# NONPERIODIC TRIGONOMETRIC POLYNOMIAL APPROXIMATION 

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#### Abstract

The suitable basis functions for approximating periodic function are periodic, trigonometric functions. When the function is not periodic, a viable alternative is to consider polynomials as basis functions. In this paper we will point out the inadequacy of polynomial approximation and suggest to switch from powers of $x$ to powers of $\sin (p x)$ where $p$ is a parameter which depends on the dimension of the approximating subspace. The new set does not suffer from the drawbacks of polynomial approximation and by using them one can approximate analytic functions with spectral accuracy. An important application of the new basis functions is related to numerical integration. A quadrature based on these functions results in higher accuracy compared to Legendre quadrature.


Key words. polynomial approximation, Fourier approximation, Chebyshev polynomials, Gaussian quadrature, spectral accuracy

AMS subject classifications. 41A05,41A10,41A25,42A15,65D05

1. Introduction . Let us consider a finite sum approximation of a function $f(x)$

$$
\begin{equation*}
f(x) \approx \sum_{k=0}^{n} a_{k} \psi_{k}(x), \quad x \in[-L, L] . \tag{1.1}
\end{equation*}
$$

If $f(x)$ and its derivatives are continuous and periodic then the suitable basis functions are

$$
\begin{equation*}
\psi_{k}(x)=e^{i\left(\frac{n}{2}-k\right) \frac{\pi x}{L}} \tag{1.2}
\end{equation*}
$$

and the error goes to zero exponentially fast. If the function is not periodic, a spectral rate of convergence can be achieved by using Chebyshev polynomials as basis functions. Approximating a function by finite sum of Chebyshev polynomials has drawbacks (see, for example [12]). Despite the spectral rate of convergence, Chebyshev polynomials have peculiar characteristics and therefore do not provide an optimal set of basis functions. This peculiarity lies in the behavior of their derivatives. While Chebyshev polynomials have uniform behavior in the interval, their derivatives are non-uniform. In order to clarify this point let us assume that the interval is $[-1,1]$. Chebyshev polynomials in this interval are defined as

$$
\begin{equation*}
T_{k}(x)=\cos (k \arccos (x)) \tag{1.3}
\end{equation*}
$$

We have

$$
\begin{equation*}
\max _{x \in[-1,1]}\left|T_{k}(x)\right|=1 \tag{1.4}
\end{equation*}
$$

and this maxima is achieved at $k-1$ points. Hence, $T_{k}(x)$ behaves uniformly. Let us consider now $\frac{d T_{k}}{d x}$. Upon defining

$$
\begin{equation*}
x=\cos (\theta) \tag{1.5}
\end{equation*}
$$

[^0]we get
\[

$$
\begin{equation*}
\frac{d T_{k}}{d x}=\frac{d T_{k}}{d \theta} \frac{d \theta}{d x}=\frac{k \sin (k \theta)}{\sin (\theta)} \tag{1.6}
\end{equation*}
$$

\]

Hence, in the vicinity of $x=0$ we have

$$
\begin{equation*}
\max \left|\frac{d T_{k}}{d x}\right| \approx k \tag{1.7}
\end{equation*}
$$

while at the end points we have

$$
\begin{equation*}
\left|\frac{d T_{k}}{d x}( \pm 1)\right|=k^{2} \tag{1.8}
\end{equation*}
$$

Thus, there is no uniformity in the extrema values of the first derivative. This discrepancy increases for each additional derivative by a power of $k$, namely

$$
\begin{equation*}
\frac{\left|T_{k}^{(j)}\left(x_{a}\right)\right|}{\left|T_{k}^{(j)}\left(x_{b}\right)\right|} \approx k^{j} \tag{1.9}
\end{equation*}
$$

where $x_{a}$ and $x_{b}$ are extrema points in the vicinity of $\pm 1$ and 0 respectively. Approximating an analytic function by Chebychev expansion is highly efficient when the approximated function exhibits a behavior similar to (1.9). As in the Fourier case, the efficiency deteriorates as the behavior of the function approximated moves away from this pattern. A reasonable target should be to look for an approximation space where the basis functions and all their derivatives exhibit uniform, or 'almost' uniform, behavior.

The peculiarity of Chebyshev approximation can be demonstrated also by considering interpolation. It is well known that polynomial interpolation in equally distributed points is not the right approach. The error function has large gradients at the boundaries and in some cases, divergence can occur(Runge phenomenon) 4]. Insisting on polynomial interpolation, one should resort to non-uniform distribution of points. A set of interpolating points which results in exponential rate of convergence (for analytic functions) is

$$
\begin{equation*}
x_{i}=\cos \left(\frac{i \pi}{n}\right) \quad i=0, \cdots, n \tag{1.10}
\end{equation*}
$$

We have

$$
\begin{gather*}
\left|\Delta x_{\max }\right|=\left|\cos \left(\frac{\pi}{2}-\frac{\pi}{n}\right)\right| \approx \frac{\pi}{n}  \tag{1.11}\\
\left|\Delta x_{\min }\right|=\left|1-\cos \left(\frac{\pi}{n}\right)\right| \approx \frac{1}{2}\left(\frac{\pi}{n}\right)^{2} \tag{1.12}
\end{gather*}
$$

The uneven distribution of points does not make sense in the general case. When the function approximated does not have large gradients at the boundaries there is no justification for concentrating points there. This inefficient strategy is more pronounced when the large gradients of the function are away from the boundaries. A way of quantifying the inefficiency of Chebychev interpolation is in the fact that we need $\pi$ points per wavelength for resolution, compared to 2 points per wavelength
which is "a law of nature" as expressed by Nyquist criteria. It seems reasonable to look for a set of basis functions where the relevant interpolating points are evenly, or almost evenly, distributed.

In this paper we present a new set of basis functions. The set is composed of nonperiodic trigonometric functions. It was first introduced in [9. The motivation in that case was to overcome the severe stability restriction which results from using Chebyshev polynomials for space discretization while solving time dependent pde's. Many researchers reported significant increases in efficiency while using the approximating method described in [9] (e.g. [1], [5, 7], [8, [11, [13]). In the present paper we would like to deal with these basis functions just from approximation viewpoint and to emphasize the advantages of using these functions as compared to polynomials. Approximating a general function by a linear combination of these basis functions overcomes the drawbacks mentioned above and, for analytic functions, the approximation is spectrally accurate. The paper is organized as follows. In Section 2 we describe the proposed approximation subspace. The basis functions depend on a parameter $p, 0<p<\frac{\pi}{2}$, and this parameter depends on $n$ ( the size of the subspace). In Section 3 we carry an analysis related to accuracy properties and in Section 4, analysis related to resolution properties. In Section 5 we use the new basis function for numerical integration. The paper is concluded in Section 6 in which we present numerical results.
2. nonperiodic Trigonometric Polynomial Subspace. Without loss of generality we will consider functions in $[-1,1]$. Let $P_{k}(y)$ be a set of polynomials, orthogonal with respect to the inner product

$$
\begin{equation*}
<f, g>=\int_{-1}^{1} f(y) g(y) w(y) d y \tag{2.1}
\end{equation*}
$$

By change of variables

$$
\begin{equation*}
y=\frac{\sin (p x)}{\sin (p)} \tag{2.2}
\end{equation*}
$$

where $p$ is a parameter in $\left(0, \frac{\pi}{2}\right)$, we get that

$$
\begin{equation*}
\psi_{k}(x)=P_{k}\left(\frac{\sin (p x)}{\sin (p)}\right) \tag{2.3}
\end{equation*}
$$

are orthogonal with respect to the inner product

$$
\begin{equation*}
<f, g>=\int_{-1}^{1} f(x) g(x) \bar{w}(x) d x \tag{2.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{w}(x)=w\left(\frac{\sin (p x)}{\sin (p)}\right) \cos (p x) . \tag{2.5}
\end{equation*}
$$

Hence, the proposed new approximating subspace is

$$
\begin{equation*}
S_{n}=\operatorname{span}\left\{\psi_{0}(x), \ldots, \psi_{n}(x)\right\} \tag{2.6}
\end{equation*}
$$

It is easily verified, by trigonometric identities, that $S_{n}$ can be written also as a subspace spanned by trigonometric polynomials. More precisely

$$
\begin{equation*}
S_{n}=\operatorname{span}\left\{Q_{0}, \ldots, Q_{n}\right\} \tag{2.7}
\end{equation*}
$$

where

$$
\begin{equation*}
Q_{k}(x)=\cos (k p x) \quad k=0,2,4, \ldots, \quad 0<p<\frac{\pi}{2} \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
Q_{k}(x)=\sin (k p x) \quad k=1,3,5, \ldots, \quad 0<p<\frac{\pi}{2} \tag{2.9}
\end{equation*}
$$

A popular set of orthogonal polynomials is the set of Jacobbi polynomials $P_{k}^{(\alpha, \beta)}(y)$. These polynomials are orthogonal under the inner product

$$
\begin{equation*}
<f, g>=\int_{-1}^{1} f(y) g(y) w(y) d y \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
w(y)=(1-y)^{\alpha}(1+y)^{\beta}, \quad-1<\alpha \text { and }-1<\beta . \tag{2.11}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\bar{w}(x)=\left(1-\frac{\sin (p x)}{\sin (p)}\right)^{\alpha}\left(1+\frac{\sin (p x)}{\sin (p)}\right)^{\beta} \cos (p x) \tag{2.12}
\end{equation*}
$$

Since

$$
\begin{equation*}
\lim _{p \rightarrow 0} \frac{\sin (p x)}{\sin (p)}=x \tag{2.13}
\end{equation*}
$$

Jacobbi polynomials can be considered as $\psi_{k}$ functions in the extremal case when $p=0$. As will be shown in the next section, in general, the parameter $p$ should be close to the other extremal point, namely $\frac{\pi}{2}$.

Two important members of the Jacobbi polynomials family are Chebyshev and Legendre polynomials. Let $T_{k}(x)$ be Chebyshev polynomial then

$$
\begin{equation*}
\psi_{k}(x)=T_{k}\left(\frac{\sin (p x)}{\sin (p)}\right) \tag{2.14}
\end{equation*}
$$

and the weight function is

$$
\begin{equation*}
\bar{w}(x)=\frac{\cos (p x)}{\sqrt{1-\frac{\sin ^{2}(p x)}{\sin ^{2}(p)}}} \tag{2.15}
\end{equation*}
$$

If $P_{k}(x)$ is Legendre polynomial then

$$
\begin{equation*}
\psi_{k}(x)=P_{k}\left(\frac{\sin (p x)}{\sin (p)}\right) \tag{2.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{w}(x)=\cos (p x) \tag{2.17}
\end{equation*}
$$

3. Approximating analytic functions by projection on $S_{n}$. Let $f(x)$ be a function continuous in $[-1,1]$. Orthogonal projection of $f$ on $S_{n}$ results in

$$
\begin{equation*}
f_{n}(x)=\sum_{k=0}^{n} a_{k} \psi_{k}(x) \tag{3.1}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{k}=\frac{<f, \psi_{k}>}{<\psi_{k}, \psi_{k}>} \tag{3.2}
\end{equation*}
$$

Since Chebyshev polynomials is the mostly used set of orthogonal polynomials, we will consider

$$
\begin{equation*}
\psi_{k}(x)=T_{k}\left(\frac{\sin (p x)}{\sin (p)}\right) \tag{3.3}
\end{equation*}
$$

in the rest of this section. Almost all the theoretical results described here are relevant to any set of orthogonal polynomials.

Since (2.10) then

$$
\begin{equation*}
<\psi_{k}, \psi_{k}>=\frac{\sin (p)}{p} \int_{-1}^{1} \frac{T_{k}^{2}(y)}{\sqrt{1-y^{2}}} d y=\beta_{k} \frac{\pi \sin (p)}{2 p} \tag{3.4}
\end{equation*}
$$

where $\beta_{0}=2$ and $\beta_{k}=1,1 \leq k$. Therefore

$$
\begin{equation*}
a_{k}=\frac{2}{\beta_{k} \pi} \int_{-1}^{1} \frac{\tilde{f}(y) T_{k}(y)}{\sqrt{1-y^{2}}} d y \tag{3.5}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{f}(y)=f(g(y ; p)) \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
g(y ; p)=\frac{1}{p} \sin ^{-1}(y \sin (p)) \tag{3.7}
\end{equation*}
$$

is the inverse of (2.2). Hence, approximating $f(x)$ by the new set of basis functions is equivalent to approximating $f(g(y ; p))$ by Chebychev polynomials. As a result, it is sufficient to deal with the latter. Observe that now, due to the singularity of $g(y ; p)$ at $y= \pm 1 / \sin (p)$, the function approximated has large gradient at the boundaries and it is justified to use Chebyshev polynomials. The relevant theory which discusses polynomial approximation of functions with singularities outside the domain of definition here follows [14.

Let $K$ be a bounded continuum in $C$ such that $K^{c}$, the complement $K$, is simply connected in the extended plane and contains the point at infinity. For such $K$ there exist a conformal mapping $\Psi(w)$ which maps the complement of the unit disc onto $K^{c}$ [14]. Let $\theta(y)$ be the inverse of $\Psi(w)$ and

$$
\begin{equation*}
B_{t}=\{y:|\theta(y)|=t\} \quad(t>1) \tag{3.8}
\end{equation*}
$$

denote the level curves in $K^{c}$ then we have the following theorem:
Theorem 2.1:Suppose $t>1$ is the largest number such that $F(y)$ is analytic inside $B_{t}$. The interpolating polynomials $P_{n}(y)$ with interpolating points $y_{i}^{n}$ that are uniformly distributed on $K$ then satisfy

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \max _{y \in K}\left|F(y)-P_{n}(y)\right|^{\frac{1}{n}}=\frac{1}{t} . \tag{3.9}
\end{equation*}
$$

Since approximating an analytic function by Chebyshev polynomials is equivalent to interpolating the function at uniformly distributed points (e.g. Chebyshev points) , the asymptotic rate of convergence can be computed by making use of this theorem. For $K=[-1,1]$, the relevant conformal mapping is 10

$$
\begin{equation*}
\theta(y)=y \pm \sqrt{y^{2}-1} \tag{3.10}
\end{equation*}
$$

$\tilde{f}(y)$ is singular at $y= \pm 1 / \sin (p)$ hence, the largest $t$ is

$$
\begin{equation*}
t=\frac{1}{\sin (p)}+\sqrt{\frac{1}{\sin ^{2}(p)}-1}=\frac{1+\cos (p)}{\sin (p)}=\cot \left(\frac{p}{2}\right) \tag{3.11}
\end{equation*}
$$

and the asymptotic rate of convergence is

$$
\begin{equation*}
\frac{1}{t}=\tan \left(\frac{p}{2}\right) \tag{3.12}
\end{equation*}
$$

Hence, approximating by the new set of basis functions, the asymptotic accuracy is $c \varepsilon$ where

$$
\begin{equation*}
\varepsilon=\left(\tan \left(\frac{p}{2}\right)\right)^{n} \tag{3.13}
\end{equation*}
$$

and $c$ is constant which depends on $\tilde{f}(y)$ but does not depend on $n$ or $y$.
By choosing

$$
\begin{equation*}
p=2 \arctan \left(\varepsilon^{\frac{1}{n}}\right) \tag{3.14}
\end{equation*}
$$

where $\varepsilon$ is the machine accuracy, we eliminate the error which results from the singular points $y= \pm 1 / \sin (p)$ and get spectral accuracy. Detailed mathematical proof of spectral accuracy is given in [3].

The choice of $p$ described above is independent of the function we are approximating. Obviously, one can do better by choosing the appropriate $p$ for each function. This can be achieved numerically by making use of a minimization algorithm which finds the parameter $p$ that minimizes the norm of the error vector defined as

$$
\begin{equation*}
E=\sum_{j=1}^{m}\left|\sum_{k=0}^{n} a_{k}(p) \psi_{k}\left(z_{j}\right)-f\left(z_{j}\right)\right| \tag{3.15}
\end{equation*}
$$

where $z_{j}, 1 \leq j \leq m$, are check points randomly distributed in the interval $[a, b]$.
Approximating $f(x)$ via interpolation, $a_{k}$ have to satisfy the following $n+1$ equations

$$
\begin{equation*}
\sum_{k=0}^{n} a_{k} \psi_{k}\left(x_{i}\right)=f\left(x_{i}\right) \quad 0 \leq i \leq n \tag{3.16}
\end{equation*}
$$

where $\left\{x_{i}\right\}_{i=0}^{n}$ is an appropriate set of interpolating points. Due to the equivalence mentioned above, a feasible set of interpolating points is

$$
\begin{equation*}
x_{i}=g\left(y_{i} ; p\right), \quad 0 \leq i \leq n \tag{3.17}
\end{equation*}
$$

where

$$
\begin{equation*}
y_{i}=\cos \left(\frac{i \pi}{n}\right) \tag{3.18}
\end{equation*}
$$

We would like to show now that, while in the Chebyshev case we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\Delta x_{\min }}{\Delta x_{\max }}=0 \tag{3.19}
\end{equation*}
$$

in the Nptp (nonperiodic trigonometric polynomial) case we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\Delta x_{\min }}{\Delta x_{\max }}=c, \quad c \neq 0 \tag{3.20}
\end{equation*}
$$

Let us address the general case

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\Delta x_{i}}{\Delta x_{\max }} \tag{3.21}
\end{equation*}
$$

We have

$$
\begin{equation*}
\frac{\Delta x_{i}}{\Delta x_{\max }}=\frac{\sin ^{-1}\left(\sin (p) y_{i+1}\right)-\sin ^{-1}\left(\sin (p) y_{i}\right)}{0-\sin ^{-1}\left(\sin (p) y_{\frac{n}{2}-1}\right)} \tag{3.22}
\end{equation*}
$$

Upon defining

$$
\begin{equation*}
\theta=\frac{\pi}{n} \tag{3.23}
\end{equation*}
$$

we have

$$
\begin{equation*}
\frac{\Delta x_{i}}{\Delta x_{\max }}=\frac{\sin ^{-1}(\sin (p) \cos (i \theta))-\sin ^{-1}(\sin (p) \cos ((i+1) \theta))}{\sin ^{-1}(\sin (p) \sin (\theta))} \tag{3.24}
\end{equation*}
$$

Computing (3.21) via l'Hopital's rule, we have to compute the derivatives of the numerator and denominator. (in what follows we will compute $\lim _{\theta \rightarrow 0}$ instead of $\left.\lim _{n \rightarrow \infty}\right)$.

We have

$$
\begin{equation*}
\frac{d}{d \theta} \sin ^{-1}(\sin (p) \cos (i \theta))=\frac{-i \sin (p) \sin (i \theta)+\cos (i \theta) \cos (p) \frac{d p}{d \theta}}{\sqrt{1-\sin ^{2}(p) \cos ^{2}(i \theta)}} \tag{3.25}
\end{equation*}
$$

One can write the r.h.s of the equation above as $h_{1}(\theta) h_{2}(\theta)$ where

$$
\begin{equation*}
h_{1}(\theta)=\frac{\sin (i \theta)}{\sqrt{1-\sin (p) \cos (i \theta)} \sqrt{1+\sin (p) \cos (i \theta)}} \tag{3.26}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{2}(\theta)=-i \sin (p)+\frac{\cos (p)}{\sin (i \theta)} \cos (i \theta) \frac{d p}{d \theta} . \tag{3.27}
\end{equation*}
$$

Since (3.14) we have

$$
\begin{equation*}
\frac{d p}{d \theta}=2 \frac{1}{1+\varepsilon^{\frac{2 \theta}{\pi}}} \varepsilon^{\frac{\theta}{\pi}} \ln (\varepsilon) \frac{1}{\pi} \tag{3.28}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \frac{d p}{d \theta}=\mu \tag{3.29}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu=\frac{\ln (\varepsilon)}{\pi} \tag{3.30}
\end{equation*}
$$

Using l'Hopital's rule we get

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} h_{1}(\theta)=\frac{i}{\sqrt{i^{2}+\mu^{2}}} \tag{3.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} h_{2}(\theta)=-\frac{i^{2}+\mu^{2}}{i} \tag{3.32}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \frac{d}{d \theta} \sin ^{-1}\left(\sin (p) y_{i}\right)=-\sqrt{i^{2}+\mu^{2}} . \tag{3.33}
\end{equation*}
$$

As to the denominator, using l'Hopital's rule again results in

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \frac{d}{d \theta}\left(\sin ^{-1}(\sin (p) \sin (\theta))\right)=1 \tag{3.34}
\end{equation*}
$$

Based on the results above we finally get

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \frac{\Delta x_{i}}{\Delta x_{\max }}=\sqrt{(i+1)^{2}+\mu^{2}}-\sqrt{i^{2}+\mu^{2}}, \quad i \geq 0 . \tag{3.35}
\end{equation*}
$$

Since $\Delta x_{0}=\Delta x_{\min }$ and using $\varepsilon=10^{-8}$, for example, we have $|\mu|=5.8635$ and therefore

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \frac{\Delta x_{\min }}{\Delta x_{\max }}=0.0847 \tag{3.36}
\end{equation*}
$$

Observing (3.35) we can conclude that the interpolating points are "almost" equally distributed as $n \rightarrow \infty$. For example, the number of points which satisfy

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} \frac{\Delta x_{i}}{\Delta x_{\max }}<0.9 \tag{3.37}
\end{equation*}
$$

is only 22 . For $n$ large, this number is negligible.
Let us look now at the behavior of the derivatives of the basis functions compared to the Chebyshev case (1.9). Upon using (2.3) and defining

$$
\begin{equation*}
\cos (\theta)=\frac{\sin (p x)}{\sin (p)} \tag{3.38}
\end{equation*}
$$

we have

$$
\begin{equation*}
\psi_{n}(x)=\cos n \theta \tag{3.39}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\frac{d \psi_{n}(x)}{d x}=\frac{d \psi_{n}}{d \theta} \frac{d \theta}{d x}=n \frac{p}{\sin (p)} \frac{\sin (n \theta)}{\sin (\theta)} \cos (p x) \tag{3.40}
\end{equation*}
$$

The maxima of the derivative is achieved at $x=1(\theta=0)$ and the minima(for $n$ odd) at $x=0\left(\theta=\frac{\pi}{2}\right)$. Since (3.14) then

$$
\begin{equation*}
\cos (p)=\frac{1-\varepsilon^{\frac{1}{n}}}{1+\varepsilon^{\frac{1}{n}}} \tag{3.41}
\end{equation*}
$$

and therefore, using l'Hopital's rule,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\max \left|\frac{d \psi_{n}(x)}{d x}\right|}{\min \left|\frac{d \psi_{n}(x)}{d x}\right|}=\frac{|\ln (\varepsilon)|}{2} \tag{3.42}
\end{equation*}
$$

Hence, the nonuniformity of the first derivative almost diminishes. In a similar way one can show almost uniformity for higher derivatives.
4. On Resolution. Let

$$
\begin{equation*}
f(x)=\sin (r \pi x) \quad-1 \leq x \leq 1 \tag{4.1}
\end{equation*}
$$

(similar analysis can be carried out for $f(x)=\cos (r \pi x)$ ).
By change of variables $y=\frac{\sin (p x)}{\sin (p)}$ we get

$$
\begin{equation*}
f(x)=\tilde{f}(y) \tag{4.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{f}(y)=\sin \left(\frac{r \pi}{p} \sin ^{-1}(y \sin (p))\right) \tag{4.3}
\end{equation*}
$$

Hence, resolving $f(x)$ by projection on subspace spanned by $\psi_{k}(x)$, where

$$
\begin{equation*}
\psi_{k}(x)=T_{k}\left(\frac{\sin (p x)}{\sin (p)}\right) \tag{4.4}
\end{equation*}
$$

is equivalent to resolving $\tilde{f}(y)$ by Chebyshev polynomials. Let $r$ be chosen such that

$$
\begin{equation*}
m=\frac{r \pi}{p} \tag{4.5}
\end{equation*}
$$

is an odd number. We will show now that

$$
\begin{equation*}
\tilde{f}(y)=(-1)^{m} T_{m}(\alpha y) \tag{4.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=\sin (p) . \tag{4.7}
\end{equation*}
$$

## Lemma 1:

Let $T_{k}$ be Chebyshev polynomial of degree $k$ then, for $k$ even, we have

$$
\begin{equation*}
\cos (k p x)=(-1)^{\frac{k}{2}} T_{k}(\sin (p x)) \tag{4.8}
\end{equation*}
$$

and for $k$ odd

$$
\begin{equation*}
\sin (k p x)=(-1)^{\frac{k-1}{2}} T_{k}(\sin (p x)) \tag{4.9}
\end{equation*}
$$

Proof:
The proof is by induction on $k$. It is easily verified for $k=0,1$.
Assume first that $k$ is even. By the recurrence relation of Chebyshev polynomials we get

$$
\begin{equation*}
T_{k+1}(\sin (p x))=2 \sin (p x) T_{k}(\sin (p x))-T_{k-1}(\sin (p x)) \tag{4.10}
\end{equation*}
$$

and by induction

$$
\begin{equation*}
T_{k+1}(\sin (p x))=(-1)^{\frac{k}{2}}(2 \sin (p x) \cos (k p x)+\sin ((k-1) p x)) \tag{4.11}
\end{equation*}
$$

Since

$$
\begin{equation*}
2 \sin (p x) \cos (k p x)=\sin ((k+1) p x)-\sin ((k-1) p x) \tag{4.12}
\end{equation*}
$$

we get

$$
\begin{equation*}
T_{k+1}(\sin (p x))=(-1)^{\frac{k}{2}} \sin ((k+1) p x) \tag{4.13}
\end{equation*}
$$

and the proof of the even case is concluded.
For $k$ odd, using the recurrence relation and induction we have

$$
\begin{equation*}
T_{k+1}(\sin (p x))=(-1)^{\frac{k-1}{2}}(2 \sin (p x) \sin (k p x)-\cos ((k-1) p x)) \tag{4.14}
\end{equation*}
$$

Since

$$
\begin{equation*}
2 \sin (p x) \sin (k p x)=\cos ((k-1) p x)-\cos ((k+1) p x) \tag{4.15}
\end{equation*}
$$

then

$$
\begin{equation*}
T_{k+1}(\sin (p x))=(-1)^{\frac{k+1}{2}} \cos ((k+1) p x) \tag{4.16}
\end{equation*}
$$

and the proof of the odd case is concluded.
Since $T_{m}(\alpha y)$ is polynomial of degree $m$ in $y$, it can be written as

$$
\begin{equation*}
T_{m}(\alpha y)=\sum_{k=0}^{m} a_{k}^{m} T_{k}(y) \tag{4.17}
\end{equation*}
$$

while

$$
\begin{equation*}
a_{k}^{m}=\frac{2}{\pi c_{k}} \int_{-1}^{1} \frac{T_{m}(\alpha y) T_{k}(y)}{\sqrt{1-y^{2}}} d y \quad c_{0}=2, c_{k}=1 \text { for } k \geq 1 \tag{4.18}
\end{equation*}
$$

Chebyshev polynomials satisfy the recurrence relation

$$
\begin{equation*}
T_{n+1}(x)=2 x T_{n}(x)-T_{n-1}(x) \tag{4.19}
\end{equation*}
$$

Hence
(4.20)

$$
a_{k}^{m}=\frac{2}{\pi c_{k}} \int_{-1}^{1} \frac{\left[2 \alpha y T_{m-1}(\alpha y)-T_{m-2}(\alpha y)\right] T_{k}(y)}{\sqrt{1-y^{2}}} d y \quad k \geq 0, m \geq 2
$$

Since (4.19) we have

$$
\begin{equation*}
2 y T_{k}(y)=T_{k+1}(y)+T_{k-1}(y) \tag{4.21}
\end{equation*}
$$

Substituting this relation in (4.20) we finally get that the coefficients satisfy

$$
\begin{gather*}
a_{0}^{0}=1, a_{0}^{1}=0, a_{1}^{1}=\alpha,  \tag{4.22}\\
a_{0}^{m}=\alpha a_{1}^{m-1}-a_{0}^{m-2}, \quad m \geq 2,  \tag{4.23}\\
a_{k}^{m}=\alpha\left(c_{k-1} a_{k-1}^{m-1}+a_{k+1}^{m-1}\right)-a_{k}^{m-2}, \quad 1 \leq k \leq m, \quad m \geq 2 . \tag{4.24}
\end{gather*}
$$

Carrying out numerical experiments we have observed that, while $k<\alpha m, a_{k}^{m}$ oscillates, and once $k \geq \alpha m,\left|a_{k}^{m}\right|$ monotonically decreases. Based on this numerical results we had conjectured, in [9], that the function $T_{m}(\alpha y)$ is resolved by $k$ terms where $k=\lceil\alpha m\rceil$ (it was proven later in [2]).

Hence, by using $\left\{\psi_{k}(x)\right\}_{k=0}^{n}$, the maximal $k$ is $n$ and therefore, since (4.5), we get

$$
\begin{equation*}
r<r_{\max } \tag{4.25}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{\max }=\frac{n p}{\pi \sin (p)} \tag{4.26}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\lim _{p \rightarrow \frac{\pi}{2}} r_{\max }=\frac{n}{2} \tag{4.27}
\end{equation*}
$$

which is exactly Nyquist criteria.
5. Numerical Integration. Approximating

$$
\begin{equation*}
I=\int_{-1}^{1} f(x) d x \tag{5.1}
\end{equation*}
$$

is an essential subject in numerical analysis. An highly accurate approach is Gaussian quadrature based on Legendre polynomials. In [6], the authors describe nonpolynomial algorithms which are aimed at overcoming the "waste" of factor $\frac{\pi}{2}$ typical to polynomial algorithms. A quadrature based on $\psi_{k}(x)$ is a member of this family of nonpolynomial algorithms.

Using quadrature based on

$$
\begin{equation*}
\psi_{k}(x)=P_{k}\left(\frac{\sin (p x)}{\sin (p)}\right) \tag{5.2}
\end{equation*}
$$

where $P_{k}$ is Legendre polynomial, is equivalent to doing first change of variables

$$
\begin{equation*}
y=\frac{\sin (p x)}{\sin (p)} \tag{5.3}
\end{equation*}
$$

which transforms (5.1) to

$$
\begin{equation*}
I=\int_{-1}^{1} \tilde{f}(y) d y \tag{5.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{f}(y)=\frac{\sin (p)}{p} \frac{f\left(\frac{1}{p} \sin ^{-1}(y \sin (p))\right)}{\sqrt{1-(y \sin (p))^{2}}} \tag{5.5}
\end{equation*}
$$

and approximating (5.4) by standard Legendre Gaussian quadrature. Hence the quadrature can be written as

$$
\begin{equation*}
\tilde{I}=\sum_{i=1}^{m} f\left(x_{i}\right) w_{i} \tag{5.6}
\end{equation*}
$$

where

$$
\begin{equation*}
x_{i}=\frac{1}{p} \sin ^{-1}\left(y_{i} \sin (p)\right) \tag{5.7}
\end{equation*}
$$

while $\left\{y_{i}\right\}_{i=1}^{m}$ are the zeros of Legendre polynomial of degree $m . \quad\left\{w_{i}\right\}_{i=1}^{m}$ are the weights defined as

$$
\begin{equation*}
w_{i}=\frac{\sin (p)}{p} \frac{\bar{w}_{i}}{\cos \left(p x_{i}\right)} \tag{5.8}
\end{equation*}
$$

while $\bar{w}_{i}$ are Legendre weights. Similarly one can write a quadrature which is related to Chebyshev polynomials or any other set of orthogonal polynomials.
6. Numerical Results. In the first set of examples we present results related to approximating functions. In this set we are comparing two algorithms :

1. Chebyshev
2. Nptp (Non periodic trigonometric polynomial).

The error presented in the tables is defined as

$$
\begin{equation*}
E r=\sqrt{\sum_{j=1}^{100}\left(f\left(y_{j}\right)-\tilde{f}\left(y_{j}\right)\right)^{2}} \tag{6.1}
\end{equation*}
$$

where $f$ is the exact function we are approximating, $\tilde{f}$ is the approximating function which results from using either Chebyshev polynomials or Nptp functions and

$$
\begin{equation*}
y_{j}=a+(j-1) \frac{b-a}{99}, \quad 1 \leq j \leq 100 \tag{6.2}
\end{equation*}
$$

are check points, equally distributed in the interval $[a, b]$.
For Nptp, the tables contain two sets of results. One set, Nptp1, is for the case where the parameter $p$ is computed according to (3.14)

$$
\begin{equation*}
p=2 \arctan \left(\varepsilon^{\frac{1}{n}}\right), \quad \varepsilon=10^{-15} \tag{6.3}
\end{equation*}
$$

and the second, Nptp2, is for the case where the parameter $p$ is computed adaptively by minimizing (3.15). The number in the brackets contains the parameter $p$ in each case.

## Example 1

In this example we approximated

$$
\begin{equation*}
f(x)=\frac{1}{2+\cos (40 x)}, \quad-1 \leq x \leq 1 \tag{6.4}
\end{equation*}
$$

This function behaves uniformly. The results are

| n | Er(Chebyshev) | Er(Nptp1) | Er(Nptp2) |
| :---: | :---: | :---: | :---: |
| 100 | $4.7562 \mathrm{e}-2$ | $1.5344 \mathrm{e}-2(1.232)$ | $1.4528 \mathrm{e}-2(1.433)$ |
| 200 | $2.2647 \mathrm{e}-3$ | $7.6117 \mathrm{e}-5(1.399)$ | $5.1861 \mathrm{e}-5(1.468)$ |
| 400 | $2.8352 \mathrm{e}-6$ | $7.9950 \mathrm{e}-9(1.485)$ | $7.8335 \mathrm{e}-9(1.390)$ |

Example 2 In this example we approximated the function

$$
\begin{equation*}
f(x)=x^{5} \cos (50 x), \quad-1 \leq x \leq 1 \tag{6.5}
\end{equation*}
$$

Due to the $x^{5}$ term, the function has large gradients close to the boundaries, nevertheless, the new algorithm outperforms Chebyshev approximation as can be seen by the results presented in the next table

| n | $\operatorname{Er}($ Chebyshev $)$ | $\operatorname{Er}($ Nptp1) | $\operatorname{Er}($ Nptp2) |
| :---: | :---: | :---: | :---: |
| 40 | $6.0165 \mathrm{e}-2$ | $3.5717 \mathrm{e}-2(0.840)$ | $3.7053 \mathrm{e}-2(0.852)$ |
| 50 | $3.7617 \mathrm{e}-3$ | $5.4146-4(0.967)$ | $2.0321 \mathrm{e}-6(1.086)$ |
| 60 | $1.6279 \mathrm{e}-4$ | $4.5186 \mathrm{e}-11(1.058)$ | $3.3845 \mathrm{e}-11(1.057)$ |

## Example 3

In this example we approximated the function

$$
\begin{equation*}
f(x)=e^{-30 x^{2}}, \quad-1 \leq x \leq 1 \tag{6.6}
\end{equation*}
$$

and the results are

| n | $\operatorname{Er}($ Chebyshev $)$ | $\operatorname{Er}(\mathrm{Nptp} 1)$ | $\operatorname{Er}(\mathrm{Nptp2)}$ |
| :---: | :---: | :---: | :---: |
| 10 | $4.8234 \mathrm{e}-1$ | $4.8138 \mathrm{e}-1(0.0796)$ | $1.3220 \mathrm{e}-1(1.5708)$ |
| 20 | $2.9417 \mathrm{e}-2$ | $2.4545 \mathrm{e}-2(0.3939)$ | $2.0958 \mathrm{e}-4(1.5708)$ |
| 40 | $2.9475 \mathrm{e}-6$ | $7.3752 \mathrm{e}-8(0.8402)$ | $4.5169 \mathrm{e}-14(1.5708)$ |

In this case, due to the fact that there is a large gradient in the center of the interval and that the function is almost 0 at the boundaries, the accuracy is significantly improved by choosing the optimal parameter (Nptp2) which, in this case, is very close to $\frac{\pi}{2}$.

## Example 4

In this example we approximated the function

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{1.1-x^{2}}}, \quad-1 \leq x \leq 1 \tag{6.7}
\end{equation*}
$$

and the results are

| n | $\operatorname{Er}($ Chebyshev $)$ | $\operatorname{Er}($ Nptp1) | $\operatorname{Er}($ Nptp2) |
| :---: | :---: | :---: | :---: |
| 20 | $2.5252 \mathrm{e}-3$ | $2.8448 \mathrm{e}-3(0.3939)$ | $2.5252 \mathrm{e}-3(0)$ |
| 40 | $3.7085 \mathrm{e}-6$ | $2.0681 \mathrm{e}-5(0.8402)$ | $3.7085 \mathrm{e}-6(0)$ |
| 80 | $9.7418 \mathrm{e}-12$ | $3.3488 \mathrm{e}-8(1.1783)$ | $9.7418 \mathrm{e}-12(0)$ |

In this case, the behavior of the function at the boundaries justifies interpolating at Chebyshev points. As expected, the minimization process resulted with $p=0$ which means choosing Chebyshev points.

## Example 5

In this example we would like to demonstrate the resolution properties of Nptp compared to Chebyshev. For this purpose we approximated the function

$$
\begin{equation*}
f(x)=\sin (100 \pi x)+\cos (100 \pi x), \quad-1 \leq x \leq 1 \tag{6.8}
\end{equation*}
$$

The results are presented in the next two tables.

| n | p | Er(Nptp1) |
| :---: | :---: | :---: |
| 220 | 1.4248 | $3.5265 \mathrm{e}-1$ |
| 240 | 1.4369 | $4.9448 \mathrm{e}-7$ |
| 260 | 1.4471 | $3.6805 \mathrm{e}-9$ |


| n | $\operatorname{Er}($ Chebyshev $)$ |
| :---: | :---: |
| 320 | $2.4532 \mathrm{e}-1$ |
| 340 | $2.6849 \mathrm{e}-4$ |
| 360 | $1.6117 \mathrm{e}-8$ |

In the next set of examples we use Nptp to approximate definite integrals

$$
\begin{equation*}
I(f)=\int_{-1}^{1} f(x) d x \tag{6.9}
\end{equation*}
$$

The tables below present the absolute value of the error while using 2 methods: Legendre and Nptp.

In the first table, the function is

$$
\begin{equation*}
f(x)=\frac{100 \cos (100 x)}{2+\sin (100 x)} \tag{6.10}
\end{equation*}
$$

The parameter $p$ is computed by (3.15) with $\varepsilon=1 . e^{-5}$.

| n | ErLegendre | ErNptp |
| :---: | :---: | :---: |
| 200 | $6.2532 \mathrm{e}-2$ | $1.0331 \mathrm{e}-3$ |
| 300 | $4.5825 \mathrm{e}-3$ | $3.7822 \mathrm{e}-6$ |
| 500 | $1.2392 \mathrm{e}-5$ | $1.8049 \mathrm{e}-9$ |

In the next table the function is

$$
\begin{equation*}
f(x)=\cos (500 x) \tag{6.11}
\end{equation*}
$$

and $p$ is computed by (3.15) with $\varepsilon=1 . e^{-15}$.

| n | ErLegendre | ErNptp |
| :---: | :---: | :---: |
| 180 | $1.9069 \mathrm{e}-1$ | $1.8320 \mathrm{e}-2$ |
| 190 | $7.3531 \mathrm{e}-2$ | $1.6238 \mathrm{e}-11$ |
| 200 | $2.2017 \mathrm{e}-1$ | $2.0517 \mathrm{e}-14$ |
| 250 | $3.1385 \mathrm{e}-1$ | $3.0422 \mathrm{e}-14$ |
| 270 | $3.0560 \mathrm{e}-6$ | $1.0923 \mathrm{e}-14$ |
| 290 | $7.3459 \mathrm{e}-15$ | $8.1304 \mathrm{e}-15$ |

Observe that, while in the Nptp case 200 points were enough to recover the solution with machine accuracy, in the standard Legendre quadrature we needed almost $50 \%$ more points in order to get machine accuracy.

Conclusions: We have presented in this paper a new set of basis functions which can be used for approximating general, smooth function defined on a real interval $[a, b]$. The new space is spanned by powers of trigonometric functions instead of powers of $x$ as in the regular polynomial case. The trigonometric functions depend on a parameter $p$ which is a function of the dimension of the approximating subspace. When one fixes $p$ to be zero he gets polynomials. Hence, polynomials can be considered as a singular member of the family where $p$ is fixed and equal to zero. As described in the paper and verified by numerical experiments, the parameter $p$ should approaches the other extreme value, $\frac{\pi}{2}$, as the dimension increases. Besides exponential accuracy, the approximating function can be computed efficiently using FFT. Approximating a function by the new set is equivalent to approximating a transformed function by polynomials. Hence, the vast literature related to polynomials can be used for analyzing algorithms which make use of the proposed set of functions.

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