# Static two-grid mixed finite-element approximations to the Navier-Stokes equations

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#### Abstract

A two-grid scheme based on mixed finite-element approximations to the incompressible Navier-Stokes equations is introduced and analyzed. In the first level the standard mixed finite-element approximation over a coarse mesh is computed. In the second level the approximation is postprocessed by solving a discrete Oseen-type problem on a finer mesh. The two-level method is optimal in the sense that, when a suitable value of the coarse mesh diameter is chosen, it has the rate of convergence of the standard mixed finite-element method over the fine mesh. Alternatively, it can be seen as a postprocessed method in which the rate of convergence is increased by one unit with respect to the coarse mesh. The analysis takes into account the loss of regularity at initial time of the solution of the Navier-Stokes equations in absence of nonlocal compatibility conditions. Some numerical experiments are shown.

#### **1** Introduction

We consider the incompressible Navier–Stokes equations

$$u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \qquad (1)$$
$$\operatorname{div}(u) = 0.$$

in a bounded domain  $\Omega \subset \mathbb{R}^d$  (d = 2, 3) with a smooth boundary subject to homogeneous Dirichlet boundary conditions u = 0 on  $\partial \Omega$ . In (1), u is the

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velocity field, p the pressure,  $\nu>0$  the diffusion coefficient and f a given force field.

In this paper we study the following two-grid mixed finite-element method for the spatial discretization of the above equations. First, for the solution (u, p)of the fully nonlinear Navier-Stokes equations (1) corresponding to a given initial condition

$$u(\cdot, 0) = u_0,\tag{2}$$

the mixed finite-element approximation  $(u_H, p_H)$  over a coarse mesh of diameter H is computed. Then, for any time t > 0, the postprocessed approximation  $(\tilde{u}_h, \tilde{p}_h)$  is obtained as the mixed finite-element approximation over a finer mesh (h < H) to the following steady Oseen-type problem:

$$-\nu\Delta\tilde{u} + (u_H(t)\cdot\nabla)\tilde{u} + \nabla\tilde{p} = f - \frac{d}{dt}u_H(t) \\ \operatorname{div}(\tilde{u}) = 0 \right\} \qquad \text{in }\Omega, \\ \tilde{u} = 0, \qquad \text{on }\partial\Omega.$$
(3)

In this paper we prove that, in terms of the fine mesh diameter h, this twogrid technique is of optimal order in the sense that, for appropriate choices of the coarse mesh diameter H, the method has the same rate of convergence of standard mixed finite element approximations in the fine mesh. On the other hand, for a suitable value of the discretization parameter h, the rate of convergence of the postprocessed approximation in terms of H increases by one unit the rate of convergence of the coarse standard approximation. The improvement in precision is achieved in both the  $H^1$  norm for the velocity and the  $L^2$  norm for the pressure in the case of linear, quadratic and cubic elements. For other than linear elements the rate of convergence in the  $L^2$  norm of the velocity is also increased by one unit. We remark that time evolution is performed only at the coarse mesh whereas at the fine grid the time appears only as a parameter (see equation (3)), thus the name of static two-grid method.

Two-grid or two-level methods are a well established technique for nonlinear steady problems, see [34]. In [25], [26] several two-level methods are considered to approximate the steady Navier-Stokes equations. They require solving a nonlinear system over a coarse mesh and, depending on the algorithm chosen, one Stokes problem, one linear Oseen problem or one Newton step over the fine mesh. The corresponding algorithms obtain the optimal rate of convergence in the fine mesh for appropriate choices of the coarse mesh diameter H.

In the case of nonlinear evolutionary equations, two-grid techniques have been proposed and studied in [1], [22], [24], [14]. In these methods, as opposed to the method studied in the present paper, time evolution is also performed over the fine mesh. The advantage of the method studied in the present paper is that since the time integration is only carried out on the coarse mesh, computations on the fine grid can be done at selected target time levels where an improved approximation is desired, with the corresponding reduction of computing time, specially if these target time levels are sufficiently spaced in time. For this reason, although some of the two-grid methods that incorporate the evolution in time of the fine mesh approximation are more accurate, the method we present can still be more efficient in terms of computational effort for a given error level.

Two-grid techniques that integrate in time only on the coarse level have previously been developed in [16], [17] (see also [27]) for spectral methods, and later extended to mixed finite-element formulations in [3], [4], [10]. In all these works the two grid method is referred to as postprocessed Galerkin method, and, instead of (3), the approximation  $(\tilde{u}_h, \tilde{p}_h)$  is found as an approximation to the following Stokes problem

$$-\nu\Delta \tilde{u} + \nabla \tilde{p} = f - \frac{d}{dt} u_H(t) - (u_H(t) \cdot \nabla) u_H(t) \\ \operatorname{div}(\tilde{u}) = 0 \right\} \qquad \text{in } \Omega,$$

$$\tilde{u} = 0, \qquad \text{on } \partial\Omega.$$

$$(4)$$

This two-grid method will be termed standard postprocessed method, to differentiate it to that studied in the present paper, which will be termed new postprocessed method. Both, the standard and the new postprocessed methods, have the same rate of convergence. However, as already noted in [12] for nonlinear convection-diffusion problems, the new postprocessing technique produces more accurate approximations than the standard postprocessed method, for moderate to small values of the diffusion parameter  $\nu$ . This will also be the case in the numerical experiments in the present paper for moderate values of the Reynolds number.

In the present paper we take into account the loss of regularity suffered by the solutions of the Navier-Stokes equations at the initial time in the absence of nonlocal compatibility conditions. Thus, for the analysis, we do not assume the solution u to have more than second-order spatial derivatives bounded in  $L^2$  up to initial time t = 0, since demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations [19], [20]. Due to the loss of regularity at t = 0, the best error bound that we can obtain is  $O(H^5 | \log(H) |)$ . For this reason we do not analyze higher than cubic finite elements. The same limit in the rate of convergence was found in [20] for standard mixed finite-element approximations and in [10], [14] for two-grid schemes.

In practice, any method to numerically solve evolutionary equations needs of some time discretization procedure. For brevity reasons, we have preferred to present the method in a semidiscrete manner without reference to any particular time discretization. However, we emphasize that being static, the method we present can be applied exactly in the same form, with any time discretization. The analysis of fully discrete procedures can be developed along the same lines that appear in [11], [13].

The rest of the paper is as follows. In Section 2 we introduce some preliminaries and notation. In Section 3 we carry out the error analysis of the new method. Finally, some numerical experiments are shown in the last section.

## 2 Preliminaries and notations

We will assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^d$ , d = 2, 3, of class  $\mathcal{C}^m$ , for  $m \geq 2$ . When dealing with linear elements  $(r = 2 \text{ below}) \Omega$  may also be a convex polygonal or polyhedral domain. We consider the Hilbert spaces

$$\begin{split} H &= \left\{ u \in L^2(\emptyset)^d \mid \operatorname{div}(u) = 0, \ u \cdot n_{\mid_{\partial\Omega}} = 0 \right\} \\ V &= \left\{ u \in H^1_0(\emptyset)^d \mid \operatorname{div}(u) = 0 \right\}, \end{split}$$

endowed with the inner product of  $L^2(\emptyset)^d$  and  $H_0^1(\emptyset)^d$ , respectively. For  $l \ge 0$ integer and  $1 \le q \le \infty$ , we consider the standard spaces,  $W^{l,q}(\Omega)^d$ , of functions with derivatives up to order l in  $L^q(\Omega)$ , and  $H^l(\Omega)^d = W^{l,2}(\Omega)^d$ . We will denote by  $\|\cdot\|_l$  the norm in  $H^l(\Omega)^d$ , and  $\|\cdot\|_{-l}$  will represent the norm of its dual space. We consider also the quotient spaces  $H^l(\Omega)/\mathbb{R}$  with norm  $\|p\|_{H^l/\mathbb{R}} =$  $\inf\{\|p+c\|_l \mid c \in \mathbb{R}\}.$ 

We recall the following Sobolev's imbeddings [2]: For  $q \in [1, \infty)$ , there exists a constant  $C = C(\Omega, q)$  such that

$$\|v\|_{L^{q'}} \le C \|v\|_{W^{s,q}}, \quad \frac{1}{q'} \ge \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty, \quad v \in W^{s,q}(\Omega)^d.$$
(5)

For  $q' = \infty$ , (5) holds with  $\frac{1}{q} < \frac{s}{d}$ .

The following inf-sup condition is satisfied (see [18]), there exists a constant  $\beta > 0$  such that

$$\inf_{q \in L^2(\Omega)/\mathbb{R}} \sup_{v \in H^1_0(\Omega)^d} \frac{(q, \nabla \cdot v)}{\|v\|_1 \|q\|_{L^2/\mathbb{R}}} \ge \beta.$$
(6)

Let  $\Pi: L^2(\emptyset)^d \longrightarrow H$  be the  $L^2(\emptyset)^d$  projection onto H. We denote by A the Stokes operator on  $\emptyset$ :

$$A: \mathcal{D}(A) \subset H \longrightarrow H, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\emptyset)^d \cap V.$$

We shall assume that u is a strong solution up to time t = T, so that

$$||u(t)||_1 \le M_1, \quad ||u(t)||_2 \le M_2, \quad 0 \le t \le T,$$
(7)

for some constants  $M_1$  and  $M_2$ . We shall also assume that there exists a constant  $\tilde{M}_2$  such that

$$||f||_1 + ||f_t||_1 + ||f_{tt}||_1 \le M_2, \quad 0 \le t \le T.$$
(8)

Finally, we shall assume that for some  $k\geq 2$ 

$$\sup_{0 \le t \le T} \left\| \partial_t^{\lfloor k/2 \rfloor} f \right\|_{k-1-2\lfloor k/2 \rfloor} + \sum_{j=0}^{\lfloor (k-2)/2 \rfloor} \sup_{0 \le t \le T} \left\| \partial_t^j f \right\|_{k-2j-2} < +\infty,$$

so that, according to Theorems 2.4 and 2.5 in [19], there exist positive constants  $M_k$  and  $K_k$  such that the following bounds hold:

$$\|u(t)\|_{k} + \|u_{t}(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \le M_{k}\tau(t)^{1-k/2},\tag{9}$$

$$\int_0^t \sigma_{k-3}(s) \left( \|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2 \right) \mathrm{d}s \le K_k^2, (10)$$

where  $\tau(t) = \min(t, 1)$  and  $\sigma_n = e^{-\alpha(t-s)}\tau^n(s)$  for some  $\alpha > 0$ . Observe that for  $t \leq T < \infty$ , we can take  $\tau(t) = t$  and  $\sigma_n(s) = s^n$ . For simplicity, we will take these values of  $\tau$  and  $\sigma_n$ .

Let  $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i \in I_h}, h > 0$  be a family of partitions of suitable domains  $\Omega_h$ , where h is the maximum diameter of the elements  $\tau_i^h \in \mathcal{T}_h$ , and  $\phi_i^h$  are the mappings of the reference simplex  $\tau_0$  onto  $\tau_i^h$ .

Let  $r \geq 2$ , we consider the finite-element spaces

$$S_{h,r} = \left\{ \chi_h \in \mathcal{C}\left(\overline{\mathcal{O}}_h\right) \mid \chi_h \mid_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0) \right\} \subset H^1(\mathcal{O}_h), \ S_{h,r}^0 = S_{h,r} \cap H^1_0(\mathcal{O}_h).$$

where  $P^{r-1}(\tau_0)$  denotes the space of polynomials of degree at most r-1 on  $\tau_0$ .

We shall denote by  $(X_{h,r}, Q_{h,r-1})$  the so-called Hood–Taylor element [7, 21], when  $r \geq 3$ , where

$$X_{h,r} = \left(S_{h,r}^{0}\right)^{d}, \quad Q_{h,r-1} = S_{h,r-1} \cap L^{2}(\mathcal{O}_{h})/\mathbb{R}, \quad r \ge 3,$$

and the so-called mini-element [8] when r = 2, where  $Q_{h,1} = S_{h,2} \cap L^2(\emptyset_h)/\mathbb{R}$ , and  $X_{h,2} = (S_{h,2}^0)^d \oplus \mathbb{B}_h$ . Here,  $\mathbb{B}_h$  is spanned by the bubble functions  $b_{\tau}, \tau \in \mathcal{T}_h$ , defined by  $b_{\tau}(x) = (d+1)^{d+1}\lambda_1(x)\cdots\lambda_{d+1}(x)$ , if  $x \in \tau$  and 0 elsewhere, where  $\lambda_1(x),\ldots,\lambda_{d+1}(x)$  denote the barycentric coordinates of x. For these elements a uniform inf-sup condition is satisfied (see [7]), that is, there exists a constant  $\beta > 0$  independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \ge \beta.$$
(11)

The approximate velocity belongs to the discrete divergence-free space

$$V_{h,r} = X_{h,r} \cap \left\{ \chi_h \in H^1_0(\mathcal{O}_h)^d \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1} \right\},$$

which is not a subspace of V.

Let  $(u, p) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega)/\mathbb{R})$  be the solution of a Stokes problem with right-hand side g, we will denote by  $s_h = S_h(u) \in V_h$  the so-called Stokes projection (see [20]) defined as the velocity component of the solution of the following problem: find  $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$  such that

$$\nu(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \qquad \forall \phi_h \in X_{h,r}, \tag{12}$$

$$(\nabla \cdot s_h, \psi_h) = 0 \qquad \qquad \forall \psi_h \in Q_{h,r-1}. \tag{13}$$

The following bound holds for  $2 \le l \le r$ :

$$||u - s_h||_0 + h||u - s_h||_1 \le Ch^l (||u||_l + ||p||_{H^{l-1}/\mathbb{R}}).$$
(14)

The proof of (14) for  $\Omega = \Omega_h$  can be found in [20]. The bound for the pressure is [18]

$$\|p - q_h\|_{L^2/\mathbb{R}} \le C_\beta h^{l-1} \big( \|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}} \big), \tag{15}$$

where the constant  $C_{\beta}$  depends on the constant  $\beta$  in the inf-sup condition (11).

We consider the semi-discrete finite-element approximation  $(u_H, p_H)$  to (u, p), solution of (1)–(2). That is, given  $u_H(0) = \Pi_H u_0$ , we compute  $u_H(t) \in X_{H,r}$ and  $p_H(t) \in Q_{H,r-1}$ ,  $t \in (0,T]$ , satisfying

$$(\dot{u}_H, \phi_H) + \nu (\nabla u_H, \nabla \phi_H) + b(u_H, u_H, \phi_H) + (\nabla p_H, \phi_H) = (f, \phi_H) \quad \forall \phi_H \in X_{H,r},$$

$$(16) \quad (\nabla \cdot u_H, \psi_H) = 0 \quad \forall \psi_H \in Q_{H,r-1}$$

$$(17) \quad (17) \quad (17)$$

where  $b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\nabla \cdot u)v, w)$  for any  $u, v, w \in H_0^1(\Omega)^d$ .

For  $2 \le r \le 5$ , provided that (14)–(15) hold for  $l \le r$ , and (9)–(10) hold for k = r, then we have

$$\|u(t) - u_H(t)\|_0 + H\|u(t) - u_H(t)\|_1 \le C \frac{H^r}{t^{(r-2)/2}}, \quad 0 \le t \le T,$$
(18)

(see, e.g., [10, 19, 20]), and also,

$$\|p(t) - p_H(t)\|_{L^2/\mathbb{R}} \le C \frac{H^{r-1}}{t^{(r'-2)/2}}, \quad 0 \le t \le T,$$
(19)

where r' = r if  $r \le 4$  and r' = r + 1 if r = 5.

#### 3 The new postprocessed method

The postprocessing technique we propose is a two-level or two-grid method. In the first level, we choose a coarse mesh of size H and compute the mixed finiteelement approximation  $(u_H, p_H)$  to (u, p) defined by (16)-(17). In the second level, the discrete velocity and pressure  $(u_H(t), p_H(t))$  are postprocessed by solving the following linear Oseen problem: find  $(\tilde{u}_h(t), \tilde{p}_h(t)) \in (X_{h,r}, Q_{h,r-1}),$ h < H, satisfying for all  $\phi_h \in X_{h,r}$  and  $\psi_h \in Q_{h,r-1}$ 

$$\nu(\nabla \tilde{u}_{h}(t),\phi_{h}) + ((u_{H}(t)\cdot\nabla)\tilde{u}_{h}(t),\phi_{h}) + (\nabla \tilde{p}_{h}(t),\phi_{h}) = (f(t) - \dot{u}_{H}(t),\phi_{h}), (20)$$

$$(\nabla \cdot \tilde{u}_{h}(t),\psi_{h}) = 0.$$
(21)

Equations (20)-(21) can also be solved over a higher order mixed finite-element space over the same grid. For simplicity in the exposition we will only consider the case in which we refine the mesh at the postprocessing step.

Let us observe that projecting equation (20) over the discretely-free space  $V_{h,r}$ , and avoiding for simplicity the dependence on t in the notation, we get that  $\tilde{u}_h \in V_{h,r}$  satisfies

$$\nu(\nabla \tilde{u}_h, v_h) + ((u_H \cdot \nabla) \tilde{u}_h, v_h) = (f - \dot{u}_H, \phi_h), \quad \forall v_h \in V_{h,r}.$$
 (22)

We now prove that equation (22) is well-posed, i.e., for H small enough there exists a unique function  $\tilde{u}_h \in V_{h,r}$  solving (22). Let us denote by  $B^H$  the bilinear form defined by

$$B^{H}(u_{h}, v_{h}) = \nu(\nabla u_{h}, \nabla v_{h}) + ((u_{H} \cdot \nabla)u_{h}, v_{h}), \quad u_{h}, v_{h} \in V_{h,r}.$$
 (23)

We proceed to show that  $B^H$  is coercive which implies that there exists a unique function  $\tilde{u}_h \in V_{h,r}$  satisfying (22). Let us also observe that once a unique  $\tilde{u}_h$  is found, using the inf-sup condition (11) one easily obtains the existence and uniqueness of the pair  $(\tilde{u}_h, \tilde{p}_h)$  satisfying (20)-(21).

**Lemma 1** Let  $B_H$  be the bilinear form defined in (23). Then, there exists a constant C such that for t > 0 the following bound holds:

$$\left| B^{H}(v_{h}, v_{h}) \right| \geq \left( \nu - C \frac{H^{r-1+\gamma}}{t^{(r-2)/2}} \right) \|v_{h}\|_{1}^{2}, \quad \forall v_{h} \in V_{h,r},$$
(24)

where  $\gamma = 1/2$  if the dimension d is d = 2, and  $\gamma = 1/4$  if d = 3.

**Proof** To prove the coercivity of  $B^H$  we follow [25, p. 2042]. Let us first observe that for any  $v_h \in V_{h,r}$ 

$$B^{H}(v_{h}, v_{h}) = \nu \|\nabla v_{h}\|_{0}^{2} - \frac{1}{2} (\nabla \cdot u_{H}, v_{h} \cdot v_{h}).$$

Let  $q_H$  be the  $L^2$  orthogonal projection of  $v_h \cdot v_h$  over  $Q_{H,r-1}$ , so that applying standard finite-element theory [9] and interpolation theory on Hilbert spaces (see e. g. [31, § II.2] we have  $||v_h \cdot v_h - q_H||_{L^2(\Omega)/\mathbb{R}} \leq CH^{\gamma} ||v_h \cdot v_h||_{\gamma}$ , for  $\gamma \in (0, 1]$ . Taking into account that the velocity u satisfies  $\nabla \cdot u = 0$  then

$$B^{H}(v_{h}, v_{h}) = \nu \|\nabla v_{h}\|_{0}^{2} - \frac{1}{2} (\nabla \cdot (u_{H} - u), v_{h} \cdot v_{h} - q_{H}).$$
(25)

And then

$$|(\nabla \cdot (u_H - u), v_h \cdot v_h - q_H)| \le C ||u_H - u||_1 ||v_h \cdot v_h - q_H||_{L^2(\Omega)/\mathbb{R}}.$$

Following [25, p. 2042] we get

$$\|v_h \cdot v_h\|_{\gamma} \le C \|v_h\|_1^2, \tag{26}$$

where  $\gamma = 1/2$  if d = 2, and  $\gamma = 1/4$  if d = 3. Using (26) together with (18) we get

$$|(\nabla \cdot (u_H - u), v_h \cdot v_h - q_H)| \le C \frac{H^{r-1}}{t^{(r-2)/2}} H^{\gamma} ||v_h||_1^2.$$

Finally, going back to (25) we reach (24).

Let us observe that, for t > 0 and  $H < (t^{(r-2)/2}\nu/C)^{1/(r-1+\gamma)}$ , as a consequence of Lemma 1, there exists a unique  $\tilde{u}_h \in V_{h,r}$  satisfying (22).

We introduce now a linearized problem that will be used in the proof of Theorem 1 where we state the rate of convergence of the new method. Let u be the velocity in the solution (u, p) of (1)-(2). We will denote by (v, j) the solution of the following linearized problem

$$-\nu\Delta v + (u \cdot \nabla)v + \nabla j = d$$

$$\operatorname{div}(v) = 0$$
(27)

in the domain  $\Omega$  subject to homogeneous Dirichlet boundary conditions. Let us observe that since the divergence of u is zero the bilinear form:

$$B(v,w) = \nu(\nabla v, \nabla w) + ((u \cdot \nabla)v, w), \quad v, w \in V.$$

associated to this problem is continuous and coercive. Since the solution  $v \in V$  of (27) satisfies

$$B(v,w) = (d,w), \quad \forall w \in V$$

by the Lax-Milgram theorem there exists a unique solution v. Due to (6) there exists also a unique pressure j.

We will assume in the sequel that both problem (27) and its dual problem satisfy the regularity assumption

$$\|v\|_{2} + \|j\|_{H^{1}(\Omega)/\mathbb{R}} \le C \|d\|_{0}.$$
(28)

The regularity assumption (28) can be proved by using the analogous regularity of the Stokes problem and a bootstrap argument, see [25, Remark 2.1].

In the following lemma we state the rate of convergence of the mixed finiteelement approximation to the solution (v, j) of (27) defined as follows: find  $(v_h, j_h) \in (X_{h,r}, Q_{h,r-1})$  such that

$$\nu(\nabla v_h, \nabla \phi_h) + ((u \cdot \nabla)v_h, \phi_h) + (\nabla j_h, \phi_h) = (d, \phi_h), \ \forall \phi_h \in X_{h,r}, \quad (29)$$
$$(\nabla \cdot v_h, \psi_h) = 0, \qquad \forall \psi_h \in Q_{h,r-1}.(30)$$

**Lemma 2** Let (v, j) be the solution of (27) and let  $(v_h, j_h)$  be its mixed finiteelement approximation. Then, the following bounds hold for  $2 \le l \le r$ 

$$\|v - v_h\|_0 + h\|v - v_h\|_1 \leq Ch^l \left(\|v\|_l + \|j\|_{H^{l-1}/\mathbb{R}}\right), \tag{31}$$

$$\|j - j_h\|_{L^2/\mathbb{R}} \leq Ch^{l-1} \left( \|v\|_l + \|j\|_{H^{l-1}/\mathbb{R}} \right).$$
(32)

**Proof** Let us denote by  $s_h = S_h(v)$  the Stokes projection of v. More precisely,  $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$  will be the solution of (12)-(13) with right-hand-side  $g = d - (u \cdot \nabla)v$ . Let us denote by  $e_h = s_h - v_h$ . Then, from (29) and (12) we get

$$\nu(\nabla e_h, \nabla w_h) + ((u \cdot \nabla)e_h, w_h) = ((u \cdot \nabla)(s_h - v), w_h), \quad \forall w_h \in V_{h,r}.$$
(33)

Taking  $w_h = e_h$  in (33) and using (5) we get

$$\nu \|e_h\|_1^2 \le C \|u\|_{L^{2d/(d-1)}} \|s_h - v\|_1 \|e_h\|_{L^{2d}} \le C \|u\|_{1/2} \|s_h - v\|_1 \|e_h\|_1,$$

so that

$$||e_h||_1 \le C||s_h - v||_1. \tag{34}$$

Since  $||v - v_h||_1 \le ||v - s_h||_1 + ||e_h||_1$  applying (14) we conclude  $h||v - v_h||_1$  is bounded by the righ-hand side of (31). The bound (32) for the pressure is

readily obtained by means of the auxiliary value  $k_h = q_h - j_h$ . Subtracting (29) from (31) and applying the inf-sup condition (11) one easily gets

$$\beta \|k_h\|_{L^2/\mathbb{R}} \le \nu \|e_h\|_1 + C \|u\|_{1/2} \|s_h - v\|_1$$

so that due to (34) and (14) it follows that  $||k_h||_{L^2/\mathbb{R}}$  is bounded by the righthand side of (32). Since  $||j - j_h||_{L^2/\mathbb{R}} \leq ||j - q_h||_{L^2/\mathbb{R}} + ||k_h||_{L^2/\mathbb{R}}$ , applying (15) we finally prove (32).

We are left with the task of proving the bound for the  $L^2$  norm of the error in the velocity. We will argue by duality. Let us observe that

$$||e_{h}||_{0} = \sup_{\varphi \in L^{2}} \sup_{\varphi \neq 0} \frac{|(e_{h}, \varphi)|}{||\varphi||_{0}}.$$
(35)

Let us fix  $\varphi \in L^2$  and let us denote by (w,k) the solution of the linearized dual problem

$$\begin{array}{c} -\nu\Delta w - (u \cdot \nabla)w + \nabla k = \varphi, \\ \operatorname{div}(w) = 0, \\ u = 0, \end{array} \end{array} in \Omega,$$
(36)

As stated before we assume that this problem satisfies the regularity assumption (28), so that

$$\|w\|_{2} + \|k\|_{H^{1}(\Omega)/\mathbb{R}} \le C \|\varphi\|_{0}.$$
(37)

We will denote by  $(w_h, k_h) \in (X_{h,r}, Q_{h,r-1})$  the mixed finite-element approximations to (w, k). Reasoning exactly as before and applying (37) we obtain

$$\|w - w_h\|_1 \leq Ch\left(\|w\|_2 + \|k\|_{H^1/\mathbb{R}}\right) \leq Ch\|\varphi\|_0, \tag{38}$$

$$||k - k_h||_{L^2/\mathbb{R}} \leq Ch \left( ||w||_2 + ||k||_{H^1/\mathbb{R}} \right) \leq Ch ||\varphi||_0.$$
(39)

Integrating by parts we reach

$$(e_h, \varphi) = \nu(\nabla e_h, \nabla w) + ((u \cdot \nabla)e_h, w) - ((\nabla \cdot e_h), k)$$
  
=  $\nu(\nabla e_h, \nabla (w - w_h)) + ((u \cdot \nabla)e_h, w - w_h) - ((\nabla \cdot e_h), k - k_h)$   
+ $\nu(\nabla e_h, \nabla w_h) + ((u \cdot \nabla)e_h, w_h).$ 

And then, applying (38) and (39) we reach

$$|(e_{h},\varphi)| \leq C\nu ||e_{h}||_{1}h||\varphi||_{0} + C||u||_{1/2}||e_{h}||_{1}h||\varphi||_{0} + C||e_{h}||_{1}Ch||\varphi||_{0} + |\nu(\nabla e_{h},\nabla w_{h}) + ((u \cdot \nabla)e_{h},w_{h})|.$$
(40)

Then, to conclude, it only remains to bound  $|\nu(\nabla e_h, \nabla w_h) + ((u \cdot \nabla)e_h, w_h)|$ which by (33) is equal to  $|((u \cdot \nabla)(s_h - v), w_h)|$ . Let us decompose

$$|((u \cdot \nabla)(s_h - v), w_h)| \le |((u \cdot \nabla)(s_h - v), w_h - w)| + |((u \cdot \nabla)(s_h - v), w)|.$$

Then, integrating by parts in the last term

$$|((u \cdot \nabla)(s_h - v), w_h)| \le C ||u||_{1/2} ||s_h - v||_1 ||w_h - w||_1 + |((u \cdot \nabla)w, s_h - v)|,$$

and the bound for the first term on the right hand side above concludes by applying (14) and (38). Finally, since

$$|((u \cdot \nabla)w, s_h - v)| \le C ||u||_{L^{2d/(d-1)}} ||\nabla w||_{L^{2d}} ||s_h - v||_0.$$

Applying Sobolev inequality (5) together with (37) and (14) we reach

$$|((u \cdot \nabla)w, s_h - v)| \le C ||u||_{1/2} ||\varphi||_0 h^l \left( ||v||_l + ||j||_{H^{l-1}/\mathbb{R}} \right),$$

so that the proof is finished.

We now state some results that will be use to get the rate of convergence of the new postprocessed method. The proof of the following lemma can be found in [15, Lemma 4] for the case r = 2 and in [10, Lemma 5.1] for r = 3, 4.

**Lemma 3** Let (u, p) be the solution of (1)-(2) and let  $u_H$  be the mixed finiteelement approximation to u. Then, there exists a positive constant C such that

$$||u_t(t) - \dot{u}_H(t)||_{-1} \le \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}, t \in (0,T], r = 2, 3, 4, (41)$$

$$\|A^{-1}\Pi\left(u_t(t) - \dot{u}_H(t)\right)\|_0 \leq \frac{C}{t^{(r-1)/2}} H^{r+1} \left|\log(H)\right|, \ t \in (0,T], \ r = 3, 4, \quad (42)$$

where r' = 2 when r = 2 and r' = 1 otherwise.

The proof of the following lemma can be found in [10, p. 226].

**Lemma 4** Let (u, p) be the solution of (1)–(2) and let  $u_H$  be the mixed finiteelement approximation to u. Then, there exists a positive constant C such that

$$||u(t) - u_H(t)||_{-1} \le \frac{C}{t^{(r-1)/2}} H^{r+1} |\log(H)|, \ t \in (0,T], \ r = 3, 4.$$
 (43)

We end this section with a theorem that states the rate of convergence of the new postprocessed method.

**Theorem 1** Let (u, p) be the solution of (1)-(2) and for r = 2, 3, 4 let (9)-(10)hold with k = r + 2. Then, there exist a positive constant C such that the new postprocessed approximation  $(\tilde{u}_h(t), \tilde{p}_h(t))$  defined by (20)-(21) satisfies the following bounds for  $t \in (0, T]$  and H small enough:

$$\|u(t) - \tilde{u}_h(t)\|_1 \leq Ch + \frac{C}{t^{1/2}} H^2 |\log(H)|^2, \quad r = 2,$$
(44)

$$\|u(t) - \tilde{u}_h(t)\|_j \leq \frac{C}{t^{(r-2)/2}} h^{r-j} + \frac{C}{t^{(r-1)/2}} H^{r+1-j} |\log(H)|, \ j = 0, 1, \ r = 3, 445)$$

$$\|p(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{(r-2)/2}} h^{r-1} + \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}, \quad r = 2, 3, 4,$$
(46)

where r' = 2 for r = 2 and r' = 1 otherwise.

**Proof** Let us consider the linearized problem (27) with right hand side  $d = f - u_t$ . Then, the solution (v, j) of (27) is the solution (u, p) of (1)–(2). Let us denote by  $(v_h, j_h)$  its mixed finite-element approximation, that is the solution of (29)–(30). This approximation satisfy the error bounds (31) and (32) for l = r. Let us decompose  $u - \tilde{u}_h = (u - v_h) + (v_h - \tilde{u}_h)$  and  $p - \tilde{p}_h = (p - j_h) + (j_h - \tilde{p}_h)$ . To bound the first terms in these two decompositions we will apply (31) and (32). In the rest of the proof we deal with the other two terms.

Let us denote by  $e_h = v_h - \tilde{u}_h$ . Subtracting (20) from (29) it is easy to see that  $e_h$  satisfies

$$\nu(\nabla e_h, \nabla \phi_h) + ((u_H \cdot \nabla) e_h, \phi_h) = (\dot{u}_H - u_t, \phi_h) + (((u_H - u) \cdot \nabla) v_h, \phi_h),$$

for all  $\phi_h \in V_{h,r}$ . Taking  $\phi_h = e_h$  in the above equation and applying (24) we get that for  $H < (t^{(r-2)/2}\nu/C)^{1/(r-1+\gamma)}$  there exists a constant C such that

$$||e_h||_1 \le C(||u_t - \dot{u}_H||_{-1} + ||u_H - u||_1 ||v_h - u||_1 + ||u_H - u||_0 ||u||_{3/2}),$$

and applying (41) from Lemma 3, (18) and (31) we get

$$\|e_h\|_1 \le \frac{C}{t^{(r-1)/2}} H^r \left|\log(H)\right|^{r'} + \frac{C}{t^{(r-2)/2}} \left(hH^{r-1} + H^r\right), \tag{47}$$

from which (44) and the case j = 1 in (45) are concluded.

We now get the error bound for the pressure. Let us denote  $r_h = j_h - \tilde{p}_h$ . Subtracting (20) from (29) and using (11) it is easy to obtain

$$\beta \|r_h\|_{L^2/\mathbb{R}} \leq \nu \|e_h\|_1 + C \|u_H\|_{1/2} \|e_h\|_1 + \|\dot{u}_H - u_h\|_{-1} + C \|u_H - u\|_1 \|v_h - u\|_1 + C \|u_H - u\|_0 \|u\|_{3/2},$$

from which we get (46) applying (47), (41) from Lemma 3, (18) and (31).

To conclude we get the error bound for the velocity in the  $L^2$  norm. We will argue as in the proof of Lemma 2, that is, recalling (35), for  $\varphi \in L^2(\Omega)$ we consider the solution (w, k) of (36), so that (40) holds, and we are left to estimate  $|\nu(\nabla e_h, \nabla w_h) + ((u \cdot \nabla) e_h, w_h)|$ . It is easy to see that

$$\nu(\nabla e_h, \nabla w_h) + ((u \cdot \nabla) e_h, w_h) = (\dot{u}_H - u_t, w_h) + (((u_H - u) \cdot \nabla) \tilde{u}_h, w_h).$$
(48)

Let us now bound the two terms on the right hand side of (48). For the first one, using (37) and (38) we get

$$\begin{aligned} (\dot{u}_H - u_t, w_h) &= (\dot{u}_H - u_t, w_h - w) + (\dot{u}_H - u_t, w) \\ &\leq \|\dot{u}_H - u_t\|_{-1} \|w_h - w\|_1 + \|A^{-1}\Pi(\dot{u}_H - u_t)\|_0 \|Aw\|_0 \\ &\leq C \|\dot{u}_H - u_t\|_{-1} h \|\varphi\|_0 + C \|A^{-1}\Pi(\dot{u}_H - u_t)\|_0 \|\varphi\|_0. \end{aligned}$$

Applying now (41) and (42) we have that  $(\dot{u}_H - u_t, w_h)$  is  $O(H^{r+1}|\log(H)|/(t^{(r-1)/2}))$  for r = 3, 4. Finally, we will bound the second term on the right hand side of (48). To this end we decompose

$$\begin{aligned} (((u_H - u) \cdot \nabla)\tilde{u}_h, w_h) &= (((u_H - u) \cdot \nabla)(\tilde{u}_h - u), w_h) + (((u_H - u) \cdot \nabla)u, w_h) \\ &\leq C \|u_H - u\|_{1/2} \|\tilde{u}_h - u\|_1 \|w_h\|_1 + (((u_H - u) \cdot \nabla)u, w_h) \\ &\leq C H \|\tilde{u}_h - u\|_1 \|\varphi\|_0 + (((u_H - u) \cdot \nabla)u, w_h), \end{aligned}$$

where in the last inequality we have applied (18) and we have bounded  $||w_h||_1 \leq C||\varphi||_0$ . Then, to conclude, it only remains to bound  $(((u_H - u) \cdot \nabla)u, w_h)$ . Adding and subtracting w we get

$$\begin{array}{rcl} (((u_H - u) \cdot \nabla)u, w_h) &= & (((u_H - u) \cdot \nabla)u, w_h - w) + (((u_H - u) \cdot \nabla)u, w) \\ &\leq & C \|u_H - u\|_0 \|u\|_{3/2} \|w_h - w\|_1 + C \|u_H - u\|_{-1} \|\nabla u \cdot w\|_1 \\ &\leq & C \|u_H - u\|_0 h \|\varphi\|_0 + C \|u_H - u\|_{-1} \|u\|_2 \|w\|_2 \\ &\leq & C \|u_H - u\|_0 h \|\varphi\|_0 + C \|u_H - u\|_{-1} \|\varphi\|_0, \end{array}$$

where we have applied (37). To conclude we apply (18) and Lemma 4.  $\Box$ 

**Remark 1** We observe from Theorem 1 that the postprocessed method increases the rate of convergence of the Galerkin method in one unit in terms of H, the size of the coarse mesh. In the case of linear elements the improvement is only achieved in the  $H^1$  norm of the velocity but it is not obtained in the  $L^2$  norm. Analogous results had been obtained for the standard postprocessing in the linear case, see [3], [15]. Let us also observe that a correct selection of the coarse and fine mesh diameters gives for the new postprocessed method the same rate of convergence than the Galerkin method over the fine mesh, although, of course, with different constants in the error bounds. The advantage of the method we propose is the saving in computational effort. For the method we propose the time integration is performed using the standard Galerkin method over the coarse mesh and only at the final time we solve one linearized Oseen-type problem over the fine mesh. Let us observe that, for example, the selection  $H = h^{1/2}$  allows to get for the new postprocessed method the rate of convergence of the fine mesh in the  $H^1$  norm when using linear elements. The selection  $H = h^{2/3}$  allows to get the rate of convergence of the fine mesh in the  $H^1$  norm when using quadratic elements, the choice  $H = h^{3/4}$  gives the rate of convergence of the fine mesh in the  $L^2$  norm also for quadratics and so on.

The reason why we have not carried out the error analysis for higher than cubic finite elements is that, as in the papers [20] and [10], due to the loss of regularity at t = 0 no better than  $O(H^5|\log(H)|)$  error bounds can be proved.

#### 4 Numerical experiments

We consider the Navier-Stokes equations (1) in the domain  $\Omega = [0, 1] \times [0, 1]$ subject to homogeneous Dirichlet boundary conditions. For the numerical experiments of this section we approximate the equations using the mini-element [8] over a regular triangulation of  $\Omega$  induced by the set of nodes (i/N, j/N),  $0 \leq i, j \leq N$ , where N = 1/H is an integer. We study the spatially semidiscrete case. Hence, in the time integration (with the trapezoidal rule) sufficiently small time steps were taken so as to ensure that errors arising from the spatial discretization were dominant. In the first experiment we take the forcing term f(t, x) such that the solution of (1)-(2) with  $\nu = 0.05$  is

$$u^{1}(x, y, t) = \pi t \sin^{2}(\pi x) \sin(2\pi y),$$
  

$$u^{2}(x, y, t) = -\pi t \sin^{2}(\pi y) \sin(2\pi x),$$
  

$$p(x, y, t) = 20tx^{2}y.$$

When using the mini-element it has been observed and reported in the literature (see for instance [32], [33], [5] [23], [28] and [29]) that the linear part of the approximation to the velocity,  $u_h^l$ , is a better approximation to the solution uthan  $u_h$  itself. The bubble part of the approximation is only introduced for stability reasons and does not improve the approximation to the velocity and pressure terms. For this reason in the numerical experiments of this section we only consider the errors in the linear approximation to the velocity. Also, following [3], we postprocess only the linear approximation to the velocity, i.e., we solve problem (20)-(21) substituting  $u_H$  and  $\dot{u}_H$  by  $u_H^l$  and  $\dot{u}_H^l$  respectively. The finite element space at the postprocessed step is the same mini-element defined over a refined mesh of size h small enough to capture the asymptotic rate of convergence in the fine grid. The coarse and fine mesh sizes in the experiments are H = 1/6, H = 1/8, H = 1/10 and H = 1/20 and h = 1/20, h = 1/26, h = 1/32 and h = 1/36 respectively. For the postprocessed approximation we also keep only the linear part. We apply the postprocessing step only once at time t = 0.5. In Figure 1 we have represented the size of the Galerkin and



Figure 1: Galerkin (solid lines) and postprocessed (dashed lines) errors in  $L^2$  (asterisks) and  $H^1$  (circles) for H = 1/6, 1/8, 1/10 and 1/12 and h = 1/20, 1/26, 1/32 and 1/36 respectively. On the left, errors for the first component of the velocity. On the right,  $L^2$  errors for the pressure.

postprocessed errors with respect to the inverse of the coarse mesh size H. On the left part of the picture we present the results corresponding to the first component of the velocity. The results obtained for the second component of the velocity are analogous. On the right part of the picture we present the errors in the pressure. In both pictures, we have used solid line for the Galerkin method and dashed line for the postprocessed method. The errors are measured in both the  $H^1$  norm and the  $L^2$  norm. In the picture, they are represented by circles ( $H^1$  norm errors) and asterisks ( $L^2$  norm errors). We can observe on the left of Figure 1 that, in agreement with the theory, the postprocessed method using the mini-element does not increase the rate of convergence in the  $L^2$  norm of the velocity although the size of the errors are reduced. In the  $H^1$  norm, however, also as predicted by the theory, the postprocessed method does increase the order of convergence by one unit (indeed, the errors of the postprocessed method in the  $H^1$  norm are slightly smaller than those of the Galerkin method in the  $L^2$  norm). The same improvement is observed for the  $L^2$  errors of the pressure on the right of Figure 1. This means that we can obtain the level of error corresponding to the fine mesh at essentially the cost of the computation in the coarse mesh because the computation on the fine mesh is performed only once at time t = 0.5. Then, the dominant computational cost is caused by the time evolution in the coarse mesh saving time when compared with the time evolution in the fine mesh that is needed in a standard approach.



Figure 2: First component of the velocity for the Galerkin method with  $\nu = 0.01$ and H = 1/10.

In the next experiment we will show that the new postprocessed method produces better results than both the Galerkin method and the standard postprocessed method (4). We consider now equations (1) with initial condition

$$u^{1}(x, y, t) = -6\sin(\pi x)^{3}\sin(\pi y)^{2}\cos(\pi y),$$
  

$$u^{2}(x, y, t) = 6\sin(\pi x)^{2}\sin(\pi y)^{3}\cos(\pi x),$$

and forcing term f = 0. We take first  $\nu = 0.01$ . In Figure 2 we have represented the linear part of the first component of the velocity for the Galerkin method with H = 1/10 at time T = 0.5. In Figure 3 we show the standard postprocessed approximation with H = 1/10 and h = 1/30. We can observe that the standard postprocessing introduces some oscillations that were not present in the Galerkin approximation. These oscillations are not reduced with a smaller value of h. Finally, in Figure 4 we have represented the linear part of the first component



Figure 3: First component of the velocity for the postprocessed method with  $\nu = 0.01$ , H = 1/10 and h = 1/30.



Figure 4: First component of the velocity for the new postprocessed method with  $\nu = 0.01$ , H = 1/10 and h = 1/30.

of the velocity for the new postprocessed approximation and the same values of coarse and fine mesh sizes, H = 1/10 and h = 1/30. We observe that this approximation does not oscillate at all and it improves the accuracy of both Galerkin and standard postprocessed approximations.

In the last experiment we repeat the experiment with a smaller value of the diffusion parameter,  $\nu = 0.005$ , and the same values of H and h as that of the previous experiment. As it was already observed in the case of convection-diffusion equations [12] the behavior of the standard postprocessed method deteriorates as the diffusion parameter decreases.

We can observe that both the Galerkin and the new postprocessed approximations, see Figures 5 and 7 respectively, do not present oscillations. As before, we can also observe the smoothing effect achieved by postprocessing with the new method proposed in this paper. On the other hand, the standard postprocessed method produces a completely wrong approximation, see Figure 6. Let us



Figure 5: First component of the velocity for the Galerkin method with  $\nu = 0.005$  and H = 1/10.



Figure 6: First component of the velocity for the postprocessed method with  $\nu = 0.005, H = 1/10$  and h = 1/30.

remark that, as it has been noted before in the literature, see [6], [30], the bubble functions used in the mini-element to generate a stable mixed finite-element satisfying the inf-sup condition (11) have also a slightly stabilizing (over-diffusive) effect for moderate values of the Reynolds number. This fact explains the nonoscillating behavior of the linear part of the approximation to the velocity in the Galerkin method of Figure 5 We can also observe, see Figure 7, that the over-diffusive effect appearing in the Galerkin approximation of Figure 5 is attenuated by postprocessing with the new method.



Figure 7: First component of the velocity for the new postprocessed method with  $\nu = 0.005$ , H = 1/10 and h = 1/30.

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