J Sci Comput (2012) 51:1–27
DOI 10.1007/s10915-011-9492-4

Optimal Error Estimates of the Local Discontinuous Galerkin Method for Surface Diffusion of Graphs on Cartesian Meshes

Liangyue Ji · Yan Xu

Received: 10 December 2010 / Revised: 21 March 2011 / Accepted: 20 April 2011 /
Published online: 7 May 2011
© Springer Science+Business Media, LLC 2011

Abstract In (Xu and Shu in J. Sci. Comput. 40:375–390, 2009), a local discontinuous Galerkin (LDG) method for the surface diffusion of graphs was developed and a rigorous proof for its energy stability was given. Numerical simulation results showed the optimal order of accuracy. In this subsequent paper, we concentrate on analyzing *a priori* error estimates of the LDG method for the surface diffusion of graphs. The main achievement is the derivation of the optimal convergence rate $k+1$ in the L^2 norm in one-dimension as well as in multi-dimensions for Cartesian meshes using a completely discontinuous piecewise polynomial space with degree $k \geq 1$.

Keywords Local discontinuous Galerkin method · Surface diffusion of graphs · Stability · Error estimates

1 Introduction

In this paper, we consider the error estimates of the local discontinuous Galerkin (LDG) method [31] for the surface diffusion of graphs

$$u_t + \nabla \cdot \left(Q \left(\boldsymbol{I} - \frac{\nabla u \otimes \nabla u}{Q^2} \right) \nabla H \right) = 0, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \tag{1.1}$$

Research of Y. Xu was supported by NSFC grant No. 10971211, No. 11031007, FANEDD No. 200916, FANEDD of CAS, NCET No. 09-0922 and the Fundamental Research Funds for the Central Universities.

L. Ji · Y. Xu (✉)
Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China
e-mail: yxu@ustc.edu.cn

L. Ji
e-mail: jlyue@mail.ustc.edu.cn

L. Ji
Delft Institute of Applied Mathematics, Delft University of Technology, 2628 CD Delft, The Netherlands

with a periodic boundary condition and a smooth enough initial condition

$$u(\boldsymbol{x},0)=u_0(\boldsymbol{x}). \tag{1.2}$$

For simplicity we always consider $\Omega=\Pi_{i=1}^{d}[a_i,b_i]\subset\mathbb{R}^d$ to be a bounded rectangular domain with dimension $d\le 3$. Here Q is the area element

$$Q=\sqrt{1+|\nabla u|^2} \tag{1.3}$$

and H is the mean curvature of the domain boundary Γ

$$H=\nabla\cdot\left(\frac{\nabla u}{Q}\right). \tag{1.4}$$

The reduced model (1.1) has an elegant divergence form, which is obtained by a (highly nonlinear) 4th order geometric partial differential equations (PDEs). Similar structure has been exploited in the continuous finite element methods by Bänsch [1] for surface diffusion of graphs and Deckelnick and by Dziuk [18] for Willmore flow of graphs.

Finite element methods have been successfully applied to solve surface diffusion of graphs. Earlier studies [12, 13] focused on the stability of the numerical scheme. Recently a second order splitting method was presented by Deckelnick, Dziuk and Elliott [15], which was proposed for Cahn-Hilliard equation by Elliott, French and Milner [20]. Subsequently Bänsch, Morin and Nochetto [1] introduced a novel variational formulation for graphs and obtained a priori quasi-optimal error estimates using continuous finite elements of degree $k\ge 1$. Later on, Deckelnick, Dziuk and Elliott [17] analyzed a fully discrete finite element approximation for anisotropic surface diffusion of graphs and proved error bounds. Geometric PDEs have many applications, such as body shape dynamics, surface construction, computer data processing, phase transitions, image processing, etc. For more computational theory we refer to the review paper [16] by Deckelnick, Dziuk and Elliott.

In [31], Xu and Shu developed a local discontinuous Galerkin (LDG) method for the surface diffusion and Willmore flow of graphs and gave a rigorous proof for its energy stability. In this method the basis functions used are discontinuous in space. The LDG discretization also results in a high order accurate, extremely local, element based discretization. In particular, the LDG method is well suited for *hp*-adaptation, which consists of local mesh refinement and/or the adjustment of the polynomial order in individual elements. The optimal error estimates results of the local discontinuous Galerkin method for Willmore flow of graphs on Cartesian meshes are given in [22].

In this paper we analyze *a priori* error estimates of the method of the surface diffusion of graphs which developed in [31]. The main achievement is the derivation of the optimal convergence rate $k+1$ in the L^2 norm on Cartesian meshes. The key idea of the LDG method for this equation is to rewrite the equation into a first order system, then choose proper fluxes. So inter-element boundary terms that due to the flux on each element and auxiliary variables which occur from the LDG discretization are the main challenges in this paper. The analysis is made for the fully nonlinear case and the results are valid for all $d\le 3$ and polynomial degree $k\ge 1$. Although the surface diffusion and Willmore flow of graphs are both fourth order nonlinear equations and has some similar nonlinear term, the energy of two equations are different. So, the technique of the proof for these two equations are different. We borrow the idea in [1] and introduce the two nonlinear operators to facilitate the proof. We generalize the analysis to fully nonlinear case comparing with analysis for linear fourth order equation in [19]. We also obtain the optimal accuracy results comparing with the sub-optimal results for continuous finite element method in [1].

The DG method is a class of finite element methods, using discontinuous, piecewise polynomials as the solution and the test space. It was first designed as a method for solving hyperbolic conservation laws containing only first order spatial derivatives, e.g. Reed and Hill [25] for solving linear equations, and Cockburn et al. [4–7] for solving nonlinear equations. It is difficult to apply the DG method directly to the equations with higher order derivatives. The LDG method is an extension of the DG method aimed at solving partial differential equations (PDEs) containing higher than first order spatial derivatives. The first LDG method was constructed by Cockburn and Shu in [8] for solving nonlinear convection diffusion equations containing second order spatial derivatives. Their work was motivated by the successful numerical experiments of Bassi and Rebay [2] for the compressible Navier-Stokes equations. The idea of the LDG method is to rewrite the equations with higher order derivatives into a first order system, then apply the DG method on the system. The design of the numerical fluxes is the key ingredient to ensure stability. The LDG techniques have been developed for convection diffusion equations (containing second derivatives) [8], nonlinear one-dimensional and two-dimensional KdV type equations [30, 33] and Cahn-Hilliard equations [28, 29]. Recently, there is a review paper on the LDG methods for high-order time-dependent partial differential equations [32]. More general information about DG methods for elliptic, parabolic and hyperbolic partial differential equations can be found in the three special journal issues devoted to the DG method [10, 11, 14], as well as in the recent books and lecture notes [21, 23, 26, 27].

The paper is organized as follows. In Sect. 2, we give some notations, definition and projections. In Sect. 3, we show LDG scheme for the surface diffusion of graphs and the main results in this paper. In Sect. 4, we give some auxiliary results which is important for our analysis. In Sect. 5, we present the proof of the error estimates. Concluding remarks are given in Sect. 6. Some of the more technical proofs of several lemmas are collected in Appendix.

2 Notations, Definitions and Projections

We first introduce notations, definitions and projections to be used later in the paper. We define some projections and present certain interpolation and inverse properties for the finite element spaces that will be used in the error analysis.

2.1 Tessellation and Function Spaces

For a rectangular partition of $\Omega = \Pi_{i=1}^{d}[a_i, b_i] \subset \mathbb{R}^d$, we denote the mesh by $L_j = [x^{(j)}_{i-\frac{1}{2}}, x^{(j)}_{i+\frac{1}{2}}]$, for $i = 1, \dots, N_j$ and $j = 0, \dots, d$. The cell lengths are denoted by $h_i^{(j)} = x^{(j)}_{i+\frac{1}{2}} - x^{(j)}_{i-\frac{1}{2}}$ with $h^{(j)} = \max_{1\le i\le N_j} h_i^{(j)}$, and $h = \max_{j=1,\dots,d} h^{(j)}$ being the maximum mesh size. We assume the mesh is regular.

Let $\mathcal{T}_h$ denote a tessellation of Ω with shape-regular elements K. Let Γ denote the union of the boundary faces of elements $K \in \mathcal{T}_h$, i.e. $\Gamma = \bigcup_{K\in\mathcal{T}_h} \partial K$, and $\Gamma_0 = \Gamma \backslash \partial\Omega$.

In order to describe the flux functions we need to introduce some notations. Let e be a face shared by the "left" and "right" elements K_L and K_R (we refer to [33] and [32] for a proper definition of "left" and "right" in our context). Define the outward unit normal vectors ν_L and ν_R on e pointing exterior to K_L and K_R, respectively. If ψ is a function on K_L and

K_R, but possibly discontinuous across e, let ψ_L denote $(\psi|_{K_L})|_e$ and ψ_R denote $(\psi|_{K_R})|_e$, the left and right trace, respectively.

Let $\mathcal{Q}^k(K)$ be the space of tensor product of polynomials of degree at most $k \geq 0$ on $K \in \mathcal{T}_h$ in each variable. The finite element spaces are denoted by

$$V_h = \{\varphi \in L^2(\Omega) : \varphi|_K \in \mathcal{Q}^k(K), \ \forall K \in \mathcal{T}_h\},$$
$$\Sigma_h = \{\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)^T \in (L^2(\Omega))^d : \eta_l|_K \in \mathcal{Q}^k(K), \ l = 1, \dots, d, \ \forall K \in \mathcal{T}_h\}.$$

For one-dimensional case, we have $\mathcal{Q}^k(K) = \mathcal{P}^k(K)$ which is the space of polynomials of degree at most $k \geq 0$ defined on K. Note that functions in V_h and Σ_h are allowed to have discontinuities across element interfaces. Here we only consider periodic boundary conditions. Notice that the assumption of periodic boundary conditions is for simplicity only and not essential: the method can be easily designed for non-periodic boundary conditions. The development of the LDG method for the non-periodic boundary conditions can be found in [24].

The definition we use for the L^2-norm, L^∞ norm in Ω and on the boundary are given by the standard definitions:

$$\|\eta\|_\Omega = \left(\int_\Omega \eta^2 \, dx\right)^{\frac{1}{2}}, \quad \|\eta\|_{L^\infty(\Omega)} = ess \sup_{x \in \Omega} |\eta|, \quad \|\eta\|_{\partial\Omega} = \left(\int_{\partial\Omega} \eta^2 \, ds\right)^{\frac{1}{2}}. \tag{2.1}$$

The $H^\ell(\Omega)$-norm over Ω is

$$\|\eta\|_{H^\ell(\Omega)} = \left(\sum_{|\alpha| \leq \ell} \|D^\alpha \eta\|_\Omega^2\right)^{\frac{1}{2}}, \quad \ell > 0. \tag{2.2}$$

We note that we simplify the notation for these norms and only designate the norm type and not the domain. Further, we need to define the inner product notation as

$$(w, v)_K = \int_K w v \, dK, \qquad (w, v)_{\partial K} = \int_{\partial K} w v \, ds, \tag{2.3}$$

$$(\boldsymbol{q}, \boldsymbol{p})_K = \int_K \boldsymbol{q} \cdot \boldsymbol{p} \, dK, \qquad (\boldsymbol{q}, \boldsymbol{p})_{\partial K} = \int_{\partial K} \boldsymbol{q} \cdot \boldsymbol{p} \, ds \tag{2.4}$$

for the scalar variables w, v and vector variables $\boldsymbol{q}, \boldsymbol{p}$ respectively. We can also get the following relations

$$\|w\|_\Omega^2 = \sum_K (w, w)_K, \qquad \|\boldsymbol{q}\|_\Omega^2 = \sum_K (\boldsymbol{q}, \boldsymbol{q})_K. \tag{2.5}$$

2.2 Notations for Different Constants

We will adopt the following convention for different constants. These constants may have a different value in each occurrence.

We will denote by C a positive constant independent of h, which may depend on the solution of the problem considered in this paper. For problems considered in this section, the exact solution is assumed to be smooth with periodic. Also, $0 \leq t \leq T$ for a fixed T. Therefore, the exact solution is always bounded.

2.3 Projection and Interpolation Properties

2.3.1 One-Dimensional Case

In what follows, we will consider the standard L^2-projection of a function ω with $k+1$ continuous derivatives into space V_h,

$$P^{\pm}: H^1(\Omega) \longrightarrow V_h,$$

which are defined as the following. Given a function $\eta \in H^1(\Omega)$ and an arbitrary subinterval $K_j = (x_{j-1}, x_j)$, the restriction of $P^{\pm}\eta$ to K_j are defined as the elements of $\mathcal{P}^k(K_j)$ that satisfy

$$\int_{K_j} (P^+\eta - \eta) w \, dx = 0, \quad \forall w \in \mathcal{P}^{k-1}(K_j), \quad \text{and} \quad P^+\eta(x_{j-1}) = \eta(x_{j-1}), \tag{2.6}$$

$$\int_{K_j} (P^-\eta - \eta) w \, dx = 0, \quad \forall w \in \mathcal{P}^{k-1}(K_j), \quad \text{and} \quad P^-\eta(x_j) = \eta(x_j). \tag{2.7}$$

For the projections mentioned above, it is easy to show (cf. [3])

$$\|\eta^e\|_\Omega + h\|\eta^e\|_{L^\infty(\Omega)} + h^{\frac{1}{2}}\|\eta^e\|_\Gamma \le Ch^{k+1}\|\eta\|_{k+1,\Omega}, \tag{2.8}$$

where $\eta^e = \pi\eta - \eta$ or $\eta^e = P^{\pm}\eta - \eta$. $\pi\eta$ is the standard L^2 projection of the function η. The positive constant C, only depending on η, is independent of h.

2.3.2 Two-Dimensional Case

To prove the error estimates for two-dimensional problems in Cartesian meshes, we need a suitable projection $P^{\pm}$ similar to the one-dimensional case. The projections P^- for scalar functions are defined as

$$P^- = P_x^- \otimes P_y^-, \tag{2.9}$$

where the subscripts x and y indicate that the one-dimensional projections defined by (2.7) are applied with respect to the corresponding variable on a two-dimensional rectangle element $I \otimes J = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

The projection Π^+ for vector-valued function $\boldsymbol{\rho} = (\rho_1(x,y), \rho_2(x,y))$ are defined as

$$\Pi^+\boldsymbol{\rho} = (P_x^+ \otimes \pi_y \rho_1, \pi_x \otimes P_y^+ \rho_2). \tag{2.10}$$

Here π_x, π_y is the standard L^2 projection in x or y direction. It is easy to see that, for any $\boldsymbol{\rho} \in [H^1(\Omega)]^2$, the restriction of $\Pi^+\boldsymbol{\rho}$ to $I \otimes J$ are elements of $[\mathcal{Q}^k(I \otimes J)]^2$ that satisfy

$$\int_I \int_J (\Pi^+\boldsymbol{\rho} - \boldsymbol{\rho}) \cdot \nabla w \, dy \, dx = 0 \tag{2.11}$$

for any $w \in \mathcal{Q}^k(I \otimes J)$, and

$$\int_J (\Pi^+\boldsymbol{\rho}(x_{i-1}, y) - \boldsymbol{\rho}(x_{i-1}, y)) \cdot \boldsymbol{\nu} w(x_{i-1}^+, y) \, dy = 0 \quad \forall w \in \mathcal{Q}^k(I \otimes J), \tag{2.12}$$

$$\int_I (\Pi^+\boldsymbol{\rho}(x,y_{j-1}) - \boldsymbol{\rho}(x,y_{j-1}))\cdot \boldsymbol{\nu} w(x,y_{j-1}^+)\,dy = 0 \quad \forall w \in \mathcal{Q}^k(I\otimes J), \tag{2.13}$$

where $\boldsymbol{\nu}$ is the outward unit normal vector to the domain integrated. For the definition of similar projection on three-dimensional case, we refer to [9].

Similar to the one-dimensional case, there are some approximation results for the projections (2.9) and (2.10) in [19]

$$\|\eta^e\|_\Omega + h^{\frac{1}{2}}\|\eta^e\|_\Gamma \le Ch^{k+1}\|\eta\|_{H^{k+1}(\Omega)}, \quad \forall \eta \in H^{k+1}(\Omega),$$

$$\|\boldsymbol{\rho}^e\|_\Omega + h^{\frac{1}{2}}\|\boldsymbol{\rho}^e\|_\Gamma \le Ch^{k+1}\|\boldsymbol{\rho}\|_{H^{k+1}(\Omega)}, \quad \forall \boldsymbol{\rho} \in [H^{k+1}(\Omega)]^d,$$

where $\eta^e = \pi\eta - \eta$, $\boldsymbol{\rho}^e = \pi\boldsymbol{\rho} - \boldsymbol{\rho}$ or $\eta^e = P^\pm\eta - \eta$, $\boldsymbol{\rho}^e = \Pi^\pm\boldsymbol{\rho} - \boldsymbol{\rho}$ and C is independent of h.

2.4 Inverse Properties and Approximation

Finally we list some inverse properties of the finite element space V_h that will be used in our error analysis. For any $\omega_h \in V_h$, there exists a positive constant C independent of ω_h and h, such that

$$\begin{aligned}
&\text{(i)} \quad \|\nabla\omega_h\|_\Omega \le Ch^{-1}\|\omega_h\|_\Omega,\\
&\text{(ii)} \quad \|\omega_h\|_\Gamma \le Ch^{-\frac{1}{2}}\|\omega_h\|_\Omega,\\
&\text{(iii)} \quad \|\omega_h\|_{L^\infty(\Omega)} \le Ch^{-\frac{d}{2}}\|\omega_h\|_\Omega,
\end{aligned} \tag{2.14}$$

where $d = 1, 2$ or 3 is the spatial dimension. For more details of these inverse properties, we refer to [3].

3 The LDG Method for the Surface Diffusion of Graphs

In this section, we consider the LDG method for the surface diffusion of graphs equation (1.1). We will give the L^2 stability and area-decreasing properties of the LDG method. The main error estimates results will be presented.

3.1 The LDG Method

To define the local discontinuous Galerkin method, we rewrite (1.1) as a first order system:

$$u_t + \nabla\cdot \boldsymbol{s} = 0, \tag{3.1a}$$

$$\boldsymbol{s} - \boldsymbol{E}(\boldsymbol{r})\boldsymbol{p} = 0, \tag{3.1b}$$

$$\boldsymbol{p} - \nabla H = 0, \tag{3.1c}$$

$$H - \nabla\cdot\boldsymbol{q} = 0, \tag{3.1d}$$

$$\boldsymbol{q} - \frac{\boldsymbol{r}}{Q} = 0, \tag{3.1e}$$

$$\boldsymbol{r} - \nabla u = 0, \tag{3.1f}$$

with

$$\boldsymbol{E}(\boldsymbol{r}) = Q\left(\boldsymbol{I} - \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{Q^2}\right), \tag{3.2}$$

$$Q = \sqrt{1+|\boldsymbol{r}|^2}, \tag{3.3}$$

where $\boldsymbol{s}$, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ are vectors, $\boldsymbol{E}(\boldsymbol{r})$ is the $d\times d$ matrix and $\boldsymbol{I}$ is the $d\times d$ identity matrix.

Applying the LDG method to the system (3.1), we have the scheme: Find u_h, $H_h \in V_h$, $\boldsymbol{s}_h$, $\boldsymbol{p}_h$, $\boldsymbol{q}_h$, $\boldsymbol{r}_h \in \Sigma_h$, such that, for all test function φ, $\vartheta \in V_h$ and $\boldsymbol{\phi}$, $\boldsymbol{\eta}$, $\boldsymbol{\rho}$, $\boldsymbol{\zeta} \in \Sigma_h$,

$$((u_h)_t, \varphi)_K - (\boldsymbol{s}_h, \nabla\varphi)_K + (\widehat{\boldsymbol{s}_h}\cdot\boldsymbol{\nu}, \varphi)_{\partial K} = 0, \tag{3.4a}$$

$$(\boldsymbol{s}_h, \boldsymbol{\phi})_K - (\boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{\phi})_K = 0, \tag{3.4b}$$

$$(\boldsymbol{p}_h, \boldsymbol{\eta})_K + (H_h, \nabla\cdot\boldsymbol{\eta})_K - (\widehat{H_h}, \boldsymbol{\nu}\cdot\boldsymbol{\eta})_{\partial K} = 0, \tag{3.4c}$$

$$(H_h, \vartheta)_K + (\boldsymbol{q}_h, \nabla\vartheta)_K - (\widehat{\boldsymbol{q}_h}\cdot\boldsymbol{\nu}, \vartheta)_{\partial K} = 0, \tag{3.4d}$$

$$(\boldsymbol{q}_h, \boldsymbol{\rho})_K - \left(\frac{\boldsymbol{r}_h}{Q_h}, \boldsymbol{\rho}\right)_K = 0, \tag{3.4e}$$

$$(\boldsymbol{r}_h, \boldsymbol{\zeta})_K + (u_h, \nabla\cdot\boldsymbol{\zeta})_K - (\widehat{u_h}, \boldsymbol{\nu}\cdot\boldsymbol{\zeta})_{\partial K} = 0, \tag{3.4f}$$

where $\boldsymbol{\nu}$ is the outward unit normal vector to ∂K and $\boldsymbol{E}(\boldsymbol{r}_h)$ and Q_h are similarly defined as follows:

$$\boldsymbol{E}(\boldsymbol{r}_h) = Q_h\left(\boldsymbol{I} - \frac{\boldsymbol{r}_h\otimes\boldsymbol{r}_h}{Q_h^2}\right), \tag{3.5}$$

$$Q_h = \sqrt{1+|\boldsymbol{r}_h|^2}. \tag{3.6}$$

The "hat" terms in (3.4) at the cell boundary obtained after integration by parts are the so-called "numerical fluxes", which are functions defined on the cell edges and should be designed based on different guiding principles for different PDEs to ensure stability. It turns out that we can take the simple choices

$$\widehat{\boldsymbol{s}_h}|_e = \boldsymbol{s}_{h,R}, \qquad \widehat{\boldsymbol{q}_h}|_e = \boldsymbol{q}_{h,R}, \qquad \widehat{H_h}|_e = H_{h,L}, \qquad \widehat{u_h}|_e = u_{h,L}, \tag{3.7}$$

which ensure L^2 stability. Numerical examples for the schemes (3.4)–(3.7) can be found in [31].

Using the numerical flux above we get the following property (see [19, Lemma 2.2])

Lemma 3.1 *Suppose e is an inter-element face shared by the elements K_1 and K_2; then*

$$\begin{aligned}
&(\hat{w}, \boldsymbol{\rho}\cdot\boldsymbol{\nu})_{\partial K_1\cap e} + (w, \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\nu})_{\partial K_1\cap e} - (w, \boldsymbol{\rho}\cdot\boldsymbol{\nu})_{\partial K_1\cap e}\\
&\quad + (\hat{w}, \boldsymbol{\rho}\cdot\boldsymbol{\nu})_{\partial K_2\cap e} + (w, \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\nu})_{\partial K_2\cap e} - (w, \boldsymbol{\rho}\cdot\boldsymbol{\nu})_{\partial K_2\cap e} = 0,
\end{aligned}$$

for any $w \in V_h$ and $\boldsymbol{\rho} \in \Sigma_h$. Here $\hat{w}_e = w_L$, $\hat{\boldsymbol{\rho}}_e = \boldsymbol{\rho}_R$ and again $\boldsymbol{\nu}$ is the outward unit normal vector to $\partial K_i \cap e$. Moreover, the periodic boundary conditions gives

$$\sum_{K\in\mathcal{T}_h} \left((\hat{w}, \boldsymbol{\rho}\cdot\boldsymbol{\nu})_{\partial K} + (w, \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\nu})_{\partial K} - (w, \boldsymbol{\rho}\cdot\boldsymbol{\nu})_{\partial K}\right) = 0.$$

The projection P^- defined in (2.9) on the Cartesian meshes has the following superconvergence property (see [19, Lemma 3.7]).

Lemma 3.2 *Suppose $\eta \in H^{k+2}(\Omega)$, $\boldsymbol{\rho} \in \Sigma_h$ and the projection P^- defined in (2.9), then we have*

$$\left| \int_\Omega (\eta - P^-\eta)\nabla \cdot \boldsymbol{\rho}\, d\Omega - \int_\Gamma (\eta - \widehat{P^-\eta})\boldsymbol{\rho} \cdot \boldsymbol{\nu}\, d\Gamma \right| \le C h^{k+1} \|\eta\|_{H^{k+2}(\Omega)} \|\boldsymbol{\rho}\|_{L^2(\Omega)}, \tag{3.8}$$

where "hat" term is numerical flux.

3.2 L^2 Stability and Area Decreasing

In this section, we give the L^2 stability and area decreasing properties of the LDG method for the surface diffusion of graphs defined in the previous section.

Proposition 3.3 (L^2 stability [31]) *The solution of the surface diffusion of graphs using the schemes (3.4)–(3.7) satisfies L^2 stability*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|_\Omega^2 + \|H_h\|_\Omega^2 = 0, \quad \forall u_h, H_h \in V_h. \tag{3.9}$$

Proposition 3.4 (Area decreasing) *The solution of the surface diffusion of graphs using the schemes (3.4)–(3.7) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Q_h\, d\Omega \le 0, \tag{3.10}$$

where Q_h is the area element.

Proof Differentiating (3.4f) with respect to time and combining with (3.4a)–(3.4e), we get

$$((u_h)_t, \varphi)_K - (\boldsymbol{s}_h, \nabla\varphi)_K + (\widehat{\boldsymbol{s}_h} \cdot \boldsymbol{\nu}, \varphi)_{\partial K} = 0, \tag{3.11a}$$

$$(\boldsymbol{s}_h, \boldsymbol{\phi})_K - (\boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{\phi})_K = 0, \tag{3.11b}$$

$$(\boldsymbol{p}_h, \boldsymbol{\eta})_K + (H_h, \nabla \cdot \boldsymbol{\eta})_K - (\widehat{H_h}, \boldsymbol{\nu} \cdot \boldsymbol{\eta})_{\partial K} = 0, \tag{3.11c}$$

$$(H_h, \vartheta)_K + (\boldsymbol{q}_h, \nabla\vartheta)_K - (\widehat{\boldsymbol{q}_h} \cdot \boldsymbol{\nu}, \vartheta)_{\partial K} = 0, \tag{3.11d}$$

$$(\boldsymbol{q}_h, \boldsymbol{\rho})_K - \left(\frac{\boldsymbol{r}_h}{Q_h}, \boldsymbol{\rho}\right)_K = 0, \tag{3.11e}$$

$$((\boldsymbol{r}_h)_t, \boldsymbol{\zeta})_K + ((u_h)_t, \nabla \cdot \boldsymbol{\zeta})_K - (\widehat{(u_h)_t}, \boldsymbol{\nu} \cdot \boldsymbol{\zeta})_{\partial K} = 0. \tag{3.11f}$$

If we take the test functions in (3.11a)–(3.11f)

$$\varphi = -H_h, \qquad \boldsymbol{\phi} = -\boldsymbol{p}_h, \qquad \boldsymbol{\eta} = \boldsymbol{s}_h, \qquad \vartheta = (u_h)_t, \qquad \boldsymbol{\rho} = -(\boldsymbol{r}_h)_t, \qquad \boldsymbol{\zeta} = \boldsymbol{q}_h,$$

then we obtain

$$-((u_h)_t, H_h)_K + (\boldsymbol{s}_h, \nabla H_h)_K - (\widehat{\boldsymbol{s}_h}\cdot\boldsymbol{\nu}, H_h)_{\partial K} = 0, \tag{3.12a}$$

$$-(\boldsymbol{s}_h, \boldsymbol{p}_h)_K + (\boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{p}_h)_K = 0, \tag{3.12b}$$

$$(\boldsymbol{p}_h, \boldsymbol{s}_h)_K + (H_h, \nabla\cdot\boldsymbol{s}_h)_K - (\widehat{H_h}, \boldsymbol{\nu}\cdot\boldsymbol{s}_h)_{\partial K} = 0, \tag{3.12c}$$

$$(H_h, (u_h)_t)_K + (\boldsymbol{q}_h, \nabla(u_h)_t)_K - (\widehat{\boldsymbol{q}_h}\cdot\boldsymbol{\nu}, (u_h)_t)_{\partial K} = 0, \tag{3.12d}$$

$$-(\boldsymbol{q}_h, (\boldsymbol{r}_h)_t)_K + \left(\frac{\boldsymbol{r}_h}{Q_h}, (\boldsymbol{r}_h)_t\right)_K = 0, \tag{3.12e}$$

$$((\boldsymbol{r}_h)_t, \boldsymbol{q}_h)_K + ((u_h)_t, \nabla\cdot\boldsymbol{q}_h)_K - (\widehat{(u_h)_t}, \boldsymbol{\nu}\cdot\boldsymbol{q}_h)_{\partial K} = 0. \tag{3.12f}$$

Summing up (3.12a)–(3.12f) we get

$$\begin{aligned}
&(\boldsymbol{s}_h, \nabla H_h)_K + (H_h, \nabla\cdot\boldsymbol{s}_h)_K - (\widehat{\boldsymbol{s}_h}\cdot\boldsymbol{\nu}, H_h)_{\partial K} - (\widehat{H_h}, \boldsymbol{\nu}\cdot\boldsymbol{s}_h)_{\partial K}\\
&\quad + (\boldsymbol{q}_h, \nabla(u_h)_t)_K + ((u_h)_t, \nabla\cdot\boldsymbol{q}_h)_K - (\widehat{\boldsymbol{q}_h}\cdot\boldsymbol{\nu}, (u_h)_t)_{\partial K} - (\widehat{(u_h)_t}, \boldsymbol{\nu}\cdot\boldsymbol{q}_h)_{\partial K}\\
&\quad + \left(\frac{\boldsymbol{r}_h}{Q_h}, (\boldsymbol{r}_h)_t\right)_K + (\boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{p}_h)_K = 0.
\end{aligned}$$

In view of (3.6) we differentiate Q_h with respect to time t to get

$$(Q_h)_t = \frac{\boldsymbol{r}_h}{Q_h}\cdot(\boldsymbol{r}_h)_t.$$

We also have

$$\begin{aligned}
(\boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{p}_h)_K &= (\boldsymbol{p}_h\cdot\boldsymbol{p}_h,\ Q_h)_K - \left(\boldsymbol{p}_h\cdot\boldsymbol{r}_h,\ \frac{\boldsymbol{p}_h\cdot\boldsymbol{r}_h}{Q_h}\right)_K\\
&\ge (\boldsymbol{p}_h\cdot\boldsymbol{p}_h,\ Q_h)_K - \left(|\boldsymbol{p}_h|^2,\ \frac{|\boldsymbol{r}_h|^2}{Q_h}\right)_K = \left(\boldsymbol{p}_h\cdot\boldsymbol{p}_h,\ \frac{1}{Q_h}\right)_K \ge 0.
\end{aligned} \tag{3.13}$$

Summing up over all elements K and Lemma 3.1 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Q_h\, d\Omega \le 0.$$

□

Now we consider the well-posedness of the LDG solution $u_h(x,t)$. On each element we get an ordinary differential equations (ODEs) by (3.4a)–(3.4f). It is a Cauchy problem. For the local existence we can get easily by the theory of the local existence and uniqueness of ODEs. Using Proposition 3.3, we can see u_h is bounded in L^2-norm. So for each h, u_h is bounded in L^∞-norm, the bound depends on h. Then we get global existence by the theory of the global existence and uniqueness of ODEs. So for each h, we can get a unique solution using this LDG scheme.

3.3 The Main Results of Error Estimates

In this section, we state the main error estimates of the semi-discrete LDG scheme (3.4) for the two-dimensional problem in Cartesian meshes.

We introduce notations

$$\boldsymbol{e}_u = u - u_h, \qquad \boldsymbol{e}_H = H - H_h, \qquad \boldsymbol{e}_{\boldsymbol{r}} = \boldsymbol{r} - \boldsymbol{r}_h,$$
$$\boldsymbol{e}_{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}_h, \qquad \boldsymbol{e}_{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{p}_h, \qquad \boldsymbol{e}_s = \boldsymbol{s} - \boldsymbol{s}_h.$$

We assume the periodic boundary conditions and the equation has a unique solution u, which satisfies

$$u \in L^\infty([0,T]; W^{3,\infty}(\Omega)) \cap L^\infty((0,T); H^{k+4}(\Omega)), \tag{3.14}$$
$$u_t \in L^\infty([0,T]; W^{1,\infty}(\Omega)) \cap L^\infty([0,T]; H^{k+3}(\Omega)), \tag{3.15}$$

which implies $\|u\|_{H^{k+4}(\Omega)}$, $\|u_t\|_{H^{k+3}(\Omega)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}_t\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|u_t\|_{L^\infty(L^\infty)}$ are all bounded. Here, $\|\cdot\|_{L^\infty(L^\infty)}$ denotes $\|\cdot\|_{L^\infty([0,T];L^\infty(\Omega))}$

Theorem 3.5 *Assume that* (3.1*a*)–(3.1*f*) *with periodic boundary conditions and smooth enough initial condition has a unique solution u, which satisfies* (3.14)–(3.15). *Let u_h be the numerical solution of the semi-discrete LDG scheme* (3.4)–(3.7) *and the initial condition $u_h(x,0) = P^- u(x,0)$. For rectangular triangulation of Ω, if the finite element space is the piecewise tensor product polynomials of degree $k \ge 1$, then for small enough h there holds the following error estimates*

$$\max_t \|e_u\|_\Omega + \max_t \|\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega + \max_t \|\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega \le Ch^{k+1}, \tag{3.16}$$
$$\int_0^T \|e_H\|_\Omega^2\, dt + \int_0^T \|\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2\, dt + \int_0^T \|\boldsymbol{e}_s\|_\Omega^2\, dt \le Ch^{2k+2}, \tag{3.17}$$

where C depends on the final time T, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}_t\|_{L^\infty(L^\infty)}$, $\|u\|_{L^\infty([0,T];H^{k+4}(\Omega))}$ and $\|u_t\|_{L^\infty([0,T];H^{k+2}(\Omega))}$.

Remark 3.1 Although the theory can only guarantee the situation when $k \ge 1$. But the numerical test shows the same optimal accuracy result for the piecewise constant polynomials in [31].

4 Auxiliary Results

In this section, we introduce two operators and some auxiliary lemmas which give the different properties of the operators defined.

4.1 Two Operators

We borrow the idea of operators from [1] and use the similar technique to prove the properties of the operators.

Let $\boldsymbol{v}$, $\boldsymbol{w}$ belong to $L^2(\mathcal{T}_h) \times L^2(\mathcal{T}_h)$. We define two operators on each element K

$$a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}) = \left(Q\left(\boldsymbol{I} - \frac{\boldsymbol{r}\otimes\boldsymbol{r}}{Q^2}\right)\boldsymbol{v}, \boldsymbol{w}\right)_K, \tag{4.1}$$
$$\tilde{a}_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}) = \left(\frac{1}{Q}\boldsymbol{v}, \boldsymbol{w}\right)_K, \tag{4.2}$$

$$a_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}) = \left(Q_h\left(\boldsymbol{I} - \frac{\boldsymbol{r}_h \otimes \boldsymbol{r}_h}{Q_h^2}\right)\boldsymbol{v}, \boldsymbol{w}\right)_K, \tag{4.3}$$

$$\tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}) = \left(\frac{1}{Q_h}\boldsymbol{v}, \boldsymbol{w}\right)_K. \tag{4.4}$$

We also use the following notations

$$a(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}) = \sum_K a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}), \qquad a(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}) = \sum_K a_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}), \tag{4.5}$$

$$\tilde{a}(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}) = \sum_K \tilde{a}_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}), \qquad \tilde{a}(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}) = \sum_K \tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}). \tag{4.6}$$

Remark 4.1 Comparing a_K and $\tilde{a}_K$

- The forms a_K and $\tilde{a}_K$ are symmetric if we fix $\boldsymbol{r}$.
- The forms a_K and $\tilde{a}_K$ are nonnegative, i.e.

$$a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{v}) \geq 0.$$

  The reason is the same as the proof in (3.13).
- Equivalence
  - If $d = 1$, $a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{v}) = \tilde{a}_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{v})$.
  - If $d > 1$, $a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{r}) = \tilde{a}_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{r})$.

They hold for $a_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w})$ and $\tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w})$.

Remark 4.2 Equivalence form of a_K. Let $\boldsymbol{\xi} := \frac{\boldsymbol{r}}{|\boldsymbol{r}|}$ if $\boldsymbol{r} \neq 0$ and be arbitrary otherwise. Let $\{\boldsymbol{\chi}_i\}_{i=1}^{d-1}$ be a normalized complementary orthogonal set perpendicular to $\boldsymbol{\xi}$. Then each vector $\boldsymbol{v}$ in R^d can be represented as follows:

$$\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{\xi}\boldsymbol{\xi} + \sum_{i=1}^{d-1} \boldsymbol{v} \cdot \boldsymbol{\chi}_i \boldsymbol{\chi}_i. \tag{4.7}$$

A simple calculation then yields

$$a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w}) = \left(\frac{1}{Q}\boldsymbol{v} \cdot \boldsymbol{\xi}, \boldsymbol{w} \cdot \boldsymbol{\xi}\right)_K + \sum_{i=1}^{d-1}(Q\boldsymbol{v} \cdot \boldsymbol{\chi}_i, \boldsymbol{w} \cdot \boldsymbol{\chi}_i)_K. \tag{4.8}$$

Let $\boldsymbol{\xi}_h := \frac{\boldsymbol{r}_h}{|\boldsymbol{r}_h|}$ if $\boldsymbol{r}_h \neq 0$ and be arbitrary otherwise. Let $\{\boldsymbol{\chi}_{hi}\}_{i=1}^{d-1}$ be a normalized complementary orthogonal set perpendicular to $\boldsymbol{\xi}_h$. The same analysis gives

$$a_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w}) = \left(\frac{1}{Q_h}\boldsymbol{v} \cdot \boldsymbol{\xi}_h, \boldsymbol{w} \cdot \boldsymbol{\xi}_h\right)_K + \sum_{i=1}^{d-1}(Q_h\boldsymbol{v} \cdot \boldsymbol{\chi}_{hi}, \boldsymbol{w} \cdot \boldsymbol{\chi}_{hi})_K. \tag{4.9}$$

4.2 Basic Geometric Formulas

We start by introducing the following notations which are also used in [22]:

$$\boldsymbol{\gamma} = \frac{(-\boldsymbol{r}, 1)^T}{Q}, \qquad \boldsymbol{\gamma}_h = \frac{(-\boldsymbol{r}_h, 1)^T}{Q_h},$$

$$N_h^K(t) = (Q_h(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h), \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)_K.$$

Here, $\boldsymbol{r}_h$ is finite element approximation to $\boldsymbol{r}$. And we denote

$$Q_h := \sqrt{1 + |\boldsymbol{r}_h|^2}, \qquad N_h(t) := \sum_K N_h^K(t).$$

Lemma 4.1 [22] *Using the notation introduced above, the follow inequalities hold*:

$$\left| \frac{1}{Q} - \frac{1}{Q_h} \right| \le |\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|, \qquad |Q - Q_h| \le Q Q_h |\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|, \tag{4.10}$$

$$\left| \frac{\boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h}{Q_h} \right| \le |\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|, \qquad \left| \frac{\boldsymbol{r} \otimes \boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h \otimes \boldsymbol{r}_h}{Q_h} \right| \le 3 Q Q_h |\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|, \tag{4.11}$$

$$|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h| \le |\boldsymbol{r} - \boldsymbol{r}_h|. \tag{4.12}$$

4.3 *A priori* Assumption

To derive the error estimates. We need to make *a priori* assumption:

- $d \le 3$

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_\Omega \le h^{\frac{7}{4}}. \tag{4.13}$$

Then we get

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^\infty(\Omega)} \le C h^{\frac{1}{4}}, \tag{4.14}$$

where C is a constant independent of h.

Recalling that $Q_h = \sqrt{1 + |\boldsymbol{r}_h|^2}$, we immediately get

$$\|Q_h\|_{L^\infty(\Omega)} = \|\sqrt{1 + |\boldsymbol{r}_h|^2}\|_{L^\infty(\Omega)} \le R, \tag{4.15}$$

where R depends on $\|\boldsymbol{r}\|_{L^\infty(\Omega)}$ and T. Without loss of generality, let us assume $\|\boldsymbol{r}\|_{L^\infty(\Omega)} < R$ and take $R = \max\{R, \|\boldsymbol{r}\|_{L^\infty(\Omega)}\}$ otherwise. This assumption will be used to get the following auxiliary estimates lemmas.

Remark 4.3 The assumption will be satisfied if $k \ge 1$. We will give the explanation in the end of the proof.

Remark 4.4 Using this assumption we simplify the proof of lemma 4.4 and Lemma 4.6 comparing with [1].

4.4 Auxiliary Lemmas

In this section, we will give some auxiliary lemmas to help prove the error estimates.

Lemma 4.2 *For every* $\epsilon > 0$,

$$\begin{aligned} |a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{w})| &\le \epsilon a_K(\boldsymbol{r}; \boldsymbol{v}, \boldsymbol{v}) + \frac{1}{4\epsilon} a_K(\boldsymbol{r}; \boldsymbol{w}, \boldsymbol{w}), \\ |a_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{w})| &\le \epsilon a_K(\boldsymbol{r}_h; \boldsymbol{v}, \boldsymbol{v}) + \frac{1}{4\epsilon} a_K(\boldsymbol{r}_h; \boldsymbol{w}, \boldsymbol{w}). \end{aligned} \tag{4.16}$$

Proof By Remark 4.2, we have

$$\begin{aligned}|a_K(\boldsymbol{r};\boldsymbol{v},\boldsymbol{w})| &= \left|\left(\frac{1}{Q}\boldsymbol{v}\cdot\boldsymbol{\xi},\boldsymbol{w}\cdot\boldsymbol{\xi}\right)_K + \sum_{i=1}^{d-1}(Q\boldsymbol{v}\cdot\boldsymbol{\chi}_i,\boldsymbol{w}\cdot\boldsymbol{\chi}_i)_K\right| \\ &= \left|\left(\frac{\boldsymbol{v}\cdot\boldsymbol{\xi}}{\sqrt{Q}},\frac{\boldsymbol{w}\cdot\boldsymbol{\xi}}{\sqrt{Q}}\right)_K + \sum_{i=1}^{d-1}(\boldsymbol{v}\cdot\boldsymbol{\chi}_i\sqrt{Q},\boldsymbol{w}\cdot\boldsymbol{\chi}_i\sqrt{Q})_K\right| \\ &\le \epsilon a_K(\boldsymbol{r};\boldsymbol{v},\boldsymbol{v}) + \frac{1}{4\epsilon}a_K(\boldsymbol{r};\boldsymbol{w},\boldsymbol{w}),\end{aligned}$$

where the last inequality is by Cauchy-Schwarz inequality. The same analysis for $a_K(\boldsymbol{r}_h;\boldsymbol{v},\boldsymbol{w})$. □

Lemma 4.3 *For every* $\epsilon>0$, *there exists a constant* $C=C(\epsilon,\|\boldsymbol{p}\|_{L^\infty(L^\infty)},\|\boldsymbol{r}\|_{L^\infty(L^\infty)})>0$, *such that*

$$|a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{w})-a_K(\boldsymbol{r}_h,\boldsymbol{p}_h,\boldsymbol{w})|\le\epsilon a_K(\boldsymbol{r}_h;\boldsymbol{e_p},\boldsymbol{e_p})+C\|\boldsymbol{w}\|_K^2+N_h^K(t). \tag{4.17}$$

Proof The proof of this lemma will be given in Appendix A.1. □

Lemma 4.4 *For every* $\epsilon>0$, *there exists a constant* $C=C(\epsilon,\|\boldsymbol{r}\|_{L^\infty(L^\infty)},\|\boldsymbol{p}\|_{L^\infty(L^\infty)})>0$, *such that*

$$|a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{w})-a_K(\boldsymbol{r}_h;\boldsymbol{p},\boldsymbol{w})|\le\epsilon a_K(\boldsymbol{r}_h;\boldsymbol{w},\boldsymbol{w})+CN_h^K(t). \tag{4.18}$$

Proof The proof of this lemma will be given in Appendix A.2. □

Corollary 4.5 (Coercivity of a_K) *There exists* $C=C(\|\boldsymbol{r}\|_{L^\infty(L^\infty)},\|\boldsymbol{p}\|_{L^\infty(L^\infty)})>0$, *such that*

$$a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{e_p})-a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\boldsymbol{e_p})\ge\frac{1}{2}a_K(\boldsymbol{r}_h;\boldsymbol{e_p},\boldsymbol{e_p})-CN_h^K(t).$$

Proof Adding and subtracting $a_K(\boldsymbol{r}_h;\boldsymbol{p},\boldsymbol{e_p})$, and using Lemma 4.4 with $\epsilon=\frac{1}{2}$, we can obtain the desired estimate. □

Lemma 4.6 *There exists* $C=C(\|\boldsymbol{r}\|_{L^\infty(L^\infty)})>0$, *such that*

$$\|\boldsymbol{e_r}\|_K^2\le CN_h^K(t). \tag{4.19}$$

Proof The proof has been given in [22] in Lemma 5.3. □

Lemma 4.7 *There exists* $C=C(\|\boldsymbol{r}_t\|_{L^\infty(L^\infty)})>0$, *such that*

$$\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{e}_{\boldsymbol{r}_t})-\tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{e}_{\boldsymbol{r}_t})\ge\frac{1}{2}\frac{d}{dt}N_h^K(t)-CN_h^K(t). \tag{4.20}$$

Proof The proof has been given in [22] in Lemma 5.9. □

Lemma 4.8 *For every* $\epsilon > 0$

$$|\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{w}) - \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{w})| \le \epsilon \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{w},\boldsymbol{w}) + \frac{1}{4\epsilon} N_h^K(t). \tag{4.21}$$

Proof Using the definition of $\tilde{a}_K$

$$\begin{aligned}
|\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{w}) - \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{w})| &= \left| \left(\frac{\boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h}{Q_h}, \boldsymbol{w} \right)_K \right| \\
&\le \left(\left| \frac{\boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h}{Q_h} \right|, |\boldsymbol{w}| \right)_K \\
&\le \epsilon \left(\frac{1}{Q_h} \boldsymbol{w}, \boldsymbol{w} \right)_K + \frac{1}{4\epsilon} (Q_h(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h), \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)_K.
\end{aligned}$$

This finally gives the proof. □

We shall use all these results in the next section.

5 Proof of the Error Estimates

In this section, we will give the proof of the main results in Sect. 3.3.

5.1 Error Equations

In order to obtain the error estimate to smooth solutions for the considered semi-discrete LDG scheme (3.4), we need to first obtain the error equation.

Notice that the scheme (3.4) is also satisfied when the numerical solutions u_h, $\boldsymbol{s}_h$, $\boldsymbol{p}_h$, H_h, $\boldsymbol{q}_h$, $\boldsymbol{r}_h$ are replaced by the exact solutions u, $\boldsymbol{s}$, $\boldsymbol{p}$, H, $\boldsymbol{q}$, $\boldsymbol{r}$. So we have the error equations

$$(u_t - (u_h)_t, \varphi)_K - (\boldsymbol{s} - \boldsymbol{s}_h, \nabla\varphi)_K + ((\widehat{\boldsymbol{s} - \boldsymbol{s}_h}) \cdot \boldsymbol{\nu}, \varphi)_{\partial K} = 0, \tag{5.1a}$$

$$(\boldsymbol{s} - \boldsymbol{s}_h, \boldsymbol{\phi})_K - (\boldsymbol{E}(\boldsymbol{r})\boldsymbol{p} - \boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{\phi})_K = 0, \tag{5.1b}$$

$$(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\eta})_K + (H - H_h, \nabla \cdot \boldsymbol{\eta})_K - (\widehat{H - H_h}, \boldsymbol{\nu} \cdot \boldsymbol{\eta})_{\partial K} = 0, \tag{5.1c}$$

$$(H - H_h, \vartheta)_K + (\boldsymbol{q} - \boldsymbol{q}_h, \nabla\vartheta)_K - ((\widehat{\boldsymbol{q} - \boldsymbol{q}_h}) \cdot \boldsymbol{\nu}, \vartheta)_{\partial K} = 0, \tag{5.1d}$$

$$(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\rho})_K - \left(\frac{\boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h}{Q_h}, \boldsymbol{\rho} \right)_K = 0, \tag{5.1e}$$

$$(\boldsymbol{r} - \boldsymbol{r}_h, \boldsymbol{\zeta})_K + (u - u_h, \nabla \cdot \boldsymbol{\zeta})_K - (\widehat{u - u_h}, \boldsymbol{\nu} \cdot \boldsymbol{\zeta})_{\partial K} = 0. \tag{5.1f}$$

Denote

$$\begin{aligned}
&B_K(u, \boldsymbol{s}, \boldsymbol{p}, H, \boldsymbol{q}, \boldsymbol{r}; \varphi, \boldsymbol{\phi}, \boldsymbol{\eta}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\
&= (u_t, \varphi)_K + (\boldsymbol{s}, \boldsymbol{\phi})_K + (\boldsymbol{p}, \boldsymbol{\eta})_K + (H, \vartheta)_K + (\boldsymbol{q}, \boldsymbol{\rho})_K + (\boldsymbol{r}, \boldsymbol{\zeta})_K \\
&\quad - (\boldsymbol{s}, \nabla\varphi)_K + (H, \nabla \cdot \boldsymbol{\eta})_K + (\boldsymbol{q}, \nabla\vartheta)_K + (u, \nabla \cdot \boldsymbol{\zeta})_K \\
&\quad + (\widehat{\boldsymbol{s}} \cdot \boldsymbol{\nu}, \varphi)_{\partial K} - (\widehat{H}, \nabla\boldsymbol{\nu} \cdot \boldsymbol{\eta})_{\partial K} - (\widehat{\boldsymbol{q}} \cdot \boldsymbol{\nu}, \vartheta)_{\partial K} - (\widehat{u}, \boldsymbol{\nu} \cdot \boldsymbol{\zeta})_{\partial K}.
\end{aligned} \tag{5.2}$$

We can easily check that B_K is linear in each variable. And we use the convention $B := \sum_K B_K$. In view of the definition of a_K and $\tilde{a}_K$, (5.1a)–(5.1f) can be simplified as follows

$$\begin{aligned} &B_K(u-u_h, \boldsymbol{s}-\boldsymbol{s}_h, \boldsymbol{p}-\boldsymbol{p}_h, H-H_h, \boldsymbol{q}-\boldsymbol{q}_h, \boldsymbol{r}-\boldsymbol{r}_h; \varphi, \boldsymbol{\phi}, \boldsymbol{\eta}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\ &\quad - (a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{\phi}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{\phi})) - (\tilde{a}_K(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{\rho}) - \tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{\rho})) = 0. \end{aligned} \tag{5.3}$$

Denote

$$\begin{aligned} \boldsymbol{e}_u &= u-u_h = u - Pu + Pu - u_h = u - Pu + P\boldsymbol{e}_u, \\ \boldsymbol{e}_H &= H - H_h = H - PH + PH - H_h = H - PH + P\boldsymbol{e}_H, \\ \boldsymbol{e}_{\boldsymbol{r}} &= \boldsymbol{r} - \boldsymbol{r}_h = \boldsymbol{r} - \Pi\boldsymbol{r} + \Pi\boldsymbol{r} - \boldsymbol{r}_h = \boldsymbol{r} - \Pi\boldsymbol{r} + \Pi\boldsymbol{e}_{\boldsymbol{r}}, \\ \boldsymbol{e}_{\boldsymbol{q}} &= \boldsymbol{q} - \boldsymbol{q}_h = \boldsymbol{q} - \Pi\boldsymbol{q} + \Pi\boldsymbol{q} - \boldsymbol{q}_h = \boldsymbol{q} - \Pi\boldsymbol{q} + \Pi\boldsymbol{e}_{\boldsymbol{q}}, \\ \boldsymbol{e}_{\boldsymbol{p}} &= \boldsymbol{p} - \boldsymbol{p}_h = \boldsymbol{p} - \Pi\boldsymbol{p} + \Pi\boldsymbol{p} - \boldsymbol{p}_h = \boldsymbol{p} - \Pi\boldsymbol{p} + \Pi\boldsymbol{e}_{\boldsymbol{p}}, \\ \boldsymbol{e}_{\boldsymbol{s}} &= \boldsymbol{s} - \boldsymbol{s}_h = \boldsymbol{s} - \Pi\boldsymbol{s} + \Pi\boldsymbol{s} - \boldsymbol{s}_h = \boldsymbol{s} - \Pi\boldsymbol{s} + \Pi\boldsymbol{e}_{\boldsymbol{s}}. \end{aligned}$$

Let P and Π be the projections onto the finite element spaces V_h and $\sum_h$, respectively, which have been defined in Sect. 2.3. In this paper we choose the projection as follows

$$(P, \Pi) = (P^-, P^+) \quad \text{in one dimension,} \tag{5.4}$$

$$(P, \Pi) = (P^-, \Pi^+) \quad \text{in multi-dimension.} \tag{5.5}$$

We choose the initial condition $u_h(x,0) = P^- u(x,0)$. Taking $\boldsymbol{\zeta} = \Pi\boldsymbol{r} - \boldsymbol{r}_h$ in (5.1f) with the help (2.8), Lemmas 3.2 and 5.7 we obtain the initial error estimates

$$\begin{aligned} \|u(x,0) - u_h(x,0)\|_\Omega &\le Ch^{k+1}, \\ \|\boldsymbol{r}(x,0) - \boldsymbol{r}_h(x,0)\|_\Omega &\le Ch^{k+1}, \\ \|\boldsymbol{q}(x,0) - \boldsymbol{q}_h(x,0)\|_\Omega &\le Ch^{k+1}. \end{aligned} \tag{5.6}$$

We can rewrite (5.3) as followings with the aid of the interpolation error

$$\begin{aligned} &B_K(P\boldsymbol{e}_u, \Pi\boldsymbol{e}_{\boldsymbol{s}}, \Pi\boldsymbol{e}_{\boldsymbol{p}}, P\boldsymbol{e}_H, \Pi\boldsymbol{e}_{\boldsymbol{q}}, \Pi\boldsymbol{e}_{\boldsymbol{r}}; \varphi, \boldsymbol{\phi}, \boldsymbol{\eta}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\ &\quad + B_K(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r}; \varphi, \boldsymbol{\phi}, \boldsymbol{\eta}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\ &\quad - (a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{\phi}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{\phi})) - (\tilde{a}_K(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{\rho}) - \tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{\rho})) = 0. \end{aligned} \tag{5.7}$$

5.2 The First Energy Equation

We try to mimic the derivation of the L^2 stability in order to gain control on $\|\boldsymbol{e}_u\|_\Omega$ and $\|\boldsymbol{e}_H\|_\Omega$.

We first choose the test functions in (5.7)

$$\varphi = P\boldsymbol{e}_u, \qquad \boldsymbol{\phi} = \Pi\boldsymbol{e}_{\boldsymbol{r}}, \qquad \boldsymbol{\eta} = \Pi\boldsymbol{e}_{\boldsymbol{q}}, \qquad \vartheta = P\boldsymbol{e}_H, \qquad \boldsymbol{\rho} = -\Pi\boldsymbol{e}_{\boldsymbol{p}}, \qquad \boldsymbol{\zeta} = -\Pi\boldsymbol{e}_{\boldsymbol{s}}.$$

Then we obtain

$$
\begin{aligned}
&B_K(P\boldsymbol{e}_u, \Pi\boldsymbol{e}_s, \Pi\boldsymbol{e}_p, P\boldsymbol{e}_H, \Pi\boldsymbol{e}_q, \Pi\boldsymbol{e}_r; P\boldsymbol{e}_u, \Pi\boldsymbol{e}_r, \Pi\boldsymbol{e}_q, P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_p, -\Pi\boldsymbol{e}_s)\\
&\qquad + B_K(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r};\\
&\qquad P\boldsymbol{e}_u, \Pi\boldsymbol{e}_r, \Pi\boldsymbol{e}_q, P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_p, -\Pi\boldsymbol{e}_s)\\
&\quad = (a_K(\boldsymbol{r}; \boldsymbol{p}, \Pi\boldsymbol{e}_r) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \Pi\boldsymbol{e}_r)) - (\tilde{a}_K(\boldsymbol{r}; \boldsymbol{r}, \Pi\boldsymbol{e}_p) - \tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{r}_h, \Pi\boldsymbol{e}_p)).
\end{aligned} \tag{5.8}
$$

In the following, we will give the estimates for each term in (5.8).

Lemma 5.1 *The following equation holds*:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2\\
&\quad = -B(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r};\\
&\qquad P\boldsymbol{e}_u, \Pi\boldsymbol{e}_r, \Pi\boldsymbol{e}_q, P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_p, -\Pi\boldsymbol{e}_s)\\
&\qquad + (a(\boldsymbol{r}; \boldsymbol{p}, \Pi\boldsymbol{e}_r) - a(\boldsymbol{r}_h; \boldsymbol{p}_h, \Pi\boldsymbol{e}_r)) - (\tilde{a}(\boldsymbol{r}; \boldsymbol{r}, \Pi\boldsymbol{e}_p) - \tilde{a}(\boldsymbol{r}_h; \boldsymbol{r}_h, \Pi\boldsymbol{e}_p)).
\end{aligned} \tag{5.9}
$$

Proof The same argument as that used for the L^2 stability in Proposition 3.3. □

Lemma 5.2 *For every* $\epsilon > 0$, *there exists a positive* $C > 0$, *such that*

$$
\begin{aligned}
&\int_0^t |B(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r};\\
&\qquad P\boldsymbol{e}_u, \Pi\boldsymbol{e}_r, \Pi\boldsymbol{e}_q, P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_p, -\Pi\boldsymbol{e}_s)|\,dt\\
&\quad \le Ch^{2k+2} + \epsilon\int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_r\|_\Omega^2 + \|\Pi\boldsymbol{e}_q\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e}_p\|_\Omega^2 + \|\Pi\boldsymbol{e}_s\|_\Omega^2)\,dt,
\end{aligned} \tag{5.10}
$$

where C *depends on* ϵ, t, $\|u\|_{H^{k+4}(\Omega)}$ *and* $\|u_t\|_{H^{k+2}(\Omega)}$.

Proof The proof of this lemma will be given in Appendix A.3. □

Lemma 5.3 *For every* $\epsilon > 0$, *there exists a positive* $C > 0$, *such that*

$$
\begin{aligned}
&\int_0^t |(a(\boldsymbol{r}; \boldsymbol{p}, \Pi\boldsymbol{e}_r) - a(\boldsymbol{r}_h; \boldsymbol{p}_h, \Pi\boldsymbol{e}_r)) - (\tilde{a}(\boldsymbol{r}; \boldsymbol{r}, \Pi\boldsymbol{e}_p) - \tilde{a}(\boldsymbol{r}_h; \boldsymbol{r}_h, \Pi\boldsymbol{e}_p))|\,dt\\
&\quad \le Ch^{2k+2} + C\int_0^t N_h(t)\,dt + \epsilon\int_0^t \|\boldsymbol{e}_p\|_\Omega^2\,dt,
\end{aligned} \tag{5.11}
$$

where C *depends on* ϵ, t, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$ *and* $\|u\|_{H^{k+4}(\Omega)}$.

Proof The proof of this lemma will be given in Appendix A.4. □

5.3 The Second Energy Equation

Next, we mimic the derivation of the area-decreasing to get control on $\|\boldsymbol{e}_p\|_\Omega$ and $N_h(t)$. We introduce the following bilinear form

$$
\begin{aligned}
&\tilde{B}_K(u, \boldsymbol{s}, \boldsymbol{p}, H, \boldsymbol{q}, \boldsymbol{r}; \varphi, \boldsymbol{\phi}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\
&\quad = (u_t, \varphi)_K + (\boldsymbol{s}, \boldsymbol{\phi})_K + (\boldsymbol{p}, \boldsymbol{\eta})_K + (H, \vartheta)_K + (\boldsymbol{q}, \boldsymbol{\rho})_K + (\boldsymbol{r}_t, \boldsymbol{\zeta})_K \\
&\qquad - (\boldsymbol{s}, \nabla\varphi)_K + (H, \nabla \cdot \boldsymbol{\eta})_K + (\boldsymbol{q}, \nabla\vartheta)_K + (u_t, \nabla \cdot \boldsymbol{\zeta})_K \\
&\qquad + (\widehat{\boldsymbol{s}} \cdot \boldsymbol{\nu}, \varphi)_{\partial K} - (\widehat{H}, \nabla\boldsymbol{\nu} \cdot \boldsymbol{\eta})_{\partial K} - (\vartheta, \widehat{\boldsymbol{q}} \cdot \boldsymbol{\nu})_{\partial K} - (\widehat{u_t}, \boldsymbol{\nu} \cdot \boldsymbol{\zeta})_{\partial K},
\end{aligned} \tag{5.12}
$$

and we can get the corresponding error equation

$$
\begin{aligned}
&\tilde{B}_K(P\boldsymbol{e}_u, \Pi\boldsymbol{e}_{\boldsymbol{s}}, \Pi\boldsymbol{e}_{\boldsymbol{p}}, P\boldsymbol{e}_H, \Pi\boldsymbol{e}_{\boldsymbol{q}}, \Pi\boldsymbol{e}_{\boldsymbol{r}}; \varphi, \boldsymbol{\phi}, \boldsymbol{\eta}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\
&\quad + \tilde{B}_K(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r}; \varphi, \boldsymbol{\phi}, \boldsymbol{\eta}, \vartheta, \boldsymbol{\rho}, \boldsymbol{\zeta}) \\
&\quad - (a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{\phi}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{\phi})) - (\tilde{a}_K(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{\rho}) - \tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{\rho})) = 0.
\end{aligned} \tag{5.13}
$$

We also use the convention $\tilde{B} := \sum_K \tilde{B}_K$.

We choose the test functions in (5.13)

$$
\begin{aligned}
&\varphi = -P\boldsymbol{e}_H, \qquad \boldsymbol{\phi} = -\Pi\boldsymbol{e}_{\boldsymbol{p}}, \qquad \boldsymbol{\eta} = \Pi\boldsymbol{e}_{\boldsymbol{s}}, \\
&\vartheta = P\boldsymbol{e}_{u_t}, \qquad \boldsymbol{\rho} = -\Pi\boldsymbol{e}_{\boldsymbol{r}_t}, \qquad \boldsymbol{\zeta} = \Pi\boldsymbol{e}_{\boldsymbol{q}}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&a(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{e}_{\boldsymbol{p}}) - a(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{e}_{\boldsymbol{p}}) + \tilde{a}(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{e}_{\boldsymbol{r}_t}) - \tilde{a}(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{e}_{\boldsymbol{r}_t}) \\
&\quad = \tilde{B}(P\boldsymbol{e}_u, \Pi\boldsymbol{e}_{\boldsymbol{s}}, \Pi\boldsymbol{e}_{\boldsymbol{p}}, P\boldsymbol{e}_H, \Pi\boldsymbol{e}_{\boldsymbol{q}}, \Pi\boldsymbol{e}_{\boldsymbol{r}}; \\
&\qquad - P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_{\boldsymbol{p}}, \Pi\boldsymbol{e}_{\boldsymbol{s}}, P\boldsymbol{e}_{u_t}, -\Pi\boldsymbol{e}_{\boldsymbol{r}_t}, \Pi\boldsymbol{e}_{\boldsymbol{q}}) \\
&\qquad + \tilde{B}(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r}; \\
&\qquad - P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_{\boldsymbol{p}}, \Pi\boldsymbol{e}_{\boldsymbol{s}}, P\boldsymbol{e}_{u_t}, -\Pi\boldsymbol{e}_{\boldsymbol{r}_t}, \Pi\boldsymbol{e}_{\boldsymbol{q}}) \\
&\qquad - (a(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{p} - \Pi\boldsymbol{p}) - a(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{p} - \Pi\boldsymbol{p})) - (\tilde{a}(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{r}_t - \Pi\boldsymbol{r}_t) \\
&\qquad - \tilde{a}(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{r}_t - \Pi\boldsymbol{r}_t)).
\end{aligned} \tag{5.14}
$$

In the following, we will give the estimates for the each term in (5.14).

Lemma 5.4 *The following equation holds*:

$$
\tilde{B}(P\boldsymbol{e}_u, \Pi\boldsymbol{e}_{\boldsymbol{s}}, \Pi\boldsymbol{e}_{\boldsymbol{p}}, P\boldsymbol{e}_H, \Pi\boldsymbol{e}_{\boldsymbol{q}}, \Pi\boldsymbol{e}_{\boldsymbol{r}}; -P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_{\boldsymbol{p}}, \Pi\boldsymbol{e}_{\boldsymbol{s}}, P\boldsymbol{e}_{u_t}, -\Pi\boldsymbol{e}_{\boldsymbol{r}_t}, \Pi\boldsymbol{e}_{\boldsymbol{q}}) = 0. \tag{5.15}
$$

Proof The same argument as that used for the area-decreasing in Proposition 3.4. □

Lemma 5.5 *For every* $\epsilon > 0$, *there exists a positive* $C > 0$, *such that*

$$
\begin{aligned}
&\bigg| \int_0^t \tilde{B}(u - Pu, \boldsymbol{s} - \Pi\boldsymbol{s}, \boldsymbol{p} - \Pi\boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi\boldsymbol{q}, \boldsymbol{r} - \Pi\boldsymbol{r}; \\
&\qquad - P\boldsymbol{e}_H, -\Pi\boldsymbol{e}_{\boldsymbol{p}}, \Pi\boldsymbol{e}_{\boldsymbol{s}}, P\boldsymbol{e}_{u_t}, -\Pi\boldsymbol{e}_{\boldsymbol{r}_t}, \Pi\boldsymbol{e}_{\boldsymbol{q}})\, dt \bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\le Ch^{2k+2} + \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi \boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2) \\
&\quad + \epsilon \int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi \boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2 + \|\Pi \boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi \boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2 + \|\Pi \boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2)\, dt,
\end{aligned}
\tag{5.16}
$$

where C depends on ϵ, t, $\|u\|_{H^{k+4}(\Omega)}$ and $\|u_t\|_{H^{k+3}(\Omega)}$.

Proof The proof of this lemma will be given in Appendix A.5. □

Lemma 5.6 *For every $\epsilon > 0$, there exists a positive $C > 0$, such that*

$$
\begin{aligned}
&\int_0^t |(a(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{p} - \Pi \boldsymbol{p}) - a(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{p} - \Pi \boldsymbol{p})) \\
&\qquad + (\tilde{a}(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{r}_t - \Pi \boldsymbol{r}_t) - \tilde{a}(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{r}_t - \Pi \boldsymbol{r}_t))|\, dt \\
&\quad \le Ch^{2k+2} + C\int_0^t N_h(t)\, dt + \epsilon \int_0^t \|\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2\, dt,
\end{aligned}
\tag{5.17}
$$

where C depends on ϵ, t, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$, $\|u\|_{H^{k+4}(\Omega)}$ and $\|u_t\|_{H^{k+2}(\Omega)}$.

Proof The proof of this lemma will be given in Appendix A.6. □

5.4 Estimates of $\|\Pi \boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2$ and $\|\Pi \boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2$

Lemma 5.7 *There exists $C = C(\|\boldsymbol{q}\|_{H^{k+1}(\Omega)}) > 0$, such that*

$$\|\Pi \boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 \le C(N_h(t) + h^{2k+2}).$$

Proof The proof has been given in [22] in Lemma 5.8. □

Lemma 5.8 *There exists $C = C(\|\boldsymbol{s}\|_{H^{k+1}(\Omega)}, \|\boldsymbol{r}\|_{L^\infty(L^\infty)}, \|\boldsymbol{p}\|_{L^\infty(L^\infty)}) > 0$, such that*

$$\|\Pi \boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2 \le C(N_h(t) + \|\boldsymbol{e}_p\|_\Omega^2 + h^{2k+2}).$$

Proof We consider (5.1b) separately.

$$((\boldsymbol{s} - \boldsymbol{s}_h), \boldsymbol{\phi})_K - (\boldsymbol{E}(\boldsymbol{r})\boldsymbol{p} - \boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \boldsymbol{\phi})_K = 0.$$

Taking $\boldsymbol{\phi} = \Pi \boldsymbol{e}_{\boldsymbol{s}}$, we obtain

$$\|\Pi \boldsymbol{e}_{\boldsymbol{s}}\|_K^2 = -(\boldsymbol{s} - \Pi \boldsymbol{s}, \Pi \boldsymbol{e}_{\boldsymbol{s}})_K + ((\boldsymbol{E}(\boldsymbol{r})\boldsymbol{p} - \boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \Pi \boldsymbol{e}_{\boldsymbol{s}})_K. \tag{5.18}$$

We observe that

$$
\begin{aligned}
|(\boldsymbol{E}(\boldsymbol{r})\boldsymbol{p} - \boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \Pi \boldsymbol{e}_{\boldsymbol{s}})_K| &= |a_K(\boldsymbol{r}; \boldsymbol{p}, \Pi \boldsymbol{e}_{\boldsymbol{s}}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \Pi \boldsymbol{e}_{\boldsymbol{s}})| \\
&= |a_K(\boldsymbol{r}; \boldsymbol{p}, \Pi \boldsymbol{e}_{\boldsymbol{s}}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}, \Pi \boldsymbol{e}_{\boldsymbol{s}})| + |a_K(\boldsymbol{r}_h; \boldsymbol{e}_{\boldsymbol{p}}, \Pi \boldsymbol{e}_{\boldsymbol{s}})|.
\end{aligned}
$$

In view of Lemma 4.2 and Lemma 4.4, we get

$$
\begin{aligned}
&|(\boldsymbol{E}(\boldsymbol{r})\boldsymbol{p} - \boldsymbol{E}(\boldsymbol{r}_h)\boldsymbol{p}_h, \Pi \boldsymbol{e}_s)_K| \\
&\quad \le 2\epsilon a_K(\boldsymbol{r}_h; \Pi\boldsymbol{e}_s, \Pi\boldsymbol{e}_s) + \frac{C}{4\epsilon} N_h^K(t) + \frac{1}{4\epsilon} a_K(\boldsymbol{r}_h; \boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{e}_{\boldsymbol{p}}) \\
&\quad \le \epsilon \|\Pi \boldsymbol{e}_s\|_K^2 + C(N_h^K(t) + a_K(\boldsymbol{r}_h; \boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{e}_{\boldsymbol{p}})) \\
&\quad \le \epsilon \|\Pi \boldsymbol{e}_s\|_K^2 + C(N_h^K(t) + \|\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2).
\end{aligned}
$$

The last inequality is according to a *priori* assumption. Finally we get the estimate with applying the Cauchy-Schwarz inequality in the first term of (5.18). □

5.5 Proof of Theorem 3.5

Now we are ready to get the estimates in Theorem 3.5.

Combining the results in Sects. 5.2 and 5.3, we can get the following estimates

$$
\begin{aligned}
&\int_0^t \frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|_\Omega^2\, dt + \int_0^t \|P\boldsymbol{e}_H\|_\Omega^2\, dt \\
&\quad + \int_0^t (a(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{e}_{\boldsymbol{p}}) - a(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{e}_{\boldsymbol{p}}) + \tilde{a}(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{e}_{r_t}) - \tilde{a}(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{e}_{r_t}))\, dt \\
&\le Ch^{2k+2} + C\int_0^t N_h(t)\, dt + \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2) \\
&\quad + \epsilon\int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_s\|_\Omega^2)\, dt. \quad (5.19)
\end{aligned}
$$

Using Corollary 4.5 and Lemma 4.7, we have

$$
\begin{aligned}
&\int_0^t \frac{1}{2}\frac{d}{dt}\|P\boldsymbol{e}_u\|_\Omega^2\, dt + \int_0^t \|P\boldsymbol{e}_H\|_\Omega^2\, dt + \int_0^t \frac{1}{2}a(\boldsymbol{r}_h; \boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{e}_{\boldsymbol{p}})\, dt + \int_0^t \frac{1}{2}\frac{d}{dt}N_h(t)\, dt \\
&\quad \le Ch^{2k+2} + C\int_0^t N_h(t)\, dt + \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2) \\
&\qquad + \epsilon\int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_s\|_\Omega^2)\, dt,
\end{aligned}
$$

where $C = C(\|\boldsymbol{r}\|_{L^\infty(L^\infty)}, \|\boldsymbol{r}_t\|_{L^\infty(L^\infty)}, \|\boldsymbol{p}\|_{L^\infty(L^\infty)}, \|u\|_{H^{k+4}(\Omega)}, \|u_t\|_{H^{k+3}(\Omega)}, \epsilon, t)$.

Form Remark 4.1, we can get

$$
\frac{1}{2}a(\boldsymbol{r}_h; \boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{e}_{\boldsymbol{p}}) \ge \frac{1}{2C}\|\boldsymbol{e}_{\boldsymbol{p}}\|^2,
$$

where C depends on *a priori* assumption constant R. So we obtain

$$
\begin{aligned}
&\frac{1}{2}\|P\boldsymbol{e}_u\|_\Omega^2 + \int_0^t \|P\boldsymbol{e}_H\|_\Omega^2\, dt + \frac{1}{2C}\int_0^t \|\boldsymbol{e}_{\boldsymbol{p}}\|^2\, dt + \frac{1}{2}N_h(t) \\
&\quad \le Ch^{2k+2} + C\int_0^t N_h(t)\, dt + \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2) \\
&\qquad + \epsilon\int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_s\|_\Omega^2)\, dt.
\end{aligned}
$$

After choosing ϵ sufficiently small, using Lemma 4.6, Lemma 5.7, Lemma 5.8 and employing the Gronwall's inequality, we get

$$\max_t \|Pe_u\|_\Omega^2 + \max_t N_h(t) + \int_0^T \|P\boldsymbol{e}_H\|_\Omega^2\, dt + \int_0^T \|\Pi \boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2\, dt \le Ch^{2k+2}$$

where C depends on T, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}_t\|_{L^\infty(L^\infty)}$, $\|u\|_{L^\infty([0,T];H^{k+4}(\Omega))}$ and $\|u_t\|_{L^\infty([0,T];H^{k+3}(\Omega))}$.

We use Lemma 4.6, Lemma 5.7, Lemma 5.8 again and obtain

$$\max_t \|e_{\boldsymbol{r}}\|_\Omega^2 + \max_t \|\Pi \boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + + \int_0^T \|\Pi \boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2\, dt \le Ch^{2k+2}.$$

Triangle inequality yields Theorem 3.5.

To complete the proof, let us verify the *a priori* assumptions (4.13). For $k \ge 1$ and $d \le 3$, we can consider h small enough so that $Ch^{k+1} < \frac{1}{2}h^{\frac{7}{4}}$, where C is the constant determined by the final time T. Then, if $t^* = \sup\{t : \|\boldsymbol{r}(s) - \boldsymbol{r}_h(s)\| \le h^{\frac{7}{4}},\ \|H(s) - H_h(s)\| \le h^{\frac{7}{4}}, s \in [0,t)\}$, we would have $\|\boldsymbol{r}(t^*) - \boldsymbol{r}_h(t^*)\| = h^{\frac{7}{4}}$, $\|H(t^*) - H_h(t^*)\| = h^{\frac{7}{4}}$ by continuity if t^* is finite. On the other hand, our proof implies that (4.13) and holds for $t \le t^*$, in particular

$$\|\boldsymbol{r}(t^*) - \boldsymbol{r}_h(t^*)\| \le Ch^{k+\frac{1}{2}} < \frac{1}{2}h^{\frac{7}{4}}, \qquad \|H(t^*) - H_h(t^*)\| \le Ch^{k+\frac{1}{2}} < \frac{1}{2}h^{\frac{7}{4}}.$$

This is a contradiction if $t^* < T$. Hence $t^* \ge T$ and our *a priori* assumptions (4.13) is justified when $d \le 3$.

6 Concluding Remarks

In this paper, we have presented the optimal error analysis for the LDG method of the surface diffusion of graphs on Cartesian meshes. The analysis is made for the fully nonlinear case and the results are valid for all space dimension $d \le 3$ and polynomial degree $k \ge 1$.

An important issue not addressed in this paper is L^2 a priori error estimates on triangular meshes. On Cartesian meshes the special projection has been used to get rid of the boundary terms. But if we follow the same proof technique in this paper for triangular meshes, we could easily lose half an order or even one order in accuracy, because we can not eliminate the inter-element boundary terms that affect the convergence rate by using the known projections. Such error estimates are left for future work.

Appendix: Proof of Several Lemmas

A.1 Proof of Lemma 4.3

We first add and subtract the term $a_K(\boldsymbol{r}_h; \boldsymbol{p}, \boldsymbol{w})$ to obtain

$$\begin{aligned} a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{w}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{w}) &= a_K(\boldsymbol{r}_h; \boldsymbol{e}_{\boldsymbol{p}}, \boldsymbol{w}) + a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{w}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}, \boldsymbol{w}) \\ &:= (I) + (II). \end{aligned}$$

We will analyze (I) and (II) separately. By Lemma 4.2, we have

$$|(I)| \le \epsilon a_K(\boldsymbol{r}_h; \boldsymbol{e_p}, \boldsymbol{e_p}) + \frac{1}{4\epsilon} a_K(\boldsymbol{r}_h; \boldsymbol{w}, \boldsymbol{w}).$$

Using the definition of a_K, we get

$$\begin{aligned} a_K(\boldsymbol{r}_h; \boldsymbol{w}, \boldsymbol{w}) &= (Q_h \boldsymbol{w}, \boldsymbol{w}) - \left(\frac{1}{Q_h} \boldsymbol{w} \cdot \boldsymbol{r}_h, \boldsymbol{w} \cdot \boldsymbol{r}_h\right)_K \\ &\le (Q_h \boldsymbol{w}, \boldsymbol{w})_K \le R \|\boldsymbol{w}\|_K^2. \end{aligned}$$

We now turn to estimate (II). It follows from the definition of a_K and Lemma 4.1

$$\begin{aligned} (II) &= |a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{w}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}, \boldsymbol{w})| \\ &= \left| \left(\left((Q - Q_h)\boldsymbol{I} - \left(\frac{\boldsymbol{r} \otimes \boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h \otimes \boldsymbol{r}_h}{Q_h} \right) \right) \boldsymbol{w}, \boldsymbol{p} \right)_K \right| \\ &\le 4(|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h| \sqrt{Q_h}, Q\sqrt{Q_h} |\boldsymbol{p}||\boldsymbol{w}|)_K \\ &\le (|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|^2, Q_h)K + 4(|\boldsymbol{p}|^2 |\boldsymbol{w}|^2, Q^2 Q_h)_K \\ &\le N_h^K(t) + C\|\boldsymbol{w}\|_K^2, \end{aligned}$$

where C depends on $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$. By the triangle inequality, we obtain (4.17).

A.2 Proof of Lemma 4.4

The proof is similar to the analysis of the (II) in Lemma 4.3. We have

$$\begin{aligned} &|a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{w}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}, \boldsymbol{w})| \\ &\quad = \left| \left(\left((Q - Q_h)\boldsymbol{I} - \left(\frac{\boldsymbol{r} \otimes \boldsymbol{r}}{Q} - \frac{\boldsymbol{r}_h \otimes \boldsymbol{r}_h}{Q_h} \right) \right) \boldsymbol{w}, \boldsymbol{p} \right)_K \right| \\ &\quad \le 4(Q Q_h |\boldsymbol{p}||\boldsymbol{w}|, |\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|)_K \\ &\quad \le 4R^2(|\boldsymbol{p}||\boldsymbol{w}|, |\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|)_K \\ &\quad \le \epsilon \left(\frac{1}{Q_h} \boldsymbol{w}, \boldsymbol{w} \right)_K + \frac{4R^4}{\epsilon} \|\boldsymbol{p}\|_{L^\infty(L^\infty)}^2 N_h^K(t). \end{aligned}$$

The desired estimate then follows by taking $C = \frac{4R^4}{\epsilon} \|\boldsymbol{p}\|_{L^\infty(L^\infty)}^2$.

A.3 Proof of Lemma 5.2

The linear form can be rewritten as the following form

$$\begin{aligned} &B(u - Pu, \boldsymbol{s} - \Pi \boldsymbol{s}, \boldsymbol{p} - \Pi \boldsymbol{p}, H - PH, \boldsymbol{q} - \Pi \boldsymbol{q}, \boldsymbol{r} - \Pi \boldsymbol{r}; \\ &P\boldsymbol{e}_u, \Pi \boldsymbol{e_r}, \Pi \boldsymbol{e_q}, P\boldsymbol{e}_H, -\Pi \boldsymbol{e_p}, -\Pi \boldsymbol{e_s}) \\ &\quad = Z_1 + Z_2, \end{aligned}$$

where

$$
\begin{aligned}
Z_1 = &- (u_t - Pu_t, P\boldsymbol{e}_u)_\Omega - (\boldsymbol{s} - \Pi\boldsymbol{s}, \Pi\boldsymbol{e}_{\boldsymbol{r}})_\Omega - (\boldsymbol{p} - \Pi\boldsymbol{p}, \Pi\boldsymbol{e}_{\boldsymbol{q}})_\Omega \\
&- (H - PH, P\boldsymbol{e}_H)_\Omega + (\boldsymbol{q} - \Pi\boldsymbol{q}, \Pi\boldsymbol{e}_{\boldsymbol{p}})_\Omega + (\boldsymbol{r} - \Pi\boldsymbol{r}, \Pi\boldsymbol{e}_{\boldsymbol{s}})_\Omega, \\
Z_2 = &(\boldsymbol{s} - \Pi\boldsymbol{s}, \nabla P\boldsymbol{e}_u)_\Omega - (H - PH, \nabla \cdot \Pi\boldsymbol{e}_{\boldsymbol{q}})_\Omega \\
&- (\boldsymbol{q} - \Pi\boldsymbol{q}, \nabla P\boldsymbol{e}_H)_\Omega + (u - Pu, \nabla \cdot \Pi\boldsymbol{e}_{\boldsymbol{s}})_\Omega \\
&- (P\boldsymbol{e}_u, (\widehat{\boldsymbol{s} - \Pi\boldsymbol{s}}) \cdot \boldsymbol{\nu})_\Gamma + (\widehat{H - PH}, \Pi\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{\nu})_\Gamma \\
&+ (P\boldsymbol{e}_H, (\widehat{\boldsymbol{q} - \Pi\boldsymbol{q}}) \cdot \boldsymbol{\nu})_\Gamma - (\widehat{u - Pu}, \Pi\boldsymbol{e}_{\boldsymbol{s}} \cdot \boldsymbol{\nu})_\Gamma.
\end{aligned}
$$

A.3.1 Estimate of $\int_0^t Z_1\, dt$

Using the approximation property of the projections, we have

$$
\begin{aligned}
|Z_1| &\le \|u_t - Pu_t\|_\Omega \|P\boldsymbol{e}_u\|_\Omega + \|\boldsymbol{s} - \Pi\boldsymbol{s}\|_\Omega \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega + \|\boldsymbol{p} - \Pi\boldsymbol{p}\|_\Omega \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega \\
&\quad + \|H - PH\|_\Omega \|P\boldsymbol{e}_H\|_\Omega + \|\boldsymbol{q} - \Pi\boldsymbol{q}\|_\Omega \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega + \|\boldsymbol{r} - \Pi\boldsymbol{r}\|_\Omega \|\Pi\boldsymbol{e}_{\boldsymbol{s}}\|_\Omega \\
&\le Ch^{k+1}(\|P\boldsymbol{e}_u\|_\Omega + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega + \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega + \|P\boldsymbol{e}_H\|_\Omega + \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega + \|\Pi\boldsymbol{e}_{\boldsymbol{s}}\|_\Omega) \\
&\le \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2) + Ch^{2k+2}.
\end{aligned}
$$

Let C which depends on t,$\|u\|_{H^{k+4}(\Omega)}$, ϵ, and $\|u_t\|_{H^{k+1}(\Omega)}$. This implies that

$$
\begin{aligned}
\left|\int_0^t Z_1\, dt\right| \le Ch^{2k+2} + \epsilon \int_0^t (&\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{r}}\|_\Omega^2 \\
&+ \|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{p}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2)\, dt.
\end{aligned}
$$

A.3.2 Estimate of $\int_0^t Z_2\, dt$

Observing the definition of the numerical fluxes and the projection, in one-dimension $Z_2 = 0$, which is analyzed in Appendix A.4 on p. 276 in [22]. In multi-dimension, we have

$$
\begin{aligned}
Z_2 = &- (H - PH, \nabla \cdot \Pi\boldsymbol{e}_{\boldsymbol{q}})_\Omega + (\widehat{H - PH}, \Pi\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{\nu})_\Gamma \\
&+ (u - Pu, \nabla \cdot \Pi\boldsymbol{e}_{\boldsymbol{s}})_\Omega - (\widehat{u - P\boldsymbol{e}_u}, \Pi\boldsymbol{e}_{\boldsymbol{s}} \cdot \boldsymbol{\nu})_\Gamma.
\end{aligned}
$$

Using of Lemma 3.2 and integrating Z_2 with respect to time between 0 and t, we obtain

$$
\left|\int_0^t Z_2\, dt\right| \le Ch^{2k+2} + \epsilon \int_0^t (\|\Pi\boldsymbol{e}_{\boldsymbol{q}}\|_\Omega^2 + \|\Pi\boldsymbol{e}_{\boldsymbol{s}}\|_\Omega^2)\, dt,
$$

where C depends on t, $\|u\|_{H^{k+4}(\Omega)}$, ϵ.

Now, we combine the estimate of **a** and **b**, we get the desired estimates.

A.4 Proof of Lemma 5.3

Assume

$$\begin{aligned}
Z_3 &= (a(\boldsymbol{r};\boldsymbol{p},\Pi\boldsymbol{e_r}) - a(\boldsymbol{r}_h;\boldsymbol{p}_h,\Pi\boldsymbol{e_r})) - (\tilde{a}(\boldsymbol{r};\boldsymbol{r},\Pi\boldsymbol{e_p}) - \tilde{a}(\boldsymbol{r}_h;\boldsymbol{r}_h,\Pi\boldsymbol{e_p}))\\
&= \sum_K (a_K(\boldsymbol{r};\boldsymbol{p},\Pi\boldsymbol{e_r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\Pi\boldsymbol{e_r}))\\
&\quad - \sum_K (\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\Pi\boldsymbol{e_p}) - \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\Pi\boldsymbol{e_p}))\\
&:= \sum_K ((IV) - (V)).
\end{aligned}$$

Next, we estimate the $\sum_K (IV)$ and $\sum_K (V)$, separately. We observe that

$$\boldsymbol{e_r} = \boldsymbol{r} - \Pi\boldsymbol{r} + \Pi\boldsymbol{e_r}.$$

We decompose (IV) into two terms

$$(IV) = (a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{e_r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\boldsymbol{e_r})) - (a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{r}-\Pi\boldsymbol{r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\boldsymbol{r}-\Pi\boldsymbol{r})).$$

Adding and subtracting $a_K(\boldsymbol{r}_h;\boldsymbol{p},\boldsymbol{e_r})$,

$$a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{e_r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\boldsymbol{e_r}) = a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{e_r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p},\boldsymbol{e_r}) + a_K(\boldsymbol{r}_h;\boldsymbol{e_p},\boldsymbol{e_r}).$$

Using Lemmas 4.2, 4.4 and 4.6,

$$\begin{aligned}
&|a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{e_r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\boldsymbol{e_r})|\\
&\quad \le \epsilon a_K(\boldsymbol{r}_h;\boldsymbol{e_r},\boldsymbol{e_r}) + CN_h^K(t) + \epsilon a_K(\boldsymbol{r}_h;\boldsymbol{e_p},\boldsymbol{e_p}) + \frac{1}{4\epsilon} a_K(\boldsymbol{r}_h;\boldsymbol{e_r},\boldsymbol{e_r})\\
&\quad \le \epsilon \|\boldsymbol{e_p}\|_K^2 + CN_h^K(t)
\end{aligned}$$

with C depending on ϵ, $\|\boldsymbol{p}\|_\infty$, $\|\boldsymbol{r}\|_\infty$. Similarly,

$$\begin{aligned}
&|a_K(\boldsymbol{r};\boldsymbol{p},\boldsymbol{r}-\Pi\boldsymbol{r}) - a_K(\boldsymbol{r}_h;\boldsymbol{p}_h,\boldsymbol{r}-\Pi\boldsymbol{r})|\\
&\quad \le \epsilon a_K(\boldsymbol{r}_h;\boldsymbol{r}-\Pi\boldsymbol{r},\boldsymbol{r}-\Pi\boldsymbol{r}) + CN_h^K(t) + \epsilon a_K(\boldsymbol{r}_h;\boldsymbol{e_p},\boldsymbol{e_p}) + \frac{1}{4\epsilon} a_K(\boldsymbol{r}_h;\boldsymbol{e_r},\boldsymbol{e_r})\\
&\quad \le \epsilon \|\boldsymbol{e_p}\|_K^2 + C(N_h^K(t) + \|\boldsymbol{r}-\Pi\boldsymbol{r}\|_K^2)
\end{aligned}$$

with C depending on ϵ, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$.

In view of

$$\boldsymbol{e_p} = \boldsymbol{p} - \Pi\boldsymbol{p} + \Pi\boldsymbol{e_p},$$

we can decompose (V) as follows

$$\begin{aligned}
(V) &= \tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\Pi\boldsymbol{e_p}) - \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\Pi\boldsymbol{e_p})\\
&= (\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{e_p}) - \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{e_p})) - (\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{p}-\Pi\boldsymbol{p}) - \tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{p}-\boldsymbol{e_p})).
\end{aligned}$$

Using Lemma 4.8, we get

$$\begin{aligned}|\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{e_p})-\tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{e_p})| &\le \epsilon\tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{e_p},\boldsymbol{e_p})+\frac{1}{4\epsilon}N_h^K(t)\\ &\le \epsilon\|\boldsymbol{e_p}\|_K^2+\frac{1}{4\epsilon}N_h^K(t).\end{aligned}$$

The same analysis yields

$$|\tilde{a}_K(\boldsymbol{r};\boldsymbol{r},\boldsymbol{p}-\Pi\boldsymbol{p})-\tilde{a}_K(\boldsymbol{r}_h;\boldsymbol{r}_h,\boldsymbol{p}-\Pi\boldsymbol{p})|\le\epsilon\|\boldsymbol{p}-\Pi\boldsymbol{p}\|_K^2+\frac{1}{4\epsilon}N_h^K(t).$$

Collecting the estimates of (IV) and (V) and summing up all the elements K, we get

$$\int_0^t Z_3\,dt\le Ch^{2k+2}+C\int_0^t N_h(t)\,dt+\epsilon\int_0^t\|\boldsymbol{e_p}\|_\Omega^2\,dt,$$

where C depends on ϵ, t, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$ and $\|u\|_{H^{k+4}(\Omega)}$.

A.5 Proof of Lemma 5.5

The linear form can be rewritten as the following form

$$\begin{aligned}&\tilde{B}(u-Pu,\boldsymbol{s}-\Pi\boldsymbol{s},\boldsymbol{p}-\Pi\boldsymbol{p},H-PH,\boldsymbol{q}-\Pi\boldsymbol{q},\boldsymbol{r}-\Pi\boldsymbol{r};\\ &\quad -P\boldsymbol{e}_H,-\Pi\boldsymbol{e_p},\Pi\boldsymbol{e_s},P\boldsymbol{e}_{u_t},-\Pi\boldsymbol{e}_{\boldsymbol{r}_t},\Pi\boldsymbol{e_q})\\ &=S_1+S_2,\end{aligned}$$

where

$$\begin{aligned}S_1=&(u_t-Pu_t,P\boldsymbol{e}_H)_\Omega+(\boldsymbol{s}-\Pi\boldsymbol{s},\Pi\boldsymbol{e_p})_\Omega-(\boldsymbol{p}-\Pi\boldsymbol{p},\Pi\boldsymbol{e_s})_\Omega\\ &-(H-PH,P\boldsymbol{e}_{u_t})_\Omega+(\boldsymbol{q}-\Pi\boldsymbol{q},\Pi\boldsymbol{e}_{\boldsymbol{r}_t})_\Omega-(\boldsymbol{r}_t-\Pi\boldsymbol{r}_t,\Pi\boldsymbol{e_q})_\Omega,\\ S_2=&-(\boldsymbol{s}-\Pi\boldsymbol{s},\nabla P\boldsymbol{e}_H)_\Omega-(H-PH,\nabla\cdot\Pi\boldsymbol{e}_s)_\Omega\\ &-(\boldsymbol{q}-\Pi\boldsymbol{q},\nabla P\boldsymbol{e}_{u_t})_\Omega-(u_t-Pu_t,\nabla\cdot\Pi\boldsymbol{e_q})_\Omega\\ &+(P\boldsymbol{e}_H,(\widehat{\boldsymbol{s}-\Pi\boldsymbol{s}})\cdot\boldsymbol{\nu})_\Gamma+(\widehat{H-PH},\Pi\boldsymbol{e_s}\cdot\boldsymbol{\nu})_\Gamma\\ &+(P\boldsymbol{e}_{u_t},(\widehat{\boldsymbol{q}-\Pi\boldsymbol{q}})\cdot\boldsymbol{\nu})_\Gamma+(\widehat{u_t-Pu_t},\Pi\boldsymbol{e_q}\cdot\boldsymbol{\nu})_\Gamma.\end{aligned}$$

A.5.1 Estimate of $\int_0^t S_1\,dt$

$$\begin{aligned}&\left|\int_0^t(-(H-PH,P\boldsymbol{e}_{u_t})_\Omega+(\boldsymbol{q}-\Pi\boldsymbol{q},\Pi\boldsymbol{e}_{\boldsymbol{r}_t})_\Omega)\,dt\right|\\ &\quad=\left|-(H-PH,P\boldsymbol{e}_u)_\Omega|_0^t+\int_0^t(H_t-PH_t,P\boldsymbol{e}_u)_\Omega\,dt\right.\end{aligned}$$

$$+ (\boldsymbol{q} - \Pi\boldsymbol{q}, \Pi\boldsymbol{e_r})_\Omega|_0^t - \int_0^t (\boldsymbol{q}_t - \Pi\boldsymbol{q}_t, \Pi\boldsymbol{e_r})_\Omega\, dt\Bigg|$$

$$\le \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e_r}\|_\Omega^2) + \epsilon\int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e_r}\|_\Omega^2)\, dt + Ch^{2k+2},$$

where C depends on $\|u\|_{H^{k+3}(\Omega)}$, ϵ, t and $\|u_t\|_{H^{k+3}(\Omega)}$. Integrating S_1 with respect to time between 0 and t. We obtain

$$\left|\int_0^t S_1\, dt\right| \le Ch^{2k+2} + \epsilon(\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e_r}\|_\Omega^2)$$
$$+ \epsilon\int_0^t (\|P\boldsymbol{e}_u\|_\Omega^2 + \|\Pi\boldsymbol{e_r}\|_\Omega^2 + \|\Pi\boldsymbol{e_q}\|_\Omega^2 + \|P\boldsymbol{e}_H\|_\Omega^2 + \|\Pi\boldsymbol{e_p}\|_\Omega^2 + \|\Pi\boldsymbol{e_s}\|_\Omega^2)\, dt.$$

A.5.2 *Estimate of* $\int_0^t S_2\, dt$

Observing the definition of the numerical fluxes and the projection, in one-dimension $S_2 = 0$. In multi-dimension, we have

$$S_2 = -(H - PH, \nabla\cdot\Pi\boldsymbol{e_s})_\Omega + (\widehat{H - PH}, \Pi\boldsymbol{e_s}\cdot\boldsymbol{\nu})_\Gamma$$
$$-(u_t - Pu_t, \nabla\cdot\Pi\boldsymbol{e_q})_\Omega + (\widehat{u_t - Pu_t}, \Pi\boldsymbol{e_q}\cdot\boldsymbol{\nu})_\Gamma.$$

Using of Lemma 3.2 and integrating S_2 with respect to time between 0 and t, we obtain

$$\left|\int_0^t S_2\, dt\right| \le Ch^{2k+2} + \epsilon\int_0^t (\|\Pi\boldsymbol{e_q}\|_\Omega^2 + \|\Pi\boldsymbol{e}_s\|_\Omega^2)\, dt,$$

where C depends on $\|u\|_{H^{k+4}(\Omega)}$, ϵ, t and $\|u_t\|_{H^{k+2}(\Omega)}$.

Now, we combine the estimate of **c** and **d**. Finally we get the desired estimates.

A.6 Proof of Lemma 5.6

Assume that

$$S_3 = \sum_K (a_K(\boldsymbol{r}; \boldsymbol{p}, \boldsymbol{p} - \Pi\boldsymbol{p}) - a_K(\boldsymbol{r}_h; \boldsymbol{p}_h, \boldsymbol{p} - \Pi\boldsymbol{p}))$$
$$+ \sum_K (\tilde{a}_k(\boldsymbol{r}; \boldsymbol{r}, \boldsymbol{r}_t - \Pi\boldsymbol{r}_t) - \tilde{a}_K(\boldsymbol{r}_h; \boldsymbol{r}_h, \boldsymbol{r}_t - \Pi\boldsymbol{r}_t)).$$

Using Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.4, Lemma 4.6 and Lemma 4.8 gives the estimate

$$\left|\int_0^t S_3\, dt\right| \le Ch^{2k+2} + C\int_0^t N_h(t)\, dt + \epsilon\int_0^t \|\boldsymbol{e_p}\|_\Omega^2\, dt,$$

where C depends on ϵ, t, $\|\boldsymbol{p}\|_{L^\infty(L^\infty)}$, $\|\boldsymbol{r}\|_{L^\infty(L^\infty)}$, $\|u\|_{H^{k+4}(\Omega)}$ and $\|u_t\|_{H^{k+2}(\Omega)}$.

References

1. Bänsch, E., Morin, P., Nochetto, R.H.: Surface diffusion of graphs: variational formulation, error analysis, and simulation. SIAM J. Numer. Anal. **42**, 773–799 (2004)
2. Bassi, F., Rebay, S.: A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations. J. Comput. Phys. **131**, 267–279 (1997)
3. Ciarlet, P.: The Finite Element Method for Elliptic Problem. North-Holland, Amsterdam (1975)
4. Cockburn, B., Hou, S., Shu, C.-W.: The Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case. Math. Comput. **54**, 545–581 (1990)
5. Cockburn, B., Lin, S.-Y., Shu, C.-W.: TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one dimensional systems. J. Comput. Phys. **84**, 90–113 (1989)
6. Cockburn, B., Shu, C.-W.: TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework. Math. Comput. **52**, 411–435 (1989)
7. Cockburn, B., Shu, C.-W.: The Runge-Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems. J. Comput. Phys. **141**, 199–224 (1998)
8. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. **35**, 2440–2463 (1998)
9. Cockburn, B., Kanschat, G., Perugia, I., Schötzau, D.: Superconvergence of the local discontinuous Galerkin method for elliptic problems on Cartesian grids. SIAM J. Numer. Anal. **39**, 264–285 (2001)
10. Cockburn, B., Shu, C.-W.: Foreword for the special issue on discontinuous Galerkin method. J. Sci. Comput. **22–23**, 1–3 (2005)
11. Cockburn, B., Shu, C.-W.: Foreword. J. Sci. Comput. **40**, 1–3 (2009)
12. Coleman, B.D., Falk, R.S., Moakher, M.: Stability of cylindrical bodies in the theory of surface diffusion. Physica D **89**, 123–135 (1995)
13. Coleman, B., Falk, R., Moakher, M.: Space-time finite element methods for surface diffusion with applications to the theory of the stability of cylinders. SIAM J. Sci. Comput. **17**, 1434–1448 (1996)
14. Dawson, C.: Foreword for the special issue on discontinuous Galerkin method. Comput. Methods Appl. Mech. Eng. **195**, 3183 (2006)
15. Deckelnick, K., Dziuk, G., Elliott, C.M.: Error analysis of a semidiscrete numerical scheme for diffusion in axially symmetric surfaces. SIAM J. Numer. Anal. **41**, 2161–2179 (2003)
16. Deckelnick, K., Dziuk, G., Elliott, C.M.: Computation of geometric partial differential equations and mean curvature flow. Acta Numer. **14**, 139–232 (2005)
17. Deckelnick, K., Dziuk, G., Elliott, C.M.: Fully discrete finite element approximation for anisotropic surface diffusion of graphs. SIAM J. Numer. Anal. **43**, 1112–1138 (2005)
18. Deckelnick, K., Dziuk, G.: Error analysis of a finite element method for the Willmore flow of graphs. Interfaces Free Bound. **8**, 21–46 (2006)
19. Dong, B., Shu, C.-W.: Analysis of a local discontinuous Galerkin method for fourth-order time-dependent problems. SIAM J. Numer. Anal. **47**, 3240–3268 (2009)
20. Elliott, C.M., French, D.A., Milner, F.A.: A second order splitting method for the Cahn-Hilliard equation. Numer. Math. **54**, 575–590 (1989)
21. Hesthaven, J., Warburton, T.: Nodal Discontinuous Galerkin Methods. Algorithms, Analysis, and Applications. Springer, Berlin (2008)
22. Ji, L., Xu, Y.: Optimal error estimates of the local discontinuous Galerkin method for Willmore flow of graphs on Cartesian meshes. Int. J. Numer. Anal. Model. **8**, 252–283 (2011)
23. Li, B.: Discontinuous Finite Elements in Fluid Dynamics and Heat Transfer. Springer, London (2006)
24. Liu, H., Yan, J.: A local discontinuous Galerkin method for the Korteweg-de Vries equation with boundary effect. J. Comput. Phys. **215**, 197–218 (2006)
25. Reed, W.H., Hill, T.R.: Triangular mesh method for the neutron transport equation. Technical report LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos, NM (1973)
26. Rivière, B.: Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations. Theory and Implementation. SIAM, Philadelphia (2008)
27. Shu, C.-W.: Discontinuous Galerkin methods: general approach and stability. In: Bertoluzza, S., Falletta, S., Russo, G., Shu, C.-W. (eds.) Numerical Solutions of Partial Differential Equations. Advanced Courses in Mathematics CRM Barcelona, pp. 149–201. Birkhäuser, Basel (2009)
28. Xia, Y., Xu, Y., Shu, C.-W.: Local discontinuous Galerkin methods for the Cahn-Hilliard type equations. J. Comput. Phys. **227**, 472–491 (2007)
29. Xia, Y., Xu, Y., Shu, C.-W.: Application of the local discontinuous Galerkin method for the Allen-Cahn/Cahn-Hilliard system. Commun. Comput. Phys. **5**, 821–835 (2009)
30. Xu, Y., Shu, C.-W.: Local discontinuous Galerkin methods for two classes of two dimensional nonlinear wave equations. Physica D **208**, 21–58 (2005)

31. Xu, Y., Shu, C.-W.: Local discontinuous Galerkin method for surface diffusion and Willmore flow of graphs. J. Sci. Comput. **40**, 375–390 (2009)
32. Xu, Y., Shu, C.-W.: Local discontinuous Galerkin methods for high-order time-dependent partial differential equations, Communications in Computational. Physics **7**, 1–46 (2010)
33. Yan, J., Shu, C.-W.: A local discontinuous Galerkin method for KdV type equations. SIAM J. Numer. Anal. **40**, 769–791 (2002)