A decoupled preconditioning technique for a mixed Stokes-Darcy model*

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Abstract

We propose an efficient iterative method to solve the mixed Stokes-Dracy model for coupling fluid and porous media flow. The weak formulation of this problem leads to a coupled, indefinite, ill-conditioned and symmetric linear system of equations. We apply a decoupled preconditioning technique requiring only good solvers for the local mixed-Darcy and Stokes subproblems. We prove that the method is asymptotically optimal and confirm, with numerical experiments, that the performance of the preconditioners does not deteriorate on arbitrarily fine meshes.

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1 Introduction

The Stokes-Darcy problem describes filtration processes that find many important applications in porous media problems. Usually, a surface free flow of a liquid is modeled by Stokes equations and the flow confined in the porous media is governed by Darcy equations. The interaction of the local models is commonly handled through the Beavers-Joseph-Saffman (BJS) interface conditions, cf. [4, 27, 21].

Recently, there has been active research on the mathematical and numerical analysis for this model. The weak formulation of this problem is generally obtained by coupling the usual velocity-pressure mixed formulation in the Stokes domain with either the primal formulation (H^1-approach, [15]) or the mixed formulation ($\mathbf{H}(\mathrm{div})$-approach, [22, 18]) in

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the Darcy domain. In both cases, optimal iterative methods are of crucial importance to solve efficiently the discrete Stokes-Darcy model since the corresponding linear systems of algebraic equations are indefinite and ill-conditioned. Several optimal iterative solvers such as the Dirichlet-Neumann or Robin-Robin domain decomposition methods [10, 11, 12, 13, 14], multi-grid methods [8, 9, 25] have been proposed for the model based on an H^1-approach in the Darcy domain. The common feature for most of these methods consists in decoupling the global model in such a way that, only independent Stokes and Darcy subproblems are involved in the iterative process. To the authors' knowledge, the only solution procedure for the $\mathbf{H}(\mathrm{div})$-conforming Darcy flux approach was proposed in [2]. In this paper, the problem is written as a global saddle point problem and a solver is implemented for the scheme using an inexact Uzawa technique relying on an expensive preconditioner. Our purpose here is to devise a decoupled preconditioning strategy that allows to apply existing optimized solvers to each local model independently.

Recently, a systematic way to obtain convergent finite element schemes for the Darcy-Stokes flow problem by combining well-known mixed finite elements that are separately convergent for the $\mathbf{H}(\mathrm{div})$-Darcy formulation and the Stokes problem was proposed in [24]. We take advantage here of a fluid-to-pressure (FtP) operator to reinterpret in this formulation the Darcy system as a nonlocal boundary condition for the Stokes problem. The corresponding discrete equations are written in terms of a symmetric and indefinite linear operator that enjoys the same spectral properties of the local discrete Stokes problem.

Many different iterative methods for solving the saddle point problems that result from the finite element discretization of the Stokes equations are known. There are, for instance, many variants of the so-called inexact Uzawa methods. Block triangular preconditioners for saddle point operators has also been discussed by many authors. An example of such preconditioners has been introduced in [5] by Bramble and Pasciak. In this strategy the original saddle point system is premultiplied by a block triangular operator and then, the resulting positive definite system is solved by a preconditioned conjugate gradient method. A possible difficulty is that this approach requires a proper choice of a critical scaling parameter to obtain a positive definite operator. Finally, we mention the block diagonal preconditioners for the minimal residual method (MINRES), cf. [16, 23] and the references therein. A comparative study of representing methods from each of these three classes is considered in [26]. The inexact Uzawa method is not feasible in our case because of the nonlocal character of the discrete flux-to-pressure operator C_{S}^h appearing in the principal block of our saddle point problem (4.1). The conclusion in [26] is that the preconditioned MINRES method may be slower than the Bramble-Pasciak method but it is more robust (it even converges without preconditioning) and it is parameter free.

Applying a preconditioned MINRES method in our case requires, at each iteration step, the solution of two local problems: a vector Laplace equation in the fluid and the mixed formulation of the Darcy problem in the porous media. This saddle point problem in the Darcy domain is again solved with a preconditioned MINRES method. The global algorithm has then the structure of an outer-inner MINRES iteration process. Thus, for this method a stopping criterion (tolerance parameter) for the inner iteration is needed. The preconditioner for the inner MINRES may be constructed by using techniques from [3, 20]. We use here the nodal auxiliary space preconditioning technique introduced in [20] by Hiptmair and Xu to solve $\mathbf{H}(\mathrm{div})$-elliptic problems. With this choice, our decoupled iterative process consists in two nested MINRES methods whose preconditioners only require the solution of several second-order H^1-elliptic problems in the Stokes and the Darcy domains. Standard multigrid techniques or domain decomposition methods can then be applied to reduce the computational effort. Theoretical analysis and numerical experiments show the optimality and efficiency of the proposed decoupled iterative solver.

The rest of the paper is organized as follows. In Section 2 we summarize the results of [24] introducing the model problem, the variational formulation, general conditions for convergence of a Galerkin discretization and examples of spaces leading to convergent methods. A reinterpretation of the continuous formulation in terms of a Fluid-to-Pressure operator and the derivation of its discrete counterpart are presented in Section 3. We take advantage of the equivalent formulation of the discrete Stokes-Darcy problem to deduce, in Section 4, a decoupled iterative solver based on a preconditioned MINRES method. Finally, numerical experiments are reported in Section 5.

Notation and background. Boldface fonts will be used to denote vectors and vector valued functions. Also, if H is a vector space of scalar functions, $\mathbf{H}$ will denote the space of $\mathbb{R}^d$ valued functions whose components are in H, endowed with the product norm.

Given an integer $m \geq 1$ and a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^d$, $(d = 2, 3)$, we denote by $\|\cdot\|_{H^m(\mathcal{O})}$ the norm in the usual Sobolev space $H^m(\mathcal{O})$, cf. [1]. For economy of notation, $(\cdot,\cdot)_{\mathcal{O}}$ stands for the inner product in $L^2(\mathcal{O})$ and $\|\cdot\|_{\mathcal{O}}$ is the corresponding norm. We recall that $H^{1/2}(\partial\mathcal{O})$ represents the image of $H^1(\mathcal{O})$ by the trace operator. Its dual with respect to the pivot space $L^2(\partial\mathcal{O})$ is denoted $H^{-1/2}(\partial\mathcal{O})$. For definition and basic properties of the spaces $\mathbf{H}(\mathrm{div},\mathcal{O})$ and $\mathbf{H}(\mathbf{curl},\mathcal{O})$, we refer to [19]. We will denote by $\mathbf{H}_0(\mathrm{div},\mathcal{O})$ the subspace of fields from $\mathbf{H}(\mathrm{div},\mathcal{O})$ with zero normal trace on the boundary $\partial\mathcal{O}$ and by $\mathbf{H}_0(\mathbf{curl},\mathcal{O})$ the subspace of fields from $\mathbf{H}(\mathbf{curl},\mathcal{O})$ with zero tangential trace on $\partial\mathcal{O}$.

For $k \geq 0$, $\mathbb{P}_k(\mathcal{O})$ will denote the space of $d-$variate polynomials of degree not greater than k defined on the set $\mathcal{O} \subset \mathbb{R}^d$ with non-trivial interior. Finally, at the discrete level, the letter h (with or without geometric meaning) will be used to denote discretization. The expression $a \lesssim b$ will be used to mean that there exists $C > 0$ independent of h such that $a \leq C\, b$ for all h. Similarly, we write $a \simeq b$ when there exist constants $C > c > 0$ independent of h such that $ca \leq b \leq Ca$.

Let us consider a linear operator $A^h : X_h \to X_h^*$ acting between a finite dimensional subspace X_h of a Hilbert space $X \subset L^2(\mathcal{O})$ and its dual X_h^*. Assume that we have chosen a basis in X_h and that the coefficients of $u_h, v_h \in X_h$ in this basis are given by $\bar{u}, \bar{v} \in \mathbb{R}^n$, where n is the dimension of X_h. We define the matrix realization $\mathbf{A}^h \in \mathbb{R}^{n\times n}$ of A^h by

$$\langle \mathbf{A}^h \bar{u}, \bar{v}\rangle_2 = \langle A^h u_h, v_h\rangle_{X_h^*\times X_h} \quad \forall u_h, v_h \in X_h, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle_2$ stands for the Euclidean scalar product in $\mathbb{R}^n$. Moreover, if $I^h : X_h \to X_h'$ is the Riesz operator given by

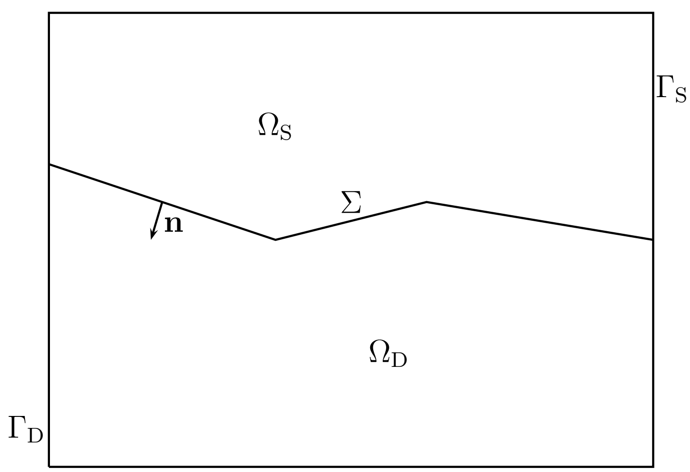
$$\langle I^h u_h, v_h\rangle_{X_h^*\times X_h} = (u_h, v_h)_{\mathcal{O}} \quad \forall u_h, v_h \in X_h,$$

then, the corresponding matrix realization $\mathbf{M}^h \in \mathbb{R}^{n\times n}$, usually referred to as the mass matrix, is defined by

$$\langle \mathbf{M}^h \bar{u}, \bar{v}\rangle_2 = \langle I^h u_h, v_h\rangle_{X_h^*\times X_h} \quad \forall u_h, v_h \in X_h.$$

2 Statement of the problem and discretization

Let us consider a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) with polyhedral Lipschitz boundary. We assume that Ω is subdivided into two subdomains by a Lipschitz polyhedral interface Σ. The subdomains are denoted Ω_{S} and Ω_{D} (S stands for Stokes and D for Darcy). We also denote $\Gamma_{\mathrm{S}} := \partial\Omega_{\mathrm{S}} \setminus \Sigma$ and $\Gamma_{\mathrm{D}} := \partial\Omega_{\mathrm{D}} \setminus \Sigma$. The normal vector field $\mathbf{n}$ on $\partial\Omega$ is chosen to point outwards. We also denote by $\mathbf{n}$ the normal vector on Σ that points from Ω_{S} to Ω_{D}.



2.1 Variational formulation

In the region Ω_{S}, the fluid flow is assumed to satisfy the Stokes system

$$-\,\mathbf{div}\,(2\nu\boldsymbol{\varepsilon}(\mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathbf{I}) = \mathbf{f}_{\mathrm{S}}, \qquad \operatorname{div}\mathbf{u}_{\mathrm{S}} = 0 \qquad \text{in } \Omega_{\mathrm{S}}, \tag{2.1}$$

where $\mathbf{I}$ is the identity in $\mathbb{R}^d$ and $\boldsymbol{\varepsilon}(\mathbf{u}_{\mathrm{S}}) := \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u}_{\mathrm{S}} + \boldsymbol{\nabla}^\top\mathbf{u}_{\mathrm{S}})$ is the deformation tensor, $\nu > 0$ is the kinematic viscosity and $\mathbf{f}_{\mathrm{S}}$ is the external body force. In the porous region Ω_{D}, the governing equations are given by the following Darcy system

$$\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}} = \mathbf{0}, \qquad \operatorname{div}\mathbf{u}_{\mathrm{D}} = f_{\mathrm{D}} \qquad \text{in } \Omega_{\mathrm{D}}, \tag{2.2}$$

where f_{D} is the source (or sink) term and the hydraulic conductivity tensor of the porous medium $\mathbf{K}(\mathbf{x})$ is symmetric and uniformly bounded and positive definite, i.e.,

$$0 < k_1|\boldsymbol{\xi}|^2 \le \mathbf{K}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \le k_2|\boldsymbol{\xi}|^2 \quad \text{for a.e. } \mathbf{x} \in \Omega_{\mathrm{D}}, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,$$

for some constants $k_2 \ge k_1 > 0$. On the outer boundaries we consider the homogeneous (essential) boundary conditions

$$\mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{S}}, \qquad \text{and} \qquad \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{D}}, \tag{2.3}$$

and on the interface between the fluid and porous media regions we impose conditions ensuring mass conservation, balance of normal forces and the Beavers-Joseph-Saffman condition [4, 27],

$$\mathbf{u}_\mathrm{S}\cdot\mathbf{n}=\mathbf{u}_\mathrm{D}\cdot\mathbf{n},\qquad 2\nu\boldsymbol{\varepsilon}(\mathbf{u}_\mathrm{S})\mathbf{n}-p_\mathrm{S}\mathbf{n}+\kappa\boldsymbol{\pi}_t\mathbf{u}_\mathrm{S}=-p_\mathrm{D}\mathbf{n}\qquad\text{on }\Sigma,\tag{2.4}$$

where $\boldsymbol{\pi}_t\mathbf{w}:=\mathbf{w}-(\mathbf{w}\cdot\mathbf{n})\mathbf{n}$ and κ is a positive and bounded function depending on $\mathbf{K}$, ν, and an experimentally determined friction constant, cf. [4, 8, 10].

Because of the mass conservation condition across Σ, the homogeneous Dirichlet boundary condition for $\mathbf{u}_\mathrm{S}$ on Γ_S and the incompressibility condition in the Stokes domain, we can easily show that

$$\int_{\Omega_\mathrm{D}} f_\mathrm{D}=0\tag{2.5}$$

is a necessary condition for existence of solution. The pressure field is defined up to an additive constant. We will normalize it by imposing that

$$\int_{\Omega_\mathrm{D}} p_\mathrm{D}=0.$$

For the velocity field, we will use the space

$$\mathbb{X}:=\{\mathbf{u}=(\mathbf{u}_\mathrm{S},\mathbf{u}_\mathrm{D})\in\mathbf{H}^1_\mathrm{S}(\Omega_\mathrm{S})\times\mathbf{H}_\mathrm{D}(\mathrm{div},\Omega_\mathrm{D})\,:\,\mathbf{u}_\mathrm{S}\cdot\mathbf{n}=\mathbf{u}_\mathrm{D}\cdot\mathbf{n}\quad\text{on }\Sigma\}\subset\mathbf{H}_0(\mathrm{div},\Omega),$$

where

$$\begin{aligned}\mathbf{H}^1_\mathrm{S}(\Omega_\mathrm{S}) &:= \{\mathbf{u}\in\mathbf{H}^1(\Omega_\mathrm{S})\,:\,\mathbf{u}=\mathbf{0}\quad\text{on }\Gamma_\mathrm{S}\} &(2.6)\\ \mathbf{H}_\mathrm{D}(\mathrm{div},\Omega_\mathrm{D}) &:= \{\mathbf{u}\in\mathbf{H}(\mathrm{div},\Omega_\mathrm{D})\,:\,\mathbf{u}\cdot\mathbf{n}=0\quad\text{on }\Gamma_\mathrm{D}\}. &(2.7)\end{aligned}$$

The space $\mathbb{X}$ will be endowed with the product norm. The space for the pressure field is $\mathbb{Q}:=L^2(\Omega_\mathrm{S})\times L^2_0(\Omega_\mathrm{D})$, where $L^2_0(\mathcal{O}):=\{p\in L^2(\mathcal{O})\,:\,(p,1)_\mathcal{O}=0\}$. The pressure field is represented as $p:=(p_\mathrm{S},p_\mathrm{D})\in\mathbb{Q}$. Adding an appropriate constant in a postprocessing step, the normalization condition $(p,1)_{\Omega_\mathrm{D}}=0$ can be modified to $(p,1)_\Omega=0$. The space $\mathbb{Q}$ is endowed with the corresponding product norm.

We consider four bilinear forms, two in the Stokes domain and two in the Darcy domain:

$$\begin{aligned}a_\mathrm{S}(\mathbf{u}_\mathrm{S},\mathbf{u}_\mathrm{S}) &:= 2\nu(\boldsymbol{\varepsilon}(\mathbf{u}_\mathrm{S}),\boldsymbol{\varepsilon}(\mathbf{v}_\mathrm{S}))_{\Omega_\mathrm{S}}+\langle\kappa\boldsymbol{\pi}_t\mathbf{u}_\mathrm{S},\boldsymbol{\pi}_t\mathbf{v}_\mathrm{S}\rangle_\Sigma, &(2.8)\\ a_\mathrm{D}(\mathbf{u}_\mathrm{D},\mathbf{u}_\mathrm{D}) &:= (\mathbf{K}^{-1}\mathbf{u}_\mathrm{D},\mathbf{v}_\mathrm{D})_{\Omega_\mathrm{D}}, &(2.9)\\ b_\mathrm{S}(\mathbf{u}_\mathrm{S},q_\mathrm{S}) &:= (\mathrm{div}\,\mathbf{u}_\mathrm{S},q_\mathrm{S})_{\Omega_\mathrm{S}}, &(2.10)\\ b_\mathrm{D}(\mathbf{u}_\mathrm{D},q_\mathrm{D}) &:= (\mathrm{div}\,\mathbf{u}_\mathrm{D},q_\mathrm{D})_{\Omega_\mathrm{D}}. &(2.11)\end{aligned}$$

These bilinear forms are combined to build the diagonal bilinear form of the mixed problem $a:\mathbb{X}\times\mathbb{X}\to\mathbb{R}$, given by

$$a(\mathbf{u},\mathbf{v}):=a_\mathrm{S}(\mathbf{u}_\mathrm{S},\mathbf{u}_\mathrm{S})+a_\mathrm{D}(\mathbf{u}_\mathrm{D},\mathbf{u}_\mathrm{D}),$$

as well as $b:\mathbb{X}\times\mathbb{Q}\to\mathbb{R}$ given by

$$b(\mathbf{u},q):=b_\mathrm{S}(\mathbf{u}_\mathrm{S},q_\mathrm{S})+b_\mathrm{D}(\mathbf{u}_\mathrm{D},q_\mathrm{D}).\tag{2.12}$$

A well posed variational form of the Darcy-Stokes problem (cf. [24, Proposition 2.3]) is: find $(\mathbf{u}, p) \in \mathbb{X} \times \mathbb{Q}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) &= (\mathbf{f}_\mathrm{S}, \mathbf{v}_\mathrm{S})_{\Omega_\mathrm{S}} \quad \forall \mathbf{v} \in \mathbb{X}, \\ b(\mathbf{u}, q) &= (f_\mathrm{D}, q_\mathrm{D})_{\Omega_\mathrm{D}} \quad \forall q \in \mathbb{Q}. \end{aligned} \tag{2.13}$$

2.2 The discrete problem

We start by creating shape-regular triangulations $\{\mathcal{T}_\mathrm{S}^h\}_h$ and $\{\mathcal{T}_\mathrm{D}^h\}_h$ of $\overline{\Omega}_\mathrm{S}$ and $\overline{\Omega}_\mathrm{D}$ respectively, consisting of triangles (tetrahedra in the three dimensional case) of diameter not larger than h. The triangulations create two inherited partitions of Σ, respectively denoted Σ_S^h and Σ_D^h. Let us consider finite dimensional subspaces of piecewise polynomial) to approximate velocity and pressure in the Stokes domain

$$\mathbf{V}^h(\Omega_\mathrm{S}) \subset \mathbf{H}^1(\Omega_\mathrm{S}), \qquad L_0^h(\Omega_\mathrm{S}) \subset L_0^2(\Omega_\mathrm{S}), \qquad L^h(\Omega_\mathrm{S}) = L_0^h(\Omega_\mathrm{S}) \oplus \mathbb{P}_0(\Omega_\mathrm{S}),$$

as well as in the Darcy domain

$$\mathbf{H}^h(\Omega_\mathrm{D}) \subset \mathbf{H}(\mathrm{div}, \Omega_\mathrm{D}), \qquad L_0^h(\Omega_\mathrm{D}) \subset L_0^2(\Omega_\mathrm{D}), \qquad L^h(\Omega_\mathrm{D}) = L_0^h(\Omega_\mathrm{D}) \oplus \mathbb{P}_0(\Omega_\mathrm{D}).$$

We also need to consider the spaces

$$\mathbf{V}_\mathrm{S}^h(\Omega_\mathrm{S}) := \mathbf{H}^h(\Omega_\mathrm{S}) \cap \mathbf{H}_\mathrm{S}^1(\Omega_\mathrm{S}), \qquad \mathbf{H}_\mathrm{D}^h(\Omega_\mathrm{D}) := \mathbf{H}^h(\Omega_\mathrm{D}) \cap \mathbf{H}_\mathrm{D}(\mathrm{div}, \Omega_\mathrm{D}), \tag{2.14}$$

as well as the discrete spaces of normal components on Σ, namely,

$$\begin{aligned} \Phi_\mathrm{S}^h &:= \{\mathbf{u}_h \cdot \mathbf{n} \,:\, \mathbf{u}_h \in \mathbf{V}_\mathrm{S}^h(\Omega_\mathrm{S})\} \subset L^2(\Sigma), \\ \Phi_\mathrm{D}^h &:= \{\mathbf{u}_h \cdot \mathbf{n} \,:\, \mathbf{u}_h \in \mathbf{H}_\mathrm{D}^h(\Omega_\mathrm{D})\} \subset L^2(\Sigma). \end{aligned}$$

We will assume that Φ_D^h contains at least the space of piecewise constant functions on Σ_D^h and denote by R_D^h the $L^2(\Sigma)$-orthogonal projection onto Φ_D^h.

The method we are proposing is a Galerkin discretization of the variational problem (2.13) using the spaces

$$\begin{aligned} \mathbb{X}^h &:= \{\mathbf{u}_h \equiv (\mathbf{u}_\mathrm{S}^h, \mathbf{u}_\mathrm{D}^h) \in \mathbf{V}_\mathrm{S}^h(\Omega_\mathrm{S}) \times \mathbf{H}_\mathrm{D}^h(\Omega_\mathrm{D}) \,:\, \mathbf{u}_\mathrm{D}^h \cdot \mathbf{n} = R_D^h(\mathbf{u}_\mathrm{S}^h \cdot \mathbf{n}) \text{ on } \Sigma\}, \\ \mathbb{Q}^h &:= L^h(\Omega_\mathrm{S}) \times L_0^h(\Omega_\mathrm{D}), \end{aligned}$$

that is, we look for $(\mathbf{u}_h, p_h) \in \mathbb{X}^h \times \mathbb{Q}^h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) &= (\mathbf{f}_\mathrm{S}, \mathbf{v}_h)_{\Omega_\mathrm{S}} \quad \forall \mathbf{v}_h \in \mathbb{X}^h, \\ b(\mathbf{u}_h, q_h) &= (f_\mathrm{D}, q_h)_{\Omega_\mathrm{D}} \quad \forall q_h \in \mathbb{Q}^h. \end{aligned} \tag{2.15}$$

Remark 2.1. *Note that $\mathbb{X}^h \not\subset \mathbb{X}$ unless $\Phi_\mathrm{S}^h \subset \Phi_\mathrm{D}^h$ in which case* (2.15) *becomes a conforming Galerkin approximation of* (2.13)*. We will say that the discretization is conforming if $\Phi_\mathrm{S}^h \subset \Phi_\mathrm{D}^h$ (and therefore $\mathbb{X}^h \subset \mathbb{X}$) and non-conforming otherwise.*

The following result is proved in [24, Proposition 3.2]. Inf-sup conditions are written in terms of the spaces

$$\mathbf{V}_0^h(\Omega_{\rm S}) := \mathbf{V}^h(\Omega_{\rm S}) \cap \mathbf{H}_0^1(\Omega_{\rm S}), \qquad \mathbf{H}_0^h(\Omega_{\rm D}) := \mathbf{H}^h(\Omega_{\rm D}) \cap \mathbf{H}_0(\mathrm{div}, \Omega_{\rm D}),$$

which arise from the application of the discretization method to problems with homogeneous boundary conditions on the entire boundary of each subdomain.

Theorem 2.1. *Let us assume that there exist a linear operator* $\mathbf{L}_h : \Phi_{\rm D} \to \mathbf{H}_{\rm D}^h(\Omega_{\rm D})$ *and a general positive constant* β*, independent of* h*, such that:*

$$\sup_{\mathbf{0}\neq\mathbf{u}_h\in\mathbf{V}_0^h(\Omega_{\rm S})} \frac{(\mathrm{div}\,\mathbf{u}_h, q_h)_{\Omega_{\rm S}}}{\|\mathbf{u}_h\|_{\mathbf{H}^1(\Omega_{\rm S})}} \geq \beta\|q_h\|_{\Omega_{\rm S}} \qquad \forall q_h \in L_0^h(\Omega_{\rm S}), \tag{2.16}$$

$$\sup_{\mathbf{0}\neq\mathbf{u}_h\in\mathbf{H}_0^h(\Omega_{\rm D})} \frac{(\mathrm{div}\,\mathbf{u}_h, q_h)_{\Omega_{\rm D}}}{\|\mathbf{u}_h\|_{\mathbf{H}(\mathrm{div},\Omega_{\rm D})}} \geq \beta\|q_h\|_{\Omega_{\rm D}} \qquad \forall q_h \in L_0^h(\Omega_{\rm D}), \tag{2.17}$$

$$\mathrm{div}\,\mathbf{H}^h(\Omega_{\rm D}) \subset L^h(\Omega_{\rm D}), \tag{2.18}$$

$$\exists \mathbf{v}_h \in \mathbf{V}_{\rm S}^h(\Omega_{\rm S}) \quad \textit{s.t.} \quad \langle \mathbf{v}_h\cdot\mathbf{n}, 1\rangle_\Sigma \geq \beta \quad \textit{and} \quad \|\mathbf{v}_h\|_{\mathbf{H}^1(\Omega_{\rm S})} \leq \beta, \tag{2.19}$$

$$(\mathbf{L}_h\phi_h)\cdot\mathbf{n} \qquad \|\mathbf{L}_h\phi_h\|_{\mathbf{H}(\mathrm{div},\Omega_{\rm D})} \leq \beta\|\phi\|_{H^{-1/2}(\Sigma)} \qquad \forall\phi_h \in \Phi_{\rm D}^h. \tag{2.20}$$

Then the discrete equations(2.15) *are uniquely solvable and the following error estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbb{X}} + \|p-p_h\|_{\Omega_{\rm D}} \lesssim \inf_{\mathbf{u}_{\rm S}^h\in\mathbf{V}_{\rm S}^h(\Omega_{\rm S})} \|\mathbf{u}_{\rm S}-\mathbf{u}_{\rm S}^h\|_{\mathbf{H}^1(\Omega_{\rm S})} + \inf_{\mathbf{u}_{\rm D}^h\in\mathbf{H}_{\rm D}^h(\Omega_{\rm D})} \|\mathbf{u}_{\rm D}-\mathbf{u}_{\rm D}^h\|_{\mathbf{H}(\mathrm{div},\Omega_{\rm D})} +$$
$$\inf_{q_h\in\mathbb{Q}} \|p-q_h\|_\Omega + \lambda(h)\left(\|p_{\rm D} - R_D^h p_{\rm D}\|_\Sigma + \|\mathbf{u}_{\rm S}\cdot\mathbf{n} - R_D^h(\mathbf{u}_{\rm S}\cdot\mathbf{n})\|_\Sigma\right).$$

Here $\lambda(h) \equiv 0$ *if* $\Phi_{\rm S}^h \subset \Phi_{\rm D}^h$ *and* $\lambda(h) \lesssim h^{1/2}$ *otherwise.*

Let us briefly discuss the five hypotheses in Theorem 2.1. The inf-sup condition (2.16) is necessary and sufficient for stability of the discretization of the Stokes equation with homogeneous boundary conditions. The inf-sup condition (2.17) and the restriction (2.18) are standard conditions for stability of the discretization of the Darcy equations with homogeneous boundary condition on the normal trace.

Condition (2.20) is the existence of a uniformly bounded right-inverse of the operator $\mathbf{H}_{\rm D}^h(\Omega_{\rm D}) \ni \mathbf{v}_h \mapsto \mathbf{v}_h\cdot\mathbf{n} \in \Phi_h^{\rm D}$. As discussed in [24, Section 5], this condition is satisfied for Brezzi-Douglas-Marini (BDM) and Raviart-Thomas (RT) elements (see below for their definitions) on general shape-regular triangulations in the plane, and on tetrahedrizations of the space that are quasi-uniform near the boundary Σ. Existence of $\mathbf{L}_h$ satisfying (2.20) for BDM and RT elements in general tetrahedrizations is an open question. Hypothesis (2.19) is a very mild condition demanding that the discrete space for the Stokes condition can provide non-trivial flow in to the Darcy domain without a blow-up of the velocity field. This condition is discussed in [24, Section 6], where it is shown that as long as the Stokes velocity space contains piecewise linear functions, this condition is satisfied.

Some examples For precise descriptions of the finite element spaces below, the reader is referred to [7], [17] and [19]. All choices below will be given with the following assumptions:

- Hypothesis (2.19) will be assumed to hold.
- The convergence orders of the Stokes and Darcy elements are chosen to match.
- If the discretization is conforming, we will assume that the Darcy partition Σ_{D}^h is either equal to or a refinement of Σ_{S}^h.

The Brezzi-Douglas-Marini (sometimes called Brezzi-Douglas-Durán-Fortin in the three dimensional case) is the mixed finite element that uses the spaces

$$\begin{aligned}\mathbf{H}^h(\Omega_{\mathrm{D}}) &:= \{\mathbf{u}_h \in \mathbf{H}(\mathrm{div},\Omega_{\mathrm{D}}) \,:\, \mathbf{u}_h|_T \in \mathbb{P}_k(T)^d \quad \forall T \in \mathcal{T}_{\mathrm{D}}^h\},\\ L^h(\Omega_{\mathrm{D}}) &:= \{p_h : \Omega_{\mathrm{D}} \to \mathbb{R} \,:\, p_h|_T \in \mathbb{P}_{k-1}(T) \quad \forall T \in \mathcal{T}_{\mathrm{D}}^h\},\end{aligned}$$

for $k \geq 1$. We will refer to it as the BDM(k) element. The BDM(1) element can be coupled in a conforming way with the MINI element and the Bernardi-Raugel element. It can also be coupled with the $\mathbb{P}_2$-iso-P_1 element in a conforming way if Σ_{D}^h is either equal to or a refinement of $\Sigma_{\mathrm{S}}^{h/2}$ and in a non-conforming way otherwise. The BDM(2) element can be coupled in a conforming way with the conforming Crouzeix-Raviart element. More generally speaking, BDM(k) can be coupled with the Taylor-Hood element of order k for any $k \geq 2$.

The Raviart-Thomas element of order k, henceforth referred to as RT(k), is defined as the pair

$$\begin{aligned}\mathbf{H}^h(\Omega_{\mathrm{D}}) &:= \{\mathbf{u}_h \in \mathbf{H}(\mathrm{div},\Omega_{\mathrm{D}}) \,:\, \mathbf{u}_h|_T \in \mathrm{RT}_k(T) \quad \forall T \in \mathcal{T}_{\mathrm{D}}^h\},\\ L^h(\Omega_{\mathrm{D}}) &:= \{p_h : \Omega_{\mathrm{D}} \to \mathbb{R} \,:\, p_h|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_{\mathrm{D}}^h\},\end{aligned}$$

where $\mathrm{RT}_k(T) = \{\mathbf{p}(\mathbf{x}) + q(\mathbf{x})\,\mathbf{x} \,:\, \mathbf{p} \in \mathbb{P}_k(T)^d, \quad q \in \mathbb{P}_k(T)\}$. The RT(0) element can be coupled in non-conforming way with the MINI element and the Bernardi-Raugel element. For $k \geq 1$, RT($k-1$) can be coupled with the Taylor-Hood element of order k.

3 An alternative point of view

In this section we propose a different way of interpreting the coupled method, based on seeing the Darcy equations as part of a generalized boundary condition for the Stokes problem.

3.1 The Darcy boundary condition

Given f_{D} satisfying the compatibility condition (2.5), we consider the solution of the Darcy problem

$$\begin{aligned}\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}^f + \nabla p_{\mathrm{D}}^f = \mathbf{0} &\qquad \text{in } \Omega_{\mathrm{D}},\\ \mathrm{div}\,\mathbf{u}_{\mathrm{D}}^f = f_{\mathrm{D}} &\qquad \text{in } \Omega_{\mathrm{D}},\\ \mathbf{u}_{\mathrm{D}}^f \cdot \mathbf{n} = 0 &\qquad \text{on } \Gamma_{\mathrm{D}} \cup \Sigma,\\ \int_{\Omega_{\mathrm{D}}} p_{\mathrm{D}}^f = 0, &\end{aligned}$$

and note that $p_{\mathrm{D}}^f \in H^1(\Omega_{\mathrm{D}})$. Also, for $\phi \in L^2(\Sigma)$, we consider the solution of

$$
\begin{aligned}
\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}^{\phi} + \nabla p_{\mathrm{D}}^{\phi} &= \mathbf{0} && \text{in } \Omega_{\mathrm{D}},\\
\operatorname{div} \mathbf{u}_{\mathrm{D}}^{\phi} &= \frac{1}{|\Omega|}\int_{\Sigma} \phi && \text{in } \Omega_{\mathrm{D}},\\
\mathbf{u}_{\mathrm{D}}^{\phi}\cdot \mathbf{n} &= 0 && \text{on } \Gamma_{\mathrm{D}},\\
\mathbf{u}_{\mathrm{D}}^{\phi}\cdot \mathbf{n} &= \phi && \text{on } \Sigma,\\
\int_{\Omega_{\mathrm{D}}} p_{\mathrm{D}}^{\phi} &= 0,
\end{aligned}
$$

and define with it the Flux-to-Pressure operator $\mathrm{FtP}(\phi) := p_{\mathrm{D}}^{\phi}|_{\Sigma}$. It is simple to prove that FtP is a linear and symmetric operator in $L^2(\Sigma)$. Indeed, the Flux-to-Pressure operator satisfies

$$\langle \mathrm{FtP}(\phi), \mathbf{v}\cdot\mathbf{n}\rangle_{\Sigma} = a_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}^{\phi}, \mathbf{v}) - b_{\mathrm{D}}(\mathbf{v}, p_{\mathrm{D}}^{\phi}) \tag{3.1}$$

for all $\mathbf{v} \in \mathbf{H}_{\mathrm{D}}(\mathrm{div}, \Omega_{\mathrm{D}})$ such that $\mathbf{v}\cdot\mathbf{n} \in L^2(\Sigma)$, which gives

$$\langle \mathrm{FtP}(\phi), \psi\rangle_{\Sigma} = \langle \mathrm{FtP}(\phi), \mathbf{u}_{\mathrm{D}}^{\psi}\cdot\mathbf{n}\rangle_{\Sigma} = a_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}^{\phi}, \mathbf{u}_{\mathrm{D}}^{\psi}) = \langle \mathrm{FtP}(\psi), \phi\rangle_{\Sigma} \quad \forall \phi, \psi \in L^2(\Sigma). \tag{3.2}$$

It is clear that, as $\int_{\Sigma} \mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} = 0$, we have the splitting

$$p_{\mathrm{D}}|_{\Sigma} = p_{\mathrm{D}}^f + \mathrm{FtP}(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n})$$

for the Darcy pressure on Σ. This allow us to write the coupling conditions (2.4) as a unilateral boundary condition for the Stokes flow on the interface Σ:

$$2\nu\boldsymbol{\varepsilon}(\mathbf{u}_{\mathrm{S}})\mathbf{n} - p_{\mathrm{S}}\mathbf{n} + \underbrace{\kappa\boldsymbol{\pi}_t\mathbf{u}_{\mathrm{S}} + \mathrm{FtP}(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n})\mathbf{n}} = -p_{\mathrm{D}}^f\mathbf{n}. \tag{3.3}$$

The underbracketed term corresponds to a symmetric positive semidefinite non-local operator that takes into account the influence of the Darcy domain on the Stokes flow, acting separately on the tangential and normal components of the Stokes flow. The Stokes system (2.1) can then be complemented with the non-local condition (3.3) and the Dirichlet condition on Γ_{S} (see (2.3)) to produce a formulation of the Stokes-Darcy problem that is equivalent to (2.13). It consists in looking for $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}_{\mathrm{S}}^1(\Omega_{\mathrm{S}})$ and $p_{\mathrm{S}} \in L^2(\Omega_{\mathrm{S}})$ such that

$$
\begin{aligned}
a_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}, \mathbf{v}) + c(\mathbf{u}_{\mathrm{S}}, \mathbf{v}) - b_{\mathrm{S}}(\mathbf{v}, p_{\mathrm{S}}) &= \ell(\mathbf{v}) && \forall \mathbf{v} \in \mathbf{H}_{\mathrm{S}}^1(\Omega_{\mathrm{S}}),\\
b_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}, q) &= 0 && \forall q \in L^2(\Omega_{\mathrm{S}}),
\end{aligned} \tag{3.4}
$$

where

$$c(\mathbf{u}, \mathbf{v}) := \langle \mathrm{FtP}(\mathbf{u}\cdot\mathbf{n}), \mathbf{v}\cdot\mathbf{n}\rangle_{\Sigma} \tag{3.5}$$

and

$$\ell(\mathbf{v}) := (\mathbf{f}_{\mathrm{S}}, \mathbf{v})_{\Omega_{\mathrm{S}}} - \langle p_{\mathrm{D}}^f, \mathbf{v}\cdot\mathbf{n}\rangle_{\Sigma}.$$

By (3.2), it follows that the bilinear form in (3.5) is symmetric and positive semidefinite. A simple argument shows that the bilinear form $a_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}, \mathbf{v}) + c(\mathbf{u}_{\mathrm{S}}, \mathbf{v})$ is coercive in $\mathbf{H}^1(\Omega_{\mathrm{S}})$. This fact gives a very simple proof of the fact that the Stokes-Darcy system is well posed and that it can be understood as a modified Stokes problem without losing any of its good properties. This will be exploited to design an effective Krylov-based iterative method to solve the algebraic linear system of equations arising from the discrete counterpart of (3.4).

3.2 The discrete flux-to-pressure operator

If we now choose discrete spaces for the Darcy problem satisfying (2.17)-(2.18), we can define a discrete version of the operator FtP as follows. Given $\phi_h \in \Phi_{\mathrm{D}}^h$ with $\int_\Sigma \phi_h = 0$, we define $\mathrm{FtP}_h(\phi_h) : \Phi_{\mathrm{D}}^h \to \mathbb{R}$ to be the functional (compare with (3.1))

$$\langle \mathrm{FtP}_h(\phi_h), \mathbf{v}_h \cdot \mathbf{n} \rangle_\Sigma := a_{\mathrm{D}}(\mathbf{u}_h^\phi, \mathbf{v}_h) - b_{\mathrm{D}}(\mathbf{v}_h, p_h^\phi), \quad \forall \mathbf{v}_h \in \mathbf{H}_{\mathrm{D}}^h(\Omega_{\mathrm{D}}), \tag{3.6}$$

where $(\mathbf{u}_h^\phi, p_h^\phi) \in \mathbf{H}_{\mathrm{D}}^h(\Omega_{\mathrm{D}}) \times L_0^h(\Omega_{\mathrm{D}})$ solves the discrete equations:

$$\mathbf{u}_h \cdot \mathbf{n} = \phi_h \qquad \text{on } \Sigma, \tag{3.7}$$

$$\mathbf{u}_h \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{3.8}$$

$$a_{\mathrm{D}}(\mathbf{u}_h^\phi, \mathbf{v}_h) - b_{\mathrm{D}}(\mathbf{v}_h, p_h^\phi) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{H}_0^h(\Omega_{\mathrm{D}}), \tag{3.9}$$

$$b_{\mathrm{D}}(\mathbf{u}_h^\phi, q_h) = 0 \qquad \forall q_h \in L_0^h(\Omega_{\mathrm{D}}). \tag{3.10}$$

With arguments similar to those used in the continuous case, it is easy to prove that

$$\langle \mathrm{FtP}_h(\phi_h), \psi_h \rangle_\Sigma = a_{\mathrm{D}}(\mathbf{u}_h^\phi, \mathbf{u}_h^\psi) = \langle \mathrm{FtP}_h(\psi_h), \phi_h \rangle_\Sigma \qquad \forall \phi_h, \psi_h \in \Phi_{\mathrm{D}}^h,$$

which shows that the discrete flux-to-pressure operator FtP_h is also symmetric and non-negative.

The discrete pressure due to sources, γ_h^f, can be similarly defined as a residual:

$$\langle \gamma_h^f, \mathbf{v}_h \cdot \mathbf{n} \rangle_\Sigma := a_{\mathrm{D}}(\mathbf{u}_h^f, \mathbf{v}_h) - b_{\mathrm{D}}(\mathbf{v}_h, p_h^f),$$

where $(\mathbf{u}_h^f, p_h^f) \in \mathbf{H}_0^h(\Omega_{\mathrm{D}}) \times L_0^h(\Omega_{\mathrm{D}})$ solve the discrete equations:

$$\begin{aligned} a_{\mathrm{D}}(\mathbf{u}_h^f, \mathbf{v}_h) - b_{\mathrm{D}}(\mathbf{v}_h, p_h^f) &= 0 && \forall \mathbf{v}_h \in \mathbf{H}_0^h(\Omega_{\mathrm{D}}), \\ b_{\mathrm{D}}(\mathbf{u}_h^f, q_h) &= (f_{\mathrm{D}}, q_h)_{\Omega_{\mathrm{D}}} && \forall q_h \in L_0^h(\Omega_{\mathrm{D}}). \end{aligned}$$

We recall that the operator R^h is the $L^2(\Sigma)$-projection on Φ_{D}^h. It is straightforward that the discrete Darcy pressure and velocity of problem (2.15) admit the splitting

$$p_{\mathrm{D}}^h = p_h^f + p_h^{R_D^h(\mathbf{u}_{\mathrm{S}}^h \cdot \mathbf{n})} \quad \text{and} \quad \mathbf{u}_{\mathrm{D}}^h = \mathbf{u}_h^f + \mathbf{u}_h^{R_D^h(\mathbf{u}_{\mathrm{S}}^h \cdot \mathbf{n})}.$$

It follows that (2.15) may be equivalently stated as follows: find $(\mathbf{u}_{\mathrm{S}}^h, p_{\mathrm{S}}^h) \in \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}}) \times L^h(\Omega_{\mathrm{S}})$ such that

$$\begin{aligned} a_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}^h, \mathbf{v}_h) + c_h(\mathbf{u}_{\mathrm{S}}^h, \mathbf{v}_h) - b_{\mathrm{S}}(\mathbf{v}_h, p_{\mathrm{S}}^h) &= \ell_h(\mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}}), \\ b_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}^h, q_h) &= 0 && \forall q_h \in L^h(\Omega_{\mathrm{S}}), \end{aligned} \tag{3.11}$$

where

$$c_h(\mathbf{u}_h, \mathbf{v}_h) := \langle \mathrm{FtP}_h(R^h(\mathbf{u}_h \cdot \mathbf{n})), R^h(\mathbf{v}_h \cdot \mathbf{n}) \rangle_\Sigma$$

and

$$\ell_h(\mathbf{v}_h) := (\mathbf{f}_{\mathrm{S}}, \mathbf{v}_h)_{\Omega_{\mathrm{S}}} - \langle p_h^f, R^h(\mathbf{v}_h \cdot \mathbf{n}) \rangle_\Sigma.$$

Inn the conforming case ($\Phi_{\mathrm{S}}^h \subset \Phi_{\mathrm{D}}^h$), the $L^2(\Sigma)$-projection operator R^h does not play any role in the formulation.

4 The decoupled iterative method

We introduce the self-adjoint operators A_{S}^h and C_{S}^h defined from $\mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})$ to its dual $\mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})^*$ by

$$\langle A_{\mathrm{S}}^h \mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})^* \times \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})} = a_{\mathrm{S}}(\mathbf{u}, \mathbf{v}) \quad \text{and} \quad \langle C_{\mathrm{S}}^h \mathbf{u}, \mathbf{v} \rangle_{\mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})^* \times \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})} = c_h(\mathbf{u}, \mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})$. Let also $B_{\mathrm{S}}^h : \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}}) \to L^h(\Omega_{\mathrm{S}})^*$ be the operator defined by

$$\langle B_{\mathrm{S}}^h \mathbf{u}, q \rangle_{L^h(\Omega_{\mathrm{S}})^* \times L^h(\Omega_{\mathrm{S}})} = -b_{\mathrm{S}}(\mathbf{u}, q)$$

for all $\mathbf{u} \in \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})$ and $q \in L^h(\Omega_{\mathrm{S}})$.

Problem (3.11) can be written in operator form as follows:

$$\mathcal{A}_{\mathrm{S}}^h \begin{pmatrix} \mathbf{u}_{\mathrm{S}}^h \\ p_{\mathrm{S}}^h \end{pmatrix} = \begin{pmatrix} \ell_h \\ 0 \end{pmatrix} \tag{4.1}$$

where

$$\mathcal{A}_{\mathrm{S}}^h := \begin{pmatrix} A_{\mathrm{S}}^h + C_{\mathrm{S}}^h & (B_{\mathrm{S}}^h)^{\mathtt{t}} \\ B_{\mathrm{S}}^h & \mathbf{0} \end{pmatrix} : \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}}) \times L^h(\Omega_{\mathrm{S}}) \to \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})^* \times L^h(\Omega_{\mathrm{S}})^*$$

and $(B_{\mathrm{S}}^h)^{\mathtt{t}}$ is the adjoint of B_{S}^h. We know from Theorem 2.1 that both $\|\mathcal{A}_{\mathrm{S}}^h\|$ and $\|(\mathcal{A}_{\mathrm{S}}^h)^{-1}\|$ are uniformly bounded in h. If we denote by $I_{\mathrm{S}}^h : L^h(\Omega_{\mathrm{S}}) \to L^h(\Omega_{\mathrm{S}})^*$ the Riesz operator defined by

$$\langle I_{\mathrm{S}}^h p, q \rangle_{L^h(\Omega_{\mathrm{S}})^* \times L^h(\Omega_{\mathrm{S}})} = (p, q)_{\Omega_{\mathrm{S}}} \quad \forall p, q \in L^h(\Omega_{\mathrm{S}}),$$

then, the positive-definite self-adjoint operator

$$\mathcal{P}_{\mathrm{S}}^h := \begin{pmatrix} A_{\mathrm{S}}^h & 0 \\ 0 & I_{\mathrm{S}}^h \end{pmatrix} : \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}}) \times L^h(\Omega_{\mathrm{S}}) \to \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})^* \times L^h(\Omega_{\mathrm{S}})^*$$

and its inverse are uniformly bounded uniformly in h. It follows that the condition number of $(\mathcal{P}_{\mathrm{S}}^h)^{-1}\mathcal{A}_{\mathrm{S}}^h$ is bounded from above by a constant independent of the mesh parameter h. Consequently, the MINRES algorithm preconditioned with $(\mathcal{P}_{\mathrm{S}}^h)^{-1}$ solves (4.1) with a reduction of the norm of the residual that is independent of the mesh size h.

Let us now discuss how action of C_{S}^h on a given $\mathbf{u}_{\mathrm{S}}^h \in \mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})$. To this end we introduce the self-adjoint operators A_{D}^h and D_{D}^h defined from $\mathbf{H}_0^h(\Omega_{\mathrm{D}})$ to its dual $\mathbf{H}_0^h(\Omega_{\mathrm{D}})^*$ by

$$\langle A_{\mathrm{D}}^h \mathbf{u}, \mathbf{v} \rangle_{\mathbf{H}_{\mathrm{D}}^h(\Omega_{\mathrm{D}})^* \times \mathbf{H}_{\mathrm{D}}^h(\Omega_{\mathrm{D}})} = a_{\mathrm{D}}(\mathbf{u}, \mathbf{v}) \quad \text{and} \quad \langle D_{\mathrm{D}}^h \mathbf{u}, \mathbf{v} \rangle_{\mathbf{H}_{\mathrm{D}}^h(\Omega_{\mathrm{D}})^* \times \mathbf{H}_{\mathrm{D}}^h(\Omega_{\mathrm{D}})} = (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_{\Omega_{\mathrm{D}}}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^h(\Omega_{\mathrm{D}})$. Let us also consider $B_{\mathrm{D}}^h : \mathbf{H}_0^h(\Omega_{\mathrm{D}}) \to L_0^h(\Omega_{\mathrm{D}})^*$ given by

$$\langle B_{\mathrm{D}}^h \mathbf{u}, q \rangle_{L_0^h(\Omega_{\mathrm{D}})^* \times L_0^h(\Omega_{\mathrm{D}})} = -b_{\mathrm{D}}(\mathbf{u}, q)$$

for all $\mathbf{u} \in \mathbf{H}_0^h(\Omega_{\mathrm{D}})$ and $q \in L_0^h(\Omega_{\mathrm{D}})$.

We compute $C_{\mathrm{S}}^h \mathbf{u}_{\mathrm{S}}^h$ through (3.6) after solving problem (3.7) with $\phi_h = R_h(\mathbf{u}_{\mathrm{S}}^h \cdot \mathbf{n})$. This is to say that we have to deal with a saddle point problem of the form

$$\mathcal{A}_{\mathrm{D}}^h \begin{pmatrix} \mathbf{u}_h^\phi \\ p_h^\phi \end{pmatrix} = \begin{pmatrix} \mathcal{F}_h \\ \mathcal{G}_h \end{pmatrix} \tag{4.2}$$

where

$$\mathcal{A}_{\mathrm{D}}^h := \begin{pmatrix} A_{\mathrm{D}}^h & (B_{\mathrm{D}}^h)^{\mathtt{t}} \\ B_{\mathrm{D}}^h & \mathbf{0} \end{pmatrix} : \mathbf{H}_0^h(\Omega_{\mathrm{D}}) \times L_0^h(\Omega_{\mathrm{D}}) \to \mathbf{H}_0^h(\Omega_{\mathrm{D}})^* \times L_0^h(\Omega_{\mathrm{D}})^*$$

and $(B_{\mathrm{D}}^h)^{\mathtt{t}}$ is the adjoint of B_{D}^h.

The stability of the pair of spaces $(\mathbf{H}_0^h(\Omega_{\mathrm{D}}), L_0^h(\Omega_{\mathrm{S}}))$ (2.17)-(2.18) ensures that both $\|\mathcal{A}_{\mathrm{D}}^h\|$ and $\|(\mathcal{A}_{\mathrm{D}}^h)^{-1}\|$ are uniformly bounded in h. If we denote by $I_{\mathrm{D}}^h : L_0^h(\Omega_{\mathrm{D}}) \to L_0^h(\Omega_{\mathrm{D}})^*$ the Riesz operator given by

$$\langle I_{\mathrm{D}}^h p, q\rangle_{L_0^h(\Omega_{\mathrm{D}})^* \times L_0^h(\Omega_{\mathrm{D}})} = (p, q)_{\Omega_{\mathrm{D}}} \quad \forall p, q \in L_0^h(\Omega_{\mathrm{D}}),$$

it is clear that the block diagonal positive-definite self-adjoint operator

$$\mathcal{P}_{\mathrm{D}}^h := \begin{pmatrix} A_{\mathrm{D}}^h + D_{\mathrm{D}}^h & 0 \\ 0 & I_{\mathrm{D}}^h \end{pmatrix} : \mathbf{H}_0^h(\Omega_{\mathrm{D}}) \times L_0^h(\Omega_{\mathrm{D}}) \to \mathbf{H}_0^h(\Omega_{\mathrm{D}})^* \times L_0^h(\Omega_{\mathrm{D}})^*$$

and its inverse are also uniformly bounded in h. It follows that we can find an inclusion set for the eigenvalues of $(\mathcal{P}_{\mathrm{D}}^h)^{-1}\mathcal{A}_{\mathrm{D}}^h$ that is independent of h. This means that the MINRES method preconditioned with $(\mathcal{P}_{\mathrm{D}}^h)^{-1}$ yields the solution of (4.2) in a number of iterations independent on the mesh size h.

Summing up, the decoupled iterative method we are proposing here to solve (2.15) consists in two nested MINRES algorithms. Computationally, the actions of the preconditioners correspond to solving two decoupled local problems. The first one is defined by the bilinear form $a_{\mathrm{S}}(\cdot,\cdot)$ in $\mathbf{V}_{\mathrm{S}}^h(\Omega_{\mathrm{S}})$ and corresponds to the block A_{S}^h. Actually, A_{S}^h is associated with the operator $-2\nu\,\mathbf{div}(\boldsymbol{\varepsilon}(\cdot))$. Therefore, the local problem in the fluid amounts to a vector Laplace equation with a Dirichlet boundary condition on Γ_{S}, a Neumann condition in the normal direction and the slip boundary condition in the tangential direction on Σ. The other local problem is defined by the bilinear form

$$(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}}, \mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}} + (\operatorname{div}\mathbf{u}_{\mathrm{D}}, \operatorname{div}\mathbf{v}_{\mathrm{D}})_{\Omega_{\mathrm{D}}}$$

on $\mathbf{H}_0^h(\Omega_{\mathrm{D}})$ corresponding to the diagonal block $A_{\mathrm{D}}^h + D_{\mathrm{D}}^h$.

For the construction of practical preconditioners for discrete systems, the computational cost of evaluating these operators and the memory requirements of these procedures are key factors. The exact inverses appearing in the canonical preconditioners should be replaced by proper cost effective, and norm equivalent operators. Let us consider self-adjoint and positive-definite operators P_{S}^h and P_{D}^h that are spectrally equivalent to A_{S}^h and $A_{\mathrm{D}}^h + D_{\mathrm{D}}^h$ respectively, i.e.,

$$\langle A_{\mathrm{S}}^h \mathbf{u}_{\mathrm{S}},\, \mathbf{u}_{\mathrm{S}}\rangle_{\mathbf{H}_{\mathrm{S}}(\Omega_{\mathrm{S}})^* \times \mathbf{H}_{\mathrm{S}}(\Omega_{\mathrm{S}})} \simeq \langle P_{\mathrm{S}}^h \mathbf{u}_{\mathrm{S}},\, \mathbf{u}_{\mathrm{S}}\rangle_{\mathbf{H}_{\mathrm{S}}(\Omega_{\mathrm{S}})^* \times \mathbf{H}_{\mathrm{S}}(\Omega_{\mathrm{S}})}$$

and

$$\langle (A_{\mathrm{D}}^h + D_{\mathrm{D}}^h)\mathbf{u}_{\mathrm{D}},\, \mathbf{u}_{\mathrm{D}}\rangle_{\mathbf{H}_0(\Omega_{\mathrm{D}})^* \times \mathbf{H}_0(\Omega_{\mathrm{D}})} \simeq \langle P_{\mathrm{D}}^h \mathbf{u}_{\mathrm{D}},\, \mathbf{u}_{\mathrm{D}}\rangle_{\mathbf{H}_0(\Omega_{\mathrm{D}})^* \times \mathbf{H}_0(\Omega_{\mathrm{D}})}.$$

Then, instead of $(\mathcal{P}_{\mathrm{S}}^h)^{-1}$ and $(\mathcal{P}_{\mathrm{D}}^h)^{-1}$, we can use respectively the preconditioners

$$\begin{pmatrix} (P_{\mathrm{S}}^h)^{-1} & 0 \\ 0 & (I_{\mathrm{S}}^h)^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (P_{\mathrm{D}}^h)^{-1} & 0 \\ 0 & (I_{\mathrm{D}}^h)^{-1} \end{pmatrix}$$

and still have an optimal decoupled iterative method for problem (2.15). Ideally, we would have the actions of $(P^h_{\mathrm{S}})^{-1}$ and $(P^h_{\mathrm{D}})^{-1}$ cost about the same as the actions of A^h_{S} and $A^h_{\mathrm{D}} + D^h_{\mathrm{D}}$. As A^h_{S} corresponds to a second-order elliptic operator in $\mathbf{H}^1_{\mathrm{S}}(\Omega_{\mathrm{S}})$, we can easily take advantage of multigrid techniques or domain decomposition methods to find a good candidate for $(P^h_{\mathrm{S}})^{-1}$. The construction of a preconditioner $(P^h_{\mathrm{D}})^{-1}$ is less obvious.

4.1 Nodal auxiliary space preconditioning in $\mathbf{H}(\mathrm{div})$

In this section we describe the construction of the nodal auxiliary space preconditioning of Hiptmair and Xu [20] for elliptic problems in $\mathbf{H}_0(\mathrm{div}, \Omega_{\mathrm{D}})$. This is our choice here for the matrix version $(\mathbf{P}^h_{\mathrm{D}})^{-1}$ of the preconditioner $(P^h_{\mathrm{D}})^{-1}$ needed in the last section. To fix the ideas, we assume that the tensor $\mathbf{K}$ is given by $\tau^{-1}\mathbf{I}$ where $\mathbf{I}$ is the identity in $\mathbb{R}^d$ and τ is a given positive constant. In our numerical experiments, $\mathbf{H}^h_0(\Omega_{\mathrm{D}})$ is derived from the $\mathrm{RT}(k-1)$ or $\mathrm{BDM}(k)$ mixed finite elements with $k = 1$ or 2. Let $\{\boldsymbol{\phi}_i; \quad i = 1, \ldots, I\}$ be the usual basis of $\mathbf{H}^h_0(\Omega_{\mathrm{D}})$, then the matrix realizations of A^h_{D} and D^h_{D} are given by

$$\mathbf{A}^h_{\mathrm{D}} := (\tau(\boldsymbol{\phi}_i, \boldsymbol{\phi}_i)_{\Omega_{\mathrm{D}}})_{1\le i,j\le I}$$

and

$$\mathbf{D}^h_{\mathrm{D}} := ((\mathrm{div}\, \boldsymbol{\phi}_i, \mathrm{div}\, \boldsymbol{\phi}_i)_{\Omega_{\mathrm{D}}})_{1\le i,j\le I}$$

respectively. Our aim is to provide a matrix $\mathbf{P}^h_{\mathrm{D}} \in \mathbb{R}^{I\times I}$ that is spectrally equivalent to $\mathbf{A}^h_{\mathrm{D}} + \mathbf{D}^h_{\mathrm{D}}$ and such that the action $(\mathbf{P}^h_{\mathrm{D}})^{-1}$ on a given vector is easier to compute then that of $(\mathbf{A}^h_{\mathrm{D}} + \mathbf{D}^h_{\mathrm{D}})^{-1}$.

We denote by $[V^h_0(\Omega_{\mathrm{D}})]^d \subset \mathbf{H}^1_0(\Omega_{\mathrm{D}})$ the standard space of piecewise $\mathbb{P}_k$ and continuous vector fields and consider its usual nodal basis $\{\boldsymbol{\varphi}_\ell,\ \ell = 1, \cdots, Ld\}$, where L is the dimension of $V^h_0(\Omega_{\mathrm{D}})$. We introduce the matrix $\mathbf{L}_h$ given by

$$(\mathbf{L}_h)_{\ell,k} = (\nabla\boldsymbol{\varphi}_\ell, \nabla\boldsymbol{\varphi}_m)_{\Omega_{\mathrm{D}}} + \tau(\boldsymbol{\varphi}_\ell, \boldsymbol{\varphi}_m)_{\Omega_{\mathrm{D}}}, \quad 1 \le \ell, m \le Ld.$$

In the three dimensional case ($d = 3$), we also need to consider the Nédélec space $\mathbf{W}^h_0(\Omega_{\mathrm{D}}) \subset \mathbf{H}_0(\mathbf{curl}, \Omega_{\mathrm{D}})$ of order k. We denote its usual basis $\{\boldsymbol{\sigma}_i,\ i = 1, \ldots, N\}$. We introduce the diagonal matrix $\mathbf{S}^{\mathrm{curl}}_h$ given by

$$(\mathbf{S}^{\mathrm{curl}}_h)_{i,i} := (\mathbf{curl}\,\boldsymbol{\sigma}_i, \mathbf{curl}\,\boldsymbol{\sigma}_i)_{\Omega_{\mathrm{D}}} \quad i = 1, \cdots, N$$

and denote the diagonal of $\mathbf{A}^h_{\mathrm{D}} + \mathbf{D}^h_{\mathrm{D}}$ by $\mathbf{S}^{\mathrm{div}}_h$.

In the three-dimensional case, we denote by $\mathbf{C}_h \in \mathbb{R}^{N\times I}$ the matrix that represents $\mathbf{curl} :\ \mathbf{W}^h_0(\Omega_{\mathrm{D}}) \to \mathbf{H}^h_0(\Omega_{\mathrm{D}})$ in the following sense,

$$\mathbf{curl}\boldsymbol{\sigma}_i = \sum_{j=1}^{I} (\mathbf{C}_h)_{i,j}\boldsymbol{\phi}_j, \quad \forall i = 1, \cdots, N.$$

In the the two-dimensional case, the matrix $\mathbf{C}_h \in \mathbb{R}^{L\times I}$ is defined similarly with respect to the operator $\mathbf{curl} :\ V^h_0(\Omega_{\mathrm{D}}) \to \mathbf{H}^h_0(\Omega_{\mathrm{D}})$ defined by $\mathbf{curl} v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}$.

We use Π_h^{curl} and Π_h^{div} to denote the canonical interpolation operators onto the finite element spaces $\mathbf{W}_0^h(\Omega_{\mathrm{D}})$ and $\mathbf{H}_0^h(\Omega_{\mathrm{D}})$ respectively. The mappings $\Pi_h^{\mathrm{div}} : [V_0^h(\Omega_{\mathrm{D}})]^d \to \mathbf{H}_0^h(\Omega_{\mathrm{D}})$ and $\Pi_h^{\mathrm{curl}} : [V_0^h(\Omega_{\mathrm{D}})]^3 \to \mathbf{W}_0^h(\Omega_{\mathrm{D}})$ will be described by the matrices $\mathbf{I}_h^{\mathrm{div}} \in \mathbb{R}^{dL\times I}$ $(d = 2, 3)$ and $\mathbf{I}_h^{\mathrm{curl}} \in \mathbb{R}^{3L\times N}$ defined by

$$\Pi_h^{\mathrm{div}}\,\boldsymbol{\varphi}_\ell = \sum_{j=1}^{I} (\mathbf{I}_h^{\mathrm{div}})_{\ell,j}\boldsymbol{\phi}_j, \quad \forall \ell = 1, \cdots, Ld$$

and

$$\Pi_h^{\mathrm{curl}}\,\boldsymbol{\varphi}_\ell = \sum_{j=1}^{N} (\mathbf{I}_h^{\mathrm{curl}})_{\ell,j}\boldsymbol{\sigma}_j, \quad \forall \ell = 1, \cdots, 3L,$$

respectively.

The 3d-$\mathbf{H}$(div) auxiliary space preconditioner of Hiptmair and Xu consists in

$$(\mathbf{P}_{\mathrm{D}}^h)^{-1} := (\mathbf{S}_h^{\mathrm{div}})^{-1} + \mathbf{I}_h^{\mathrm{div}}(\mathbf{L}_h)^{-1}(\mathbf{I}_h^{\mathrm{div}})^{\mathtt{t}} + \tau^{-1}\mathbf{C}_h\Big((\mathbf{S}_h^{\mathrm{curl}})^{-1} + \mathbf{I}_h^{\mathrm{curl}}(\mathbf{L}_h)^{-1}(\mathbf{I}_h^{\mathrm{curl}})^{\mathtt{t}}\Big)\mathbf{C}_h^{\mathtt{t}}$$

and the 2-d version of this preconditioner is defined by

$$(\mathbf{P}_{\mathrm{D}}^h)^{-1} := (\mathbf{S}_h^{\mathrm{div}})^{-1} + \mathbf{I}_h^{\mathrm{div}}(\mathbf{L}_h)^{-1}(\mathbf{I}_h^{\mathrm{div}})^{\mathtt{t}} + \tau^{-1}\mathbf{C}_h\,(-\Delta_h)^{-1}\,\mathbf{C}_h^{\mathtt{t}}$$

where the matrix Δ_h stands for the discrete Laplacian on the finite element space $V_0^h(\Omega_{\mathrm{D}})$.

Notice that the transfer matrices $\mathbf{C}_h$, $\mathbf{I}_h^{\mathrm{div}}$ and $\mathbf{I}_h^{\mathrm{curl}}$ corresponding to the **curl** operator and the interpolations are sparse matrices that can be computed in a straightforward manner. The evaluation the preconditioner is then essentially reduced to several second-order elliptic operators. Hence, standard multigrid techniques domain decomposition methods for H^1 equations can be applied.

5 Numerical results

This section is devoted to the description of numerical experiments validating the effectiveness of the decoupled iterative method. We will show results for two dimensional problems, considering three examples of pairs of stable elements for the Darcy-Stokes problem. The first two examples correspond to the conforming Galerkin schemes based on the combination of the MINI and $\mathbb{P}_2$-iso-$\mathbb{P}_1$ elements for the Stokes problem with the lowest order Brezzi-Douglas-Marini element BDM(1). The third one is the nonconforming scheme resulting from the Taylor-Hood element and the second order Raviart-Thomas element RT(1).

5.1 Convergence rates

We begin by introducing some notation. The variable DOF stands for the total number of degrees of freedom defining the finite element subspaces $\mathbb{X}^h$ and $\mathbb{Q}^h$, and the individual errors are denoted by:

$$\mathrm{e}(\mathbf{u}_{\mathrm{D}}) := \|\mathbf{u}_{\mathrm{D}} - \mathbf{u}_{\mathrm{D}}^h\|_{\mathbf{H}(\mathrm{div},\Omega_{\mathrm{D}})}, \qquad \mathrm{e}(\mathbf{u}_{\mathrm{S}}) := \|\mathbf{u}_{\mathrm{S}} - \mathbf{u}_{\mathrm{S}}^h\|_{\mathbf{H}^1(\Omega_{\mathrm{S}})},$$

and

$$\mathrm{e}(p_\mathrm{D}) := \|p_\mathrm{D} - p_\mathrm{D}^h\|_{\Omega_\mathrm{D}}, \qquad \mathrm{e}(p_\mathrm{S}) := \|p_\mathrm{S} - p_\mathrm{S}^h\|_{\Omega_\mathrm{S}},$$

where $\mathbf{u}_\mathrm{D}^h := \mathbf{u}_h|_{\Omega_\mathrm{D}}$, $\mathbf{u}_\mathrm{S}^h := \mathbf{u}_h|_{\Omega_\mathrm{D}}$, $p_\mathrm{D}^h := p_h|_{\Omega_\mathrm{D}}$ and $p_\mathrm{S}^h := p_h|_{\Omega_\mathrm{S}}$ with $(\mathbf{u}_h, p_h) \in \mathbb{X}^h \times \mathbb{Q}^h$ being the solution of (2.15). We also let $\mathtt{r}(\mathbf{u}_\mathrm{D})$, $\mathtt{r}(\mathbf{u}_\mathrm{S})$, $\mathtt{r}(p_\mathrm{D})$ and $\mathtt{r}(p_\mathrm{S})$ be the experimental rates of convergence given by

$$\mathtt{r}(\mathbf{u}_\mathrm{D}) := \frac{\log(\mathrm{e}(\mathbf{u}_\mathrm{D})/\mathrm{e}'(\mathbf{u}_\mathrm{D}))}{\log(h/h')}, \qquad \mathtt{r}(\mathbf{u}_\mathrm{S}) := \frac{\log(\mathrm{e}(\mathbf{u}_\mathrm{S})/\mathrm{e}'(\mathbf{u}_\mathrm{S}))}{\log(h/h')},$$

and

$$\mathtt{r}(p_\mathrm{D}) := \frac{\log(\mathrm{e}(p_\mathrm{D})/\mathrm{e}'(p_\mathrm{D}))}{\log(h/h')}, \qquad \mathtt{r}(p_\mathrm{S}) := \frac{\log(\mathrm{e}(p_\mathrm{S})/\mathrm{e}'(p_\mathrm{S}))}{\log(h/h')},$$

where h and h' are two consecutive mesh sizes with errors e and e$'$.

We now describe the data of the example. We consider the domains $\Omega_\mathrm{D} := (0,1) \times (0,1/2)$ and $\Omega_\mathrm{S} := (0,1) \times (1/2,1)$. We take $\nu = 1$, $\kappa = 1$ and $\mathbf{K} = \mathbf{I}$, the identity of $\mathbb{R}^{2\times 2}$. The right hand side functions are selected in the model in such a way that the exact solution is given by:

$$p_\mathrm{D}(\boldsymbol{x}) := 6\pi \left(\frac{x_2}{2} - \frac{1}{4\pi}\sin(2\pi x_2)\right) \sin^2(2\pi x_1)\cos(2\pi x_1),$$

in the porous media and by

$$\mathbf{u}_\mathrm{S}(\boldsymbol{x}) := 2\pi \begin{pmatrix} \sin(\pi x_2)\cos(\pi x_2)\sin^3(2\pi x_1) \\ -3\sin^2(2\pi x_1)\cos(2\pi x_1)\sin^2(\pi x_2) \end{pmatrix}$$

and

$$p_\mathrm{S}(\boldsymbol{x}) := -\frac{\pi}{4}\cos\left(\frac{\pi}{2}x_1\right)\left(x_2 + 0.5 - 2\cos^2\left(\frac{\pi}{2}(x_2 + 0.5)\right)\right)$$

in the Stokes domain.

DOF	h	$\mathrm{e}(\mathbf{u}_\mathrm{S})$	$\mathtt{r}(\mathbf{u}_\mathrm{S})$	$\mathrm{e}(p_\mathrm{S})$	$\mathtt{r}(p_\mathrm{S})$	$\mathrm{e}(\mathbf{u}_\mathrm{D})$	$\mathtt{r}(\mathbf{u}_\mathrm{D})$	$\mathrm{e}(p_\mathrm{D})$	$\mathtt{r}(p_\mathrm{D})$
543	1/8	1.86E+01	–	9.26E−00	–	4.73E+01	–	1.60E−01	–
2043	1/16	1.01E+01	0.87	3.04E−00	1.60	2.48E+01	0.92	8.10E−02	0.98
7923	1/32	5.17E−00	0.97	8.80E−01	1.79	1.26E+01	0.98	3.99E−02	1.02
31203	1/64	2.59E−00	0.99	2.49E−01	1.82	6.31E−00	0.99	1.98E−02	1.00
123843	1/128	1.29E−00	0.99	7.56E−02	1.72	3.16E−00	0.99	9.92E−03	1.00

Table 1: Convergence rates: MINI–BDM(1)

We begin by providing a numerical exploration of the asymptotic convergence rates of the three examples. In Tables 1, 2 and 3, we summarize the convergence history of the Galerkin scheme (2.15) for a sequence of nested uniform meshes of the computational domain $\Omega := (0,1)^2$ by means of triangles. All the results are obtained by applying our decoupled preconditioning technique. In each case we display the numerical rates of convergence versus the degrees of freedom *DOF*. We observe that, as expected, in the case of the MINI–BDM(1) and the $\mathbb{P}_2$-iso-$\mathbb{P}_1$–BDM(1) couplings, the convergence is linear for

DOF	h	$\mathrm{e}(\mathbf{u}_\mathrm{S})$	$\mathtt{r}(\mathbf{u}_\mathrm{S})$	$\mathrm{e}(p_\mathrm{S})$	$\mathtt{r}(p_\mathrm{S})$	$\mathrm{e}(\mathbf{u}_\mathrm{D})$	$\mathtt{r}(\mathbf{u}_\mathrm{D})$	$\mathrm{e}(p_\mathrm{D})$	$\mathtt{r}(p_\mathrm{D})$
385	1/8	1.86E+01	–	4.10E−00	–	4.73E+01	–	1.60E−01	–
1423	1/16	1.01E+01	0.87	2.14E−00	0.94	2.48E+01	0.92	8.11E−02	0.98
5467	1/32	5.17E−00	0.97	6.03E−01	1.82	1.26E+01	0.98	3.99E−02	1.02
21427	1/64	2.59E−00	0.99	1.57E−01	1.94	6.32E−00	0.99	1.98E−02	1.00
84835	1/128	1.30E−00	0.99	4.25E−02	1.88	3.16E−00	0.99	9.92E−03	1.00

Table 2: Convergence rates: $\mathbb{P}2$-iso-$\mathbb{P}1$–BDM(1)

DOF	h	$\mathrm{e}(\mathbf{u}_\mathrm{S})$	$\mathtt{r}(\mathbf{u}_\mathrm{S})$	$\mathrm{e}(p_\mathrm{S})$	$\mathtt{r}(p_\mathrm{S})$	$\mathrm{e}(\mathbf{u}_\mathrm{D})$	$\mathtt{r}(\mathbf{u}_\mathrm{D})$	$\mathrm{e}(p_\mathrm{D})$	$\mathtt{r}(p_\mathrm{D})$
887	1/8	4.09E−00	–	1.08E−00	–	1.48E+01	–	5.23E−02	–
3371	1/16	9.56E−01	2.09	8.88E−02	3.60	4.03E−00	1.87	1.35E−02	1.95
13139	1/32	2.37E−01	2.01	7.06E−03	3.65	1.07E−00	1.90	3.40E−03	1.99
51875	1/64	5.93E−02	1.99	7.85E−04	3.17	2.94E−01	1.87	8.50E−04	2.00
206147	1/128	1.48E−02	1.99	1.98E−04	1.98	8.43E−02	1.80	2.12E−04	2.00

Table 3: Convergence rates: Taylor-Hood–RT(1)

the velocities in both the Stokes and the Darcy domains. The Taylor-Hood–RT(1) scheme provides a quadratic convergence for the Stokes and Darcy velocity unknowns. We fixed the tolerance parameter for the outer MINRES method at 10^{-6} and checked empirically that the largest inner MINRES tolerance parameter that provides a convergence in agreement with the rates predicted by the theory is 10^{-2}. All the results displayed here are obtained with this combination of tolerance parameters.

5.2 Performance of the iterative method

In the following, we will denote by $\mathbf{A}_\mathrm{S}^h$, $\mathbf{B}_\mathrm{S}^h$, $\mathbf{C}_\mathrm{S}^h$ and $\mathbf{M}_\mathrm{S}^h$ the matrix realizations of A_S^h, B_S^h, C_S^h, and I_S^h respectively. Similarly, $\mathbf{A}_\mathrm{D}^h$, $\mathbf{B}_\mathrm{D}^h$, $\mathbf{D}_\mathrm{D}^h$ and $\mathbf{M}_\mathrm{D}^h$ are the matrix realizations of A_D^h, B_D^h, D_D^h, and I_D^h respectively.

The numerical results were obtained using Matlab's own MINRES routine. For all experiments, the convergence is attained when the Euclidean norm of the relative residual is reduced by 10^{-6} for the outer MINRES while (as indicated above) the tolerance for the inner MINRES method is set to 10^{-2}. The outer MINRES is applied to a linear system of equations with matrix

$$\begin{pmatrix} \mathbf{A}_\mathrm{S}^h + \mathbf{C}_\mathrm{S}^h & (\mathbf{B}_\mathrm{S}^h)^\mathtt{t} \\ \mathbf{B}_\mathrm{S}^h & \mathbf{0} \end{pmatrix}.$$

It is initialized with the solution of the Stokes problem with a non slip boundary condition Γ_S and an homogeneous Neumann boundary condition on Σ. The MINRES algorithm is accelerated with one of the following preconditioners:

$$\mathcal{P}_\mathrm{S}^{\diagdown} := \begin{pmatrix} (\mathbf{A}_\mathrm{S}^h)^{-1}_{\diagdown} & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}_\mathrm{S}^h)^{-1} \end{pmatrix}, \qquad \mathcal{P}_\mathrm{S}^{\mathtt{BPX}} := \begin{pmatrix} (\mathbf{A}_\mathrm{S}^h)^{-1}_{\mathtt{BPX}} & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}_\mathrm{S}^h)^{-1} \end{pmatrix},$$

where the notation $(\mathbf{A}_\mathrm{S}^h)^{-1}_{\diagdown}$ means that the linear systems of equations with matrix $\mathbf{A}_\mathrm{S}^h$ are solved by a direct solver (with the Matlab backslash command) while $(\mathbf{A}_\mathrm{S}^h)^{-1}_{\mathtt{BPX}}$ means that we use the Bramble-Pasciak-Xu [6, 28] preconditioner corresponding to the SPD vectorial Laplace matrix $\mathbf{A}_{\mathrm{S}}^h$.

On the other hand, each application of $\mathbf{C}_{\mathrm{S}}^h$ to a vector requires the solution of a saddle point problem with matrix

$$\begin{pmatrix} \mathbf{A}_{\mathrm{D}}^h & (\mathbf{B}_{\mathrm{D}}^h)^{\mathtt{t}} \\ \mathbf{B}_{\mathrm{D}}^h & \mathbf{0} \end{pmatrix}.$$

We again accomplish this task applying the MINRES method preconditioned with one of the following symmetric and block diagonal matrices:

$$\mathcal{P}_{\mathrm{D}}^0 := \begin{pmatrix} (\mathbf{A}_{\mathrm{D}}^h + \mathbf{D}_{\mathrm{D}}^h)_{\diagdown}^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}_{\mathrm{D}}^h)^{-1} \end{pmatrix}, \quad \mathcal{P}_{\mathrm{D}}^{\diagdown} := \begin{pmatrix} (\mathbf{P}_{\mathrm{D}}^h)_{\diagdown}^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}_{\mathrm{D}}^h)^{-1} \end{pmatrix},$$

$$\mathcal{P}_{\mathrm{D}}^{\mathtt{BPX}} := \begin{pmatrix} (\mathbf{P}_{\mathrm{D}}^h)_{\mathtt{BPX}}^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}_{\mathrm{D}}^h)^{-1} \end{pmatrix}.$$

In the definition of the preconditioner $\mathcal{P}_{\mathrm{D}}^0$, $(\mathbf{A}_{\mathrm{D}}^h+\mathbf{D}_{\mathrm{D}}^h)_{\diagdown}^{-1}$ means that we simply use a direct solver for $\mathbf{A}_{\mathrm{D}}^h + \mathbf{D}_{\mathrm{D}}^h$ with the aid of the backslash Matlab command. The preconditioners $\mathcal{P}_{\mathrm{D}}^{\diagdown}$ and $\mathcal{P}_{\mathrm{D}}^{\mathtt{BPX}}$ are obtained by substituting $(\mathbf{A}_{\mathrm{D}}^h + \mathbf{D}_{\mathrm{D}}^h)^{-1}$ in $\mathcal{P}_{\mathrm{D}}^0$ by the Hiptmair and Xu preconditioner $(\mathbf{P}_{\mathrm{D}}^h)^{-1}$. The subscript $\diagdown$ in $(\mathbf{P}_{\mathrm{D}})_{\diagdown}^{-1}$ means that we solve the underlying Laplace problems with a direct solver, with the Matlab backslash command, and $(\mathbf{P}_{\mathrm{D}}^h)_{\mathtt{BPX}}^{-1}$ means that we use the well-known BPX-preconditioner [6, 28] for $(\mathbf{L}_h)^{-1}$ and $(-\Delta_h)^{-1}$.

In the cases where the mass matrix is diagonal the action of its inverse can be explicitly computed. In the other cases, one simple and effective strategy consists in substituting the action of the inverse of the mass matrix by one sweep of the symmetric Gauss-Seidel method.

DOF	h	$\mathcal{P}_{\mathrm{S}}^{\diagdown}(\mathcal{P}_{\mathrm{D}}^0)$	$\mathcal{P}_{\mathrm{S}}^{\diagdown}(\mathcal{P}_{\mathrm{D}}^{\diagdown})$	$\mathcal{P}_{\mathrm{S}}^{\diagdown}(\mathcal{P}_{\mathrm{D}}^{\mathtt{BPX}})$	$\mathcal{P}_{\mathrm{S}}^{\mathtt{BPX}}(\mathcal{P}_{\mathrm{D}}^0)$	$\mathcal{P}_{\mathrm{S}}^{\mathtt{BPX}}(\mathcal{P}_{\mathrm{D}}^{\diagdown})$	$\mathcal{P}_{\mathrm{S}}^{\mathtt{BPX}}(\mathcal{P}_{\mathrm{D}}^{\mathtt{BPX}})$
543	1/8	26(4)	26(26)	26(29)	56(4)	56(26)	56(29)
2043	1/16	32(4)	32(30)	32(46)	84(4)	84(30)	84(46)
7923	1/32	40(4)	40(33)	40(62)	121(4)	121(33)	121(62)
31203	1/64	46(4)	46(38)	46(77)	144(4)	144(38)	144(77)
123843	1/128	50(4)	50(42)	50(91)	158(4)	158(42)	158(91)

Table 4: Number of iterations: MINI–BDM(1)

DOF	h	$\mathcal{P}_{\mathrm{S}}^{\diagdown}(\mathcal{P}_{\mathrm{D}}^0)$	$\mathcal{P}_{\mathrm{S}}^{\diagdown}(\mathcal{P}_{\mathrm{D}}^{\diagdown})$	$\mathcal{P}_{\mathrm{S}}^{\diagdown}(\mathcal{P}_{\mathrm{D}}^{\mathtt{BPX}})$	$\mathcal{P}_{\mathrm{S}}^{\mathtt{BPX}}(\mathcal{P}_{\mathrm{D}}^0)$	$\mathcal{P}_{\mathrm{S}}^{\mathtt{BPX}}(\mathcal{P}_{\mathrm{D}}^{\diagdown})$	$\mathcal{P}_{\mathrm{S}}^{\mathtt{BPX}}(\mathcal{P}_{\mathrm{D}}^{\mathtt{BPX}})$
385	1/8	24(4)	24(26)	24(29)	50(4)	50(26)	50(29)
1423	1/16	30(4)	30(29)	30(46)	80(4)	80(29)	80(46)
5467	1/32	36(4)	36(34)	36(62)	107(4)	107(34)	107(62)
21427	1/64	42(4)	42(38)	42(76)	130(4)	130(38)	130(76)
84835	1/128	44(4)	44(41)	44(91)	146(4)	146(41)	146(91)

Table 5: Number of iterations: $\mathbb{P}2$-iso-$\mathbb{P}1$–BDM(1)

In tables 4, 5 and 6, we list the number of iterations of the two nested MINRES methods with different combinations of preconditioners. The preconditioner in brackets

DOF	h	$\mathcal{P}_{\mathrm{S}}^{\setminus}(\mathcal{P}_{\mathrm{D}}^{0})$	$\mathcal{P}_{\mathrm{S}}^{\setminus}(\mathcal{P}_{\mathrm{D}}^{\setminus})$
887	1/8	28(5)	28(28)
3371	1/16	34(5)	34(32)
13139	1/32	38(5)	38(36)
31203	1/64	42(5)	42(40)
206147	1/128	44(5)	44(44)

Table 6: Number of iterations: Taylor-Hood–RT(1)

is the one used for the inner MINRES. We show the number of outer MINRES iterations and the number in brackets is an average of the number of inner MINRES iterations. We observe that for different mesh sizes, the iterative method results in a uniform number of MINRES iterations. Therefore, the preconditioners are robust with respect to the mesh size, which agrees with the theoretical results.

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