rho)(t) - \mathcal{S}(u, \lambda)(t)\|_X + \|h_\rho(t) - h(t)\|_X). \end{aligned} \tag{3.9}$$

Next, we remark that

$$\|f_\rho(t) - f(t)\|_X \leq \max_{s \in [0,n]} \|f_\rho(s) - f(s)\|_X := \delta_{\rho n}, \tag{3.10}$$

$$\|h_\rho(t) - h(t)\|_X \leq \max_{s \in [0,n]} \|h_\rho(s) - h(s)\|_X := \omega_{\rho n} \tag{3.11}$$

and, in order to simplify the writing we denote

$$\varphi_\rho(t) := \|u_\rho(t) - u(t)\|_X + \|\lambda_\rho(t) - \lambda(t)\|_Y. \tag{3.12}$$

Then, inequalities (3.9)–(3.11) imply that

$$\varphi_\rho(t) \le c_0(\delta_{\rho n} + \omega_{\rho n} + \|\mathcal{S}_\rho(u_\rho, \lambda_\rho)(t) - \mathcal{S}(u, \lambda)(t)\|_X). \tag{3.13}$$

On the other hand

$$\begin{aligned}\|\mathcal{S}_\rho(u_\rho, \lambda_\rho)(t) - \mathcal{S}(u, \lambda)(t)\|_X \le\ & \|\mathcal{S}_\rho(u_\rho, \lambda_\rho)(t) - \mathcal{S}_\rho(u, \lambda)(t)\|_X \\ & + \|\mathcal{S}_\rho(u, \lambda)(t) - \mathcal{S}(u, \lambda)(t)\|_X,\end{aligned}$$

and, therefore, assumptions (3.3) and (3.5)(a) combined with definition (3.12) show that

$$\|\mathcal{S}_\rho(u_\rho, \lambda_\rho)(t) - \mathcal{S}(u, \lambda)(t)\|_X \le d_{\rho n}\varphi_\rho(t) + r_{\rho n}\int_0^t \varphi_\rho(s)\, ds + H_n(\rho)J_n(u, \lambda). \tag{3.14}$$

We now use (3.13) and (3.14) to see that

$$\varphi_\rho(t) \le c_0(\delta_{\rho n} + \omega_{\rho n}) + c_0 d_{\rho n}\varphi_\rho(t) + c_0 r_{\rho n}\int_0^t \varphi_\rho(s)\, ds + c_0 H_n(\rho)J_n(u, \lambda).$$

Therefore, the hypothesis (3.6) allows us to write

$$(1 - c_0\widetilde{d}_0)\varphi_\rho(t) \le c_0(\delta_{\rho n} + \omega_{\rho n}) + c_0 r_{\rho n}\int_0^t \varphi_\rho(s)\, ds + c_0 H_n(\rho)J_n(u, \lambda).$$

Recall now that $d_0 = \dfrac{1}{c_0}$, as shown in the proof of Lemma 2.4. Thus, assumption (3.5)(b) combined with inequality $\widetilde{d}_0 < d_0$ in (3.6) imply that

$$\varphi_\rho(t) \le c\,(\delta_{\rho n} + \omega_{\rho n}) + c\,R_n\int_0^t \varphi_\rho(s)\, ds + c\,H_n(\rho)J_n(u, \lambda)$$

where c is a positive constant independent of ρ and n. Using now a Gronwall argument we obtain

$$\varphi_\rho(t) \le c\,(\delta_{\rho n} + \omega_{\rho n} + H_n(\rho)J_n(u, \lambda))e^{cR_n t}$$

and, therefore,

$$\max_{t\in[0,n]} \varphi_\rho(t) \le c(\delta_{\rho n} + \omega_{\rho n} + H_n(\rho)J_n(u, \lambda))e^{cnR_n}. \tag{3.15}$$

We now use assumption (3.7), the equivalence (1.3) and the definitions (3.10), (3.11) to see that

$$\delta_{\rho n} \to 0, \quad \omega_{\rho n} \to 0 \quad \text{as} \quad \rho \to 0. \tag{3.16}$$

Therefore, passing to the limit in (3.15) as $\rho \to 0$ with a fixed positive integer n, using the convergences (3.5)(c), (3.16) we deduce that

$$\max_{t\in[0,n]} \varphi_\rho(t) \to 0 \quad \text{as} \quad \rho \to 0.$$

Using now notation (3.12) we obtain that

$$\max_{t\in[0,n]} \|u_\rho(t) - u(t)\|_X \to 0, \quad \max_{t\in[0,n]} \|\lambda_\rho(t) - \lambda(t)\|_Y \to 0 \quad \text{as} \quad \rho \to 0. \tag{3.17}$$

Finally, we use (3.17) and (1.3) to see that the convergences (3.8) hold, which concludes the proof. □

4 A history-dependent contact problem

In this section we introduce a model of frictionless contact which can be studied by using the abstract results presented in Section 2. The physical setting is as follows. A viscoplastic body occupies the bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$), with the boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,\Gamma_1 > 0$. We assume that the boundary Γ is Lipschitz continuous and we denote by $\boldsymbol{\nu}$ its unit outward normal, defined almost everywhere. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in Ω, surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and, finally, we assume that the body is in contact with a deformable foundation on Γ_3. The contact is frictionless and we model it with a normal compliance condition with unilateral constraint, in which the stiffness coefficient depends on the history of the penetration. The process is quasistatic and we study it in the unbounded interval of time $[0, +\infty)$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable. Then, the classical formulation of the contact problem is the following.

Problem 3 Find a displacement field $\boldsymbol{u}:\Omega \times [0, +\infty) \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, +\infty) \to \mathbb{S}^d$ such that

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{4.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{4.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{4.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g(t), \ \ \sigma_\nu(t) + k(\zeta \boldsymbol{u}(t))p(u_\nu(t)) \le 0, \\ (u_\nu(t) - g(t))(\sigma_\nu(t) + k(\zeta \boldsymbol{u}(t))p(u_\nu(t))) = 0, \\ \zeta \boldsymbol{u}(t) = \displaystyle\int_0^t u_\nu^+(s)\, ds \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{4.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{4.6}$$

for all $t \in \mathbb{R}_+$ and, moreover,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{4.7}$$

We now provide a brief description of the equations and conditions in Problem 3 and refer the reader to [9,26,28] for details and additional comments on the classical formulation of the contact problems. First, Eq. (4.1) represents the viscoplastic constitutive law of the material, in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized stress tensor, $\mathcal{E}$ is the elasticity tensor and $\mathcal{G}$ is a given constitutive function. Moreover, the dot above represents the derivative with respect to the time variable t. Quasistatic frictionless and frictional contact problems for such kind of materials were studied in various works, see for instance [3,4,9,26] and the references therein.

Equation (4.2) is the equilibrium equation in which Div represents the divergence operator for tensor-valued functions; we use it here since the process is assumed to be quasistatic. Conditions (4.3) and (4.4) are the displacement and traction boundary conditions, respectively, and condition (4.5) represents a new version of the normal compliance condition with unilateral constraint; here u_ν and σ_ν represent the normal component of the displacement and the stress field, respectively; $g \geq 0$ is a time-dependent bound for the penetration, p represents a given normal compliance function, k is a positive function and $\zeta\boldsymbol{u}(t)$ represents the accumulated contact penetration depth at time t, u_ν^+ being the positive part of u_ν. We interpret $k = k(\zeta\boldsymbol{u})$ as a stiffness coefficient which, clearly, depends on the history of penetration. Note that such kind of dependence models the surface hardening or softening which appears in various applications, when cycles of contact and no contact arise. Details can be found in [25]. Condition (4.5) was introduced for the first time in [18] in the case when g is a constant and $k \equiv 1$. In this particular form it was recently used in [3,4], in the study of a quasistatic viscoplastic problem. Condition (4.6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (4.7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

We turn now to the variational formulation of Problem 3. To this end, we need further notation and preliminaries. First, we use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by ν_i the components of $\boldsymbol{\nu}$, i.e. $\boldsymbol{\nu} = (\nu_i)$. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i/\partial x_j$. Recall that the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \ , \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \ , \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

We use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we consider the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},$$
$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,\Gamma_1 > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$.

Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_{tr} which depends only on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_{tr} \|\boldsymbol{v}\|_V \quad \text{for all } \ \boldsymbol{v} \in V. \tag{4.8}$$

Inequality (4.8) represents a consequence of the Sobolev trace theorem. We also consider the space

$$S = \{ \boldsymbol{w} = \boldsymbol{v}_{|\Gamma_3} : \boldsymbol{v} \in V \},$$

where $\boldsymbol{v}_{|\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{v} \in V$ to Γ_3. Thus, $S \subset H^{1/2}(\Gamma_3; \mathbb{R}^d)$ where $H^{1/2}(\Gamma_3; \mathbb{R}^d)$ is the space of the restrictions on Γ_3 of traces on Γ of functions of $H^1(\Omega)^d$. It is known that S can be organized as a Hilbert space, in a canonical way, see for instance [1,8,17]. The dual of the space S will be denoted by D and the duality paring between D and S will be denoted by $\langle \cdot, \cdot \rangle_{\Gamma_3}$. Nevertheless, for simplicity, we write $\langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3}$ instead of $\langle \boldsymbol{\mu}, \boldsymbol{v}_{|\Gamma_3} \rangle_{\Gamma_3}$, when $\boldsymbol{\mu} \in D$ and $\boldsymbol{v} \in V$.

For a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that the divergence operator is defined by the equality $\mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j})$ and, in addition, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall \boldsymbol{v} \in V. \tag{4.9}$$

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \left\{ \mathcal{E} = (\mathcal{E}_{ijkl}) \ : \ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq d \right\},$$

and we recall that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1 \leq i,j,k,l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \leq d\, \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \quad \forall \mathcal{E} \in \mathbf{Q}_\infty, \ \boldsymbol{\tau} \in Q. \tag{4.10}$$

In the study of the mechanical problem (4.1)–(4.7) we assume that the elasticity tensor $\mathcal{E}$, the nonlinear constitutive function $\mathcal{G}$, the normal compliance function and the stiffness function k satisfy the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega), \ 1 \leq i, j, k, l \leq d. \\ \text{(c) There exists } m_\mathcal{E} > 0 \text{ such that} \\ \quad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_\mathcal{E} \|\boldsymbol{\tau}\|^2 \ \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{array} \right. \tag{4.11}$$

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_\mathcal{G} > 0 \text{ such that} \\ \quad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \qquad \leq L_\mathcal{G} \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \right) \\ \qquad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable in } \Omega, \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Q. \end{array} \right. \tag{4.12}$$

$$\begin{cases} \text{(a) } p : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \quad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) There exists } p_0 > 0 \text{ such that } |p(r)| \le p_0 \quad \forall\, r \in \mathbb{R}. \end{cases} \tag{4.13}$$

$$\begin{cases} \text{(a) } k : \mathbb{R}_+ \to \mathbb{R}_+. \\ \text{(b) There exists } L_k > 0 \text{ such that} \\ \quad |k(r_1) - k(r_2)| \le L_k |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) There exists } k_0 > 0 \text{ such that } |k(r)| \le k_0 \quad \forall\, r \in \mathbb{R}. \end{cases} \tag{4.14}$$

We also assume that the densities of the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C\left(\mathbb{R}_+; L^2(\Omega)^d\right), \quad \boldsymbol{f}_2 \in C\left(\mathbb{R}_+; L^2(\Gamma_2)^d\right), \tag{4.15}$$

the penetration bound satisfies

$$g \in C(\mathbb{R}_+; \mathbb{R}_+), \tag{4.16}$$

and the initial data are such that

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \tag{4.17}$$

Finally, we assume that

$$\text{there exists } \widetilde{\boldsymbol{\theta}} \in V \text{ such that } \widetilde{\theta}_\nu = 1 \text{ a.e. on } \Gamma_3 \tag{4.18}$$

where, recall, $\widetilde{\theta}_\nu = \widetilde{\boldsymbol{\theta}} \cdot \boldsymbol{\nu}$.

Next, we define the sets $K \subset V$ and $\Lambda \subset D$, the bilinear form $b : V \times D \to \mathbb{R}$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$K = \{\, \boldsymbol{v} \in V \,:\, v_\nu \le 0 \quad \text{a.e. on } \Gamma_3 \,\}, \tag{4.19}$$

$$\Lambda = \{\, \boldsymbol{\mu} \in D \,:\, \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \le 0 \quad \forall\, \boldsymbol{v} \in K \,\}, \tag{4.20}$$

$$b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v} \rangle_{\Gamma_3} \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D, \tag{4.21}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{4.22}$$

Assume now that $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are regular functions which verify (4.1)–(4.7), $t \in \mathbb{R}_+$, $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in \Lambda$. Then, we integrate (4.1) with the initial condition (4.7) to find that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{4.23}$$

Next, using Green's formula (4.9) and the equation of equilibrium (4.2) we have

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v}\, da. \tag{4.24}$$

Then, since $\boldsymbol{v} = \boldsymbol{0}$ on Γ_1, using (4.4), (4.6) and (4.22) we obtain that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\boldsymbol{f}(t), \boldsymbol{v})_V + \int_{\Gamma_3} \sigma_\nu(t) v_\nu\, da. \tag{4.25}$$

Let $\boldsymbol{\lambda}(t) \in D$ be the Lagrange multiplier defined by

$$\langle \boldsymbol{\lambda}(t), \boldsymbol{w} \rangle_{\Gamma_3} = -\int_{\Gamma_3} (\sigma_\nu(t) + k(\zeta \boldsymbol{u}(t)) p(u_\nu(t)) w_\nu \, da \quad \forall \boldsymbol{w} \in S. \tag{4.26}$$

Then, taking into account (4.21) we can write

$$\int_{\Gamma_3} \sigma_\nu(t) v_\nu \, da = -b(\boldsymbol{v}, \boldsymbol{\lambda}(t)) - \int_{\Gamma_3} k(\zeta \boldsymbol{u}(t)) p(u_\nu(t)) v_\nu \, da \quad \forall \boldsymbol{v} \in V \tag{4.27}$$

and, combining this equality with (4.25) we obtain that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + b(\boldsymbol{v}, \boldsymbol{\lambda}(t)) + \int_{\Gamma_3} k(\zeta \boldsymbol{u}(t)) p(u_\nu(t)) v_\nu \, da = (\boldsymbol{f}(t), \boldsymbol{v})_V. \tag{4.28}$$

On the other hand, using (4.5), (4.19) and (4.20) we deduce that $\boldsymbol{\lambda}(t) \in \Lambda$. Moreover, using assumption (4.18) and the definition (4.21) of the bilinear form b it it is easy to see that

$$\begin{aligned} b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) &= b(\boldsymbol{u}(t) - g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) + b(g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \\ &= \langle \boldsymbol{\mu} - \boldsymbol{\lambda}(t), \boldsymbol{u}(t) - g(t)\widetilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} + b(g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \end{aligned}$$

and, therefore,

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) = \langle \boldsymbol{\mu}, \boldsymbol{u}(t) - g(t)\widetilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} - \langle \boldsymbol{\lambda}(t), \boldsymbol{u}(t) - g(t)\widetilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} + b(g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)). \tag{4.29}$$

In addition, (4.5) and (4.18) imply that

$$\boldsymbol{u}(t) - g(t)\widetilde{\boldsymbol{\theta}} \in K, \quad \langle \boldsymbol{\lambda}(t), \boldsymbol{u} \rangle_{\Gamma_3} = \langle \boldsymbol{\lambda}(t), g(t)\widetilde{\boldsymbol{\theta}} \rangle_{\Gamma_3},$$

which show that

$$\langle \boldsymbol{\mu}, \boldsymbol{u}(t) - g(t)\widetilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} \leq 0, \quad \langle \boldsymbol{\lambda}(t), \boldsymbol{u} - g(t)\widetilde{\boldsymbol{\theta}} \rangle_{\Gamma_3} = 0. \tag{4.30}$$

We combine now (4.29) and (4.30) to deduce that

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)). \tag{4.31}$$

We now gather equalities (4.23), (4.28) and inequality (4.31) to obtain the following variational formulation of the mechanical problem $\mathcal{P}$.

Problem 4 Find a displacement field $\boldsymbol{u}:\mathbb{R}_+ \to V$, a stress field $\boldsymbol{\sigma}:\mathbb{R}_+ \to Q$ and a Lagrange multiplier $\boldsymbol{\lambda}:\mathbb{R}_+ \to \Lambda$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{4.32}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + b(\boldsymbol{v}, \boldsymbol{\lambda}(t)) + \int_{\Gamma_3} k(\zeta \boldsymbol{u}(t)) p(u_\nu(t)) v_\nu \, da = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, \tag{4.33}$$

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \leq b(g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \forall \boldsymbol{\mu} \in \Lambda, \tag{4.34}$$

for all $t \in \mathbb{R}_+$.

Note that Problem 4 represents a mixed variational formulation which couples a nonlinear implicit integral equation for the stress field, a history-dependent variational equation for the displacement field, and a first-order time-dependent variational inequality for the Lagrange multiplier. This formulation is quite different to that in Problem 1. Nevertheless, we shall see in the next section that we can associate to Problem 4 a mixed variational formulation of the form (1.1)–(1.2) and, therefore, the analysis of Problem (4.32)–(4.34) can be carried out by using the abstract results we obtained in the previous two sections of this paper.

5 Weak solvability

In the study of Problem 4 we have the following existence and uniqueness result.

Theorem 5.1 *Assume* (4.11)–(4.18). *There exists $e_0 > 0$ which depends only on $\mathcal{E}$, Ω, Γ_1 and Γ_3 such that, if $k_0 L_p < e_0$, then Problem* 4 *has a unique solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$. *Moreover, the solution satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; X), \quad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q), \quad \boldsymbol{\lambda} \in C(\mathbb{R}_+; \Lambda). \tag{5.1}$$

The proof of Theorem 5.1 will be carried out in several steps. To this end, we assume in what follows that (4.11)–(4.18) hold. The first step is given by the following result.

Lemma 5.2 *For each function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ there exists a unique function $\boldsymbol{\sigma}^I(\boldsymbol{u}) \in C(\mathbb{R}_+; Q)$ such that*

$$\boldsymbol{\sigma}^I(\boldsymbol{u})(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}^I(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+. \tag{5.2}$$

Moreover, the operator $\boldsymbol{\sigma}^I : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ satisfies the following property: for every $n \in \mathbb{N}$ there exists $\widetilde{r}_n > 0$ such that

$$\|\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(t) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(t)\|_Q \le \widetilde{r}_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \tag{5.3}$$
$$\forall\, \boldsymbol{u}_1,\ \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].$$

Proof Let $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ and consider the operator $\mathcal{L} : C(\mathbb{R}_+; Q) \to C(\mathbb{R}_+; Q)$ defined as follows

$$\mathcal{L}\boldsymbol{\tau}(t) = \int_0^t \mathcal{G}(\boldsymbol{\tau}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{5.4}$$
$$\forall\, \boldsymbol{\tau} \in C(\mathbb{R}_+; Q),\ t \in \mathbb{R}_+.$$

The operator $\mathcal{L}$ depends on $\boldsymbol{u}$ but, for simplicity, we do not indicate explicitly this dependence.

Let $\boldsymbol{\tau}_1,\ \boldsymbol{\tau}_2 \in C(\mathbb{R}_+;\ Q)$ and let $t \in \mathbb{R}_+$. Then, using (5.4) and (4.12) we have

$$\begin{aligned}
&\|\mathcal{L}\boldsymbol{\tau}_1(t) - \mathcal{L}\boldsymbol{\tau}_2(t)\|_Q \\
&\quad \le \int_0^t \|\mathcal{G}(\boldsymbol{\tau}_1(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) - \mathcal{G}(\boldsymbol{\tau}_2(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\|_Q \, ds \\
&\quad \le L_{\mathcal{G}} \int_0^t \|\boldsymbol{\tau}_1(s) - \boldsymbol{\tau}_2(s)\|_Q \, ds.
\end{aligned}$$

Next, we use Theorem 1.1 to see that $\mathcal{L}$ has a unique fixed point in $C(\mathbb{R}_+;\ Q)$, denoted $\boldsymbol{\sigma}^I(\boldsymbol{u})$. And, finally, we combine (5.4) with equality $\mathcal{L}\boldsymbol{\sigma}^I(\boldsymbol{u}) = \boldsymbol{\sigma}^I(\boldsymbol{u})$ to see that (5.2) holds.

To proceed, let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C(\mathbb{R}_+;\ V)$, $n \in \mathbb{N}$ and let $t \in [0, n]$. Then, using (5.2) and taking into account (4.10)–(4.12) we write

$$\begin{aligned}
&\|\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(t) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(t)\|_Q \\
&\quad \le L_{\mathcal{G}}\Big(\int_0^t d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds + \int_0^t \|\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(s) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(s)\|_Q \, ds\Big) \\
&\quad = \omega\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds + \int_0^t \|\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(s) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(s)\|_Q \, ds\Big),
\end{aligned}$$

where $\omega = L_{\mathcal{G}}(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + 1)$. Using now a Gronwall argument we deduce that

$$\|\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(t) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(t)\|_Q \le \omega\, e^{n\,\omega} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds.$$

This inequality shows that (5.3) holds with $\widetilde{r}_n = \omega\, e^{n\,\omega}$. □

We now use the Riesz's representation theorem and Lemma 5.2 to define the operators $A : V \to V$ and $\mathcal{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by equalities

$$(A\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \forall\, \boldsymbol{v},\ \boldsymbol{w} \in V, \tag{5.5}$$

$$\begin{aligned}(\mathcal{R}\boldsymbol{u}(t), \boldsymbol{v})_V &= (\boldsymbol{\sigma}^I(\boldsymbol{u})(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \\ &\quad + \int_{\Gamma_3} k(\zeta\boldsymbol{u}(t))p(u_\nu(t))v_\nu \, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+.\end{aligned} \tag{5.6}$$

Then, we have the following equivalence result.

Lemma 5.3 *Let* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ *be a triple of functions with regularity* (5.1). *Then* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ *is a solution of Problem* 4 *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \boldsymbol{\sigma}^I(\boldsymbol{u})(t), \tag{5.7}$$

$$(A\boldsymbol{u}(t), \boldsymbol{v})_V + (\mathcal{R}\boldsymbol{u}(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}(t)) = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V, \tag{5.8}$$

$$b(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \le b(g(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)) \quad \forall\, \boldsymbol{\mu} \in \Lambda, \tag{5.9}$$

for all $t \in \mathbb{R}_+$.

Proof Assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ is a solution of Problem 4 and let $t \in \mathbb{R}_+$. Then, using (4.32) we have

$$\begin{aligned}&\boldsymbol{\sigma}(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\\&\quad = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\end{aligned}$$

and, using the definition (5.2) of the operator $\boldsymbol{\sigma}^I$, we obtain (5.7). Equality (5.8) follows from (4.33) combined with (5.7) and the definition of the operators A and $\mathcal{R}$ and, finally, (5.9) coincides with (4.34).

Conversely, assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ satisfies (5.7)–(5.9). Then by (5.7) and the definition (5.2) of $\boldsymbol{\sigma}^I(\boldsymbol{u})$ we obtain (4.32). Moreover, using (5.7), (5.8) and the definition of the operators A and $\mathcal{S}$ we obtain (4.33), which concludes the proof. □

We now proceed with the following existence and uniqueness result.

Lemma 5.4 *There exists $e_0 > 0$ which depends only on $\mathcal{E}$, Ω, Γ_1 and Γ_3 such that, if $k_0 L_p < e_0$, then there exists a unique couple of functions $(\boldsymbol{u}, \boldsymbol{\lambda})$, which satisfies* (5.8)–(5.9) *for all $t \in \mathbb{R}_+$. Moreover, $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ and $\boldsymbol{\lambda} \in C(\mathbb{R}_+; \Lambda)$.*

Proof We shall apply Theorem 2.1, with $X = V$, $Y = D$, $h = g\boldsymbol{\theta}$ and $\mathcal{S} : C(\mathbb{R}_+; V) \times C(\mathbb{R}_+; D) \to C(\mathbb{R}_+; V)$ given by

$$\mathcal{S}(\boldsymbol{u}, \boldsymbol{\lambda}) = \mathcal{R}(\boldsymbol{u}) \quad \forall\, (\boldsymbol{u}, \boldsymbol{\lambda}) \in C(\mathbb{R}_+; V) \times C(\mathbb{R}_+; D). \tag{5.10}$$

To this end, we use assumption (4.11) to see that the operator A defined by (5.5) verifies condition (2.1). Moreover, the bilinear form $b(\cdot, \cdot)$ is continuous and satisfies the "inf-sup" condition, i.e. there exists $\alpha > 0$ which depends only on Ω, Γ_1 and Γ_3 such that

$$\inf_{\boldsymbol{\mu} \in D,\ \boldsymbol{\mu} \neq \mathbf{0}_D} \ \sup_{\boldsymbol{v} \in V,\ \boldsymbol{v} \neq \mathbf{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V \|\boldsymbol{\mu}\|_D} \geq \alpha,$$

see [21], for instance. We conclude from here that condition (2.3) holds. Also, taking into account (4.15) and (4.22) it follows that $\boldsymbol{f} \in C(\mathbb{R}_+, V)$. Finally, since $h = g\boldsymbol{\theta}$, it follows from (4.16) that $h \in C(\mathbb{R}_+, V)$ and we conclude that condition (2.4) holds, too.

Let us now check (2.2). To this end, let $n \in \mathbb{N}$, $t \in [0, n]$ and $\boldsymbol{v} \in V$. According to the definition (5.6) of the operator $\mathcal{R}$ we have

$$\begin{aligned}(\mathcal{R}\boldsymbol{u}_1(t) - \mathcal{R}\boldsymbol{u}_2(t), \boldsymbol{v})_V &= (\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(t) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q\\&+ \int_{\Gamma_3} (k(\zeta \boldsymbol{u}_1(t)) p(u_{1\nu}(t)) - k(\zeta \boldsymbol{u}_2(t)) p(u_{2\nu}(t))) v_\nu \, da.\end{aligned}$$

Then, by a standard calculus based on the trace inequality (4.8) and the properties of the functions p and k we deduce that

$$\begin{aligned}&|(\mathcal{R}\boldsymbol{u}_1(t) - \mathcal{R}\boldsymbol{u}_2(t), \boldsymbol{v})_V| \leq \|\boldsymbol{\sigma}^I(\boldsymbol{u}_1)(t) - \boldsymbol{\sigma}^I(\boldsymbol{u}_2)(t)\|_Q \|\boldsymbol{v}\|_V\\&\quad + c_{tr}^2 k_0 L_p \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}\|_V + c_{tr}\, p_0 L_k \|\zeta \boldsymbol{u}_1(t) - \zeta \boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)^d} \|\boldsymbol{v}\|_V.\end{aligned}$$

Using now estimate (5.3) in Lemma 5.2 and the definition of the function ζ we can write

$$|(\mathcal{R}\boldsymbol{u}_1(t) - \mathcal{R}\boldsymbol{u}_2(t), \boldsymbol{v})_V| \leq \widetilde{r}_n\left(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|\, ds\right)\|\boldsymbol{v}\|_V$$
$$+ c_{tr}^2\left(k_0 L_p\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + p_0 L_k\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\right)\|\boldsymbol{v}\|_V$$

and, therefore,

$$\|\mathcal{R}\boldsymbol{u}_1(t) - \mathcal{R}\boldsymbol{u}_2(t)\|_V \leq c_{tr}^2 k_0 L_p\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + (\widetilde{r}_n + c_{tr}^2 p_0 L_k)\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds. \tag{5.11}$$

Inequality (5.11) combined with definition (5.10) shows that the operator $\mathcal{S}$ satisfies condition (2.2) with $d_n = c_{tr}^2 k_0 L_p$ and $r_n = \widetilde{r}_n + c_{tr}^2 p_0 L_k$.

We are now in position to apply Theorem 2.1. According to this theorem there exists $d_0 > 0$ which depends only on A and b such that if $d_n < d_0$ for all positive integers n, then there exists a unique couple of functions $(\boldsymbol{u}, \boldsymbol{\lambda})$ which satisfies (5.8)–(5.9) for all $t \in \mathbb{R}_+$. We now take

$$e_0 = d_0 c_{tr}^{-2} \tag{5.12}$$

which, clearly, depends only on $\mathcal{E}$, Ω, Γ_1 and Γ_3. We note that $d_n < d_0$ iff $k_0 L_p \leq e_0$ which concludes the proof. □

We now have all the ingredients to prove Theorem 5.1.

Proof Let e_0 be defined by (5.12) and assume that $k_0 L_p < e_0$. Under this condition, Lemma 5.4 implies that there exists a unique couple of functions $(\boldsymbol{u}, \boldsymbol{\lambda})$, such that (5.8)–(5.9) hold, for all $t \in \mathbb{R}_+$. Define $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\sigma}^I(\boldsymbol{u})$ and note that, obviously, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Then the triple $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ represents a solution to problem (5.7)–(5.9) with regularity (5.1). The existence part of the theorem follows now from Lemma 5.3. The uniqueness part follows from Lemma 5.3 combined with the uniqueness of the solution of the system (5.8)–(5.9) guaranteed by Lemma 5.4. □

We end this section with the remark that the inequality $k_0 L_p < e_0$, which guarantees uniqueness solvability of Problem 4, represents a smallness condition on the normal compliance function p and the stiffness function k. It is satisfied if, for instance, either the Lipschitz constant L_p or the bound k_0 is small enough.

6 Continuous dependence with respect to the data

In this section we study the behavior of the solution of Problem 4 with respect to a perturbation of the data. To this end we assume in what follows that (4.11)–(4.18) hold and $k_0 L_p < e_0$, where e_0 is defined in Theorem 5.1. Also, we denote by $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ the solution of Problem 4 obtained in Theorem 5.1. In addition, for each $\rho > 0$ we denote by $\mathcal{G}_\rho$, p_ρ, k_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, g_ρ, $\boldsymbol{u}_{0\rho}$, $\boldsymbol{\sigma}_{0\rho}$ a perturbation of $\mathcal{G}$, p, k, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, g, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$, respectively, which satisfies the following conditions.

$$\begin{cases} \text{(a) } \mathcal{G}_\rho : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_\mathcal{G}^\rho > 0 \text{ such that} \\ \quad \|\mathcal{G}_\rho(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}_\rho(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \qquad \le L_\mathcal{G}^\rho \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right) \\ \qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}_\rho(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable in } \Omega, \\ \quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}_\rho(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{6.1}$$

$$\begin{cases} \text{(a) } p_\rho : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p^\rho > 0 \text{ such that} \\ \quad |p_\rho(r_1) - p_\rho(r_2)| \le L_p^\rho |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}. \\ \text{(c) There exists } p_0^\rho > 0 \text{ such that } |p_\rho(r)| \le p_0^\rho \quad \forall\, r \in \mathbb{R}. \end{cases} \tag{6.2}$$

$$\begin{cases} \text{(a) } k_\rho : \mathbb{R}_+ \to \mathbb{R}_+. \\ \text{(b) There exists } L_k^\rho > 0 \text{ such that} \\ \quad |k_\rho(r_1) - k_\rho(r_2)| \le L_k^\rho |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}. \\ \text{(c) There exists } k_0^\rho > 0 \text{ such that } |k_\rho(r)| \le k_0^\rho \quad \forall\, r \in \mathbb{R}. \end{cases} \tag{6.3}$$

$$\boldsymbol{f}_{0\rho} \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_{2\rho} \in C(\mathbb{R}_+; L^2(\Gamma_2)^d), \tag{6.4}$$

$$g_\rho \in C(\mathbb{R}_+; \mathbb{R}_+) \tag{6.5}$$

$$\boldsymbol{u}_{0\rho} \in V, \quad \boldsymbol{\sigma}_{0\rho} \in Q. \tag{6.6}$$

With these data we define the function $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equality

$$(\boldsymbol{f}_\rho(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0\rho}(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_{2\rho}(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+ \tag{6.7}$$

and we consider the following problem.

Problem 5 Find a displacement field $\boldsymbol{u}_\rho : \mathbb{R}_+ \to V$, a stress field $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$ and a Lagrange multiplier $\boldsymbol{\lambda}_\rho : \mathbb{R}_+ \to \Lambda$ such that

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \int_0^t \mathcal{G}_\rho(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)))\, ds + \boldsymbol{\sigma}_{0\rho} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}), \tag{6.8}$$

$$(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + b(\boldsymbol{v}, \boldsymbol{\lambda}_\rho(t)) + \int_{\Gamma_3} k_\rho(\zeta \boldsymbol{u}_\rho(t)) p_\rho(u_{\rho\nu}(t)) v_\nu\, da \tag{6.9}$$

$$= (\boldsymbol{f}_\rho(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,$$

$$b(\boldsymbol{u}_\rho(t), \boldsymbol{\mu} - \boldsymbol{\lambda}_\rho(t)) \le b(g_\rho(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}_\rho(t)) \quad \forall\, \mu \in \Lambda, \tag{6.10}$$

for all $t \in \mathbb{R}_+$.

Note that, here and below, $u_{\rho\nu}(t)$ represents the normal component of the function $\boldsymbol{u}_\rho(t)$, i.e. $u_{\rho\nu}(t) = \boldsymbol{u}_\rho(t) \cdot \boldsymbol{\nu}$, for all $t \in \mathbb{R}_+$.

Under the assumptions above, if $k_0^\rho L_p^\rho < e_0$, Theorem 5.1 guarantees the existence of a unique solution $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\lambda}_\rho)$ to Problem 5 such that

$$\boldsymbol{u}_\rho \in C(\mathbb{R}_+; V), \quad \boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q), \quad \boldsymbol{\lambda}_\rho \in C(\mathbb{R}_+; \Lambda). \tag{6.11}$$

Our interest in what follows lies in the behavior of the solution as ρ tends to zero. To this end we consider the following additional assumptions.

$$\left\{\begin{array}{l} \text{There exist } L_0 > 0,\ \widetilde{p}_0 > 0,\ \widetilde{k}_0 > 0,\ \widetilde{e}_0 > 0 \text{ and } K_0 > 0 \text{ such that} \\ \text{(a) } L_{\mathcal{G}}^{\rho} < L_0 \text{ for all } \rho > 0. \\ \text{(b) } p_0^{\rho} \le \widetilde{p}_0 \text{ for all } \rho > 0. \\ \text{(c) } k_0^{\rho} \le \widetilde{k}_0 \text{ for all } \rho > 0. \\ \text{(d) } k_0^{\rho} L_p^{\rho} \le \widetilde{e}_0 < e_0 \text{ for all } \rho > 0. \\ \text{(e) } L_k^{\rho} \le K_0 \text{ for all } \rho > 0. \end{array}\right. \tag{6.12}$$

$$\left\{\begin{array}{l} \text{There exist } M,\ N,\ P : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \\ \text{(a) } \|\mathcal{G}_\rho(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})\| \le M(\rho)(\|\boldsymbol{\sigma}\| + \|\boldsymbol{\varepsilon}\| + 1) \\ \qquad \text{for all } \boldsymbol{\sigma},\ \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } |p_\rho(r) - p(r)| \le N(\rho)(|r| + 1) \quad \text{for all } r \in \mathbb{R}. \\ \text{(c) } |k_\rho(r) - k(r)| \le P(\rho)(|r| + 1) \quad \text{for all } r \in \mathbb{R}. \\ \text{(d) } \lim_{\rho\to 0} M(\rho) = 0, \quad \lim_{\rho\to 0} N(\rho) = 0, \quad \lim_{\rho\to 0} P(\rho) = 0. \end{array}\right. \tag{6.13}$$

$$\left\{\begin{array}{l} \text{(a) } \boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0. \\ \text{(b) } \boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \\ \text{(c) } g_\rho \to g \quad \text{in} \quad C(\mathbb{R}_+; \mathbb{R}) \quad \text{as} \quad \rho \to 0. \\ \text{(d) } \boldsymbol{u}_{0\rho} \to \boldsymbol{u}_0 \quad \text{in} \quad V \quad \text{as} \quad \rho \to 0. \\ \text{(e) } \boldsymbol{\sigma}_{0\rho} \to \boldsymbol{\sigma}_0 \quad \text{in } Q \quad \text{as} \quad \rho \to 0. \end{array}\right. \tag{6.14}$$

Our main result in this section is the following.

Theorem 6.1 *Assume* (6.12)–(6.14). *Then the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\lambda}_\rho)$ *of Problem* 5 *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\lambda})$ *of Problem* 4, *i.e.*

$$\left\{\begin{array}{l} \text{(a) } \boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V) \quad \text{as} \quad \rho \to 0. \\ \text{(b) } \boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in} \quad C(\mathbb{R}_+; Q) \quad \text{as} \quad \rho \to 0. \\ \text{(c) } \boldsymbol{\lambda}_\rho \to \boldsymbol{\lambda} \quad \text{in} \quad C(\mathbb{R}_+; D) \quad \text{as} \quad \rho \to 0. \end{array}\right. \tag{6.15}$$

Proof We use the operators A, $\boldsymbol{\sigma}^I$, $\mathcal{R}$ and $\mathcal{S}$ defined by (5.5), (5.2), (5.6) and (5.10), respectively. Moreover, for each $\rho > 0$ we define the operators $\boldsymbol{\sigma}_\rho^I : C(\mathbb{R}_+, V) \to C(\mathbb{R}_+, Q)$, $\mathcal{R}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ and $\mathcal{S}_\rho : C(\mathbb{R}_+; V) \times C(\mathbb{R}_+; D) \to C(\mathbb{R}_+; V)$ by equalities

$$\boldsymbol{\sigma}_\rho^I(\boldsymbol{u})(t) = \int_0^t \mathcal{G}_\rho(\boldsymbol{\sigma}_\rho^I(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_{0\rho} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}), \tag{6.16}$$

$$(\mathcal{R}_\rho \boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{\sigma}_\rho^I(\boldsymbol{u})(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} k_\rho(\zeta \boldsymbol{u}(t)) p_\rho(u_\nu(t)) v_\nu\, da, \tag{6.17}$$

$$\mathcal{S}_\rho(\boldsymbol{u}, \boldsymbol{\lambda}) = \mathcal{R}_\rho(\boldsymbol{u}) \tag{6.18}$$

for all $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $t \in \mathbb{R}_+$, $\boldsymbol{v} \in V$ and $\boldsymbol{\lambda} \in C(\mathbb{R}_+; D)$. Then, Lemma 5.3 states that (5.7)–(5.9) hold for all $t \in \mathbb{R}_+$ and, moreover,

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \boldsymbol{\sigma}_\rho^I(\boldsymbol{u}_\rho)(t), \tag{6.19}$$

$$(A\boldsymbol{u}_\rho(t), \boldsymbol{v})_V + (\mathcal{R}_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}_\rho(t)) = (\boldsymbol{f}_\rho(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, \tag{6.20}$$

$$b(\boldsymbol{u}_\rho(t), \boldsymbol{\mu} - \boldsymbol{\lambda}_\rho(t)) \le b(g_\rho(t)\widetilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}_\rho(t)) \quad \forall \boldsymbol{\mu} \in \Lambda, \tag{6.21}$$

for all $t \in \mathbb{R}_+$. We note that the system (5.8)–(5.9) is of the form (1.1)–(1.2) with $\mathcal{S}$ given by (5.10) while the system (6.20)–(6.21) is of the form (3.1)–(3.2) with $\mathcal{S}_\rho$ given by (6.18).

Therefore, in order to apply Theorem 3.1, we check in what follows the validity of the conditions (3.5)–(3.7).

To start, we fix $\rho > 0$, $n \in \mathbb{N}$, $t \in [0, n]$, $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\lambda} \in C(\mathbb{R}_+; D)$ and $\boldsymbol{v} \in V$. We write

$$|(\mathcal{R}_\rho \boldsymbol{u}(t) - \mathcal{R}\boldsymbol{u}(t), \boldsymbol{v})_V| \le \|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(t) - \boldsymbol{\sigma}^I(\boldsymbol{u})(t)\|_Q \|\boldsymbol{v}\|_V + \int_{\Gamma_3} \Big(k_\rho(\zeta \boldsymbol{u}(t)) p_\rho(u_\nu(t)) - k(\zeta \boldsymbol{u}(t)) p(u_\nu(t))\Big) v_\nu \, da. \tag{6.22}$$

We now use (6.16), (5.2) to see that

$$\|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(t) - \boldsymbol{\sigma}^I(\boldsymbol{u})(t)\|_Q \le \int_0^t \|\mathcal{G}_\rho(\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) - \mathcal{G}(\boldsymbol{\sigma}^I(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\|_Q \, ds + \|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_Q$$

and, therefore, (4.10) yields

$$\begin{aligned} &\|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(t) - \boldsymbol{\sigma}^I(\boldsymbol{u})(t)\|_Q \\ &\quad \le \int_0^t \|\mathcal{G}_\rho(\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) - \mathcal{G}_\rho(\boldsymbol{\sigma}^I(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\|_Q \, ds \\ &\qquad + \int_0^t \|\mathcal{G}_\rho(\boldsymbol{\sigma}^I(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))) - \mathcal{G}(\boldsymbol{\sigma}^I(\boldsymbol{u})(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\|_Q \, ds \\ &\qquad + \|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + d\, \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V. \end{aligned}$$

Using now (6.1), (6.12)(a), (6.13)(a) and (4.10) we obtain

$$\|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(t) - \boldsymbol{\sigma}^I(\boldsymbol{u})(t)\|_Q \le L_0 \int_0^t \|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(s) - \boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q \, ds + c\, M(\rho) \int_0^t (\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q + \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\|_Q + 1)\, ds + c\left(\|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V\right)$$

where, here and below, c is a positive constant which does not depend on ρ and n, and whose value will change from line to line. Applying now a Gronwall argument and using the inequality $t \le n$ we have

$$\|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u})(t) - \boldsymbol{\sigma}^I(\boldsymbol{u})(t)\|_Q \le c\Bigg(\|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V + M(\rho) \int_0^n (\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q + \|\boldsymbol{u}(s)\|_V + 1)\, ds\Bigg) e^{L_0 n}. \tag{6.23}$$

We turn now to the second term into the right side of the inequality (6.22). We write

$$\int_{\Gamma_3}\Big(k_\rho(\zeta\boldsymbol{u}(t))p_\rho(u_\nu(t))-k(\zeta\boldsymbol{u}(t))p(u_\nu(t))\Big)v_\nu\,da$$
$$\leq\int_{\Gamma_3}\Big(k_\rho(\zeta\boldsymbol{u}(t))p_\rho(u_\nu(t))-k_\rho(\zeta\boldsymbol{u}(t))p(u_\nu(t))\Big)v_\nu\,da$$
$$+\int_{\Gamma_3}\Big(k_\rho(\zeta\boldsymbol{u}(t))p(u_\nu(t))-k(\zeta\boldsymbol{u}(t))p(u_\nu(t))\Big)v_\nu\,da$$

and, using (6.3)(c), (6.12)(c) and (4.13)(c) we find that

$$\int_{\Gamma_3}\Big(k_\rho(\zeta\boldsymbol{u}(t))p_\rho(u_\nu(t))-k(\zeta\boldsymbol{u}(t))p(u_\nu(t))\Big)v_\nu\,da$$
$$\leq c\int_{\Gamma_3}|p_\rho(u_\nu(t))-p(u_\nu(t))||v_\nu|\,da+c\int_{\Gamma_3}|k_\rho(\zeta\boldsymbol{u}(t))-k(\zeta\boldsymbol{u}(t))||v_\nu|\,da.$$

Then, using (6.13)(b),(c) and the trace inequality (4.8), after some algebra we obtain that

$$\int_{\Gamma_3}\Big(k_\rho(\zeta\boldsymbol{u}(t))p_\rho(u_\nu(t))-k(\zeta\boldsymbol{u}(t))p(u_\nu(t))\Big)v_\nu\,da \tag{6.24}$$
$$\leq c\Big(N(\rho)+P(\rho)+N(\rho)\max_{r\in[0,n]}\|\boldsymbol{u}(r)\|_V+P(\rho)\int_0^n\|\boldsymbol{u}(s)\|_V\,ds\Big)\|\boldsymbol{v}\|_V.$$

We now combine the inequalities (6.22)–(6.24) to deduce that

$$|(\mathcal{R}_\rho\boldsymbol{u}(t)-\mathcal{R}\boldsymbol{u}(t),\boldsymbol{v})_V|\leq c\Big(\|\boldsymbol{\sigma}_{0\rho}-\boldsymbol{\sigma}_0\|_Q+\|\boldsymbol{u}_{0\rho}-\boldsymbol{u}_0\|_V$$
$$+M(\rho)\int_0^n(\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q+\|\boldsymbol{u}(s)\|_V+1)\,ds\Big)e^{L_0n}\|\boldsymbol{v}\|_V$$
$$+c\Big(N(\rho)+P(\rho)+N(\rho)\max_{r\in[0,n]}\|\boldsymbol{u}(r)\|_V+P(\rho)\int_0^n\|\boldsymbol{u}(s)\|_V\,ds\Big)\|\boldsymbol{v}\|_V.$$

Therefore, since $\boldsymbol{v}$ is an arbitrary element in V, we find that

$$\|\mathcal{R}_\rho\boldsymbol{u}(t)-\mathcal{R}\boldsymbol{u}(t)\|_V\leq c\Big(\|\boldsymbol{\sigma}_{0\rho}-\boldsymbol{\sigma}_0\|_Q+\|\boldsymbol{u}_{0\rho}-\boldsymbol{u}_0\|_V+N(\rho)+P(\rho)\Big)$$
$$+c\,M(\rho)\Big(\int_0^n(\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q+\|\boldsymbol{u}(s)\|_V+1)\,ds\Big)e^{L_0n}$$
$$+c\Big(N(\rho)\max_{r\in[0,n]}\|\boldsymbol{u}(r)\|_V+P(\rho)\int_0^n\|\boldsymbol{u}(s)\|_V\,ds\Big).$$

Denote

$$\widetilde{J}_n(\boldsymbol{u}) = 1 + e^{L_0 n} \int_0^n (\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q + \|\boldsymbol{u}(s)\|_V + 1)\, ds$$
$$+ \max_{r\in[0,n]} \|\boldsymbol{u}(r)\|_V + \int_0^n \|\boldsymbol{u}(s)\|_V\, ds.$$

Then, it follows from the previous inequality that

$$\|\mathcal{R}_\rho \boldsymbol{u}(t) - \mathcal{R}\boldsymbol{u}(t)\|_V$$
$$\leq c\left(\|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V + M(\rho) + N(\rho) + P(\rho)\right)\widetilde{J}_n(\boldsymbol{u}).$$

Therefore, denoting

$$\widetilde{H}(\rho) = c\left(\|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V + M(\rho) + N(\rho) + P(\rho)\right), \tag{6.25}$$

we deduce that

$$\|\mathcal{R}_\rho \boldsymbol{u}(t) - \mathcal{R}\boldsymbol{u}(t)\|_V \leq \widetilde{H}(\rho)\widetilde{J}_n(\boldsymbol{u}). \tag{6.26}$$

We now combine equalities (6.18), (5.10) and inequality (6.26) to conclude that condition (3.5)(a) holds with $H_n(\rho) = \widetilde{H}(\rho)$ and $J_n(\boldsymbol{u}, \boldsymbol{\lambda}) = \widetilde{J}_n(\boldsymbol{u})$.

Also, the proofs of Lemmas 5.4 and 5.2 show that the operator $\mathcal{S}_\rho$ satisfies condition (3.3) with

$$d_{\rho n} = c_{tr}^2 k_0^\rho L_p^\rho \qquad \text{and} \qquad r_{\rho n} = \widetilde{r}_{\rho n} + c_{tr}^2 p_0^\rho L_k^\rho \tag{6.27}$$

where

$$\widetilde{r}_{\rho n} = L_{\mathcal{G}}^\rho (d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + 1)e^{n\,L_{\mathcal{G}}^\rho (d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}+1)}.$$

Therefore, using assumption (6.12)(a),(b) and (e) it follows that

$$r_{\rho n} \leq L_0(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + 1)e^{nL_0(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}+1)} + c_{tr}^2\,\widetilde{p}_0 K_0.$$

We conclude from above that condition (3.5)(b) holds with

$$R_n = L_0(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + 1)e^{nL_0(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}+1)} + c_{tr}^2\,\widetilde{p}_0 K_0.$$

In addition, using assumptions (6.13)(d), (6.14)(d), (e), we deduce that the function $H_n(\rho) = \widetilde{H}(\rho)$ defined by (6.25) is such that

$$\lim_{\rho\to 0} H_n(\rho) = 0.$$

Therefore, condition (3.5)(c) is satisfied, too.

Moreover, using assumption (6.12)(d) and equality (5.12) we deduce that

$$c_{tr}^2 k_0^\rho L_p^\rho \leq c_{tr}^2 \widetilde{e}_0 < d_0 \quad \forall\, n \in \mathbb{N},\ \forall\, \rho > 0$$

and, using the first equality in (6.27) we obtain that

$$d_{\rho n} \leq c_{tr}^2 \widetilde{e}_0 < d_0 \quad \forall\, n \in \mathbb{N},\ \forall\, \rho > 0.$$

We conclude from here that (3.6) holds with $\widetilde{d}_0 = c_{tr}^2 \widetilde{e}_0$.

Finally, we note that assumption (6.14)(a)–(c) imply that

$$\boldsymbol{f}_\rho \to \boldsymbol{f} \quad \text{in} \quad C(\mathbb{R}_+; V), \quad g_\rho \widetilde{\boldsymbol{\theta}} \to g\widetilde{\boldsymbol{\theta}} \quad \text{in} \quad C(\mathbb{R}_+; V)$$

as $\rho \to 0$ and, therefore, (3.7) holds.

The convergence (6.15)(a) and (c) represent now a direct consequence of Theorem 3.1, applied for the systems (6.20)–(6.21) and (5.8)–(5.9).

Next, to provide the convergence (6.15)(b) we use equalities (6.16), (5.2), assumptions (6.1)(b), (6.12)(a) and arguments similar to those used in the proof of (6.23) to find that

$$\|\boldsymbol{\sigma}^I_\rho(\boldsymbol{u}_\rho)(t) - \boldsymbol{\sigma}^I(\boldsymbol{u})(t)\|_Q \le c\Bigg(\|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V + \int_0^n \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\, ds$$

$$+ M(\rho)\int_0^n (\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q + \|\boldsymbol{u}(s)\|_V + 1)\, ds\Bigg)e^{L_0 n}.$$

Then, using (6.19), (5.7) and (4.10) we deduce that

$$\begin{aligned}
&\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \le c\,\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \\
&\quad + c\Bigg(\|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V + \int_0^n \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\, ds \\
&\qquad + M(\rho)\int_0^n \big(\|\boldsymbol{\sigma}^I(\boldsymbol{u})(s)\|_Q + \|\boldsymbol{u}(s)\|_V + 1\big)\, ds\Bigg)e^{L_0 n}.
\end{aligned} \tag{6.28}$$

We now combine inequality (6.28) with assumptions (6.14)(d), (e), (6.13)(d) and the convergence (6.15)(a). As a result we obtain that

$$\max_{t\in[0,n]} \|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \quad \text{as} \quad \rho \to 0. \tag{6.29}$$

Finally, we use (6.29) and (1.3) to deduce that (6.15)(b) holds, which concludes the proof. □

In addition to the mathematical interest in the convergence result (6.15) it is of importance from mechanical point of view, since it states that the weak solution of the problem (4.1)–(4.7) depends continuously on the viscoplastic function, the normal compliance function, the stiffness coefficient, the penetration bound, the densities of body forces and surface tractions, and the initial data, as well.

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