SELF-CONCORDANCE IS NP-HARD

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ABSTRACT. We give an elementary proof of a somewhat curious result, namely, that deciding whether a convex function is self-concordant is in general an intractable problem.

1. Introduction

Nesterov and Nemirovskii [20] famously showed that the optimal solution of a conic programming problem can be computed to ε -accuracy in polynomial time if the cone has a self-concordant barrier function whose gradient and Hessian are both computable in polynomial time. Their work established self-concordance as a singularly important notion in modern optimization theory.

We show in this article that deciding whether a convex function is self-concordant at a point is nonetheless an NP-hard problem. In fact we will prove that deciding the self-concordance of a convex function defined locally by a cubic polynomial (which cannot be convex on all of \mathbb{R}^n), arguably the simplest non-trivial instance, is already an NP-hard problem. This is not so surprising given that deciding the convexity of a quartic polynomial, arguably the simplest non-trivial instance of deciding convexity, is also NP-hard — a much harder recent result [1]. In addition to the NP-hardness of self-concordance, we will see that there is no fully polynomial time approximation scheme for the optimal self-concordant parameter and that deciding second-order self-concordance [14] of a quartic polynomial is also an NP-hard problem.

These hardness results are intended only to add to our understanding of self-concordance. They do not in anyway detract from the usefulness of the notion since in practice self-concordant barriers are *constructed* at the outset to have the requisite property [20, Chapter 5]. It is unlikely that one would ever need to generate random functions and then test them for self-concordance.

We deduce the NP-hardness of self-concordance using a result of Nesterov himself, namely, minimizing a cubic form over a sphere is in general an NP-hard problem. Nesterov's result, which appears in an unpublished manuscript [19], contains some minor errors that have unfortunately been widely reproduced. We take the opportunity here to correct them and also to prove a variant of Nesterov's result directly from Motzkin–Straus Theorem [18].

2. Self-concordance in terms of tensors

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f: \Omega \to \mathbb{R}$ be in $C^d(\Omega)$, i.e., has continuous partials up to at least order d. Recall that the dth order derivative of f at $x \in \Omega$, denoted $\nabla^d f(x)$, is a *tensor* of order d [15]. To be more precise, this simply means that $\nabla^d f(x)$ is a multilinear functional on $T_x(\Omega)$, the tangent space of Ω at x, that is,

$$\nabla^d f(x) : \underbrace{T_x(\Omega) \times \cdots \times T_x(\Omega)}_{d \text{ copies}} \to \mathbb{R},$$

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where

$$\nabla^d f(x)(h_1, \dots, \alpha h_i + \beta h'_i, \dots, h_n) = \alpha \nabla^d f(x)(h_1, \dots, h_i, \dots, h_n) + \beta \nabla^d f(x)(h_1, \dots, h'_i, \dots, h_n)$$
 for $i = 1, \dots, n$.

With respect to the standard basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ of $T_x(\Omega)$, we may identify $T_x(\Omega) \cong \mathbb{R}^n$ and $\nabla^d f(x)$ may be regarded as a 'd-dimensional matrix' (d-hypermatrix for short),

$$\nabla^d f(x) = [a_{i_1 \cdots i_d}]_{i_1, \dots, i_d = 1}^n \in \mathbb{R}^{n \times \cdots \times n}.$$

Indeed, we must have

$$a_{i_1\cdots i_d} = \frac{\partial^d f(x)}{\partial x_{i_1}\cdots \partial x_{i_d}},$$

and since $f \in C^d(\Omega)$, we get that $a_{i_1 \cdots i_d} = a_{i_{\sigma(1)} \cdots i_{\sigma(d)}}$, i.e., $\nabla^d f(x)$ is a *symmetric d*-hypermatrix. Every symmetric *d*-hypermatrix $A = [a_{i_1 \cdots i_d}]_{i_1, \dots, i_d=1}^n \in \mathbb{R}^{n \times \dots \times n}$ defines a homogeneous polynomial of degree *d*, denoted

$$A(h,...,h) := \sum_{i_1,...,i_d=1}^n a_{i_1...i_d} h_{i_1} \cdots h_{i_d} \in \mathbb{R}[h_1,...,h_n]_d.$$

Readers should be able to infer from the above discussion that d-hypermatrices are coordinate representations of d-tensors, just as matrices are coordinate representations of linear operators and bilinear forms (both are 2-tensors). We refer the reader to [17] for more information.

The usual definition of self-concordance requires that $f \in C^3(\Omega)$ and in which case it is given by a condition involving the matrix $\nabla^2 f(x) \in \mathbb{R}^{n \times n}$ and the 3-hypermatrix $\nabla^3 f(x) \in \mathbb{R}^{n \times n \times n}$.

Definition 2.1 (Nesterov–Nemirovskii). Let $\Omega \subseteq \mathbb{R}^n$ is a convex open set. Then $f: \Omega \to \mathbb{R}$ is said to be **self-concordant** with parameter $\sigma > 0$ at $x \in \Omega$ if

$$(2.1) \nabla^2 f(x)(h,h) \ge 0$$

and

$$(2.2) [\nabla^3 f(x)(h, h, h)]^2 \le 4\sigma [\nabla^2 f(x)(h, h)]^3$$

for all $h \in \mathbb{R}^n$; f is self-concordant on Ω if (2.1) and (2.2) hold for all $x \in \Omega$. The set of self-concordant functions on Ω with parameter σ is denoted by $S_{\sigma}(\Omega)$.

By (2.1), a function self-concordant on Ω is necessarily convex on Ω . A minor deviation from [20] is that σ above is really the reciprocal of the self-concordance parameter as defined in [20, Definition 2.1.1]. Our hardness results would be independent of the choice of σ . Note that

$$\nabla^2 f(x)(h,h) = \sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} h_i h_j, \quad \nabla^3 f(x)(h,h,h) = \sum_{i,j,k=1}^n \frac{\partial^3 f(x)}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k.$$

So for a fixed $x \in \Omega$, $\nabla^2 f(x)(h,h)$ is a quadratic form in h and $\nabla^3 f(x)(h,h,h)$ is a cubic form in h. It is well-known that deciding (2.1) at any fixed x is a polynomial-time problem (but not so for deciding it over all $x \in \Omega$, see [1]). Hence given a $\sigma > 0$, deciding self-concordance at x essentially boils down to (2.2): Is the square of a given cubic form globally bounded above by the cube of a given quadratic form? We shall see in the next sections that this decision problem is NP-hard.

While we shall think of $\nabla^2 f(x)$ as a matrix and $\nabla^3 f(x)$ as a hypermatrix thoughout this article, we nonetheless wish to highlight that condition (2.2) is really a condition on $\nabla^2 f(x)$ regarded as a 2-tensor and $\nabla^3 f(x)$ regarded as a 3-tensor; that is, (2.2) is independent of the choice of coordinates, a property that follows from the *affine invariance* of self-concordance [20, Proposition 2.1.1]. Self-concordance on Ω is then a global condition about the tensor fields $\nabla^3 f$ and $\nabla^2 f$.

3. Maximizing a cubic form over a sphere

We will include a proof that the clique and stability numbers of a graph with n vertices and m edges may be expressed as the maximal values of cubic forms (in n+m variables) over the unit sphere \mathbb{S}^{n+m-1} . This, or at least the stability number version, is known but the reference [19, Theorem 4] usually cited contains some slight errors that have been reproduced several times elsewhere¹. We take the opportunity to provide a corrected version below. Our proof follows Motzkin–Straus Theorem [18] and the similar result of Nesterov [19, Theorem 4] for stability number.

Let G = (V, E) be an undirected graph with n vertices and m edges. We shall require that $E \neq \emptyset$ throughout, so $n \geq 2$ and $m \geq 1$. Recall that $S \subseteq V$ is a *clique* in G if $\{i, j\} \in E$ for all $i, j \in S$ and $S \subseteq V$ is *stable* in G if $\{i, j\} \notin E$ for all $i, j \in S$. The *clique number* and *stability number* of G are respectively:

$$\omega(G) = \max\{|S| : S \subseteq V \text{ is clique}\}, \quad \alpha(G) = \max\{|S| : S \subseteq V \text{ is stable}\}.$$

Motzkin and Straus [18] showed that $\omega(G)$ may be expressed in terms of the maximal value of a simple quadratic polynomial over the unit simplex. Although not in [18], it is straightforward to see that essentially the same proof also yields a similar expression for $\alpha(G)$.

Theorem 3.1 (Motzkin–Straus). Let $\Delta^n = \{x \in \mathbb{R}^n : x_1 + \dots + x_n = 1, x_i \geq 0\}$ denote the unit simplex in \mathbb{R}^n . Then the clique number $\omega(G)$ and stability number $\alpha(G)$ may be determined via quadratic optimization over simplices:

(3.1)
$$1 - \frac{1}{\omega(G)} = 2 \max_{x \in \Delta^n} \sum_{\{i,j\} \in E} x_i x_j, \quad 1 - \frac{1}{\alpha(G)} = 2 \max_{x \in \Delta^n} \sum_{\{i,j\} \notin E} x_i x_j.$$

Since deciding if a clique of a given size exists is an NP-complete problem [11], an immediate consequence is that clique number is NP-hard and by Motzkin–Straus Theorem, so is quadratic maximization over a simplex.

In an unpublished manuscript [19, Theorem 4], Nesterov used Motzkin–Straus Theorem to obtain an alternate expression (3.3) for stability number involving the maximal value of a cubic form over a sphere. In the following we derive a similar expression (3.2) for the clique number, which yields slightly simpler expressions for our discussions in Sections 5 and 7, and may perhaps be of independent interest.

Theorem 3.2 (Nesterov). Let G = (V, E) be an undirected graph with n vertices and m edges. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : ||x|| = 1\}$ denote the unit ℓ^2 -sphere in \mathbb{R}^d . The clique number $\omega(G)$ and stability number $\alpha(G)$ may be determined via cubic optimization over spheres:

(3.2)
$$1 - \frac{1}{\omega(G)} = \frac{27}{2} \max_{(u,v) \in \mathbb{S}^{n+m-1}} \left[\sum_{\{i,j\} \in E} u_i u_j w_{ij} \right]^2,$$

(3.3)
$$1 - \frac{1}{\alpha(G)} = \frac{27}{2} \max_{(u,v) \in \mathbb{S}^{n+m-1}} \left[\sum_{\{i,j\} \notin E} u_i u_j w_{ij} \right]^2.$$

Proof. This follows from Motzkin–Straus Theorem and the equalities

(3.4)
$$\max_{x \in \Delta^n} \sum_{\{i,j\} \in E} x_i x_j = \max_{u \in \mathbb{S}^{n-1}} \sum_{\{i,j\} \in E} u_i^2 u_j^2$$

(3.5)
$$= \max_{u \in \mathbb{S}^{n-1}, w \in \mathbb{S}^{m-1}} \left[\sum_{\{i,j\} \in E} u_i u_j w_{ij} \right]^2$$

(3.6)
$$= \frac{27}{4} \max_{(u,v) \in \mathbb{S}^{n+m-1}} \left[\sum_{\{i,j\} \in E} u_i u_j w_{ij} \right]^2.$$

¹For example, [5, Theorem 3.4]. To see that the expression is incorrect, take a graph with three vertices and one edge, the left-hand side gives $1/\sqrt{2}$ and the right-hand side gives 1.

(3.4) comes from substituting $x_i = u_i^2$, i = 1, ..., n. Cauchy-Schwartz yields

$$\sum_{\{i,j\}\in E} u_i u_j w_{ij} \le \left(\sum_{\{i,j\}\in E} u_i^2 u_j^2\right)^{1/2} \left(\sum_{\{i,j\}\in E} w_{ij}^2\right)^{1/2}$$

and so

$$\max_{\|u\|=\|w\|=1} \sum_{\{i,j\}\in E} u_i u_j w_{ij} \leq \max_{\|u\|=1} \left(\sum_{\{i,j\}\in E} u_i^2 u_j^2 \right)^{1/2} \max_{\|w\|=1} \left(\sum_{\{i,j\}\in E} w_{ij}^2 \right)^{1/2} \\
= \max_{\|u\|=1} \left(\sum_{\{i,j\}\in E} u_i^2 u_j^2 \right)^{1/2} =: \alpha.$$
(3.7)

Let the maximal value α be attained at $\overline{u} \in \mathbb{S}^{n-1}$. We set $\overline{w}_{ij} = \overline{u}_i \overline{u}_j / \alpha$ for all $\{i, j\} \in E$ (note that $\alpha > 0$ if $E \neq \emptyset$). Observe that

$$\sum_{\{i,j\}\in E} \overline{u}_i \overline{u}_j \overline{w}_{ij} = \frac{1}{\alpha} \sum_{\{i,j\}\in E} \overline{u}_i^2 \overline{u}_j^2 = \alpha,$$

and $\overline{w} \in \mathbb{S}^{m-1}$ since

$$\sum\nolimits_{\{i,j\}\in E}\overline{w}_{ij}^2=\frac{1}{\alpha^2}\sum\nolimits_{\{i,j\}\in E}\overline{u}_i^2\overline{u}_j^2=1.$$

Hence equality is attained in (3.7) and we have (3.5). We deduce (3.6) from

$$\max_{\|(u,w)\|=1} \sum_{\{i,j\}\in E} u_{i}u_{j}w_{ij} = \max_{\|u\|^{2} + \|w\|^{2} = 1} \sum_{\{i,j\}\in E} u_{i}u_{j}w_{ij}$$

$$= \sup_{\beta \in (0,1)} \left[\max_{\|u\|^{2} = \beta, \|w\|^{2} = 1 - \beta} \sum_{\{i,j\}\in E} u_{i}u_{j}w_{ij} \right]$$

$$= \sup_{\beta \in (0,1)} \left[\max_{\|u/\sqrt{\beta}\|^{2} = 1, \|w/\sqrt{1-\beta}\|^{2} = 1} \sum_{\{i,j\}\in E} \frac{u_{i}}{\sqrt{\beta}} \frac{u_{j}}{\sqrt{\beta}} \frac{w_{ij}}{\sqrt{1-\beta}} \times \beta \sqrt{1-\beta} \right]$$

$$= \left[\max_{\|u\|^{2} = 1, \|w\|^{2} = 1} \sum_{\{i,j\}\in E} u_{i}u_{j}w_{ij} \right] \times \sup_{\beta \in (0,1)} \beta \sqrt{1-\beta}$$

$$= \frac{2}{3\sqrt{3}} \max_{\|u\|=\|w\|=1} \sum_{\{i,j\}\in E} u_{i}u_{j}w_{ij}.$$

Note that the maximal value of $\sum_{\{i,j\}\in E} u_i u_j w_{ij}$, whether over $\mathbb{S}^{n-1}\times \mathbb{S}^{m-1}$ or over \mathbb{S}^{n+m-1} , can always be attained with $u\geq 0$ and $w\geq 0$, thereby allowing one to take square in (3.5) and (3.6). The same proof works word-for-word for stability number with the replacement of index of summation ' $\{i,j\}\in E$ ' by ' $\{i,j\}\notin E$ '.

4. Complexity theory for casual users

In this article, we use complexity classes defined in the standard classical framework: Time complexity measured in bits with the Cook-Karp-Levin notions of reducibility [4, 11, 16] on a Turing machine [24]. This is also the most common framework for discussing complexity issues in optimization [25]. We briefly recall the intuitive ideas behind the complexity classes used in this article for our readers. This is of course not meant to be anywhere near a rigorous treatment; for that, see [21, 23].

A decision problem, i.e., answer is YES or NO, is in NP if one can verify an YES answer in polynomial time; it is said to be in co-NP if one can verify a NO answer in polynomial time. A decision problem in NP is said to be NP-complete if one can reduce any other problem in NP to it. Likewise, a decision problem in co-NP is said to be co-NP-complete if one can reduce any other problem in co-NP to it. NP-complete and co-NP-complete problems are believed to be intractable.

Example 4.1 (Subset sum problem [8]). Let S be a finite set of integers. The problem "Does S have a non-empty subset with a zero sum?" is NP-complete — one can check a purported YES answer, i.e., a nonempty subset of integers, is indeed a YES answer, i.e., has a zero sum. The problem "Does every non-empty subset of S have a nonzero sum?" is co-NP-complete — one can check a purported NO answer, i.e., a nonempty subset of integers, is indeed a NO answer, i.e., has a nonzero sum. Note that the two problems are logical complements of each other. This is in fact another way to define the co-NP class, namely, it comprises problems that are logical complements of NP problems.

A problem is said to be *NP-hard* [12, 13] if one can reduce any NP-complete decision problem to it. A problem is said to be *co-NP-hard* if one can reduce any co-NP-complete decision problem to it. In other words, if one can solve an NP-hard problem, then one can solve any NP (including NP-complete) problems; if one can solve a co-NP-hard problem, then one can solve any co-NP (including co-NP-complete) problems. So NP-hard problems are believe to be even harder than NP-complete problem and co-NP-hard problems are believed to be even more intractable than co-NP-complete problems. An NP-hard problem need not be in NP, i.e., one need not be able to check an YES answer in polynomial time; an NP-hard problem that is in NP is by definition NP-complete. Likewise a co-NP-hard problem need not be in co-NP, i.e., one need not be able to check a NO answer in polynomial time; a co-NP-hard problem that is in co-NP is by definition co-NP-complete.

We have used the term 'reduce' without stating what it meant. There are in fact two different notions: The *Cook reduction* (also known as polynomial-time Turing reduction) and the *Karp reduction* (also known as polynomial-time many-one reduction). For our purpose, all we need to know is that Cook reduction is believed to be a stronger notion of reducibility than Karp reduction: Under Cook reduction, every decision problem can be reduced to its complement. A consequence is that with respect to Cook reducibility, there is no distinction between NP and co-NP, NP-complete and co-NP-complete, or NP-hard and co-NP-hard. Under Karp reduction, it is believed (although unknown) that these classes are all distinct.

Given an NP-hard maximization problem, e.g., the ones in Theorems 3.1 and 3.2, a polynomial time approximation scheme (PTAS) is an algorithm that, for any fixed $\varepsilon > 0$, would produce an approximate solution within a factor of $1-\varepsilon$ of the maximal value, and has running time polynomial in input size and ε .

5. Complexity of deciding self-concordance

The recent resolution of Shor's conjecture by Ahmadi, Olshevsky, Parrilo, and Tsitsiklis [1] shows that deciding the convexity of a quartic polynomial globally over \mathbb{R}^n is NP-hard. So the self-concordance of a function that is not a priori known to be convex is NP-hard in a trivial way since deciding whether (2.1) holds for all $x \in \Omega$ in Definition 2.1 is already an NP-hard problem. Our complexity result requires more stringent conditions: (i) Our functions are assumed to be convex in Ω and so (2.1) is always satisfied and self-concordance reduces to checking (2.2); (ii) We show that (2.2) is already NP-hard to check at a single point $x \in \Omega$.

Throughout the following we shall require the *inputs* to our problems to take rational or finite-extensions² of rational values, e.g., $A \in \mathbb{Q}^{n \times n \times n}$, $q \in \mathbb{Q}$, to ensure a finite bit-length input. Note however that an NP-hard problem may not be in the class NP; so an NP-hard decision problem can be posed over the reals, e.g., 'Is there an $h \in \mathbb{R}^n$ such that $[A(h,h,h)]^2 \leq q[h^\top h]^3$ holds?' without causing any issue since it is not required to have a polynomial-time checkable certificate.

²In this article we only encounter simple quadratic and cubic extensions, cf. (7.4) and (5.3). Note that for positive $q \in \mathbb{Q}$, elements of $\mathbb{Q}(\sqrt{q})$ and $\mathbb{Q}(\sqrt[3]{q})$ may be written as $a + b\sqrt{q}$ and $a + b\sqrt[3]{q} + c(\sqrt[3]{q})^2$ respectively. Therefore they may be represented by pairs and triples of rational numbers.

We will now formulate a decision problem that will lead us to the requisite NP-hardness of self-concordance. Let G = (V, E) be an undirected graph with n vertices and m edges where $n \ge 2$ and $m \ge 1$. Let $E = \{\{i_k, j_k\} : k = 1, \ldots, m\}$. Define $A_G = [a_{ijk}]_{i,j,k=1}^{n+m} \in \mathbb{Q}^{(n+m)\times(n+m)\times(n+m)}$ by

$$a_{ijk} = \begin{cases} 1 & \{i_k, j_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Note that A_G is a symmetric hypermatrix, i.e.,

$$a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji}$$

for all $i, j, k \in \{1, \dots, n+m\}$. Let us denote the coordinates of $h \in \mathbb{R}^{n+m}$ as

$$h = (u_1, \dots, u_n, w_{i_1 j_1}, \dots, w_{i_m j_m}).$$

In which case,

$$A_G(h, h, h) = \sum_{k=1}^{m} u_{i_k} u_{j_k} w_{i_k j_k} = \sum_{\{i, j\} \in E} u_i u_j w_{ij},$$

and so by Theorem 3.2,

(5.1)
$$\max_{h \neq 0} \left[\frac{A_G(h, h, h)}{\|h\|^3} \right]^2 = \max_{\|h\|=1} [A_G(h, h, h)]^2 = \frac{2}{27} \left(1 - \frac{1}{\omega(G)} \right).$$

The CLIQUE problem asks if for a given graph G and a given $k \in \mathbb{N}$, whether G has a clique of size k? CLIQUE is well-known to be NP-complete [11]. In other words, deciding if $\omega(G) \geq k$, or equivalently, $\omega(G) > k - 1$, is an NP-hard problem; and by (5.1), so is deciding if

(5.2)
$$\max_{h \neq 0} \left[\frac{A_G(h, h, h)}{\|h\|^3} \right]^2 > \frac{2}{27} \left(1 - \frac{1}{k - 1} \right).$$

Strictly speaking, we have restricted ourselves to the subclass of undirected graphs with at least two vertices and one edge — clearly CLIQUE is still NP-complete for this slightly smaller class. But with this restriction, we may assume that $k \geq 2$ and therefore the right-hand side of (5.2) is always defined.

Hence deciding if there exists an $h \in \mathbb{R}^{n+m}$ for which

$$[A_G(h, h, h)]^2 > \frac{2}{27} \left(1 - \frac{1}{k-1}\right) [h^{\top} h]^3$$

is an NP-hard problem. Note that $q = \frac{2}{27}[1-(k-1)^{-1}] \in \mathbb{Q}$ and so this problem is of the form:

Problem 5.1. Given a symmetric $A \in \mathbb{Q}^{(n+m)\times(n+m)\times(n+m)}$ and a positive $q \in \mathbb{Q}$, is it true that there exists $h \in \mathbb{R}^{n+m}$ for which $[A(h,h,h)]^2 > q[h^\top h]^3$?

Let $\sigma \in \mathbb{Q}$, $\sigma > 0$, be a self-concordance parameter and let

(5.3)
$$\gamma := \frac{1}{3} \left[\frac{1}{2\sigma} \left(1 - \frac{1}{k-1} \right) \right]^{1/3}.$$

We follow the notation in Section 2. Let Ω be the ε -ball $B_{\varepsilon}(0) = \{x \in \mathbb{R}^n : ||x|| < \varepsilon\}$ where $\varepsilon > 0$ is to be chosen later. We are interested in deciding self-concordance at x = 0 of the cubic polynomial $f: \Omega \to \mathbb{R}$ defined by

$$f(x) = \frac{\gamma}{2} x^{\top} x + A_G(x, x, x) = \frac{\gamma}{2} \sum_{i=1}^{n+m} x_i^2 + \sum_{i,j,k=1}^{n+m} a_{ijk} x_i x_j x_k.$$

We have $\nabla^2 f(0) = \gamma I$ where I is the $(n+m) \times (n+m)$ identity matrix. Since $\gamma > 0$, i.e., $\nabla^2 f(x)$ is strictly positive definite in a neighborhood of x = 0 and so there exists some $B_{\varepsilon}(0)$ on which f is convex — this gives us our choice of ε . Also, $\nabla^3 f(0) = A_G$.

Hence $\nabla^2 f(0)(h,h) = \gamma h^{\top} h = \gamma ||h||_2^2$, $\nabla^3 f(0)(h,h,h) = A_G(h,h,h)$, and f is self-concordant at the origin with parameter $\sigma \in \mathbb{Q}$ if and only if

$$[A_G(h, h, h)]^2 \le 4\sigma\gamma^3[h^\top h]^3 = \frac{2}{27}\left(1 - \frac{1}{k-1}\right)[h^\top h]^3$$

for all $h \in \mathbb{R}^{n+m}$. Note that this problem is of the form:

Problem 5.2. Given a symmetric $A \in \mathbb{Q}^{(n+m)\times(n+m)\times(n+m)}$ and a positive $q \in \mathbb{Q}$, is it true that for every $h \in \mathbb{R}^{n+m}$, we have $[A(h,h,h)]^2 \leq q[h^\top h]^3$?

Mathematically, Problems 5.1 and 5.2 are of course equivalent, being logical complements of each other. However they may or may not have the same computational complexity. By our discussion in Section 4, if our notion of reduction is the Cook reduction, then we may indeed conclude that Problems 5.1 and 5.2 are equivalent in terms of computational complexity, i.e., deciding self-concordance is NP-hard. However, if our notion of reduction is the Karp reduction, then what we may deduce from the NP-hardness of Problem 5.1 is that Problem 5.2 is co-NP-hard. In either case, our conclusion is that self-concordance is intractable.

Theorem 5.3. Deciding whether a cubic polynomial is self-concordant at the origin is NP-hard under Cook reduction and co-NP-hard under Karp reduction.

The argument in this section clearly works not just for cubic polynomials but for any $f \in C^3(\Omega)$ as long as $0 \in \Omega$, $\nabla^2 f(0) = \gamma I$, and $\nabla^3 f(0) = A_G$ — other derivatives and the remainder term in the Taylor expansion of f at x = 0 may be chosen arbitrarily as long as f stays convex in Ω . This liberty allows one to extend the construction above to functions with other desired properties. For instance, we may want an example where $\Omega = \mathbb{R}^n$ and since cubic polynomials cannot be convex on the whole of \mathbb{R}^n , we will need a quartic f and therefore need to choose $\nabla^4 f(0)$ accordingly; or we may want an example where f is a barrier function, which is equivalent to f having an epigraph $\{(x,t) \in \mathbb{R}^{n+1} : x \in \Omega, f(x) \leq t\}$ that is closed. One may trivially replace 0 by any point $a \in \mathbb{R}^n$ by considering the function $f_a(x) = f(x-a)$ on $\Omega = B_{\varepsilon}(a)$.

While we have proved our hardness result for functions on $\Omega \subseteq \mathbb{R}^n$, it is easy to extend this to any \mathbb{R} -vector space, for example, symmetric matrices $\mathbb{S}^{n \times n}$ or polynomials $\mathbb{R}[x_1, \dots, x_n]$, or even Riemannian manifolds with a non-trivial class of geodesically convex functions (i.e., not just the constant functions). Since self-concordance at a point is a local property, a choice of coordinate patch would transform the problem to one over \mathbb{R}^n ; and by our remark at the end of Section 2, it will in fact be independent of our choice of coordinates.

Deciding self-concordance on the whole of Ω is of course at least as hard as deciding self-concordance at a point in Ω and hence we have the following.

Corollary 5.4. For any Ω and any $\sigma > 0$, deciding membership in $S_{\sigma}(\Omega)$ is NP-hard.

One may wonder why our conclusion in Theorem 5.3 is stated as NP- and co-NP-hardness as opposed to NP- and co-NP-completeness. It may appear that given a NO certificate $h \in \mathbb{R}^n$, it would be easy (i.e., requires polynomial time) to decide whether $[A(h,h,h)]^2 \leq q[h^{\top}h]^3$ is indeed violated. But observe that it is only easy to compute the quantities $[A(h,h,h)]^2$, $q[h^{\top}h]^3$, and compare their magnitudes when we measure time complexity in units of real operations (i.e., arithmetic and ordering in \mathbb{R}). Since we measure time complexity in units of bit operations, even if the certificate h is in \mathbb{Q}^n , it could well have an exponential number of bits and thus it is not at all clear that we may check $[A(h,h,h)]^2 \leq q[h^{\top}h]^3$ easily.

6. Inapproximability of optimal self-concordance parameter

Let $A \in \mathbb{Q}^{n \times n \times n}$ be symmetric and $f: \Omega \to \mathbb{R}$ be defined by the cubic polynomial $f(x) = \frac{1}{2}x^{\top}x + A(x, x, x)$. As in Section 5, Ω is chosen to be a neighborhood of the origin so that f is

convex on Ω . The condition (2.2) for self-concordance of f at x=0 with parameter $\sigma>0$ may be written as

$$(6.1) |A(h, h, h)| \le 2\sqrt{\sigma} ||h||_2^3$$

for all $h \in \mathbb{R}^n$. This is equivalent to requiring

(6.2)
$$\max_{h \neq 0} \frac{A(h, h, h)}{\|h\|_2^3} \le 2\sqrt{\sigma},$$

as A(-h, -h, -h) = -A(h, h, h) and we may drop the absolute value in (6.1). Since $A \in \mathbb{R}^{n \times n \times n}$ is a symmetric 3-hypermatrix, the *spectral norm* [7, 17] of A,

$$||A||_{2,2,2} := \max_{h_1,h_2,h_3 \neq 0} \frac{A(h_1,h_2,h_3)}{||h_1||_2 ||h_2||_2 ||h_3||_2} = \max_{h \neq 0} \frac{A(h,h,h)}{||h||_3^3}.$$

For the interested reader, the second equality above follows from Banach's result on the polarization constant of Hilbert spaces [2, 22]. Hence the optimal self-concordance parameter of f at x = 0, i.e., the smallest value of σ so that (6.2) holds, is given by

(6.3)
$$\sigma_{\text{opt}} = \frac{1}{4} ||A||_{2,2,2}^2.$$

The spectral norm of a 3-hypermatrix is NP-hard to approximate to within a certain constant factor by [7, Theorem 1.11], which we state here for easy reference.

Theorem 6.1 (Hillar–Lim). Let $A \in \mathbb{Q}^{n \times n \times n}$ and N be the input size of A in bits. Then it is NP-hard to approximate $||A||_{2,2,2}$ to within a factor of $1 - \varepsilon$ where

$$\varepsilon = 1 - \left(1 + \frac{1}{N(N-1)}\right)^{-1/2} = \frac{1}{2N(N-1)} + O\left(\frac{1}{N^4}\right).$$

By (6.3) and Theorem 6.1, σ_{opt} is NP-hard to approximate to within a factor of $\frac{1}{4}(1-\varepsilon)^2$ and consequently we have the following inapproximability result.

Corollary 6.2. There is no polynomial time approximation scheme for determining the optimal self-concordance parameter σ_{opt} unless P = NP.

We refer the reader to [6, 9] for more extensive approximability results and approximation algorithms (that are not PTAS). In particular, the results in [9] for quartic polynomials would apply to the optimal second-order self-concordance parameter (see the next section).

7. Complexity of deciding second-order self-concordance

There is also an interesting notion of second-order self-concordance due to Jarre [14]. This requires that $f \in C^4(\Omega)$ and is given by a condition involving the matrix $\nabla^2 f(x) \in \mathbb{R}^{n \times n}$ and the 4-hypermatrix $\nabla^4 f(x) \in \mathbb{R}^{n \times n \times n \times n}$.

Definition 7.1 (Jarre). If $\Omega \subseteq \mathbb{R}^n$ is a convex open set, then $f: \Omega \to \mathbb{R}$ is said to be **self-concordant of order two** with parameter $\tau > 0$ at $x \in \Omega$ if

(7.1)
$$\nabla^2 f(x)(h,h) \ge 0$$

and

(7.2)
$$\nabla^4 f(x)(h,h,h,h) \le 6\tau \left[\nabla^2 f(x)(h,h)\right]^2$$

for all $h \in \mathbb{R}^n$; f is self-concordant of order two on Ω if (7.1) and (7.2) hold for all $x \in \Omega$.

Note that

$$\nabla^4 f(x)(h,h,h,h) = \sum_{i,j,k,l=1}^n \frac{\partial^4 f(x)}{\partial x_i \partial x_j \partial x_k \partial x_l} h_i h_j h_k h_l,$$

is a quartic polynomial in h for any fixed $x \in \Omega$.

We follow the same argument in Section 5 to show that deciding (7.2) is NP-hard. This time the result would be deduced from Motzkin–Strass Theorem except that for better parallelism with Section 5, we will use the quartic-maximization-over-sphere form (3.4) instead of the quadratic-maximization-over-simplex form (3.1).

Given a graph G=(V,E) with n vertices and m edges where $n \geq 2$ and $m \geq 1$, we define $A_G \in \mathbb{R}^{n \times n \times n \times n}$ by

$$a_{ijkl} = \begin{cases} 1 & i = k, \ j = l, \ \text{and} \ \{i, j\} \in E, \\ 0 & \text{otherwise}. \end{cases}$$

So $A = [a_{ijkl}]_{i,j,k,l=1}^n \in \mathbb{Q}^{n \times n \times n \times n}$ is a symmetric 4-hypermatrix. Now observe that, as in (3.4),

$$\max_{\|h\|=1} A_G(h, h, h, h) = \max_{h \in \mathbb{S}^{n-1}} \sum_{\{i, j\} \in E} h_i^2 h_j^2 = \frac{1}{2} \left(1 - \frac{1}{\omega(G)} \right)$$

by Motzkin-Strass Theorem. As in Section 5, given an integer $k \geq 2$, deciding if a k-clique exists in G is equivalent to deciding if $\omega(G) > k - 1$. Hence deciding if there exists $h \in \mathbb{R}^n$ with

(7.3)
$$A_G(h, h, h, h) > \frac{1}{2} \left(1 - \frac{1}{k-1} \right) [h^{\top} h]^2$$

is NP-hard.

Given a second-order self-concordance parameter $\tau \in \mathbb{Q}$, $\tau > 0$, let

(7.4)
$$\gamma := \left[\frac{1}{12\tau} \left(1 - \frac{1}{k-1} \right) \right]^{1/2}.$$

We may now define $f:\Omega\to\mathbb{R}$ accordingly as the quartic polynomial

$$f(x) = \frac{\gamma}{2} x^{\top} x + A_G(x, x, x, x) = \frac{\gamma}{2} \sum_{i=1}^n x_i^2 + \sum_{i,j,k,l=1}^n a_{ijkl} x_i x_j x_k x_l.$$

Hence $\nabla^2 f(0)(h,h) = \gamma h^{\top} h = \gamma ||h||_2^2$ and $\nabla^4 f(0)(h,h,h,h) = A_G(h,h,h,h)$. Again we choose Ω to be a neighborhood of the origin so that f is convex on Ω as we did in Section 5. So the function f is second-order self-concordant at x = 0 with parameter τ if and only if

(7.5)
$$[A_G(h, h, h, h) \le 6\tau\gamma^2 [h^\top h]^2 = \frac{1}{2} \left(1 - \frac{1}{k-1}\right) [h^\top h]^2$$

is satisfied for all $h \in \mathbb{R}^n$. As in Section 5, we observe that the problem of deciding if there exists an $h \in \mathbb{R}^n$ satisfying (7.3) and the problem of deciding if (7.5) is satisfied for all $h \in \mathbb{R}^n$ are logical complements. Since the former is NP-hard, we arrive at the following conclusion:

Theorem 7.2. Deciding if a quartic polynomial is second-order self-concordant at the origin is NP-hard under Cook reduction and co-NP-hard under Karp reduction.

It has recently been shown that deciding various seemingly innocuous properties of quartic polynomials [1, 10] all fall into the NP-hard category, Theorem 7.2 provides yet another such example.

8. Conclusion

As we have mentioned in the introduction, the hardness results here are intended to shed light on the properties of self-concordance. They do not in anyway invalidate the usefulness of the notion in practice since there are basic principles that one may use to construct self-concordant (and second-order self-concordant) functions for use as barriers in cone programming — see "How to construct self-concordant barriers" in [20, Chapter 5] for an extensive discussion or "Self-concordant calculus" in [3, Section 9.6] for a summary.

Self-concordance and second-order self-concordance are conditions involving high-order tensors (orders 3 and 4 respectively), which is a topic of great interest to the author. In particular, their NP-hardness serves as yet reminder of the complexity of tensor problems [7].

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